Nonnegativity Problems for Matrix Semigroups

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Abstract

The matrix semigroup membership problem asks, given square matrices $M, M_1, \dots, M_k$ of the same dimension, whether M lies in the semigroup generated by $M_1, \dots, M_k$. It is classical that this problem is undecidable in general, but decidable in case $M_1, \dots, M_k$ commute. In this paper we consider the problem of whether, given $M_1, \dots, M_k$, the semigroup generated by $M_1, \dots, M_k$ contains a non-negative matrix. We show that in case $M_1, \dots, M_k$ commute, this problem is decidable subject to Schanuel's Conjecture. We show also that the problem is undecidable if the commutativity assumption is dropped. A key lemma in our decidability proof is a procedure to determine, given a matrix M, whether the sequence of matrices $(M^n)_{n=0}^{\infty}$ is ultimately nonnegative. This answers a problem posed by S. Akshay [1]. The latter result is in stark contrast to the notorious fact that it is not known how to determine, for any specific matrix index (i, j), whether the sequence $(M^n)_{i,j}$ is ultimately nonnegative. Indeed the latter is equivalent to the Ultimate Positivity Problem for linear recurrence sequences, a longstanding open problem.

2012 ACM Subject Classification Computing methodologies → Symbolic and algebraic manipulation; Theory of computation → Formal languages and automata theory

Keywords and phrases Decidability, Linear Recurrence Sequences, Schanuel's Conjecture

Digital Object Identifier 10.4230/LIPIcs.STACS.2024.27

Related Version *Full Version*: `https://arxiv.org/abs/2311.06241`

Funding *Julian D'Costa*: emmy.network foundation under the aegis of the Fondation de Luxembourg, UKRI Frontier Research Grant EP/X033813/1.
Joël Ouaknine: DFG grant 389792660 as part of TRR 248 (see `https://perspicuous-computing.science`). Joël Ouaknine is also affiliated with Keble College, Oxford as `emmy.network` Fellow.
James Worrell: UKRI Frontier Research Grant EP/X033813/1.

1 Introduction

The *Membership Problem* for finitely generated matrix semigroups asks, given square matrices $M, M_1, \dots, M_k$ of the same dimension and with rational entries, whether M lies in the semigroup generated by $M_1, \dots, M_k$. The problem was shown to be undecidable by Markov in the 1940s [15], thereby becoming one of the first instances of a natural undecidable mathematical problem. The problem, however, becomes decidable under the assumption that the matrices $M_1, \dots, M_k$ commute [2].

There are many variants of the Membership Problem. In the *Mortality Problem* one asks whether the zero matrix lies in a finitely generated matrix semigroup. This problem is undecidable already for 3×3 matrices [21]. Meanwhile, the *Identity Problem* asks to determine whether the identity matrix lies in a given finitely generated matrix semigroup. The latter problem is undecidable in general but decidable for certain nilpotent and low-order matrix groups [3, 9, 10, 12].

The present paper is concerned with the *Non-negative Membership Problem*, which asks to determine whether a given finitely generated matrix semigroup contains a non-negative matrix, i.e., a matrix all of whose entries are non-negative. We show that this problem is undecidable in general but is decidable in the commutative case subject to Schanuel's Conjecture, a well-known unifying conjecture in transcendence theory. Our reliance on Schanuel's Conjecture arises because we reduce the commutative case of the Non-negative Membership Problem to the decision problem for the first-order theory of real-closed fields with exponential. As shown by Macintyre and Wilkie [14], the latter theory is decidable if one assumes Schanuel's Conjecture for the real exponential.

A key lemma in our main decidability result involves determining, for a given matrix M, whether for all but finitely many $n \in \mathbb{N}$ the matrix power M^n is non-negative. In such a case we say that M is eventually non-negative. We give an effective characterisation of eventually non-negative matrices, answering a question posed by S. Akshay [1]. The characterisation is relatively straightforward and relies on classical results about rational sequences over the semi-ring of non-negative rational numbers. We note that the problem of determining whether, for some fixed index (i, j), the sequence of scalars $\langle (M^n)_{i,j} : n \in \mathbb{N} \rangle$ is ultimately non-negative is equivalent to the Ultimate Positivity Problem for linear recurrence sequences, decidability of which is a longstanding open problem [20].

It is immediate that a matrix semigroup contains a non-negative matrix if and only if it contains an eventually non-negative matrix. Using a symbolic version of our criterion characterizing eventually non-negative matrices, we reduce the Non-negative Membership Problem to a version of integer programming with certain transcendental constants (namely logarithms of algebraic numbers). In turn we reduce the solution of such integer programs to the decision problem for the first-order theory of real closed fields with exponential.

A simpler variant of our main result concerns the problem of deciding whether a finitely generated matrix semigroup contains a positive matrix, i.e., a matrix all of whose entries are strictly positive. Here, to show decidability, we rely on a known characterisation of eventually positive matrices, due to Noutsos [18]. While we still need to invoke Schanuel's Conjecture in this case, we do so through the use of a procedure of Richardson [22] for deciding equality of elementary numbers (which is much more straightforward than the result of Macintyre and Wilkie mentioned above).

As far as we are aware, the Non-negative Membership Problem has not been directly addressed before. We note however that decidability of the version of this problem for sub-semigroups of the group $\mathbf{GL}(2, \mathbb{Z})$ of 2×2 invertible integer matrices follows directly from [7, Theorem 13].

2 Mathematical Background

Here we state some number-theoretic results that we will need in the sequel.

First stated in the 1960s, Schanuel's conjecture is a unifying conjecture in transcendental number theory that generalizes many of the classical results in the field.

▶ **Conjecture 1** (Schanuel's conjecture [13]). *If $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ are rationally linearly independent, then some k-element subset of $\{\alpha_1, \ldots, \alpha_k, e^{\alpha_1}, \ldots, e^{\alpha_k}\}$ is algebraically independent.*

An elementary point is an element of $\mathbb{C}^n$ that arises as an isolated, nonsingular solution of n equations in n variables $x_1, \ldots, x_n$, with each equation being either of the form $P(x_1, \ldots, x_n) = 0$, where $P \in \mathbb{Q}[x_1, \ldots, x_n]$ is a polynomial with rational coefficients, or of the form $x_j - e^{x_i} = 0$ for $i, j \in \{1, \ldots, n\}$. An elementary number is the polynomial image of an elementary point.

Roughly speaking, an elementary number is obtained by starting with the rationals and applying addition, subtraction, multiplication, division, exponentiation, and taking natural logarithms. Elementary numbers may be transcendental and, unsurprisingly, it is non-trivial to determine whether an elementary number is equal to zero.

▶ **Proposition 2** (Richardson [22]). *The problem of determining zeroness of an elementary number is semi-decidable. The problem is moreover decidable if one assumes Schanuel's conjecture.*

We will also need the following theorem due to Masser.

▶ **Theorem 3** (Multiplicative relations among algebraic numbers [16]). *Let m be fixed, and let $\lambda_1, \ldots, \lambda_m$ be complex algebraic numbers. Consider the free abelian group L under addition given by*

$$L = \{(v_1, \ldots, v_m) \in \mathbb{Z}^m : \lambda_1^{v_1} \ldots \lambda_m^{v_m} = 1\}.$$

Then L has a basis $\{w_1, \ldots, w_p\} \subseteq \mathbb{Z}^m$ (with $p \leq m$), where the entries of each of the w_j are all polynomially bounded in the sum of the heights and degrees of the minimal polynomials of $\lambda_1, \ldots, \lambda_m$.

3 Linear Recurrence Sequences

First, we recall some basic terminology and results about linear recurrence sequences.

A sequence $\boldsymbol{u} = (u_n)_{n=0}^{\infty}$ of elements of a semiring K is called K*-rational* if there exists $d \geq 1$, $v, w \in K^d$ and $M \in K^{d \times d}$ such that $u_n = v^\top M^n w$ for all n. When K is a field, a sequence is K-rational if and only if it satisfies a linear recurrence relation

$$u_n = a_1 u_{n-1} + \cdots a_d u_{n-d} \quad (n \geq d)$$

where $a_1, \ldots, a_d \in K$. In this case we also call $\boldsymbol{u}$ a *linear recurrence sequence (LRS).*

With the unique minimal order recurrence satisfied by $\boldsymbol{u}$ we associate the *characteristic polynomial*

$$P(X) = X^d - a_1 X^{d-1} - \cdots - a_d\,.$$

The roots of $P(X)$ are called the *characteristic roots* of $\boldsymbol{u}$. Writing $\lambda_1, \ldots, \lambda_m$ for the distinct characteristic roots, in non-increasing order of absolute value, the sequence $\boldsymbol{u}$ admits a closed-form *exponential-polynomial* representation:

$$u_n = \sum_{i=1}^{m} P_i(n) \lambda_i^n\,,$$

where the P_i are univariate polynomials whose coefficients are algebraic over K. We say that $\boldsymbol{u}$ is *non-degenerate* if no quotient of two distinct characteristic roots is a root of unity. We also say that a matrix $M \in \mathbb{Q}^{d \times d}$ is non-degenerate if no quotient of two distinct eigenvalues is a root of unity.

In this paper we exclusively consider sequences with rational entries. We say that an LRS $\boldsymbol{u}$ is *dominated* if λ_1 is the unique characteristic root of maximum modulus. Note that in this case λ_1 is necessarily real. We have the following three straightforward propositions concerning dominated LRS.

▶ **Proposition 4.** *If an LRS $\boldsymbol{u}$ is dominated then $\{n \in \mathbb{N} : u_n \geq 0\}$ is an effectively computable ultimately periodic set.*

Proof. Consider the closed form representation $u_n = \sum_{i=1}^{m} P_i(n)\lambda_i^n$ with unique dominant root λ_1, necessarily real. Suppose that P_1 has degree k and leading coefficient a. Then we have $\frac{u_n}{n^k|\lambda_1|^n} = a(\lambda_1/|\lambda_1|)^n + o(1)$. Hence for sufficiently large n the sign of u_n is determined by the sign of a and the parity of n. ◀

▶ **Proposition 5.** *If $\boldsymbol{u}$ is a non-degenerate LRS such that some subsequence $(u_{cn+d})_{n=0}^{\infty}$ is dominated, where c is a positive integer and $d \in \{0, 1 \ldots, c-1\}$, then $\boldsymbol{u}$ itself is dominated.*

Proof. The sequence $\boldsymbol{u}$ admits a closed-form representation $u_n = \sum_{i=1}^{m} P_i(n)\lambda_i^n$, where $\lambda_1, \ldots, \lambda_m$ are the characteristic roots and $P_1, \ldots, P_m$ are polynomials. Then

$$\begin{aligned} u_{cn+d} &= \sum_{i=1}^{m} P_i(cn+d)\lambda_i^{cn+d} \\ &= \sum_{i=1}^{m} Q_i(n)(\lambda_i^c)^n \end{aligned}$$

where $Q_i(n) := P_i(cn+d)\lambda_i^d$ for $i \in \{1, \ldots, m\}$.

Note that the polynomials $Q_1, \ldots, Q_m$ are non-zero and, by non-degeneracy of $\boldsymbol{u}$, the numbers $\lambda_1^c, \ldots, \lambda_m^c$ are pairwise distinct. Since the sequence $(u_{cn+d})_{n=0}^{\infty}$ is dominated, we have that λ_1^c is its unique characteristic root of maximum modulus. But then λ_1 is the unique characteristic root of $\boldsymbol{u}$. ◀

▶ **Proposition 6.** *An LRS that is both non-degenerate and rational over the semiring $\mathbb{Q}_+$ of nonnegative rational numbers is dominated.*

Proof. Berstel [4] showed that if a sequence $\boldsymbol{u}$ is $\mathbb{Q}_+$-rational then its characteristic roots of maximum modulus all have the form $\rho\omega$ for some non-negative real number ρ and root of unity ω. Since $\boldsymbol{u}$ is non-degenerate it follows that it has a unique dominant root. For an exposition, see [5, Chap. 8, Thm 1.1]. We provide an alternate proof based on the Perron-Frobenius theorem in Appendix B. ◀

▶ **Theorem 7.** *Given $M \in \mathbb{Q}^{d\times d}$ the set $S := \{n \in \mathbb{N} : M^n \geq 0\}$ is ultimately periodic and effectively computable.*

Proof. Recall that for some (effectively computable) strictly positive integer L the matrix M^L is non-degenerate. It will suffice to show that for each $l \in \{0, \ldots, L-1\}$ we can compute the set $S_l := \{n \in S : n \equiv l \bmod L\}$. Our procedure to do this is as follows. First, check for every pair of indices $i, j \in \{1, \ldots, d\}$, whether the sequence $(u_n^{(i,j)})_{n=0}^{\infty}$ given by $u_n^{(i,j)} = (M^{Ln+l})_{i,j}$, is dominated. If yes then by Proposition 4 we can compute S_l as the intersection over all pairs (i, j) of the sets $\{n \in \mathbb{N} : u_n^{(i,j)} \geq 0\}$. If no, then we claim that S_l is empty.

Indeed, suppose $n_0 \in S_l$. Then for each pair of indices $i, j \in \{1, \ldots, d\}$, the LRS $(v_n^{(i,j)})_{n=0}^{\infty}$ defined by $v_n^{(i,j)} = (M^{n_0(1+Ln)})_{i,j} = e_i^\top M^{n_0}(M^{Ln_0})^n e_j^\top$ is both non-degenerate and $\mathbb{Q}^+$-rational. By Proposition 6 each sequence $(v_n^{(i,j)})_{n=0}^{\infty}$ is dominated. Moreover, since $(v_n^{(i,j)})_{n=0}^{\infty}$ is a subsequence of $(u_n^{(i,j)})_{n=0}^{\infty}$, the latter is also dominated by Proposition 5. This proves (the contrapositive of) our claim. ◀

▶ Remark 8. We can extract from the proof of Theorem 7 an effective characterisation of those matrices M such that $M^n \geq 0$ for some positive integer n. Let L be the least positive integer such that M^L is non-degenerate. Then some positive power of M is a non-negative matrix iff some positive power of M^L is non-negative iff for all indices (i,j) the sequence $(u_n^{(i,j)})_{n=0}^{\infty}$ defined by $u_n^{(i,j)} := (M^{Ln})_{i,j}$ is dominated and not ultimately negative.

4 The Positive Membership Problem

4.1 Eventually Positive Matrices

Recall that a positive matrix is one whose entries are all strictly positive. In this section we show decidability of the following problem.

▶ **Problem 9** (Positive Membership for Commutative Semigroup). *Given a set of commuting $d \times d$ matrices $\{A_1, \ldots, A_k\}$ with rational entries, decide whether the generated semigroup contains a positive matrix.*

We approach this problem through the notion of eventually positive matrix.

▶ **Definition 10** (Eventually Positive Matrix). *We call a matrix M* eventually positive *if there exists a natural number n_0 such that for all $n \geq n_0$, the matrix M^n is positive.*

We will need the following definition and result, adapted from Noutsos [18], which characterizes eventual positivity of a matrix by proving a converse of the Perron-Frobenius theorem.

▶ **Definition 11** (Strong Perron-Frobenius property). *A matrix $A \in \mathbb{R}^{n\times n}$ has the* strong Perron-Frobenius property *if there exists an eigenvalue λ with the following properties:*

1. *λ is real and positive,*
2. *λ is the unique dominant eigenvalue,*
3. *λ is simple,*
4. *λ has a corresponding eigenvector, all of whose entries are positive.*

We now have:

▶ **Theorem 12** (Characterizing eventual positivity [18]). *A matrix A is eventually positive iff A and $A^\top$ both have the strong Perron-Frobenius property.*

Clearly a semigroup contains a positive matrix if and only if it contains an eventually positive matrix. By the theorem above, this holds if and only if the semigroup contains a matrix A such that both A and $A^\top$ have the strong Perron-Frobenius property.

4.2 Reduction to Integer Programming

We consider a direct-sum decomposition of $\mathbb{C}^d$ induced by a collection of commuting matrices $A_1, \ldots, A_k \in \mathbb{C}^{d\times d}$. Let $\sigma(A_i)$ denote the set of eigenvalues of A_i and consider a tuple of eigenvalues

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \sigma(A_1) \times \cdots \times \sigma(A_k).$$

Recall that $\ker(A_i - \lambda_i I)^d$ is the generalized eigenspace of A_i corresponding to λ_i for $i \in \{1, \ldots, k\}$. We say that the tuple $\boldsymbol{\lambda}$ is a *joint eigenvalue* of $A_1, \ldots, A_k$ if

$$V_{\boldsymbol{\lambda}} := \bigcap_{i=1}^{k} \ker(A_i - \lambda_i I)^d$$

is non null. Intuitively, joint eigenvalues are tuples of eigenvalues of the A_i whose respective generalized eigenspaces have non-trivial intersection. The set of joint eigenvalues is called the *joint spectrum* of $A_1, \ldots, A_k$, denoted Σ. Using the fact that commuting matrices preserve each other's generalized eigenspaces, it can be shown (see, e.g., [19, Theorem 2.4]) that

$$\mathbb{C}^d = \oplus_{\boldsymbol{\lambda} \in \Sigma} V_{\boldsymbol{\lambda}} ,$$

and that, for all $i \in \{1, \ldots, k\}$ and $\boldsymbol{\lambda} \in \Sigma$, A_i preserves $V_{\boldsymbol{\lambda}}$, and the restriction of A_i to $V_{\boldsymbol{\lambda}}$ has spectrum $\{\lambda_i\}$. The commuting matrices $A_1^\top, \ldots, A_k^\top$ induce an analogous decomposition $\mathbb{C}^d = \oplus_{\boldsymbol{\lambda} \in \Sigma} W_{\boldsymbol{\lambda}}$, where $W_{\boldsymbol{\lambda}} := \cap_{i=1}^k \ker(A_i^\top - \lambda_i I)$.

Consider a matrix $A := A^{m_1} \cdots A_k^{m_k}$ for $m_1, \ldots, m_k \in \mathbb{N}$. Given $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Sigma$, by the Spectral Mapping Theorem the restriction of A to the subspace $V_{\boldsymbol{\lambda}}$ has a single eigenvalue $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$. Thus all non-zero vectors in $V_{\boldsymbol{\lambda}}$ are generalized eigenvectors of A for this eigenvalue. It follows that if A is eventually positive, then Conditions 1 and 2 of Theorem 12 imply that there exists $\boldsymbol{\lambda} \in \Sigma$ such that $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ is real positive and is the unique dominant eigenvalue of A. Meanwhile, by Conditions 3 and 4 we can furthermore choose $\boldsymbol{\lambda}$ such that the spaces $V_{\boldsymbol{\lambda}}$ and $W_{\boldsymbol{\lambda}}$ are both one dimensional and contain a positive vector. In such a situation we call $\boldsymbol{\lambda}$ a *dominant joint eigenvalue* for A, $V_{\boldsymbol{\lambda}}$ a *positive right eigenspace*, and $W_{\boldsymbol{\lambda}}$ a *positive left eigenspace*.

Conversely, suppose that there exist $m_1, \ldots, m_k \in \mathbb{N}$ and $\boldsymbol{\lambda} \in \Sigma$ such that $\boldsymbol{\lambda}$ is a dominant joint eigenvalue of $A := A_1^{m_1} \cdots A_k^{m_k}$, and $V_{\boldsymbol{\lambda}}$ and $W_{\boldsymbol{\lambda}}$ are positive right and left joint eigenspaces respectively. Then Theorem 12 implies that A is eventually positive.

In summary, the semigroup generated by $A_1, \ldots, A_k$ contains a positive matrix if and only if there exists $\boldsymbol{\lambda} \in \Sigma$, such that $V_{\boldsymbol{\lambda}}$ and $W_{\boldsymbol{\lambda}}$ are positive joint eigenspaces, and there are $m_1, \ldots, m_d \in \mathbb{N}$ with

1. $\prod_{i=1}^k \lambda_i^{m_i} > 0$,
2. $\forall \boldsymbol{\mu} \in \Sigma \setminus \{\boldsymbol{\lambda}\}$, $\prod_{i=1}^k \lambda_i^{m_i} > \left| \prod_{i=1}^k \mu_i^{m_i} \right|$.

It is clear that the Positive Membership Problem reduces to the restricted version of the problem in which one asks for the existence of a positive matrix formed by a product in which all generators of the semigroup are used at least once. Thus we can assume that the desired exponents $m_1, \ldots, m_k$ in Conditions 1 and 2 are all strictly positive. In this situation we may rewrite Condition 1 as $\sum_{i=1}^k m_i \arg(\lambda_i) = 0 \bmod 2\pi$ and Condition 2 as either $\mu_1 \cdots \mu_k = 0$ or $\sum_{i=1}^k m_i \log |\mu_i/\lambda_i| < 0$ for all $\boldsymbol{\mu} \in \Sigma \setminus \{\boldsymbol{\lambda}\}$. Let $\boldsymbol{c} = (\arg(\lambda_1), \ldots, \arg(\lambda_k)) \in \mathbb{R}^k$ and let A be the $(|\Sigma| - 1 \times k)$-matrix defined by writing, for all $i \in \{1, \ldots, k\}$ $\boldsymbol{\mu} \in \Sigma \setminus \{\boldsymbol{\lambda}\}$, $a_{\boldsymbol{\mu},i} := \log |\mu_i/\lambda_i|$ if $\mu_1 \cdots \mu_k \neq 0$ and $a_{\boldsymbol{\mu},i} = -1$ if $\mu_1 \cdots \mu_k = 0$. Then the two conditions above are equivalent to the existence of a solution $x \in \mathbb{N}^k$ of the following integer program:

$$(c^\top x = 0 \bmod 2\pi) \wedge Ax < 0 .$$

Note that by incorporating positivity constraints $-x_i < 0$ in A we can assume without loss of generality that x ranges over $\mathbb{Z}^k$. We now show that the satisfiability of such an integer program is decidable, subject to Schanuel's conjecture.

4.3 Integer Programming with Logs of Algebraic Numbers

▶ **Problem 13** (IP-log). *An instance of the IP-log problem consists of a matrix $A \in \mathbb{R}^{m \times n}$ whose entries are sums of logarithms of non-zero real algebraic numbers and a vector $c \in \mathbb{R}^n$ whose entries are arguments of non-zero algebraic numbers. The problem asks to determine whether there exists a vector $x \in \mathbb{Z}^n$ such that $c^\top x = 0 \mod 2\pi$ and $Ax < 0$.*

Let us start by observing that one can eliminate the equation in the IP-log problem by an effective linear change of variables. Indeed, exponentiation turns the linear relation $c^\top x = 0 \mod 2\pi$ into a multiplicative relation among algebraic numbers. We can then use Theorem 3 to find a basis $\{v_1, \ldots, v_l\} \subseteq \mathbb{Z}^n$ of the group of integer solutions of the equation $c^\top x = 0 \mod 2\pi$. Let $B \in \mathbb{Z}^{n\times l}$ be the matrix that has these basis vectors as columns. Then $c^\top x = 0 \mod 2\pi$ iff $x = By$ for some integer vector y. Hence the instance of IP-log has a solution iff there exists a vector $y \in \mathbb{Z}^l$ such that $ABy < 0$. In other words, we have reduced the initial instance of IP-log to another instance with a trivial linear constraint.

▶ **Theorem 14.** *The strict homogenous IP-log problem is decidable, assuming Schanuel's conjecture.*

Proof. An instance of the strict homogenous IP-log problem asks to determine the truth of the sentence:

$$\exists x \in \mathbb{Z}^n : (c^\top x = 0 \mod 2\pi) \wedge Ax < 0\,.$$

As described above, we may assume without loss of generality that the equality constraint is trivial (i.e., $c = 0$). It thus suffices to determine whether the system of inequalities $Ax < 0$ admits a solution $x \in \mathbb{Z}^n$. But, by scaling, this system of inequalities has a solution in $\mathbb{Z}^n$ iff it has a solution in $\mathbb{Q}^n$. In turn, by strictness of the constraints, there is a solution in $\mathbb{Q}^n$ iff there is a solution in $\mathbb{R}^n$. We are thus left with the task of determining whether there exists $x \in \mathbb{R}^n$ such that $Ax < 0$. Here we can apply Fourier-Motzkin elimination [8] (that is to say, quantifier elimination in linear real arithmetic).

Recall that that Fourier-Motzkin elimination solves a system of linear inequalities by eliminating the variables sequentially until one obtains an equisatisfiable variable-free system of inequalites between constants. In the case at hand these constants will be rational expressions in a fixed number of logarithms of real algebraic numbers. As such, they are elementary numbers and so, using Richardson's algorithm (Proposition 2), we can determine which coefficients are zero. For a coefficient which is not zero, we need merely compute it to sufficient precision to determine its sign. ◀

The following example illustrates the main points of the argument above: Let $\omega = e^{2\pi i/3}$ be a primitive cube root of unity and let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be real positive algebraic numbers. Consider the following system of inequations and an equation:

$$\begin{aligned}
&x_1 \arg(\omega) + x_2 \arg(\omega^2) = 0 \mod 2\pi\\
&x_1 \log(\lambda_1) + x_2 \log(\lambda_2) < 0\\
&x_1 \log(\lambda_3) + x_2 \log(\lambda_4) < 0
\end{aligned}$$

Clearly the equation above is satisfied when $x_1 = 3y_1 + 2y_2$ and $x_2 = 3y_1 - y_2$ for some $y_1, y_2 \in \mathbb{Z}$. Thus we eliminate the equation and obtain

$$\begin{aligned}
&(3y_1 + 2y_2)\log\lambda_1 + (3y_1 - y_2)\log\lambda_2 < 0\\
&(3y_1 + 2y_2)\log\lambda_3 + (3y_1 - y_2)\log\lambda_4 < 0\,,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&y_1 \log(\lambda_1^3\lambda_2^3) + y_2 \log(\lambda_1^2/\lambda_2) < 0\\
&y_1 \log(\lambda_3^3\lambda_4^3) + y_2 \log(\lambda_3^2/\lambda_4) < 0\,.
\end{aligned}$$

Assuming $\lambda_1\lambda_2 > 1 > \lambda_3\lambda_4$ we can isolate y_1 by dividing out the coefficients (with known signs) to get the following lower and upper bound on y_1:

$$y_1 < -y_2\frac{\log(\lambda_1^2/\lambda_2)}{\log(\lambda_1^3\lambda_2^3)} \;\wedge\; y_1 > -y_2\frac{\log(\lambda_3^2/\lambda_4)}{\log(\lambda_3^3\lambda_4^3)}\,.$$

Evidently the system above has a solution iff

$$-y_2\frac{\log(\lambda_3^2/\lambda_4)}{\log(\lambda_3^3\lambda_4^3)} < -y_2\frac{\log(\lambda_1^2/\lambda_2)}{\log(\lambda_1^3\lambda_2^3)},$$

i.e., iff

$$\frac{\log(\lambda_1^2/\lambda_2)}{\log(\lambda_1^3\lambda_2^3)} < \frac{\log(\lambda_3^2/\lambda_4)}{\log(\lambda_3^3\lambda_4^3)}\,.$$

The truth of the latter inequality can be checked by first using Richardson's algorithm to verify that the left-hand and right-hand expressions are unequal and then calculating to sufficient precision to determine which of the two is greater. (In this case existence of a solution was equivalent to the matrix A having non-zero determinant, but this will not hold for general systems with more constraints.)

4.4 Algorithm

In summary, we have the following algorithm for the positive membership problem in the commutative case:

INPUT: A set of commuting rational matrices $\{A_1,\dots,A_k\}$.

1. Find all joint eigenvalues $\boldsymbol{\lambda}\in\Sigma$ for which both the corresponding right eigenspace $V_{\boldsymbol{\lambda}}$ and the left eigenspace $W_{\boldsymbol{\lambda}}$ are one-dimensional and contain a positive vector. This can be done, e.g., using a decision procedure for the theory of real-closed fields.
2. For each joint eigenvalue $\boldsymbol{\lambda}$ identified in Step 1, compute the corresponding IP-log problem as in Section 4.2 to see if $\boldsymbol{\lambda}$ is dominant. If the IP-log problem is satisfiable then output "YES" and halt.
3. Output "NO".

We conclude:

▶ **Theorem 15.** *The positive membership problem is decidable for commutative semigroups, assuming Schanuel's conjecture is true.*

5 The Non-negative Membership Problem for Commutative Semigroups

We now combine the ideas in Sections 3 and 4 to solve the problem of deciding whether a semigroup of commuting matrices contains a non-negative matrix. For ease of exposition we will assume that the matrices are simultaneously diagonalizable. The general commuting case involves more complicated algebra and is proven in Appendix A.

▶ **Problem 16** (Non-negative Matrix in Commutative Simultaneously Diagonalizable Semigroup)**.** *Given a set of commuting simultaneously diagonalizable $d\times d$ matrices $\{A_1,\dots,A_k\}$ with rational entries, decide whether the semigroup generated by multiplying these matrices together contains a matrix with all its entries greater than or equal to zero.*

First, we refine our notion of dominated recurrences.

▶ **Definition 17** (Positively dominated by p). *Let $\boldsymbol{u} = (u_n)_{n=0}^{\infty}$ be a linear recurrence sequence which is non-degenerate and does not have any polynomial terms in its exponential-polynomial form. Then $\boldsymbol{u} = \sum_{l=1}^{d} c_l \lambda_l^n$ for complex numbers $\lambda_1, \ldots, \lambda_d$ and coefficients c_l. We say $\boldsymbol{u}$ is* positively dominated by term p *(here p (for positive) refers to the index) if*

1. $c_p > 0$,
2. $\lambda_p > 0$,
3. $\forall l \neq p,\ |\lambda_p| > |\lambda_l|$.

Call this predicate $PD_p(\boldsymbol{u})$.

Given a single matrix M, we consider the d^2 recurrences $\boldsymbol{u}^{ij}$ defined by $(u^{ij})_n := e_i^\top M^n e_j$.

We have shown in Section 3 that M being eventually non-negative is equivalent to the decidable condition

$$\forall i \forall j (\boldsymbol{u}^{ij} \text{ is ultimately zero} \vee \exists p\ PD_p(\boldsymbol{u}^{ij})).$$

We now show a similar construction for multiple matrices. Let $A_1, \ldots, A_k \in \mathbb{Q}^{d \times d}$ be a set of commuting simultaneously diagonalizable matrices. The idea is to search for an eventually non-negative matrix $A_1^{m_1} \ldots A_k^{m_k}$.

Let $\boldsymbol{m}$ denote the tuple $(m_1, \ldots, m_k)$. Define the integer parameterized matrix entry recurrence $\boldsymbol{u}^{ij}(\boldsymbol{m})$ by $[\boldsymbol{u}^{ij}(\boldsymbol{m})]_n := e_i^\top [A_1^{m_1} \ldots A_k^{m_k}]^n e_j$.

The existence of an eventually non-negative matrix (and thus, a non-negative matrix) in the semigroup is equivalent to the decidable condition

$$ENN := \exists \boldsymbol{m}\ \forall i \forall j (\boldsymbol{u}^{ij}(\boldsymbol{m}) \text{ is ultimately zero} \vee \exists p\ PD_p(\boldsymbol{u}^{ij}(\boldsymbol{m}))).$$

Let S be a matrix that simultaneously diagonalizes the matrices $A_1, \ldots, A_k$ such that $A_r = S^{-1} D_r S = S^{-1} \operatorname{diag}(\lambda_{r1}, \ldots, \lambda_{rd}) S$. The notation λ_{rl} denotes the lth eigenvalue of A_r (with multiplicity).

Then

$$\begin{aligned}
u_n^{ij}(m_1, \ldots, m_k) &:= e_i^\top [A_1^{m_1} \ldots A_k^{m_k}]^n e_j \\
&= e_i^\top S^{-1} [D_1^{m_1} \ldots D_k^{m_k}]^n S e_j \\
&= e_i^\top S^{-1} [\operatorname{diag}(\lambda_{11}, \ldots, \lambda_{1d})^{m_1} \ldots \operatorname{diag}(\lambda_{k1}, \ldots, \lambda_{kd})^{m_k}]^n S e_j \\
&= e_i^\top S^{-1} \left[\operatorname{diag} \left(\prod_{r=1}^{k} \lambda_{r1}^{m_r}, \ldots, \prod_{r=1}^{k} \lambda_{rd}^{m_r} \right) \right]^n S e_j \\
&= \sum_{l=1}^{d} (S^{-1})_{il} (S)_{lj} \left[\prod_{r=1}^{k} \lambda_{rl}^{m_r} \right]^n .
\end{aligned}$$

Now we have an exponential-polynomial representation for $\boldsymbol{u}^{ij}(\boldsymbol{m})$ with coefficients $(S^{-1})_{il}(S)_{lj}$ and roots $\prod_{r=1}^{k} \lambda_{r1}^{m_r}, \ldots, \prod_{r=1}^{k} \lambda_{rd}^{m_r}$.

We see that $\boldsymbol{u}^{ij}(\boldsymbol{m})$ is positively dominated by p if

1. $(S^{-1})_{ip} (S)_{pj} > 0$,
2. $\prod_{r=1}^{k} \lambda_{rp}^{m_r} > 0$,
3. $\forall l \neq p$ such that $(S^{-1})_{ip} (S)_{pj} \neq 0$, $\left| \prod_{r=1}^{k} \lambda_{rp}^{m_r} \right| > \left| \prod_{r=1}^{k} \lambda_{rl}^{m_r} \right|$.

Let $c(p) = (\arg(\lambda_{1p}), \ldots, \arg(\lambda_{kp}))$ and let $A(p)$ be the (at most) $(d-1) \times k$-matrix defined by $a_{lr} = \log|\lambda_{rl}/\lambda_{rp}|$. Here l runs over elements of $\{1, \ldots, d\}$ apart from p such that $(S^{-1})_{il}\,(S)_{lj} \neq 0$. Let $\boldsymbol{m}$ be the vector $(m_1, \ldots, m_k) \in \mathbb{N}^k$.

Now Condition 2 is equivalent to $c(p)^\top \boldsymbol{m} = 0 \bmod 2\pi$ and Condition 3 is equivalent to $A(p)\boldsymbol{m} < 0$.

For simplicity we introduce the following new notation:

$$\begin{aligned}
Z(i,j,\boldsymbol{m}) &:= \boldsymbol{u}^{ij}(\boldsymbol{m}) \text{ is identically zero}\\
S(i,j,p) &:= (S^{-1})_{ip}\,(S)_{pj} > 0\\
C(p,\boldsymbol{m}) &:= c(p)^\top \boldsymbol{m} = 0 \bmod 2\pi\\
A(i,j,p,\boldsymbol{m}) &:= A(p)\boldsymbol{m} < 0\\
\mathit{NND}(i,j,p,m) &:= Z(i,j,\boldsymbol{m}) \vee (S(i,j,p) \wedge C(p,\boldsymbol{m}) \wedge A(i,j,p,\boldsymbol{m})).
\end{aligned}$$

Note that $A(i,j,p,\boldsymbol{m})$ depends on i and j because we only care about the eigenvalue blocks where the coefficient is non-zero.

Writing out the predicate ENN in full with new notation, we have

$$\begin{aligned}
\mathit{ENN} &:= \exists \boldsymbol{m}\ \forall i \forall j (\boldsymbol{u}^{ij}(\boldsymbol{m}) \text{ is identically zero} \vee \exists p\ \mathit{PD}_p(\boldsymbol{u}^{ij}(\boldsymbol{m})))\\
&\equiv \exists \boldsymbol{m}\ \forall i \forall j (Z(i,j,\boldsymbol{m}) \vee \exists p\ (S(i,j,p) \wedge C(p,\boldsymbol{m}) \wedge A(i,j,p,\boldsymbol{m})))\\
&\equiv \exists \boldsymbol{m}\ \forall i \forall j\ \exists p\ \mathit{NND}(i,j,p,\boldsymbol{m})
\end{aligned}$$

Now observe that the quantifications over i, j and p are finite and range over $\{1, \ldots, d\}$. Thus we can replace the quantifications with finite disjunctions and conjunctions.

Our goal is to move all disjunctions outside the existential quantifier on $\boldsymbol{m}$ so that we can use the IP-log algorithm from Section 4.3 on conjunctions of terms of the form $A\boldsymbol{m} < 0$.

Let f range over functions assigning a particular choice of p to each sequence $\boldsymbol{u}^{ij}(\boldsymbol{m})$. There are $d^{(d \times d)}$ such functions.

$$\begin{aligned}
\mathit{ENN} &\equiv \exists \boldsymbol{m}\ \forall i \forall j (\exists p\ \mathit{NND}(i,j,p,\boldsymbol{m}))\\
&\equiv \exists \boldsymbol{m}\ \wedge_{ij} (\vee_p\ \mathit{NND}(i,j,p,\boldsymbol{m}))\\
&\equiv \exists \boldsymbol{m}\ \vee_f (\wedge_{ij} \mathit{NND}(i,j,p = f(i,j),\boldsymbol{m}))\\
&\equiv \vee_f (\ \exists \boldsymbol{m}\ \wedge_{ij}\ \mathit{NND}(i,j,p = f(i,j),\boldsymbol{m})).
\end{aligned}$$

Essentially this means we non-deterministically choose a p for each sequence $\boldsymbol{u}^{ij}(\boldsymbol{m})$ and then check if there is an $\boldsymbol{m}$ that works for all of them – that makes all the selected p's into real positively dominating terms.

Analysing the elements of $\mathit{NND}(i,j,f(i,j),\boldsymbol{m})$, we see that $Z(i,j,\boldsymbol{m})$ and $S(i,j,f(i,j))$ constraints are trivially checkable. Constraints $\wedge_{ij} C(f(i,j),\boldsymbol{m})$ can be removed iteratively using Masser's theorem. The remaining conjunctions are terms of the form $\wedge_{ij} A(i,j,f(i,j),\boldsymbol{m})$, but since these are linear programs $A(i,j,p)\boldsymbol{m} < \mathbf{0}$ we can simply concatenate the various matrices $A(i,j,p)$ together. We can then use the IP-log algorithm from Section 4.3 to check if there exists an integer solution to the conjunction of these terms.

Of course in practice we would only need to check d different matrices of size $d - 1 \times k$ since the eigenvalues are the same for all sequences.

Thus we have reduced the diagonalizable case to a finite disjunction of predicates, each of which can be reduced to strict homogenous IP-log. More generally, we have:

▶ **Theorem 18.** *The non-negative membership problem is decidable for commutative semigroups, assuming Schanuel's conjecture is true.*

Proof. See Appendix A. ◀

6 Undecidability in the Non-commutative Case

To complement the decidability results for commuting matrices in the preceding sections, in this section we show undecidability of the full version of the membership problem, in which commutativity is not assumed:

▶ **Problem 19** (Non-negative Membership). *Given a set of $d \times d$ matrices with rational entries, decide whether the generated semigroup contains a non-negative matrix.*

The proof of undecidability is by reduction from the threshold problem for probabilistic automata, which is well-known to be undecidable [11].

▶ **Problem 20** (Threshold Problem for Probabilistic Automata). *Given vectors u and v in $\mathbb{Q}^d$ and a matrix semigroup $\mathcal{S}$ generated by stochastic matrices $\{A_1, \ldots, A_k\} \in \mathbb{Q}^{d\times d}$, decide whether there exists a matrix $A \in \mathcal{S}$ such that $u^\top Av \geq 1/2$.*

▶ **Theorem 21.** *The Non-negative Membership Problem is undecidable.*

Proof. Given non-negative integers m, n, write $0^{m\times n}$ for the zero matrix of dimension $m \times n$. Suppose that we are given an instance of the threshold problem for probabilistic automata, defined by vectors $u, v \in \mathbb{Q}^d$ and a matrix semigroup $\mathcal{S}$ generated by stochastic matrices $A_1, \ldots, A_k \in \mathbb{Q}^{d\times d}$. Now consider the semigroup $\mathcal{S}'$ generated by the following matrices of dimension $(d + 2) \times (d + 2)$:

$$U := \begin{bmatrix} 1 & -1/2 & \boldsymbol{u}^\top \\ 0 & 0 & 0^{1\times d} \\ 0^{d\times 1} & 0^{d\times 1} & 0^{d\times d} \end{bmatrix},$$

$$A'_i := \begin{bmatrix} 1 & -1/2 & 0^{1\times d} \\ 0 & 0 & 0^{1\times d} \\ 0^{d\times 1} & 0^{d\times 1} & A_i \end{bmatrix} \quad (i = 1, \ldots, k),$$

$$\text{and } V := \begin{bmatrix} 1 & -1/2 & 0^{1\times d} \\ 0 & 0 & 0^{1\times d} \\ 0^{d\times 1} & \boldsymbol{v} & 0^{d\times d} \end{bmatrix}.$$

Note that matrix A'_i incorporates A_i for $i = 1, \ldots, k$, while the matrices U and V respectively incorporate the initial and final vectors u and v of the probabilistic automaton.

We claim that the semigroup $\mathcal{S}'$ contains a non-negative matrix if and only if there exists a matrix $A \in \mathcal{S}$ such that $u^\top Av \geq \frac{1}{2}$. To this end, consider a string of matrices chosen from the set $\{U, V\} \cup \{A'_1, \ldots, A'_k\}$. Any product B of such a string that does not end with a suffix $UA'_{i_1} \cdots A'_{i_s}V$, for some $i_1, \ldots, i_s \in \{1, \ldots, k\}$, has $B_{1,2} = -\frac{1}{2}$ and hence cannot be a non-negative matrix. It remains to consider products B of strings that do have such a suffix. In this case we have

$$B_{1,2} = (UA'_{i_1} \cdots A'_{i_s}V)_{1,2} = u^\top A_{i_1} \cdots A_{i_s} v - \frac{1}{2},$$

and hence B is only non-negative if $u^\top A_{i_1} \cdots A_{i_s} v \geq \frac{1}{2}$. Since it further holds that $B_{1,1} = 1$ and $B_{i,j} = 0$ for all other entries (i, j), we conclude that exists a non-negative matrix in the semigroup $\mathcal{S}'$ if and only if there exists a matrix $A \in \mathcal{S}$ such that $u^\top Av \geq \frac{1}{2}$. ◀

7 Further work

We leave open the question of quantitative refinements of our decidability results. These include giving complexity upper bounds for the Non-negative and Positive Membership Problems as well as the related question of giving upper bounds on the length of the shortest string of generators that yields a non-negative or positive matrix in a given semigroup. Both questions would seem to be difficult owing to the use of Schanuel's Conjecture in our proofs. Characterising the complexity of determining whether a matrix is eventually non-negative would seem to be more straightforward. We claim that the decision procedure can be implemented in non-deterministic polynomial time. Note that the analogous Eventual Positivity problem is in **PTIME** [18].

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A Non-diagonalizable Case

▶ **Problem 22** (Non-negative Membership for Commutative Semigroup). *Given a set of commuting $d \times d$ matrices $\{A_1, \ldots, A_k\}$ with rational entries, decide whether the semigroup generated by multiplying these matrices together contains a matrix with all its entries greater than or equal to zero.*

It is known that unfortunately simultaneous Jordanization of commuting matrices is not always possible [6]. However, a slightly weaker block diagonal form [19, Thm 12] is possible. Here we put the matrices into block diagonal form, where each block is of the form $\lambda_i I + N$ where N is strictly upper triangular and thus nilpotent.

Let $A_1, \ldots, A_k \in \mathbb{Q}^{d\times d}$ be a set of commuting matrices.

As in Section 5, define the integer parameterized matrix entry recurrence $\boldsymbol{u}^{ij}(\boldsymbol{m})$ by $u^{ij}_n(m_1, \ldots, m_k) := e_i^\top [A_1^{m_1} \ldots A_k^{m_k}]^n e_j$.

Let S be a matrix that simultaneously block-diagonalizes the matrices such that $A_r = S^{-1} BD_r S = S^{-1} \operatorname{diag}(B_{r1}, \ldots, B_{rb}) S$. Here $B_{rl} = \lambda_{rl} I + N_{rl}$ denotes the lth block of A_r, where N_{rl} is strictly upper triangular. Note that for fixed l the various N_{rl} inherit commutativity from the original matrices. Then we have that

$$\begin{aligned}
u^{ij}_n(m_1, \ldots, m_k) &:= e_i^\top [A_1^{m_1} \ldots A_k^{m_k}]^n e_j \\
&= e_i^\top S^{-1} [BD_1^{m_1} \ldots BD_k^{m_k}]^n S e_j \\
&= e_i^\top S^{-1} [\operatorname{diag}(B_{11}, \ldots, B_{1b})^{m_1} \ldots \operatorname{diag}(B_{k1}, \ldots, B_{kb})^{m_k}]^n S e_j \\
&= e_i^\top S^{-1} \left[\operatorname{diag}\left(\prod_{r=1}^{k} B_{r1}^{m_r}, \ldots, \prod_{r=1}^{k} B_{rb}^{m_r} \right) \right]^n S e_j \\
&= [s_{i1}^{-1}, \ldots, s_{id}^{-1}] \operatorname{diag}\left(\left[\prod_{r=1}^{k} B_{r1}^{m_r} \right]^n, \ldots, \left[\prod_{r=1}^{k} B_{rb}^{m_r} \right]^n \right)^n [s_{1j}, \ldots, s_{dj}]^\top.
\end{aligned}$$

Let us examine the structure of the submatrix $\left[\prod_{r=1}^{k} B_{rb}^{m_r}\right]^n$. Recall that $B_{rl} = \lambda_{rl} I + N_{rl}$. Note that any power or product of powers of the nilpotents can be non-zero only upto total degree at most d. We adopt the convention that a zero power indicates the identity matrix of appropriate size. For ease of notation we drop the block subscript and expand out

$$\begin{aligned}\left[\prod_{r=1}^{k} B_{rb}^{m_r}\right]^n &= \left[\prod_{r=1}^{k} (\lambda_r I + N_r)^{m_r}\right]^n \\ &= \prod_{r=1}^{k} \lambda_r^{nm_r} (I + N_r/\lambda_r)^{nm_r} \\ &= \prod_{r=1}^{k} \lambda_r^{nm_r} \cdot \prod_{r=1}^{k} \left(\sum_{i_r=0}^{d} \binom{nm_r}{i_r} (N_r/\lambda_r)^{i_r}\right) \\ &= \prod_{r=1}^{k} \lambda_r^{nm_r} \cdot \left[\sum_{(i_1,\ldots,i_k)=(0,\ldots,0)}^{i_1+\cdots+i_k=d} \left(\prod_{r=1}^{k} \binom{nm_r}{i_r} (N_r/\lambda_r)^{i_r}\right)\right] \\ &= \mathrm{MPoly}(n\mathbf{m}) \prod_{r=1}^{k} \lambda_r^{nm_r}.\end{aligned}$$

Here $\mathrm{MPoly}(n\mathbf{m})$ is shorthand for the block-matrix with entries which are polynomial of degree at most d in the variables $(m_1, \ldots, m_k)$ (scaled by n) with coefficients arising from the nilpotent submatrices.

Substituting this back into our original expression for $u_n^{ij}(m_1, \ldots, m_k)$ we get

$$\begin{aligned}&u_n^{ij}(m_1, \ldots, m_k) := e_i^\top [A_1^{m_1} \ldots A_k^{m_k}]^n e_j \\ &= [s_{i1}^{-1}, \ldots, s_{id}^{-1}] \,\mathrm{diag}\left(\left[\prod_{r=1}^{k} B_{r1}^{m_r}\right]^n, \ldots, \left[\prod_{r=1}^{k} B_{rb}^{m_r}\right]^n\right)^n [s_{1j}, \ldots, s_{dj}]^\top \\ &= [s_{i1}^{-1}, \ldots, s_{id}^{-1}] \,\mathrm{diag}\left(\mathrm{MPoly}_1(n\mathbf{m}) \prod_{r=1}^{k} \lambda_{r1}^{nm_r}, \ldots, \mathrm{MPoly}_b(n\mathbf{m_r}) \prod_{r=1}^{k} \lambda_{rd}^{nm_r}\right)^n [s_{1j}, \ldots, s_{dj}]^\top \\ &= \sum_{l=1}^{d} \left(\mathrm{poly}_l^{ij}(n\mathbf{m}) \prod_{r=1}^{k} \lambda_{rl}^{nm_r}\right) \text{ (after folding in constants).}\end{aligned}$$

Notice that the asymptotic top term in n in the polynomial $\mathrm{poly}_l^{ij}(n\mathbf{m})$ is the homogenous subpolynomial of highest degree in $(m_1, \ldots, m_k)$ - call it $h_l^{ij}(\mathbf{m})$. Once we pick a particular p to be our positively dominating term, we only need the following three conditions for the recurrence to be positively dominated by p:

1. $h_p^{ij}(\mathbf{m}) > 0$,
2. $\prod_{r=1}^{k} \lambda_{rp}^{m_r} > 0$,
3. $\forall l \neq p$ such that $\mathrm{poly}_l^{ij}(n\mathbf{m})$ is not the zero polynomial, $\left|\prod_{r=1}^{k} \lambda_{rp}^{m_r}\right| > \left|\prod_{r=1}^{k} \lambda_{rl}^{m_r}\right|$.

Via the same algebraic manipulations as in the diagonalizable case, checking

$$ENN := \exists m \; \forall i \forall j (\boldsymbol{u}^{ij}(\boldsymbol{m}) \text{ is identically zero} \vee \exists p \; PD_p(\boldsymbol{u}^{ij}(\boldsymbol{m}))),$$

reduces to solving conjunctions of the form

$$\wedge_{ij} (h_l^{ij}(\mathbf{m}) > 0 \wedge \mathbf{c}(p)^\top \mathbf{m} = 0 \bmod 2\pi \wedge \mathbf{A}(p)\mathbf{m} < 0)$$

over the integers.

We can eliminate the second conjunct using Masser's theorem. Now suppose there exists a real solution on the unit sphere to $h_l^{ij}(\mathbf{m}) > 0 \wedge \mathbf{A}(p)\mathbf{m} < 0$. By openness, there exists a rational solution nearby. Since both these conjuncts are homogenous, $\mathbf{m}$ is a solution iff $n\mathbf{m}$ is a solution, for all real $n > 0$. Thus we may clear denominators from the rational solution to obtain an integer solution $\mathbf{m}$ to *ENN*.

The sentence

$$\exists \mathbf{m} \in \mathbb{R}^k : h_l^{ij}(\mathbf{m}) > 0 \wedge \mathbf{A}(p)\mathbf{m} < 0$$

can be written in the first order theory of the reals with exponentiation, which is decidable assuming Schanuel's conjecture as shown by Wilkie and Macintyre [14].

We now need to prove that failing to find such a real solution implies that the semigroup does not contain a non-negative matrix. Suppose that there exists some $m = (m_1, \dots, m_k)$ such that $A_1^{m_1} \dots A_k^{m_k}$ is non-negative. Then $(A_1^{m_1} \dots A_k^{m_k})^n$ is non-negative for all n. By Proposition 6, each individual recurrence is ultimately zero or must have a strictly dominant top term. Thus m satisfies $Am < 0$ and $c^\top m = 0 \bmod 2\pi$ for the appropriate A and c. The top term (as a function of n) has a polynomial coefficient which is the homogenous polynomial we identified above. Thus the homogenous polynomial is non-negative for m, completing the requirements necessary for the sentence $\exists \mathbf{m} \in \mathbb{R}^k : h_l^{ij}(\mathbf{m}) > 0 \wedge \mathbf{A}(p)\mathbf{m} < 0$ to be true.

We conclude:

▶ **Theorem 23.** *The non-negative membership problem is decidable for commutative semigroups, assuming Schanuel's conjecture is true.*

B Positive Rational Sequences are Dominated

We need the following classical results from Perron-Frobenius theory (see, e.g., [17, Chap. 8]).

▶ **Theorem 24.**

1. *If $A \geq 0$ is irreducible then it has* cyclic peripheral spectrum*, i.e., its eigenvalues of maximum modulus have the form $\{\rho, \rho\omega, \dots, \rho\omega^{k-1}\}$, where $\rho > 0$, k is a positive integer, and ω is a primitive k-th root of unity.*
2. *If a non-negative irreducible matrix A has only one eigenvalue on the spectral circle it is called a primitive matrix. If A is primitive, the pointwise limit $\lim_{n\to\infty}(A/\rho(A))^n$ exists and is a strictly positive matrix.*

We now prove Berstel's result for matrix entry recurrences.

▶ **Proposition 25.** *Let $M \in \mathbb{Q}^{d\times d}$ be a non-negative non-degenerate matrix. Then for all $i, j \in \{1, \dots, d\}$ the LRS $u_n = (M^n)_{i,j}$ is dominated.*

Proof. Since $M \geq 0$ there exists a permutation matrix P such that M can be written in the form $M = PUP^{-1}$, where $U \geq 0$ is block upper triangular. It follows that there exist $i', j' \in \{1, \dots, d\}$ such that

$$(M^n)_{i,j} = (PU^nP^{-1})_{i,j} = (U^n)_{i',j'}$$

for all $n \in \mathbb{N}$. Now write

$$U = \begin{pmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,e} \\ 0 & B_{2,2} & \dots & B_{2,e} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & B_{e,e} \end{pmatrix},$$

where all the blocks in U are non-negative and the diagonal blocks $B_{1,1}, B_{2,2}, \dots, B_{e,e}$ are irreducible. Then

$$(U^n)_{i',j'} = \sum_{\substack{l_1<l_2<\dots<l_m \\ n_1+n_2+\cdots n_m = n-(m-1)}} e_{i'}^\top B_{l_1,l_1}^{n_1} B_{l_1,l_2} B_{l_2,l_2}^{n_2} \cdots B_{l_{m-1},l_m} B_{l_m,l_m}^{n_m} e_{j'}\,, \tag{1}$$

where the sum runs over all positive integers m and strictly increasing sequences of block indices $l_1 < \cdots < l_m$.

Consider a single block $B_{l,l}$ along the diagonal. Since it is irreducible and non-negative, it has cyclic peripheral spectrum. By our assumption of non-degeneracy, $r_l := \rho(B_{l,l}) \geq 0$ is the only eigenvalue on the spectral circle. Thus $B_{l,l}$ is primitive and by our second Perron-Frobenius result above, asymptotically $B_{l,l}^n / r_l^n \sim C_l$ where C_l is a positive matrix and the asymptotic equivalence relation $\sim$ applies entry-wise. Let r_{max} be the maximum spectral radius of a block $B_{l,l}$ lying on a path from i' to j'.

We now analyse the asymptotic behavior of the normalized recurrence $(U^n)_{i',j'}/r_{max}^n$. Consider a summand S_n in $(U^n)_{i',j'}/r_{max}^n$. Replacing the diagonal blocks with their asymptotic limits,

$$S_n \sim \left(\frac{r_{l_1}}{r_{max}}\right)^{n_1} \left(\frac{r_{l_2}}{r_{max}}\right)^{n_2} \cdots \left(\frac{r_{l_m}}{r_{max}}\right)^{n_m} e_{i'}^{\top} C_{l_1} B_{l_1,l_2} C_{l_2} \cdots B_{l_{m-1},l_m} C_{l_m}\, e_{j'}.$$

Although the number of terms in the sum grows polynomially in n, we see that each term with some $r_{l_k} < r_{max}$ that does not have n_k constant tends to zero exponentially quickly. The remaining summands in $(U^n)_{i',j'}/r_{max}^n$ are thus those where n_k is non-constant only for blocks with $\rho(B_{l_k,l_k}) = r_{max}$. Let K be the sum of the constant powers for non-maximal blocks in such a summand Q_n, and P be the product of the powers of non-maximal r_{l_k}. Then

$$Q_n \sim r_{max}^n \cdot (P/r_{max}^{K+m-1}) \cdot e_{i'}^{\top} C_{l_1} B_{l_1,l_2} C_{l_2} \cdots B_{l_{m-1},l_m} C_{l_m}\, e_{j'}.$$

Observe that the coefficient of r_{max}^n is a product of spectral radii and non-negative matrices, and is thus non-negative. This implies different such terms containing r_{max}^n cannot cancel out. So $e_{i'}^{\top} U^n e_{j'} \sim A(n) \cdot r_{max}^n$ for some polynomial A with positive coefficients depending on the non-diagonal blocks and the spectral radii of the diagonal blocks. Thus $(M^n)_{i,j}$ is either dominated by r_{max} or ultimately zero (the latter in case $r_{max} = 0$ or the sum in (1) is empty). ◀