A note on signed and minus domination in graphs*

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Abstract

In this paper, we give upper bounds on the upper signed domination number of $[l,k]$ graphs, which generalize some results obtained in other papers. Further, good lower bounds are established for the minus k-subdomination number γ_{ks}^{-101} and signed k-subdomination number γ_{ks}^{-11}.

1. Introduction

For a graph $G=(V,E)$ and $M \subset V$, we let $d_M(v)=\{u \in M : uv \in E(G)\}$. Let $l \le k$ be two positive integers. If $l \le d(v) \le k$ for all $v \in V$, then we call G an $[l,k]$ graph. If $d(v)=k-1$ or k for all $v \in V$, then we call G a nearly k-regular graph. For $A, B \subset V$, and $A \cap B = \emptyset$, let $e(A,B)=|\{xy \in E(G) : x \in A, y \in B\}|$.

For any real-valued function $f : V \to R$ and $S \subseteq V$, let $f(S)=\sum_{u \in S} f(u)$ and $f[v]=f(N[v])$, where $N[v]$ is the closed neighborhood of v. The weight of f is defined as $f(V)$. A dominating function $g : V \to R$ is a minimal dominating function if every dominating function h satisfies $g(v) \le h(v)$ for every $v \in V$. A signed dominating function of G is a function $g : V \to \{-1,1\}$ such that for every $v \in V$, $f[v] \ge 1$. The upper signed domination number of G is $\Gamma_s(G)=\max\{f(V) : f$ is a minimal signed dominating function on $G\}$. Many results on signed domination in graphs have been presented by various authors ([3],[5],[7–9]).

Let k be a positive integer such that $1 \le k \le |V|$. A minus k-subdominating function is a function $f : V \to \{-1,0,1\}$ such that the closed neighborhood sum $f(N[v]) \ge 1$ for at least k vertices of G. The minus k-subdomination number, denoted by γ_{ks}^{-101}, is equal to $\min\{f(V) : f$ is a minus k-subdominating function on $G\}$. A signed k-subdominating function is a function $f : V \to \{-1,1\}$ such that the closed neighborhood sum $f(N[v]) \ge 1$ for at least k vertices of G. The signed k-subdomination number, denoted by γ_{ks}^{-11}, is equal to $\min\{f(V) : f$ is a signed k-subdominating function on $G\}$. A majority dominating function is defined in [1]

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as a function $f : V \to \{-1, 1\}$ such that $f[v] \geq 1$ for at least half the vertices $v \in V$. The majority domination number, denoted by $\gamma_{maj}(G)$, is equal to $\min\{f(V) : f$ is a majority dominating function on $G\}$. For other terminology we follow [5].

Paper [1] deals exclusively with majority domination in graphs. Hattingh et al. [4] provided the exact value of the minus k-subdomination number of cycles and a lower bound of minus k-subdomination number of trees. Cockayne et al. [2] got the exact value or lower bound of signed k-subdomination domination number of paths and trees.

Theorem 1 [1] *If $n \geq 2$ is an integer and $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-101}(P_n) = \lceil \frac{n}{3} \rceil + k - n + 1.$$

Theorem 2 [4] *If $n \geq 3$ is an integer and $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-101}(C_n) = \begin{cases} \lceil \frac{(n-2)}{3} \rceil & \text{if} \quad k = n-1 \quad \text{and} \quad (k = 0 \quad \text{or} \quad k = 1 (mod\ 3)), \\ 2\lfloor \frac{2k+4}{3} \rfloor - n & \text{otherwise.} \end{cases}$$

Theorem 3 [4] *If T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-101}(T) \geq k - n + 2.$$

Theorem 4 [2] *For $n \geq 2$ and $1 \leq k \leq n$,*

$$\gamma_{ks}^{-11}(P_n) = 2\lfloor \frac{2k+4}{3} \rfloor - n.$$

Theorem 5 [2] *If T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n$, then*

$$\gamma_{ks}^{-11}(T) \geq 2\lfloor \frac{2k+4}{3} \rfloor - n,$$

with equality for $T = P_n$.

2. Upper bound on Γ_s

Theorem 6 *Let $2 \leq l \leq k+1$. If G is an $[l, k+1]$ graph of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \dfrac{k^2+5k-l+4}{k^2+5k+l+4} n & \text{if } k \text{ is even and } l \text{ is even,} \\ \dfrac{k^2+5k-l+5}{k^2+5k+l+3} n & \text{if } k \text{ is even and } l \text{ is odd,} \\ \dfrac{k^2+4k-l+3}{k^2+4k+l+3} n & \text{if } k \text{ is odd and } l \text{ is even,} \\ \dfrac{k^2+4k-l+4}{k^2+4k+l+2} n & \text{if } k \text{ is odd and } l \text{ is odd.} \end{cases}$$

Proof Let g be a minimal signed dominating function of weight $g(V(G)) = \Gamma_s(G)$. Let $M = \{x \in V : g(x) = -1\}, P = \{x \in V : g(x) = 1\}$. For $l \le i \le k+1$, denote $H_i = \{v \in V : d(v) = i\}$ and $|M \cap H_i| = u_i$. Clearly, if $v \in P \cap H_i$, then $g[v] = i + 1 - 2d_M(v)$ and $d_M(v) \le \lfloor \frac{i}{2} \rfloor$. Let $A_{i,j} = \{v \in P \cap H_i : d_M(v) = j\}$ and $|A_{i,j}| = a_{i,j}$. Obviously,

$$n = \sum_{i=l}^{k+1} |M \cap H_i| + \sum_{i=l}^{k+1} |P \cap H_i| = \sum_{i=l}^{k+1} u_i + \sum_{i=l}^{k+1} \sum_{j=0}^{\lfloor i/2 \rfloor} a_{i,j}. \tag{1}$$

And

$$e(M, P) \le \sum_{i=l}^{k+1} i u_i. \tag{2}$$

Hence,

$$\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} j a_{i,j} \le \sum_{i=l}^{k+1} i u_i. \tag{3}$$

On the other hand, since g is minimal, for every vertex $v \in \bigcup_{i=l}^{k+1} A_{i,0}$, there is a vertex $x \in N[v]$ with $g[x] = 1$ or 2. Since $N(v) \cap M = \emptyset$, and $g[v_j] = i + 1 - 2j$ for every $v_j \in A_{i,j}$, we have

$$e(\bigcup_{i=l}^{k+1} A_{i,0}, \bigcup_{i=l}^{k+1} A_{i,\lfloor \frac{i}{2} \rfloor}) \ge \sum_{i=l}^{k+1} a_{i,0}. \tag{4}$$

Every vertex $v \in A_{i,\lfloor \frac{i}{2} \rfloor}$ has at most $\lceil \frac{i}{2} \rceil$ neighbors in $\bigcup_{i=l}^{k+1} A_{i,0}$. We deduce that

$$e(\bigcup_{i=l}^{k+1} A_{i,0}, \bigcup_{i=l}^{k+1} A_{i,\lfloor \frac{i}{2} \rfloor}) \le \sum_{i=l}^{k+1} \lceil \frac{i}{2} \rceil a_{i,\lfloor \frac{i}{2} \rfloor}. \tag{5}$$

Combining (1)–(5), we find that

$$\begin{aligned} n &= \sum_{i=l}^{k+1} u_i + \sum_{i=l}^{k+1} a_{i,0} + \sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} a_{i,j} \\ &\le \sum_{i=l}^{k+1} u_i + \sum_{i=l}^{k+1} \lceil \frac{i}{2} \rceil a_{i,\lfloor \frac{i}{2} \rfloor} + \sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} a_{i,j} \\ &= \sum_{i=l}^{k+1} u_i + \sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} a_{i,j} + \sum_{i=l}^{k+1} (\lceil \frac{i}{2} \rceil + 1) a_{i,\lfloor \frac{i}{2} \rfloor}. \end{aligned} \tag{6}$$

Case 1 *k, l is even.*

For $l \le i \le k+1$, it is easy to show that

$$\lceil \frac{i}{2} \rceil + 1 \le \frac{k+4}{l} \lfloor \frac{i}{2} \rfloor.$$

Thus by (6), we have

$$\begin{aligned} n \le\ & \sum_{i=l}^{k+1} u_i + \frac{k+4}{l}\Big(\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} a_{i,j} + \sum_{i=l}^{k+1} \lfloor \frac{i}{2} \rfloor a_{i,\lfloor \frac{i}{2} \rfloor}\Big) \\ \le\ & \sum_{i=l}^{k+1} u_i + \frac{k+4}{l}\Big(\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} j a_{i,j} + \sum_{i=l}^{k+1} \lfloor \frac{i}{2} \rfloor a_{i,\lfloor \frac{i}{2} \rfloor}\Big) \\ =\ & \sum_{i=1}^{k+1} u_i + \frac{k+4}{l} \sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} j a_{i,j} \\ \le\ & \sum_{i=l}^{k+1} u_i + \frac{k+4}{l} \sum_{i=l}^{k+1} i u_i \qquad (by \quad (3)) \\ \le\ & \frac{k^2+5k+l+4}{l} \sum_{i=l}^{k+1} u_i, \end{aligned}$$

which gives

$$\sum_{i=l}^{k+1} u_i \ge \frac{l}{k^2+5k+l+4} n,$$

and

$$\begin{aligned} \Gamma_s(G) \quad &= \quad n - 2\sum_{i=l}^{k+1} u_i \\ &\le \quad \frac{k^2+5k-l+4}{k^2+5k+l+4} n. \end{aligned}$$

Case 2 *k is even and l is odd.*

For $l \le i \le k+1$, it is easy to show that

$$\lceil \frac{i}{2} \rceil + 1 \le \frac{k+4}{l-1} \lfloor \frac{i}{2} \rfloor.$$

Thus by (6), we have

$$n \le \sum_{i=l}^{k+1} u_i + \frac{k+4}{l-1}\Big(\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} a_{i,j} + \sum_{i=l}^{k+1} \lfloor \frac{i}{2} \rfloor a_{i,\lfloor \frac{i}{2} \rfloor}\Big)$$

$$
\begin{aligned}
&\leq \sum_{i=l}^{k+1} u_i + \frac{k+4}{l-1}\left(\sum_{i=l}^{k+1}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor-1} ja_{i,j} + \sum_{i=l}^{k+1}\lfloor\frac{i}{2}\rfloor a_{i,\lfloor\frac{i}{2}\rfloor}\right)\\
&= \sum_{i=1}^{k+1} u_i + \frac{k+4}{l-1}\sum_{i=l}^{k+1}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor} ja_{i,j}\\
&\leq \sum_{i=l}^{k+1} u_i + \frac{k+4}{l-1}\sum_{i=l}^{k+1} iu_i \qquad (by \quad (3))\\
&\leq \frac{k^2+5k+l+3}{l-1}\sum_{i=l}^{k+1} u_i,
\end{aligned}
$$

which gives

$$\sum_{i=l}^{k+1} u_i \geq \frac{l-1}{k^2+5k+l+3}\, n,$$

and

$$
\begin{aligned}
\Gamma_s(G) &= n - 2\sum_{i=l}^{k+1} u_i\\
&\leq \frac{k^2+5k-l+5}{k^2+5k+l+3}\, n.
\end{aligned}
$$

Case 3 *k is odd and l is even.*

For $l \leq i \leq k+1$, it is easy to show that

$$\lceil\frac{i}{2}\rceil + 1 \leq \frac{k+3}{l}\lfloor\frac{i}{2}\rfloor.$$

Thus by (6), we have

$$
\begin{aligned}
n &\leq \sum_{i=l}^{k+1} u_i + \frac{k+3}{l}\left(\sum_{i=l}^{k+1}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor-1} a_{i,j} + \sum_{i=l}^{k+1}\lfloor\frac{i}{2}\rfloor a_{i,\lfloor\frac{i}{2}\rfloor}\right)\\
&\leq \sum_{i=l}^{k+1} u_i + \frac{k+3}{l}\left(\sum_{i=l}^{k+1}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor-1} ja_{i,j} + \sum_{i=l}^{k+1}\lfloor\frac{i}{2}\rfloor a_{i,\lfloor\frac{i}{2}\rfloor}\right)\\
&= \sum_{i=1}^{k+1} u_i + \frac{k+3}{l}\sum_{i=l}^{k+1}\sum_{j=1}^{\lfloor\frac{i}{2}\rfloor} ja_{i,j}\\
&\leq \sum_{i=l}^{k+1} u_i + \frac{k+3}{l}\sum_{i=l}^{k+1} iu_i \qquad (by \quad (3))\\
&\leq \frac{k^2+4k+l+3}{l}\sum_{i=l}^{k+1} u_i,
\end{aligned}
$$

which gives

$$\sum_{i=l}^{k+1} u_i \geq \frac{l}{k^2+4k+l+3}\, n,$$

and

$$\begin{aligned} \Gamma_s(G) &= n - 2\sum_{i=l}^{k+1} u_i \\ &\leq \frac{k^2+4k-l+3}{k^2+4k+l+3}\, n. \end{aligned}$$

Case 4 *k is odd and l is odd.*

For $l \leq i \leq k+1$, it is easy to show that

$$\lceil \frac{i}{2} \rceil + 1 \leq \frac{k+3}{l-1} \lfloor \frac{i}{2} \rfloor.$$

Thus by (6), we have

$$\begin{aligned} n \leq & \sum_{i=l}^{k+1} u_i + \frac{k+3}{l-1}\Big(\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} a_{i,j} + \sum_{i=l}^{k+1} \lfloor \frac{i}{2} \rfloor a_{i,\lfloor \frac{i}{2} \rfloor}\Big) \\ \leq & \sum_{i=l}^{k+1} u_i + \frac{k+3}{l-1}\Big(\sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor - 1} j a_{i,j} + \sum_{i=l}^{k+1} \lfloor \frac{i}{2} \rfloor a_{i,\lfloor \frac{i}{2} \rfloor}\Big) \\ = & \sum_{i=1}^{k+1} u_i + \frac{k+3}{l-1} \sum_{i=l}^{k+1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} j a_{i,j} \\ \leq & \sum_{i=l}^{k+1} u_i + \frac{k+3}{l-1} \sum_{i=l}^{k+1} i u_i \qquad (by \quad (3)) \\ \leq & \frac{k^2+4k+l+2}{l-1} \sum_{i=l}^{k+1} u_i, \end{aligned}$$

which gives

$$\sum_{i=l}^{k+1} u_i \geq \frac{l-1}{k^2+4k+l+2} n,$$

and

$$\begin{aligned} \Gamma_s(G) &= n - 2\sum_{i=l}^{k+1} u_i \\ &\leq \frac{k^2+4k-l+4}{k^2+4k+l+2}\, n. \end{aligned}$$

This completes the proof of Theorem 1. ∎

Corollary 1 [7] *If G is a nearly $(k+1)$-regular graph of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \frac{(k+2)^2}{k^2+6k+4}\, n & \text{if } k \text{ is even,} \\ \frac{k^2+3k+4}{k^2+5k+2} & \text{if } k \text{ is odd.} \end{cases}$$

Corollary 2 *If G is a graph with $\delta(G) \geq 2$, then*

$$\Gamma_s(G) \leq \begin{cases} \frac{\Delta^2+3\Delta-\delta}{\Delta^2+3\Delta+\delta}\, n & \text{if } \delta \text{ is even and } \Delta \text{ is odd,} \\ \frac{\Delta^2+3\Delta-\delta+1}{\Delta^2+3\Delta+\delta-1}\, n & \text{if } \delta \text{ is odd and } \Delta \text{ is odd,} \\ \frac{\Delta^2+2\Delta-\delta}{\Delta^2+2\Delta+\delta}\, n & \text{if } \delta \text{ is even and } \Delta \text{ is even,} \\ \frac{\Delta^2+2\Delta-\delta+1}{\Delta^2+2\Delta+\delta-1}\, n & \text{if } \delta \text{ is odd and } \Delta \text{ is even.} \end{cases}$$

Corollary 3 [3] *If G is a k-regular graph, $k \geq 1$, of order n, then*

$$\Gamma_s(G) \leq \begin{cases} \frac{k+1}{k+3} n & \text{if } k \text{ is even,} \\ \frac{(k+1)^2}{k^2+4k-1}\, n & \text{if } k \text{ is odd.} \end{cases}$$

3. Lower bounds on γ_{ks}^{-101} and γ_{ks}^{-11}

Theorem 7 *If G is a graph of order n and size ϵ, then*

$$\gamma_{ks}^{-101}(G) \geq \frac{(\delta - \Delta - 1)n + (\Delta + 2)k - 2\epsilon}{\delta + 1}.$$

Proof Let g be a minus k-subdominating function on G such that $g(V) = \gamma_{ks}^{-101}(G)$ and

$$\begin{aligned} P &= \{v \in V \mid g(v) = 1\}, \\ M &= \{v \in V \mid g(v) = -1\}, \\ Q &= \{v \in V \mid g(v) = 0\}. \end{aligned}$$

Further, we let

$$\begin{aligned} P_1 &= \{v \in P | g[v] \geq 1\} \\ P_2 &= \{v \in P | g[v] < 1\} \\ M_1 &= \{v \in M | g[v] \geq 1\} \\ M_2 &= \{v \in M | g[v] < 1\} \\ Q_1 &= \{v \in Q | g[v] \geq 1\} \\ Q_2 &= \{v \in Q | g[v] < 1\} \\ V_1 &= P_1 \cup M_1 \cup Q_1 \\ V_2 &= P_2 \cup M_2 \cup Q_2. \end{aligned}$$

Let $t(v)$ denote the number of vertices of weight 0 in $N(v)$. And let $p_i = |P_i|$, $m_i = |M_i|$ and $q_i = |Q_i|$ for $i = 1, 2$. Put $p = |P|$ and $m = |M|$. Then we have

$$|N(v) \cap M| \leq \begin{cases} \frac{d(v)-t(v)}{2} & \text{if} \quad v \in P_1, \\ \frac{d(v)-t(v)}{2} - 1 & \text{if} \quad v \in M_1, \\ \frac{d(v)-t(v)-1}{2} & \text{if} \quad v \in Q_1, \\ d(v) - t(v) & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} \sum_{v \in M} d(v) &\leq \sum_{v \in P_1} \frac{d(v) - t(v)}{2} + \sum_{v \in M_1} (\frac{d(v) - t(v)}{2} - 1) + \sum_{v \in Q_1} \frac{d(v) - t(v) - 1}{2} \\ &\quad + \sum_{v \in P_2} (d(v) - t(v)) + \sum_{v \in M_2} (d(v) - t(v)) + \sum_{v \in Q_2} (d(v) - t(v)) \\ &= \frac{1}{2} \sum_{v \in V} d(v) - \frac{1}{2} \sum_{v \in V} t(v) + \frac{1}{2} \sum_{v \in V_2} (d(v) - t(v)) - m_1 - \frac{1}{2} q_1 \\ &\leq \frac{1}{2} \sum_{v \in V} d(v) - \frac{1}{2} \sum_{v \in V} t(v) + \frac{1}{2} \sum_{v \in V_2} d(v) - m_1 - \frac{1}{2} q_1. \end{aligned}$$

Noting that $\sum_{v \in V} t(v) = \sum_{v \in Q} d(v) \geq \delta q$ and $\sum_{v \in M} d(v) \geq \delta m$, we have

$$\begin{aligned} \delta m &\leq \epsilon - \frac{1}{2} \delta q + \frac{1}{2} \Delta (p_2 + m_2 + q_2) - m_1 - \frac{1}{2} q_1 \\ &= \epsilon - m - \frac{1}{2} \delta q - \frac{1}{2} q + \frac{1}{2} \Delta (p_2 + m_2 + q_2) + \frac{1}{2} q_2 + m_2 \\ &\leq \epsilon - m - \frac{1}{2} \delta q - \frac{1}{2} q + \frac{\Delta + 2}{2} (p_2 + m_2 + q_2). \end{aligned} \tag{7}$$

Since g is a minus k-subdominating function, we have

$$p_2 + m_2 + q_2 \leq n - k. \tag{8}$$

Combining (7) and (8) we have

$$2m + q \leq \frac{2\epsilon + (\Delta + 2)(n - k)}{\delta + 1}.$$

Therefore

$$\gamma_{ks}^{-101}(G) = n - (2m + q) \geq \frac{(\delta - \Delta - 1)n + (\Delta + 2)k - 2\epsilon}{\delta + 1}.$$

∎

Then, since $\gamma_{ks}^{-101} \leq \gamma_{ks}^{-11}(G)$ for all graphs G, as an immediate corollary of Theorem 7, we have:

Corollary 4 *If G is a graph of order n and size ϵ, then*

$$\gamma_{ks}^{-11}(G) \geq \frac{(\delta - \Delta - 1)n + (\Delta + 2)k - 2\epsilon}{\delta + 1}.$$

In the special case when $k \geq n/2$, we have

Corollary 5 [6] *If G is a graph of order n and size ϵ, then*

$$\gamma_{maj}(G) \geq \frac{n(2\delta - \Delta) - 4\epsilon}{2(\delta + 1)}.$$

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