Combining Topological and Directional Information for Spatial Reasoning *

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Abstract

Current research on qualitative spatial representation and reasoning usually focuses on one single aspect of space. However, in real world applications, several aspects are often involved together. This paper extends the well-known RCC8 constraint language to deal with both topological and directional information, and then investigates the interaction between the two kinds of information. Given a topological (RCC8) constraint network and a directional constraint network, we ask when the joint network is satisfiable. We show that when the topological network is over one of the three maximal tractable subclasses of RCC8, the problem can be reduced into satisfiability problems in the RCC8 algebra and the rectangle algebra (RA). Therefore, reasoning techniques developed for RCC8 and RA can be used to solve the satisfiability problem of a joint network.

1 Introduction

Originating from Allen's work on temporal interval algebra (IA) [Allen, 1983], the qualitative approach to temporal as well as spatial information is popular in Artificial Intelligence and related research fields. This is mainly because in many applications precise numerical information is usually unavailable or not necessary.

While Allen's interval algebra is the principal formalism of qualitative temporal reasoning, there are more than a dozen of formalisms that deal with different aspects of space in qualitative spatial reasoning (QSR). Spatial relations are usually classified as *topological*, *directional*, and *metric*. Metric relations have a nature of semi-quantitative and fuzziness. In this paper, we are concerned with topological and directional relations between plane regions.

Most earlier research on topological and directional relations focuses on one single aspect. The most influential formalism for topological relations is the Region Connection Calculus (RCC) [Randell *et al.*, 1992]. As for directional relations, there are several well-known formalisms, e.g. [Frank, 1991; Goyal and Egenhofer, 2001].

In natural language and many practical applications, topological and directional relations are used together. For example, when describing the location of Titisee, a famous tourist sight in Germany, we might say "Titisee is *in* the Black Forest and is *east* of the town of Freiburg."

In this paper we extend the RCC8 constraint language to deal with topological as well as directional information. We first formalize the four cardinal directional relations between plane regions, viz. *west*, *east*, *north*, *south*, and then define nine basic relations by using the usual relational operations of intersection and complementation.

An important reasoning problems is to decide when a network of topological and directional constraints is *satisfiable* (or *consistent*). Given a network of topological (RCC8) constraints Θ and a network of directional constraints Δ, we try to decide when the joint network $\Theta \uplus \Delta$ is satisfiable.

Since topological and directional information is not independent, $\Theta \uplus \Delta$ may be unsatisfiable despite that both Θ and Δ are satisfiable. Our main result states that, if topological constraints are all in one of the three maximal tractable subclasses of RCC8 [Renz, 1999], then the satisfiability of the joint network can be determined by considering the satisfiability of two related networks in, resp., RCC8 and the rectangle algebra (RA), where RA is the two-dimensional counterpart of IA [Balbiani *et al.*, 1999].

The rest of this paper proceeds as follows. Section 2 introduces basic notions and well-known examples of qualitative calculi. The cardinal direction calculus DIR9 is introduced in Section 3. Section 4 describes and proves the main result. A new subclass, $\mathcal{A}_{\max}$, of IA is identified, which is closed under converse, intersection, and composition. We show *satisfiable* interval networks over $\mathcal{A}_{\max}$ has a maximal instantiation in the sense of [Ligozat, 1994]. Section 5 concludes the paper.

2 Qualitative calculi

The establishment of a proper qualitative calculus is the key to the success of the qualitative approach to temporal and spatial reasoning. This section introduces basic notions and examples of qualitative calculi (cf. [Ligozat and Renz, 2004]).

*This work was partly supported by the Alexander von Humboldt Foundation, the National Natural Science Foundation of China (60305005, 60673105, 60321002), and a Microsoft Research Professorship.

2.1 Basic notions

Let $\mathbb{U}$ be the universe of temporal/spatial/spatial-temporal entities, and set **Rel**($\mathbb{U}$) to be the set of binary relations on $\mathbb{U}$. With the usual relational operations of intersection, union, and complement, **Rel**($\mathbb{U}$) is a Boolean algebra.

A finite set $\mathcal{B}$ of nonempty relations on $\mathbb{U}$ is *jointly exhaustive and pairwise disjoint* (JEPD for short) if any two entities in $\mathbb{U}$ are related by one and only one relations in $\mathcal{B}$. Write $\langle\mathcal{B}\rangle$ for the subalgebra of **Rel**($\mathbb{U}$) generated by $\mathcal{B}$. Clearly, relations in $\mathcal{B}$ are atoms in the algebra $\langle\mathcal{B}\rangle$. We call $\langle\mathcal{B}\rangle$ a *qualitative calculus* on $\mathbb{U}$, and call relations in $\mathcal{B}$ *basic* relations of the calculus.

There are several general (but optional) restrictions on the choice of $\mathcal{B}$. Write $id_{\mathbb{U}}$ for the identity relation on $\mathbb{U}$. For a relation $\alpha \in \mathbf{Rel}(\mathbb{U})$, write $\alpha^\sim$ for the converse of α, which is defined as $(x, y) \in \alpha^\sim$ iff $(y, x) \in \alpha$. Many qualitative calculi require the identity relation $id_{\mathbb{U}}$ to be a basic relation. In this paper, we relax this restriction, and require that (i) $id_{\mathbb{U}}$ is contained in one basic relation; and (ii) $\mathcal{B}$ is closed under converse, i.e. if α is a basic relation, then so is $\alpha^\sim$.

Note that the composition of two relations in $\langle\mathcal{B}\rangle$ is not necessarily in $\langle\mathcal{B}\rangle$. For $\alpha, \beta \in \langle\mathcal{B}\rangle$, the *weak composition* [Ligozat and Renz, 2004] of α and β, written as $\alpha \circ_w \beta$, is defined to be the smallest relation in $\langle\mathcal{B}\rangle$ which contains $\alpha \circ \beta$.

An important reasoning problem in a qualitative calculus is to determine the *satisfiability* or *consistency* of a network Γ of constraints of the form $x\gamma y$, where γ is a relation in $\langle\mathcal{B}\rangle$. A constraint network $\Gamma = \{v_i\gamma_{ij}v_j\}_{i,j=1}^n$ is *satisfiable* (or *consistent*) if there is an instantiation $\{a_i\}_{i=1}^n$ in $\mathbb{U}$ such that $(a_i, a_j) \in \gamma_{ij}$ holds for all $1 \leq i, j \leq n$.

Consistency of a network can be approximated by using an $O(n^3)$ time path-consistency algorithm (PCA). A network $\Gamma = \{v_i\gamma_{ij}v_j\}_{i,j=1}^n$ is *path-consistent* if (i) γ_{ii} is the basic relation in $\mathcal{B}$ that contains $id_{\mathbb{U}}$; (ii) $\varnothing \neq \gamma_{ij} = \gamma_{ji}{}^\sim$; and (iii) $\gamma_{ij} \subseteq \gamma_{ik} \circ_w \gamma_{kj}$ for all i, j, k.

The essence of a PCA is to apply the following rules for any three i, j, k until the network is stable.

$$\gamma_{ij} \leftarrow \gamma_{ij} \cap \gamma_{ji}{}^\sim \quad (1)$$

$$\gamma_{ij} \leftarrow \gamma_{ij} \cap \gamma_{ik} \circ_w \gamma_{kj} \quad (2)$$

If the empty relation occurs during the process, then the network is inconsistent, otherwise the resulting network is path-consistent.

2.2 Interval algebra and rectangle algebra

The interval algebra IA [Allen, 1983] is generated by a set $\mathcal{B}_{int}$ of 13 basic relations between time intervals (see Table 1). IA is closed under composition, i.e. the composition of any two interval relations is in IA. Nebel and Bürckert [Nebel and Bürckert, 1995] identified a maximal tractable subclass $\mathcal{H}$ of IA, called *ORD-Horn* subclass, and showed that applying PCA is sufficient for deciding satisfiability for $\mathcal{H}$.

Ligozat [Ligozat, 1994] introduced a partial order $\preceq$ on $\mathcal{B}_{int}$ (see Fig. 1), and termed $(\mathcal{B}_{int}, \preceq)$ the *interval lattice*. For $\alpha, \beta \in \mathcal{B}_{int}$, we write $[\alpha, \beta]$ for the union of relations between α and β. Note that $[\alpha, \beta]$ is nonempty iff $\alpha \preceq \beta$. For example, $[\mathsf{o}, \mathsf{oi}]$ is the relation $\cup\{\mathsf{o}, \mathsf{s}, \mathsf{f}, \mathsf{d}, \mathsf{eq}, \mathsf{di}, \mathsf{fi}, \mathsf{si}, \mathsf{oi}\}$. An interval relation is called *convex* if it is of the form $[\alpha, \beta]$

Table 1: The set of basic interval relations $\mathcal{B}_{int}$, where $x = [x^-, x^+], y = [y^-, y^+]$ are two intervals.

Relation	Symb.	Conv.	Meaning	Dim
precedes	p	pi	$x^+ < y^-$	2
meets	m	mi	$x^+ = y^-$	1
overlaps	o	oi	$x^- < y^+ < x^+ < y^+$	2
starts	s	si	$x^- = y^- < x^+ < y^+$	1
during	d	di	$x^- < y^- < y^+ < x^+$	2
finishes	f	fi	$y^- < x^- < x^+ = y^+$	1
equals	eq	eq	$x^- = y^- < x^+ = y^+$	0

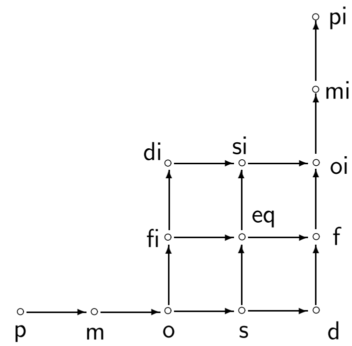

Figure 1: The interval lattice $(\mathcal{B}_{int}, \preceq)$.

with $\alpha, \beta \in \mathcal{B}_{int}$. For $\alpha \in \mathcal{B}_{int}$, $\dim(\alpha)$, the dimension of α, is defined [Ligozat, 1994] (see Table 1). The dimension of a non-basic relation is the maximal dimension of its basic relations.

The following convex relations are of particular importance in this paper:

$$⋒ = \cup\{\mathsf{m}, \mathsf{o}, \mathsf{s}, \mathsf{f}, \mathsf{d}, \mathsf{eq}, \mathsf{di}, \mathsf{fi}, \mathsf{si}, \mathsf{oi}, \mathsf{mi}\} \quad (3)$$

$$\Subset = \cup\{\mathsf{s}, \mathsf{d}, \mathsf{eq}, \mathsf{f}\} \quad (4)$$

$$\Supset = \cup\{\mathsf{fi}, \mathsf{di}, \mathsf{eq}, \mathsf{si}\} \quad (5)$$

The rectangle algebra (RA) is the two-dimensional counterpart of IA. RA is generated by a set of 169 JEPD relations between rectangles.[1] Write $\mathcal{B}_{rec}$ for this set, i.e.

$$\mathcal{B}_{rec} = \{\alpha \otimes \beta : \alpha, \beta \in \mathcal{B}_{int}\} \quad (6)$$

For a rectangle r, write $I_x(r)$ and $I_y(r)$ as, resp., the x- and y-projection of r. The basic rectangle relation between two rectangles a, b is $\alpha \otimes \beta$ iff $(I_x(a), I_x(b)) \in \alpha$ and $(I_y(a), I_y(b)) \in \beta$. The dimension of a basic rectangle relation $\alpha \otimes \beta$ is defined as $\dim(\alpha \otimes \beta) = \dim(\alpha) + \dim(\beta)$. The dimension of a non-basic rectangle relation is defined to be the maximal dimension of its basic relations.

Note that if $\mathcal{S}$ is a tractable subclass of IA, then $\mathcal{S} \otimes \mathcal{S} = \{\alpha \otimes \beta : \alpha, \beta \in \mathcal{S}\}$ is also tractable in RA. A tractable subclass of RA larger than $\mathcal{H} \otimes \mathcal{H}$ is also obtained in [Balbiani *et al.*, 1999], where $\mathcal{H}$ is the ORD-Horn subclass of IA.

[1]In this paper we always assume that the two sides of a rectangle are parallel to the axes of some predefined orthogonal basis in the Euclidean plane.

Table 2: The set of RCC8 basic relations $\mathcal{B}_{top}$, where a, b are two plane regions and a° and b° are, resp., their interiors.

Relation	Symb.	Meaning
equals	**EQ**	$a = b$
disconnected	**DC**	$a \cap b = \varnothing$
externally connected	**EC**	$a \cap b \neq \varnothing \wedge a^\circ \cap b^\circ = \varnothing$
partially overlap	**PO**	$a^\circ \cap b^\circ \neq \varnothing \wedge$ $a \not\subseteq b \wedge a \not\supseteq b$
tangential proper part	**TPP**	$a \subset b \wedge a \not\subset b^\circ$
non-tangential proper part	**NTPP**	$a \subset b^\circ$

2.3 RCC8 algebra

A *plane region* (or *region*) is a nonempty regular closed subset of the real plane. The relations in Table 2 and the converses of **TPP** and **NTPP** form a JEPD set [Randell *et al.*, 1992]. These are RCC8 basic relations. Write $\mathcal{B}_{top}$ for this set. The RCC8 algebra is $\langle \mathcal{B}_{top} \rangle$. We write $\mathbf{P}$ and $\mathbf{PP}$, resp., for $\mathbf{TPP} \cup \mathbf{NTPP} \cup \mathbf{EQ}$ and $\mathbf{TPP} \cup \mathbf{NTPP}$.

Renz [Renz, 1999] showed that there are only three maximal tractable subclasses of RCC8 that contain all basic relations. These subclasses are denoted as $\widehat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8$. For these subclasses, applying PCA is sufficient for deciding the satisfiability of a network, and there is an $O(n^2)$ algorithm for finding an atomic refinement of any path-consistent network.

3 The Cardinal Direction Calculus DIR9

For a bounded plane region a, define $\sup_x(a) = \sup\{x \in \mathbb{R} : (\exists y)(x, y) \in a\}$, $\inf_x(a) = \inf\{x \in \mathbb{R} : (\exists y)(x, y) \in a\}$. The definitions of $\inf_y(a)$ and $\sup_y(a)$ are similar. We call $\mathcal{M}(a) = I_x(a) \times I_y(a)$ the *minimum bounding rectangle* (MBR) of a, where $I_x(a) = [\inf_x(a), \sup_x(a)]$ and $I_y(a) = [\inf_y(a), \sup_y(a)]$ are the x- and y-projection of a.

For two bounded plane regions a, b, if $\sup_x(a) < \inf_x(b)$, then we say a is *west* of b and b is *east* of a, written as $a\mathsf{W}b$ and $b\mathsf{E}a$; and if $\sup_y(a) < \inf_y(b)$ then we say a is *south* of b and b is *north* of a, written as $a\mathsf{S}b$ and $b\mathsf{N}a$.

When a is neither west nor east of b, then $I_x(a) \cap I_x(b) \neq \varnothing$. In this case, we say a is in *x-contact* with b, written as $a\mathsf{Cx}b$. Similarly, if a is neither north nor south of b, then we say a is in *y-contact* with b, written as $a\mathsf{Cy}b$.

The Boolean algebra generated by N,S,W,E, written as DIR9, has nine atoms (see Table 3). Although it is very simple, DIR9 is sufficient for expressing directional information in many situations. Moreover, DIR9 is a subclass of RA.

Remark 3.1. Let $\mathcal{B}^3_{int}$ be the JEPD set $\{\mathsf{p}, \Cap, \mathsf{pi}\}$. The algebra $\langle \mathcal{B}^3_{int} \rangle$ is in fact the interval algebra $\mathcal{A}_3$ studied in [Golumbic and Shamir, 1993]. Let $\mathcal{H}_3 = \{\mathsf{p}, \Cap, \mathsf{pi}, \mathsf{p} \cup \Cap, \Cap \cup \mathsf{pi}, \top\}$, where $\top$ is the universal relation. Note that $\mathcal{H}_3 = \mathcal{A}_3 \cap \mathcal{H}$. Golumbic and Shamir proved that $\mathcal{H}_3$ is a maximal tractable subclass of $\mathcal{A}_3$. Since $\mathcal{A}_3 \otimes \mathcal{A}_3$ is exactly DIR9 (see Table 3), $\mathcal{H}_3 \otimes \mathcal{H}_3$ is a tractable subclass of DIR9.[2] It will be our future work to find maximal tractable subclasses of DIR9.

[2] It is worth noting that the intersection of DIR9 and $\mathcal{N}$ is $\mathcal{H}_3 \otimes \mathcal{H}_3$, where $\mathcal{N}$ is the new tractable subset of RA of [Balbiani *et al.*, 1999].

Table 3: Nine cardinal directional relations, where a, b are two bounded plane regions.

Relation	Symb.	Meaning	RA
northwest	NW	$a\mathsf{N}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \mathsf{pi}$
north and x-contact	NC	$a\mathsf{N}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \mathsf{pi}$
northeast	NE	$a\mathsf{N}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \mathsf{pi}$
y-contact and west	CW	$a\mathsf{Cy}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \Cap$
y-contact and x-contact	CC	$a\mathsf{Cy}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \Cap$
y-contact and east	CE	$a\mathsf{Cy}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \Cap$
southwest	SW	$a\mathsf{S}b$ and $a\mathsf{W}b$	$\mathsf{p} \otimes \mathsf{p}$
south and x-contact	SC	$a\mathsf{S}b$ and $a\mathsf{Cx}b$	$\Cap \otimes \mathsf{p}$
southeast	SE	$a\mathsf{S}b$ and $a\mathsf{E}b$	$\mathsf{pi} \otimes \mathsf{p}$

4 Main result

Suppose $V = \{v_i\}_{i=1}^n$ is a collection of spatial variables, $\Theta = \{\theta_{ij}\}_{i=1}^n$ and $\Delta = \{\delta_{ij}\}_{i=1}^n$ are, resp., a topological (RCC8) and a directional (DIR9) constraint network over V. Our problem is when the joint network $\Theta \uplus \Delta$ is satisfiable.

Without loss of generality, in this section we assume (i) $\theta_{ii} = \mathbf{EQ}$ for all i and $\mathbf{EQ} \neq \theta_{ij} = \theta_{ji}^\sim$ for all $i \neq j$; and (ii) $\delta_{ii} = \mathsf{CC}$ and $\delta_{ij} = \delta_{ji}^\sim$ for all i, j.

The following example shows that topological and directional constraints are not independent.

Example 4.1. Let $V = \{v_1, v_2\}$, $\theta_{12} = \mathbf{EC}$, $\delta_{12} = \mathsf{NC}$. Both Θ and Δ are trivially satisfiable. For any two regions a, b, if $a\mathbf{EC}b$, then by $a \cap b \neq \varnothing$ we know $\mathcal{M}(a) \cap \mathcal{M}(b) \neq \varnothing$, hence $a\mathsf{NC}b$ cannot hold. Therefore $\Theta \uplus \Delta$ is inconsistent.

Given Θ and Δ, we define an RCC8 network $\overline{\Theta}$ and an RA network $\overline{\Delta}$ as follows:

$$\overline{\theta}_{ij} = \begin{cases} \theta_{ij} \cap \mathbf{DC}, & \text{if } \mathsf{CC} \cap \delta_{ij} = \varnothing; \\ \theta_{ij}, & \text{otherwise.} \end{cases}$$

$$\overline{\delta}_{ij} = \begin{cases} \delta_{ij} \cap \mathsf{eq} \otimes \mathsf{eq}, & \text{if } \theta_{ij} = \mathbf{EQ}; \\ \delta_{ij} \cap \Subset \otimes \Subset, & \text{if } \mathbf{EQ} \neq \theta_{ij} \subseteq \mathbf{P}; \\ \delta_{ij} \cap \Supset \otimes \Supset, & \text{if } \mathbf{EQ} \neq \theta_{ij} \subseteq \mathbf{P}^\sim; \\ \delta_{ij} \cap \Cap \otimes \Cap, & \text{if } \mathbf{DC} \cap \theta_{ij} = \varnothing; \\ \delta_{ij}, & \text{otherwise.} \end{cases}$$

Note that $\overline{\theta}_{ij} \subseteq \theta_{ij}$ and $\overline{\delta}_{ij} \subseteq \delta_{ij}$ for any two i, j. This means $\overline{\Theta}$ is a refinement of Θ and $\overline{\Delta}$ is a refinement of Δ (in RA).

Lemma 4.1. *$\Theta \uplus \Delta$ is satisfiable iff $\overline{\Theta} \uplus \overline{\Delta}$ is satisfiable.*

As for Example 4.1, by $\delta_{12} = \mathsf{NC}$ we know $\overline{\theta}_{12} = \mathbf{EC} \cap \mathbf{DC} = \varnothing$. Therefore the joint network is inconsistent.

The main contribution of this paper is to show that if Θ is a path-consistent RCC8 network over one of the three maximal tractable subclasses of RCC8, viz. $\widehat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8$, then $\Theta \uplus \Delta$ is satisfiable iff $\overline{\Theta}$ and $\overline{\Delta}$ are satisfiable.

To prove this result, we need two further lemmas.

For $r = [x^-, x^+] \times [y^-, y^+]$ and $\epsilon > 0$, we write $r + \epsilon$ for $[x^- - \epsilon/2, x^+ + \epsilon/2] \times [y^- - \epsilon/2, y^+ + \epsilon/2]$. For two rectangles r_1, r_2, if no edges of r_1 are in line with any edge of r_2, then we can expand the two rectangles a little without changing their rectangle relation. In general, we have

Lemma 4.2. *Let $\{r_i = [x_i^-, x_i^+] \times [y_i^-, y_i^+]\}_{i=1}^n$ be a collection of rectangles, where no two points in either $X =$*

$\{x_i^-, x_i^+\}_{i=1}^n$ *or* $Y = \{y_i^-, y_i^+\}_{i=1}^n$ *are identical. Set* ϵ *to be the smaller one of* $\min\{|x - x'| : x \neq x',\ x, x' \in X\}$ *and* $\min\{|y - y'| : y \neq y',\ y, y' \in Y\}$*. Suppose* $\{a_i\}_{i=1}^n$ *is a collection of bounded plane regions such that* $r_i \subset \mathcal{M}(a_i) \subset r_i + \epsilon/4$*. Then the basic rectangle relation between* $\mathcal{M}(a_i)$ *and* $\mathcal{M}(a_j)$ *is the same as that between* r_i *and* r_j *for any* i, j*.*

For an RCC8 basic network Θ and a collection of rectangles $\{r_i\}_{i=1}^n$, if $\{r_i\}_{i=1}^n$ is in a sense 'compatible' with Θ, then a realization $\{a_i^*\}_{i=1}^n$ of Θ can be found such that $\mathcal{M}(a_i^*)$ is *almost* identical with r_i for each i.

Definition 4.1. A collection of rectangles $\{r_i\}_{i=1}^n$ are *compatible* with an RCC8 basic network $\Theta = \{v_i\theta_{ij}v_j\}_{i,j=1}^n$ if for any i, j we have

- If $\theta_{ij} = \mathbf{EQ}$, then $r_i = r_j$;
- If $\theta_{ij} \neq \mathbf{DC}$, then $r_i \cap r_j$ is a rectangle, i.e. the interior of $r_i \cap r_j$ is nonempty;
- If $\theta_{ij} \subseteq \mathbf{PP}$, then r_i is contained in the interior of r_j.

Lemma 4.3. *Let* $\Theta = \{v_i\theta_{ij}v_j\}_{i,j=1}^n$ *be a satisfiable RCC8 basic network. Suppose* $\{r_i\}_{i=1}^n$ *is a collection of rectangles that are compatible with* Θ*. Then we have a realization* $\{a_i^*\}_{i=1}^n$ *of* Θ *such that each* a_i^* *is a bounded region and* $r_i \subset \mathcal{M}(a_i^*) \subset r_i + \epsilon/4$*, where* $\epsilon > 0$ *is the positive number as defined in Lemma 4.2.*

Proof. The proof is similar to that given for RCC8 (cf. [Renz, 1998; Li, 2006b; Li and Wang, 2006]). First, we define $l(i)$, the ntpp-level of v_i, inductively as follows:

- $l(i) = 1$ if there is no j such that $\theta_{ji} = \mathbf{NTPP}$;
- $l(i) = k + 1$ if there is a variable v_j such that (a) $l(j) = k$ and $\theta_{ji} = \mathbf{NTPP}$, and (b) $\theta_{mi} = \mathbf{NTPP}$ implies $l(m) \leq k$ for any variable v_m.

Write E_{ik} ($k = 1, \cdots, 4$) for the four corner points of r_i. For $i \neq j$, if θ_{ij} is $\mathbf{EC}$ or $\mathbf{PO}$, then choose two new points P_{ij} and P_{ji} in the interior of $r_i \cap r_j$. Set N to be the set of these points, and set δ to be the smallest distance between two points in N. Clearly $0 < \delta < \epsilon$. For each point P in N, construct a system of squares $\{p^-, p^+, p^{(1)}, \cdots, p^{(n)}\}$ as in Fig. 2, where $p^{(n)}$ is a square centered at P with the length of $\delta/2$, and p^- and p^+ are two smaller squares that meet at P.

Now we construct n bounded regions $\{a_i^*\}_{i=1}^n$ as follows.

- $a_i = \bigcup_{k=1}^4 e_{ik}^{(1)}$;
- $a_i' = a_i \cup \bigcup\{p_{ij}^{(-)} \cup p_{ji}^{(+)} : \theta_{ij} = \mathbf{EC}\} \cup \bigcup\{p_{ij}^{(1)} \cup p_{ji}^{(1)} : \theta_{ij} = \mathbf{PO}\}$;
- $a_i'' = a_i' \cup \{a_k' : \theta_{ki}$ is $\mathbf{TPP}$ or $\mathbf{NTPP}\}$;
- $a_i^* = a_i'' \cup \bigcup\{p^{(l(i))} : p \in N$ and $(\exists j)[\theta_{ji} = \mathbf{NTPP}$ and $p^{(1)} \cap a_j'' \neq \varnothing]\}$.

Then $\{a_i^*\}_{i=1}^n$ is a realization of Θ. Moreover, we have $r_i \subset \mathcal{M}(c_i) \subset r_i + \delta/4 \subset r_i + \epsilon/4$. □

Now we prove our main result.

Theorem 4.1. *Let* Θ *be a path-consistent RCC8 network over* $\widehat{\mathcal{H}}$ *(or* $\mathcal{C}_8$*,* $\mathcal{Q}_8$*), and let* Δ *be a DIR9 network. Then* $\Theta \uplus \Delta$ *is satisfiable iff* $\overline{\Theta}$ *and* $\overline{\Delta}$ *are satisfiable.*

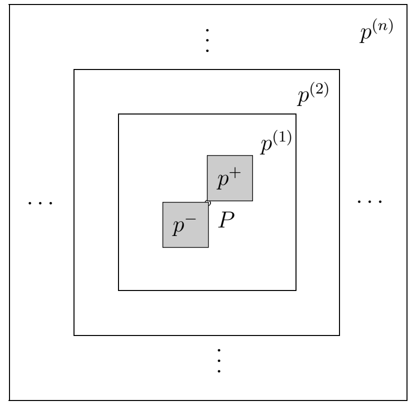

Figure 2: An illustration of the ntpp-chain centered at P_{ij}.

Proof. We take $\widehat{\mathcal{H}}_8$ as an example. The proofs for the other two maximal tractable subclasses are similar.

Suppose $\overline{\Theta}$ and $\overline{\Delta}$ are satisfiable. Since Θ is a path-consistent network over $\widehat{\mathcal{H}}_8$, we can construct an atomic RCC8 network Θ^* as follows [Renz, 1999].

- if $\mathbf{DC} \subseteq \theta_{ij}$, then set $\theta_{ij}^* = \mathbf{DC}$;
- else if $\mathbf{EC} \subseteq \theta_{ij}$, then set $\theta_{ij}^* = \mathbf{EC}$;
- else if $\mathbf{PO} \subseteq \theta_{ij}$, then set $\theta_{ij}^* = \mathbf{PO}$;
- else if $\mathbf{TPP} \subseteq \theta_{ij}$, then set $\theta_{ij}^* = \mathbf{TPP}$;
- else if $\theta_{ij} = \mathbf{NTPP}$ then set $\theta_{ij}^* = \mathbf{NTPP}$;
- else if $\mathbf{TPP}^\sim \subseteq \theta_{ij}$ then set $\theta_{ij}^* = \mathbf{TPP}^\sim$;
- else if $\theta_{ij} = \mathbf{NTPP}^\sim$ then set $\theta_{ij}^* = \mathbf{NTPP}^\sim$.

By $\theta_{ij}^* \subseteq \theta_{ij}$ for any i, j, we know Θ^* is an atomic refinement of Θ and $\overline{\Theta}$. Moreover, for any two i, j, $\theta_{ij} \subseteq \mathbf{P}$ iff $\theta_{ij}^* \subseteq \mathbf{P}$.

We claim that the satisfiable RA network $\overline{\Delta}$ has a realization $\{r_i\}_{i=1}^n$ that is compatible with Θ^*. In other words, Θ^* and $\{r_i\}$ satisfy the conditions of Lemma 4.3. Since the proof is quite complex, we defer it to the next subsection.

Now, by Lemma 4.3 we can find a realization $\{c_i\}_{i=1}^n$ of Θ^* such that $r_i \subset \mathcal{M}(c_i) \subset r_i + \epsilon/4$. By Lemma 4.2, $\{c_i\}$ is also a realization of $\overline{\Delta}$. Therefore, $\Theta \uplus \Delta$ is satisfiable. □

Recall that applying PCA is sufficient for deciding satisfiability for RCC8 subclasses $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, and $\mathcal{Q}_8$, and for RA subclass $\mathcal{H} \otimes \mathcal{H}$. We have the following corollary, where $\mathcal{H}_3$ is defined in Remark 3.1.

Corollary 4.1. *Let* Θ *be an RCC8 network over either one of* $\widehat{\mathcal{H}}_8$*,* $\mathcal{C}_8$*,* $\mathcal{Q}_8$*, and let* Δ *be a DIR9 network over* $\mathcal{H}_3 \otimes \mathcal{H}_3$*. Then deciding the satisfiability of* $\Theta \uplus \Delta$ *is of cubic complexity.*

Proof. It is of quadratic complexity to compute $\overline{\Theta}$ and $\overline{\Delta}$. Note that $\overline{\Delta}$ is a rectangle network over $\mathcal{A}_{\max} \otimes \mathcal{A}_{\max} \subset \mathcal{H} \otimes \mathcal{H}$, and applying PCA to $\overline{\Theta}$ and $\overline{\Delta}$ is of cubic complexity. □

The next example shows, however, if Θ is not over any one of $\widehat{\mathcal{H}}$, $\mathcal{C}_8$, and $\mathcal{Q}_8$, then the result may not hold.

Example 4.2. Let $V = \{v_1, v_2, v_3\}$, $\theta_{12} = \theta_{13} = \mathbf{DC}$, $\theta_{23} = \mathbf{TPP} \cup \mathbf{TPP}^\sim$, $\delta_{12} = \mathsf{NC}$, $\delta_{13} = \mathsf{CW}$, $\delta_{23} = \mathsf{CC}$. Both Θ and Δ are satisfiable, but not $\Theta \uplus \Delta$.

4.1 Maximal instantiation of rectangle networks

For an RCC8 network Θ and a DIR9 network Δ, suppose $\overline{\Theta}$ and $\overline{\Delta}$ are satisfiable, and Θ^* is the atomic refinement of Θ constructed in the proof of Theorem 4.1. We show Δ has a rectangle realization $\{r_i\}_{i=1}^n$ that is compatible with Θ^*.

We first define an auxiliary subclass $\mathcal{A}_{\max}$ of IA.

Definition 4.2. An interval relation α is in $\mathcal{A}_{\max}$ if

- α is convex;
- $\mathsf{m} \subseteq \alpha$ only if $\mathsf{o} \subseteq \alpha$, and $\mathsf{mi} \subseteq \alpha$ only if $\mathsf{oi} \subseteq \alpha$;
- $\mathsf{s} \subseteq \alpha$ only if $\mathsf{d} \subseteq \alpha$, and $\mathsf{si} \subseteq \alpha$ only if $\mathsf{di} \subseteq \alpha$;
- $\mathsf{f} \subseteq \alpha$ only if $\mathsf{d} \subseteq \alpha$, and $\mathsf{fi} \subseteq \alpha$ only if $\mathsf{di} \subseteq \alpha$.

Note that all $\mathcal{A}_{\max}$ relations except eq are of dimension 2.

By the definition of $\mathcal{A}_{\max}$, it is easy to see that $\mathcal{A}_{\max}$ is closed under converse and intersection. The following lemma further asserts that $\mathcal{A}_{\max}$ is closed under composition.

Lemma 4.4. *If $\alpha, \beta \in \mathcal{A}_{\max}$, then $\alpha \circ \beta \in \mathcal{A}_{\max}$.*

Proof. (Sketch.) Since the composition of two convex relations is still convex, $\alpha \circ \beta$ is convex too.

If $\mathsf{m} \subseteq \alpha \circ \beta$, then there exist two basic relations a, b such that $a \subseteq \alpha$, $b \subseteq \beta$, and $\mathsf{m} \subseteq a \circ b$. By checking the composition table of IA [Allen, 1983], we have $\mathsf{o} \not\subseteq a \circ b$ iff (i) $a = \mathsf{m}$ and $b \in \{\mathsf{s}, \mathsf{eq}, \mathsf{si}\}$; or (ii) $a \in \{\mathsf{f}, \mathsf{eq}, \mathsf{fi}\}$ and $b = \mathsf{m}$. By $a \subseteq \alpha \in \mathcal{A}_{\max}$ and $b \subseteq \beta \in \mathcal{A}_{\max}$, we know $\mathsf{o} \subseteq \alpha$ or $\mathsf{o} \subseteq \beta$. By the composition table we have $\mathsf{o} \subseteq \mathsf{o} \circ b$ or $\mathsf{o} \subseteq a \circ \mathsf{o}$, hence $\mathsf{o} \subseteq \alpha \circ \beta$. This means $\mathsf{m} \subseteq \alpha \circ \beta$ only if $\mathsf{o} \subseteq \alpha \circ \beta$. Symmetrically, we can show $\mathsf{mi} \subseteq \alpha \circ \beta$ only if $\mathsf{oi} \subseteq \alpha \circ \beta$. The proofs for the remaining four cases are similar. In this way we know $\alpha \circ \beta$ is also in $\mathcal{A}_{\max}$. □

We next characterize when a composition of two relations in $\mathcal{A}_{\max}$ can contain eq.

Lemma 4.5. *For two interval relations $\alpha, \beta \in \mathcal{A}_{\max}$, if $\mathsf{eq} \subseteq \alpha \circ \beta$, then $\mathsf{eq} \subseteq \alpha \cap \beta$ or $[\mathsf{o}, \mathsf{oi}] \subseteq \alpha \circ \beta$.*

Proof. By Lemma 4.4 we know $\alpha \circ \beta$ is in $\mathcal{A}_{\max}$. If $\mathsf{eq} \in \alpha \circ \beta$, then there exist two basic relations a, b such that $a \subseteq \alpha$, $b \subseteq \beta$, and $\mathsf{eq} \subseteq a \circ b$. This is possible iff b is the converse of a. Suppose $a \neq \mathsf{eq}$. There are two possible situations. If $a \in \{\mathsf{p}, \mathsf{o}, \mathsf{d}, \mathsf{di}, \mathsf{oi}, \mathsf{pi}\}$, then by the composition table we know $[\mathsf{o}, \mathsf{oi}] \subseteq a \circ b$. If $a \in \{\mathsf{m}, \mathsf{s}, \mathsf{f}, \mathsf{fi}, \mathsf{si}, \mathsf{mi}\}$, then $a \circ b$ is $\mathsf{f} \cup \mathsf{eq} \cup \mathsf{fi}$ or $\mathsf{s} \cup \mathsf{eq} \cup \mathsf{si}$. Since $\alpha \circ \beta \in \mathcal{A}_{\max}$, we know d, di are contained in $\alpha \circ \beta$. Checking Fig. 1, we know $[\mathsf{o}, \mathsf{oi}]$ is the smallest convex relation which contains both d and di. Therefore $[\mathsf{o}, \mathsf{oi}]$ is contained in the convex relation $\alpha \circ \beta$. □

The following lemma characterizes when an eq relation can be obtained as the intersection of two relations in $\mathcal{A}_{\max}$.

Lemma 4.6. *For two $\mathcal{A}_{\max}$ relations $\alpha \neq \mathsf{eq}$ and $\beta \neq \mathsf{eq}$, if $\alpha \cap \beta = \mathsf{eq}$, then $\alpha = \beta^\sim$ and α is either $\Subset$ or $\Supset$.*

Proof. If α is neither $\Subset$ nor $\Supset$, then $[\mathsf{o}, \mathsf{oi}]$ is contained in α. Note that in this case, by $\alpha \cap \beta = \mathsf{eq}$, we have $\beta = \mathsf{eq}$. This is a contradiction. So α is $\Subset$ or $\Supset$. The same conclusion also holds for β. Clearly, α is the converse of β. □

We will next show when a $\Subset$ or $\Supset$ relation can be generated by the composition of two $\mathcal{A}_{\max}$ relations.

Lemma 4.7. *For two $\mathcal{A}_{\max}$ relations $\alpha \neq \mathsf{eq}$ and $\beta \neq \mathsf{eq}$, if $\alpha \circ \beta = \Subset$, then $\alpha = \beta = \Subset$; if $\alpha \circ \beta = \Supset$, then $\alpha = \beta = \Supset$.*

Proof. By Lemma 4.5 we know $\mathsf{eq} \subseteq \alpha \cap \beta$. Since α is not eq, either $\Subset$ or $\Supset$ is contained in α. The same conclusion holds for β. By $\alpha \circ \beta = \Subset$ and $\mathsf{eq} \subseteq \alpha \cap \beta$, we know α and β can only be $\Subset$. □

As a corollary of the above two lemmas, we have

Corollary 4.2. *For three $\mathcal{A}_{\max}$ relations $\alpha \neq \mathsf{eq}$, $\beta \neq \mathsf{eq}$, $\gamma \neq \mathsf{eq}$, $\alpha \cap \beta \circ \gamma = \mathsf{eq}$ iff $\alpha = \beta^\sim = \gamma^\sim$ and α is $\Subset$ or $\Supset$.*

Since $[\mathsf{o}, \mathsf{oi}]$ is the smallest convex relation in $\mathcal{A}_{\max}$ which strictly contains $\Subset$, we have the following result.

Lemma 4.8. *For $\alpha, \beta \in \mathcal{A}_{\max}$, if $\alpha \cap \beta = \Subset$, then either one is $\Subset$; if $\alpha \cap \beta = \Supset$, then either one is $\Supset$.*

Let $\Gamma = \{v_i \gamma_{ij} v_j\}_{i,j=1}^n$ be a rectangle (or interval) network, and let $\{a_i\}_{i=1}^n$ be a realization of Γ. We say $\{a_i\}$ is a *maximal instantiation* of Γ iff $\dim(\gamma_{ij})$ is equal to the dimension of the basic relation between a_i and a_j for all i, j [Ligozat, 1994; Balbiani *et al.*, 1999].

Theorem 4.2 ([Ligozat, 1994])**.** *Let Λ be an interval network that contains only preconvex relations. If Λ is path-consistent, then there is a maximal instantiation of Λ.*[3]

The path-consistency condition in the above theorem is necessary. For example, if Λ is the constraint network $\{x \Subset y \Subset z \Subset x\}$, then Λ is satisfiable but has no maximal instantiation: by applying PCA we obtain $x\ \mathsf{eq}\ y\ \mathsf{eq}\ z\ \mathsf{eq}\ x$.

But when only relations in $\mathcal{A}_{\max}$ are concerned, we have the following result.

Theorem 4.3. *Let Λ be an interval network that contains only relations in $\mathcal{A}_{\max}$. Suppose Λ is satisfiable and for all i, j, k we have (a) $\lambda_{ij} = \mathsf{eq}$ iff $i = j$; (b) $\lambda_{ij} = \lambda_{ji}^\sim$; (c) $\lambda_{ik} = \Subset \wedge \lambda_{kj} = \Subset \rightarrow \lambda_{ij} = \Subset$. Then Λ has a maximal instantiation.*

Proof. Suppose by applying PCA the network will be stable after m steps. We prove by induction that at no step this introduces new $\Subset$ or $\Supset$ or eq constraints. This means that, for each step $1 \leq p \leq m$, $\lambda_{ij}^p \in \{\mathsf{eq}, \Subset, \Supset\}$ only if $\lambda_{ij}^p = \lambda_{ij}$, where λ_{ij}^p is the relation between x_i and x_j after the p-th step.

Suppose this holds for $p \geq 1$. We show it holds for step $p+1$. Note that $\lambda_{ij}^{p+1} = \lambda_{ij}^p \cap \lambda_{ik}^p \circ \lambda_{kj}^p$ for some k or $\lambda_{ij}^{p+1} = \lambda_{ij}^p \cap {\lambda_{ji}^p}^\sim$. We take the first case as an example.

If $\lambda_{ij}^{p+1} = \mathsf{eq}$, then by Corollary 4.2 we know $\lambda_{ik}^p = \lambda_{kj}^p = {\lambda_{ij}^p}^\sim$ is $\Subset$ or $\Supset$. By the induction hypothesis we know $\lambda_{ik} = \lambda_{kj} = \lambda_{ji}$ are all $\Subset$ or $\Supset$. This contradicts the third condition. Therefore $\lambda_{ij}^{p+1} \neq \mathsf{eq}$.

If $\lambda_{ij}^{p+1} = \Subset \neq \lambda_{ij}^p$, then by Lemma 4.8 we have $\lambda_{ik}^p \circ \lambda_{kj}^p = \Subset$. But according to Lemma 4.7, this is possible iff both λ_{ik}^p and λ_{kj}^p are $\Subset$. By the induction hypothesis and the third condition, we know $\lambda_{ik} = \lambda_{kj} = \Subset$ and $\lambda_{ij} = \Subset$. This is a contradiction. Hence $\lambda_{ij}^{p+1} \neq \Subset$.

[3]We remind here that a convex relation is preconvex, and an interval relation is preconvex iff it is an ORD-Horn relation.

Similarly, we can show that if $\lambda_{ij} \neq \ni$, then $\lambda_{ij}^{p+1} \neq \ni$.

Now, since Λ is satisfiable, applying PCA on Λ will obtain a path-consistent network Λ^*, which is still over $\mathcal{A}_{\max}$ and satisfies the following condition: For any $i \neq j$, λ_{ij}^* is contained in λ_{ij} and of dimension 2. By Theorem 4.2 we know Λ^*, hence Λ, has a maximal instantiation. □

Since each λ_{ij} $(i \neq j)$ is of dimension 2, by the above theorem, we know Λ has a realization $\{I_i = [x_i^-, x_i^+]\}_{i=1}^n$ such that $x_i^-, x_i^+, x_j^-, x_j^+$ are different for any $i \neq j$.

Theorem 4.4. *Suppose Θ is an RCC8 network, and Δ is a DIR9 network. If $\overline{\Delta}$ is satisfiable, then $\overline{\Delta}$ has a maximal instantiation.*

Proof. Suppose $\{a_i\}_{i=1}^n$ is a realization of $\overline{\Delta}$. Define an interval network $\Delta' = \{v_i \delta'_{ij} v_j\}_{i,j=1}^n$ as follows. For all i set $\delta'_{ii} = \mathsf{eq} \otimes \mathsf{eq}$. For all $i \neq j$, if $\overline{\delta}_{ij} \in \{\Subset \otimes \Subset, \Supset \otimes \Supset\}$ then set $\delta'_{ij} = \overline{\delta}_{ij}$; otherwise, set δ'_{ij} to be the basic DIR9 relation in which a_i and a_j are related.

We note that Δ' satisfies (i) $\delta'_{ii} = \mathsf{eq} \otimes \mathsf{eq}$; (ii) $\delta'_{ij} = {\delta'_{ji}}^{\sim}$; (iii) $\delta'_{ik} = \Subset \otimes \Subset \wedge \delta'_{kj} = \Subset \otimes \Subset \rightarrow \delta'_{ij} = \Subset \otimes \Subset$.

Moreover, each δ'_{ij} has the form $\lambda_{ij}^x \otimes \lambda_{ij}^y$, where $\lambda_{ij}^x, \lambda_{ij}^y$ are interval relations in $\mathcal{A}_{\max}$. Write $\Lambda_x = \{x_i \lambda_{ij}^x x_j\}_{i,j=1}^n$ and $\Lambda_y = \{y_i \lambda_{ij}^y y_j\}_{i,j=1}^n$. For all $i \neq j$ we have $\dim(\lambda_{ij}^x) = \dim(\lambda_{ij}^y) = 2$ and $\dim(\delta'_{ij}) = 4$.

It is easy to see that Δ' is satisfiable iff both Λ_x and Λ_y are satisfiable. Since they satisfy the conditions given in Theorem 4.3, both Λ_x and Λ_y have maximal instantiations. Suppose $\mathcal{I}_x = \{I_i^x\}_{i=1}^n$ and $\mathcal{I}_y = \{I_i^y\}_{i=1}^n$ are, resp., maximal instantiations of Λ_x and Λ_y. Set $r_i = I_i^x \times I_i^y$ for $i = 1, \cdots, n$. Then $\mathcal{I} = \{r_i\}_{i=1}^n$ is a rectangle realization of Δ', hence a realization of $\overline{\Delta}$ and Δ. For $i \neq j$, the basic relation between r_i and r_j is of dimension 4. Therefore $\mathcal{I}$ is maximal. □

By the definition of $\overline{\Delta}$ and the maximality of $\mathcal{I}$ (as constructed above), we know $\mathcal{I}$ is compatible with Θ^*.

Theorem 4.5. *Suppose Θ is a path-consistent RCC8 network over either of $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, $\mathcal{Q}_8$, and suppose Δ is a DIR9 network. Let Θ^* be the atomic refinement of Θ as in the proof of Theorem 4.1. If $\overline{\Theta}$ and $\overline{\Delta}$ are satisfiable, then $\overline{\Delta}$ has a realization that is compatible with Θ^*.*

5 Conclusion

In this paper, we have investigated the interaction between topological and directional constraints. We have showed that, for the three maximal tractable subclasses of RCC8, the problem of deciding the satisfiability of a joint network of topological and directional constraints can be reduced to two simple satisfiability problems in RCC8 and RA.

An earlier attempt to combining topological and directional information was reported in [Li, 2006a], where we introduced a hybrid calculus that combines DIR9 with RCC5. A preliminary result was obtained, which states that the consistency of atomic networks in the hybrid calculus can be decided in polynomial time.

Sistla and Yu [Sistla and Yu, 2000] considered a similar problem of reasoning about spatial relations in picture retrieval systems, where mereological relations *inside*, *outside*, *overlaps*, and directional relations *left*, *above*, *behind* are considered. The constraint language considered there is rather restrictive. It can express neither *genuine* topological relations (e.g. *tangential proper part* and *externally connected to*) nor negations of some relations (e.g. *neither left nor right*).

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