One n Remains to Settle the Tree Conjecture

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Abstract

In the famous *network creation game* of Fabrikant et al. [11] a set of agents play a game to build a connected graph. The n agents form the vertex set V of the graph and each vertex $v \in V$ buys a set E_v of edges inducing a graph $G = (V, \bigcup\limits_{v \in V} E_v)$. The private objective of each vertex is to minimize the sum of its building cost (the cost of the edges it buys) plus its connection cost (the total distance from itself to every other vertex). Given a cost of α for each individual edge, a long-standing conjecture, called the *tree conjecture*, states that if $\alpha > n$ then every Nash equilibrium graph in the game is a spanning tree. After a plethora of work, it is known that the conjecture holds for any $\alpha > 3n - 3$. In this paper we prove the tree conjecture holds for $\alpha > 2n$. This reduces by half the open range for α with only $(n - 3, 2n)$ remaining in order to settle the conjecture.

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1 Introduction

A foundational motivation for the field of algorithmic game theory was to understand the evolution and functionality of networks, specifically, the Internet; see Papadimitriou [15]. Of particular fascination concerned how the actions of self-motivated agents affected the structure of the World Wide Web and social networks more generally. An early attempt to study this conundrum was undertaken by Fabrikant et al. [11] with their now classical *network creation game*. Despite its extreme simplicity, their model (detailed below) has become highly influential for two reasons. First, it inspired the development of a wide range of network formation models, [2],[5]. Second, it has lead to one of the longest-standing open problems in algorithmic game theory, namely, the tree conjecture. The latter motivates this research. So let us begin by describing the model and the conjecture.

1.1 The Network Creation Game

Consider a set of vertices, $V = \{1, 2, \ldots, n\}$, who attempt to construct a connected graph between themselves. To do this, each vertex (agent) can purchase individual edges for a fixed cost of α each. Consequently, a strategy for vertex $v \in V$ is a set of (incident) edges E_v. Together the strategies of the agents forms a graph $G = (V, E)$, where $E = E_1 \cup E_2 \cup \cdots \cup E_n$. In the *network creation game*, the objective of each vertex is to minimize the sum of its building and its connection cost. The *building cost* for vertex v is $\alpha \cdot |E_v|$, the cost of all the edges it buys. The *connection cost* is $D(v) = \sum_{u:u \neq v} d_G(u, v)$, the sum of the distances in G of v to every other vertex, where $d_G(u, v) = \infty$ if there is no path between u and v. That is, the total cost to the vertex is $c_v(E) = \alpha \cdot |E_v| + D(v)$.

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Given the different objectives of the agents, we study Nash Equilibria (NE) in the network creation game. A Nash equilibrium graph is a graph $G = (V, E)$ in which no vertex v can reduce its total cost by changing its strategy, that is, by altering the set of edges it personally buys, given the strategies of the other vertices remain fixed. Thus E_v is a best response to $(E_u)_{u \neq v}$, for every vertex v. Observe that every Nash equilibrium graph must be a connected graph. Attention in the literature has focused on whether or not every Nash equilibrium graph is minimal, that is, a spanning tree.

1.2 The Tree Conjecture

The network creation game was designed by Fabrikant et al. [11]. They proved that any network equilibrium graph which forms a tree costs at most 5 times that of a star, i.e. the optimal network. They proposed that for α greater than some constant, every Nash equilibrium graph is a spanning tree. This was the original *tree conjecture* for network creation games.

In the subsequent twenty years, incremental progress has been made in determining the exact range of α for which all Nash equilibria are trees. Albers et al. [1] demonstrated that the conjecture holds for $\alpha \geq 12n \log n$. However, they also provided an counterexample to the original tree conjecture. Moreover, Mamageishvili et al. [13] proved the conjecture is false for $\alpha \leq n - 3$.

As a result, a revised conjecture took on the mantle of the *tree conjecture*, namely that every Nash equilibrium is a tree if $\alpha > n$. Mihalák and Schlegel [14] were the first to show that this tree conjecture holds for $\alpha \geq c\dot{n}$ for a large enough constant c, specifically, $c > 273$. Since then, the constant has been improved repeatedly, by Mamageishvili et al. [13], then Àlvarez and Messegué [3], followed by Bilò and Lenzner [6], and finally by Dippel and Vetta [10] who proved the result for $\alpha > 3n - 3$.

We remark that extensions and variations of the network creation game have also been studied; we refer the interested reader to [4, 7, 8, 9, 16, 17].

1.3 Our Contribution

In this paper we improve the range in which the tree conjecture is known to hold from $\alpha > 3n - 3$ [10] to $\alpha > 2n$:

▶ **Theorem 1.** *If G is a Nash equilibrium graph for the network creation game (n, α) and $\alpha > 2n$, then G is a tree.*

Our high-level strategy to prove Theorem 1 is straightforward: we assume the existence of a biconnected component H in a Nash equilibrium and then prove, via consideration of a collection of strategy deviations, that some vertex has a better strategy, provided $\alpha > 2n$. This contradicts the best response conditions and proves that every Nash equilibrium graph is a spanning tree.

Our approach applies some prior methodologies and combines them with some original tools. One important technique we exploit is that of *min-cycles*, introduced by Lenzner [12]. A min-cycle is a cycle in a graph with the property that, for every pair of vertices in the cycle, the cycle contains a shortest path between the pair. For certain values of α, currently $\alpha > 2n - 3$, min-cycles are known to have the nice property that each vertex in the cycle buys exactly one edge of the cycle. This property has been leveraged in various ways [3, 6, 10] to show that no min-cycles can exist in a Nash equilibrium graph for large enough α. Importantly, the smallest cycle in any graph must be a min-cycle. Consequently, for large enough α, there can be no smallest cycle and, thus, no cycle at all. This implies that, for such an α, all Nash equilibrium graphs are trees.

The techniques we develop concern the analysis of the presupposed biconnected component H in the Nash equilibrium. The existence of H implies the existence of a special vertex, called r, which has the lowest connection cost amongst all vertices in H. That is, $D(r) \leq D(v)$ for all $v \in H$. Given r, we take a shortest path tree T in G rooted at r. The key is to exploit the structural properties inherent in T. To derive these properties, we design a new class of strategy deviations available to vertices in H. Because these deviations must be non-improving responses, they impose numerous beneficial restrictions on the edges in H. In particular, we can then show that the sum of degrees of vertices within H exceeds 2 times the number of edges within H. This absurdity proves the Nash equilibrium graph is a tree.

1.4 Overview of Paper

This paper has three main sections. Section 2 consists of preliminaries and contains three things. First, we examine the structure of Nash equilibrium graphs which contain a hypothesized biconnected component H. Second, we discuss min-cycles, which appear in several previous papers, each of which contains a useful lemma that we take advantage of in this paper. Third, we introduce some new lemmas pertaining to T, the shortest path tree mentioned above.

Section 3 has two parts. The first presents a set of three deviation strategies. That is, three specific ways a vertex in a biconnected component might consider changing which edges it buys in order to decrease its personal cost. For a Nash equilibrium graph these alternate strategies cannot reduce the personal cost, and this allows us to derive deviation bounds for that vertex and the edges it buys. The second part uses these deviations bounds to prove claims about the structure of the graph. In particular, we show that vertices which buy edges outside T cannot be too close to one another.

Finally, Section 4 presents two main lemmas. The first shows that, in a set U of three vertices adjacent in T, all at different depths, where the lowest buys an edge outside T, only one vertex in U can have degree two in H. The second lemma bounds the number of times such sets can intersect. These two lemmas are the core of the proof of Theorem 1. After proving the main result, the paper culminates with concluding remarks on the future of the tree conjecture.

2 Preliminaries

Recall our basic approach is to prove by contradiction the non-existence of a biconnected component in a Nash equilibrium graph. Accordingly, we begin in Section 2.1 by introducing the notation necessary to analyse biconnected components in network creation game graphs. To disprove the existence of a biconnected component it suffices to disprove the existence of a cycle. Of particular importance here is the concept of a min-cycle. So in Section 2.2 we present a review of min-cycles along with a corresponding set of fundamental lemmas which appear in [12, 6, 13]. Finally, in Section 2.3, we prove a collection of technical results concerning the shortest path tree T that we will utilize throughout the rest of the paper. Lemma 9 is an especially useful tool for this method, and could prove valuable in future work on this problem.

2.1 Biconnected Components

Given a subgraph W of a graph G, we let $d_W(v, u)$ denote the distance between v and u in W. In particular, $d(u, v) = d_G(u, v)$ is the distance from v to u in the whole graph. Let $D(v) = \sum_{u:u \neq v} d_G(u, v)$ be the *connection cost* for vertex v, the sum of the distances from v to every other vertex.

In a Nash equilibrium graph G containing biconnected components, we will refer to the largest biconnected component as H. For a vertex $v \in H$, we denote by $C(v)$ a smallest cycle containing v, breaking ties arbitrarily. Let $r \in H$ be a vertex whose connection cost is smallest amongst all the vertices in H. Once built, an edge uv of G can be traversed in either direction. Thus G is an undirected graph. However, it will often be useful to view G as a directed graph. Specifically, we may orient uv is from u to v if the edge was bought by u and orient it from v to u if it was bought by v.

Given r, we will make heavy use of the shortest path tree T rooted at r. There may be multiple options for the choice of T. So we insist our choice of T has the following property: For all shortest paths P between r and any vertex u which is directed from r to u, we have $P \subseteq T$. The proof of Lemma 3 shows that we will never have two shortest paths directed from r to the same vertex v, so this choice of T is well-defined. Next, given an edge uv where u is the parent of v in T, we say that uv is a *down-edge* of T if the edge was bought by u; otherwise we say it is an *up-edge* of T. We denote by $T^{\downarrow}$ the set of down-edges and by $T^{\uparrow}$ the set of up-edges of T. Let $T(v)$ be the subtree rooted at v in the tree T rooted at r. If $uv \in T^{\downarrow}$ then we define $T(uv) = T(v)$. Otherwise, if $uv \notin T^{\downarrow}$ then we define $T(uv) = \emptyset$.

Finally, there remain two types of sets we need to define. The first type is the "S-sets" , which are sets of vertices. In a graph G, with a subgraph W, $S_W(v)$ is the set of vertices w with the following property: a shortest path in G from w to W contains v. Therefore $S_W(v)$ will always contain v, and it will contain no vertices of $W \setminus v$. W can even be a single vertex u, i.e. $S_u(v)$ is the set of all vertices in G with a shortest path to u containing v. The second type is the "X-sets" , which are sets of edges. For largest biconnected component H of G and the shortest path tree T, the set X_0 is the set of all edges in $H \setminus T$, or *out-edges*. For all integers $i \geq 1$, X_i contains the set X_{i-1} as well as all edges $uv \in H \cap T^{\downarrow}$ bought by the parent u and where the child v buys an edge of X_{i-1}. The X_i sets can be thought of as the set of all out-edges and all down-edges in directed paths of length $\leq i$ to a vertex which buys an out-edge. Sometimes it will be useful to include in the X-sets all the edges $vu \in H \cap T^{\uparrow}$, bought by child v to parent u. To do this, we add superscript $+$ to the X set. For instance $X_0^+ = X_0 \cup T^{\uparrow}$ is the set of all up-edges and out-edges, which happens to be the set of all edges e with $T(e) = \emptyset$. Similarly, $X_i^+ = X_i \cup T^{\uparrow}$.

2.2 Min-Cycles

Recall, a *min-cycle* is a cycle in a graph with the property that, for every pair of vertices in the cycle, the cycle contains a shortest path between the pair. This concept, introduced by Lenzner [12], has proved very fruitful in the study of Nash equilibria in network creation games. In particular, we now present three useful lemmas concerning min-cycles that appear in [12, 6, 13], respectively. For completeness and because some of the ideas used are informative, we include short proofs of these three lemmas; similar proofs appeared in [10]

▶ **Lemma 2** ([12]). *The smallest cycle C containing an edge e is a min-cycle.*

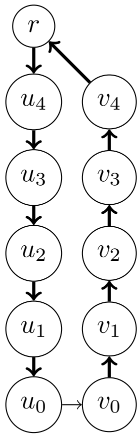


Figure 1 In the diagram, edges of T are bold. Vertex u_0 buys an edge $u_0v_0 \in X_j$ for all $j \geq 0$. Similarly, each vertex u_i buys edge $u_iu_{i-1} \in X_j$ for all $4 \geq i \geq 1$, $j \geq i$. Each vertex v_i buys $v_iv_{i+1} \in X_j^+$ for all $3 \geq i \geq 0$, $j \geq 0$. v_4 buys $v_4r \in X_j^+$ and r buys $ru_4 \in X_k$ for $j \geq 0$, $k \geq 5$.

Proof. Consider the smallest cycle C containing an edge e. Suppose for the sake of contradiction that there are two vertices $u, v \in C$ such $d_G(u, v) < d_C(u, v)$. Without loss of generality, suppose the shortest path between u and v, labelled P, lies entirely outside C. Note that C contains two paths between u and v. Let Q be the path of C from u to v that contains e. Then $P \cup Q$ is a cycle containing e that is strictly smaller than C, a contradiction. ◀

For an edge e or vertex u in a biconnected component H, we may write $C(e), C(u)$ to denote the smallest cycle containing edge e, vertex u respectively, breaking ties arbitrarily. A nice property of min-cycles is that every vertex in the min-cycle buys exactly one edge of the cycle.

▶ **Lemma 3** ([6]). *Let $\alpha > 2(n-1)$. Every min-cycle in a Nash equilibrium graph G is directed.*

Proof. Let C be a min-cycle that is *not* directed. Then there is a vertex v that buys two edges of the cycle, say vx and vy. Let $uw \in C$ be an edge furthest from v (if $|C|$ is odd, there is a unique choice, otherwise choose one of two furthest edges). Let u buy uw. We've chosen uw so that both u and w have a shortest path to v which does not contain uw, therefore every vertex in G has a shortest path to v without uw. Without loss of generality, let $d(u, y) < d(u, v)$. Then because of how we chose uw, both v and x have a shortest path to u which does not contain vx, therefore every vertex in G has a shortest path to u without vx.

First, consider that if u sells uw and buys uv, no vertices become farther from v. It follows, by the Nash equilibrium conditions, that

$$D(u) \leq D(v) + n - 1 \tag{1}$$

because $D(u)$ must be less than the cost to swap uw for uv and use the edge uv in followed by the shortest path from v to each of the $n-1$ vertices.

On the other hand, v can sell *both* vx and vy and instead buy the edge vu without increasing the distance from u to any other vertex. All of the vertices which used vy in their shortest paths to u become closer to u. It follows, by the Nash equilibrium conditions, that

$$D(v) \leq D(u) + n - 1 - \alpha \tag{2}$$

Together (1) and (2) give $D(v) \leq D(v) + 2(n-1) - \alpha < D(v)$. This contradiction implies that C must be a directed cycle. ◀

Furthermore, we may now obtain a lower bound on the girth of a Nash equilibrium graph in terms of the cost α.

▶ **Lemma 4** ([13]). *Any cycle C in a Nash equilibrium graph has $|C| \geq \frac{2\alpha}{n} + 2$.*

Proof. Take a minimum length cycle $C = v_0, v_1 \dots, v_{k-1}, v_k = v_0$ in G. First assume $e_i = v_i v_{i+1}$ is bought by v_i, for each $0 \leq i \leq k-1$. Now, for each vertex $u \in V$, we define a set $L_u \subseteq \{0, 1, \dots, k-1\}$ as follows. We have $i \in L_u$ if and only if *every* shortest path from $v_i \in C$ to u uses the edge e_i.

We claim $|L_u| \leq \frac{|C|-1}{2}$ for every vertex u. If not, take a vertex u with $|L_u| > \frac{|C|-1}{2}$. Let $d(v_i, u)$ be the shortest distance between u and $v_i \in C$. Next give v_i a label $\ell_i = d(v_i, u) - d(v_{i+1}, u)$. Observe that $\ell_i \in \{-1, 0, 1\}$. Furthermore, the labels sum to zero as

$$\sum_{i=0}^{|C|-1} \ell_i \;=\; \sum_{i=0}^{|C|-1} (d(v_i, u) - d(v_{i+1}, u)) \;=\; \sum_{i=0}^{|C|-1} d(v_i, u) - \sum_{i=1}^{|C|} d(v_i, u) \;=\; 0$$

Now take a vertex v_i in C that uses e_i in *every* shortest path to u; that is, $i \in L_u$. Then $\ell_i = 1$ and $\ell_{i-1} \geq 0$. In particular, if $|L_u| > \frac{|C|-1}{2}$ then there are $> \frac{|C|-1}{2}$ positive labels and $> 1 + \frac{|C|-1}{2}$ non-negative labels. Hence, there are $< |C| - \left(1 + \frac{|C|-1}{2}\right) = \frac{|C|-1}{2}$ negative labels. But then the sum of the labels is strictly positive, a contradiction.

Now, as $|L_u| \leq \frac{|C|-1}{2}$ for every vertex u, there must exist a v_i, e_i pair where v_i needs e_i for its shortest paths to $\leq \frac{n}{2}$ vertices. For the vertices that do require e_i in their shortest paths, deleting e_i increases their distance by $\leq |C| - 2$, as we can replace e_i with $C \setminus e_i$ in those paths. Therefore, if v_i sells e_i its cost increases by $\leq (|C| - 2) \cdot \frac{n}{2} - \alpha$. This must be non-negative by the Nash equilibrium conditions. Rearranging, we have $\frac{2\alpha}{n} + 2 \leq |C|$.

Now consider the other case, where some vertex v_i buys $e_1 = v_i v_{i+1}$ and $e_2 = v_i v_{i-1}$. Without loss of generality, v_i needs e_1 for its shortest paths to $\leq \frac{n}{2}$ vertices. Therefore, just as in the previous case, if v_i sells e_1 its cost increases by $\leq (|C| - 2) \cdot \frac{n}{2} - \alpha$. This must be non-negative by the Nash equilibrium conditions. Again, rearranging, we have $\frac{2\alpha}{n} + 2 \leq |C|$. ◀

An important immediate consequence of Lemma 4 is that we may assume that the girth of any Nash equilibrium graph of interest here is at least 7.

▶ **Corollary 5.** *Let $\alpha > 2n$. All cycles in a Nash equilibrium graph have length at least* 7.

2.3 The Shortest Path Tree T

Let us now consider the shortest path tree T rooted at the vertex r with smallest connection cost amongst the vertices in the biconnected component H. In this section, we present some technical lemmas that give useful insights into the structure of this tree T. We begin by upper bounding the size of any subtree in T.

▶ **Lemma 6.** *If $v \neq r$ then $|T(v)| \leq \frac{n}{2}$.*

Proof. Recall T is the shortest path tree of the vertex r with the smallest connection cost and $T(v)$ is the subtree of T rooted at v. For any vertex $x \in T(v)$, it immediately follows that $d(v, x) = d(r, x) - d(v, r)$. On the other hand, for any vertex $x \notin T(v)$, the triangle inequality implies that $d(v, x) \leq d(r, x) + d(v, r)$. Thus

$$
\begin{aligned}
D(v) &\leq D(r) - d(v,r) \cdot |T(v)| + d(v,r) \cdot (n - |T(v)|) \\
&= D(r) + d(v,r) \cdot (n - 2 \cdot |T(v)|)
\end{aligned}
$$

But $D(r) \leq D(v)$. Therefore $d(v,r) \cdot (n - 2 \cdot |T(v)|) \geq 0$ and rearranging gives $|T(v)| \leq \frac{n}{2}$. ◀

The next two lemmas concern the properties of paths related to any vertex u that buys an edge $uv \in X_i$ where $i \leq 2$.

▶ **Lemma 7.** *Let $\alpha > 2n$. If u buys $uv \in X_i$, for some $i \leq 2$, then all $w \in T(uv)$ have a path to r of length $\leq d(w,r) + 2i$ which does not contain u.*

Proof. If $uv \in X_0$ then, as $uv \notin T^{\downarrow}$, we have $|T(uv)| = \emptyset$. Thus the claim is trivially true.

So assume $uv \notin X_0$. Then uv is the first edge in a path of length i in T from u to a vertex x which buys $xy \in H \setminus T$. Thus, there is a path P_0 in T of length $\ell(P_0) = i - 1$ from v to x. Now let P_1 be the path from y to r in T. Observe that P_1 does not contain u; otherwise xy and T would define a cycle of length ≤ 6 (because $i \leq 2$), contradicting Corollary 5. Note also that P_1 has length $\ell(P_1) \leq d(r,v) + i$ because T is a shortest path tree.

Next, let P_2 be the path from w to v. Note P_2 has length $d(w,r) - d(v,r)$ and also does not contain u. Consequently P_2, P_0, xy, P_1 is a path from w to r. This path has length at most

$$
\begin{aligned}
\ell(P_2) + \ell(P_0) + 1 + \ell(P_1) &\leq (d(w,r) - d(v,r)) + (i-1) + 1 + (d(v,r) + i) \\
&= d(w,r) + 2i
\end{aligned}
$$

Furthermore this path does not contain u. ◀

▶ **Lemma 8.** *Let $\alpha > 2n$. If u buys $uv \in X_i$, for some $i \leq 2$, then $d(u,r) \geq \lfloor \frac{|C(u)|}{2} \rfloor - i$.*

Proof. So $uv \in X_i$ is associated with an edge $xy \in X_0$, where $xy = uv$ if $uv \in X_0$. Let C be the cycle defined by $T + xy$. Because T is a shortest path tree it cannot be the case that y is an ancestor of x in T. Neither is $y \in T(u)$; if so, $d(u,y) \leq 3$ and therefore $\ell(C) \leq 6$, contradicting Corollary 5. Thus $y \notin T(u)$ and, consequently, $uv \in C$.

Let z be the lowest common ancestor of x and y. Then C is a path from z to x, plus the edge xy, plus a path from y to z. So it has length

$$
d(x,z) + 1 + d(y,z) \leq 2d(x,r) + 2 = 2d(u,r) + 2i + 2
$$

Observe that to achieve this maximum length, we must have $d(x,z) = d(x,r) = d(y,r) - 1$, and C must consist of two shortest paths from y to r, meaning C will also contain r.

Next consider the smallest cycle $C(u)$ containing u. Suppose for the sake of contradiction that $d(u,r) \leq \lfloor \frac{|C(u)|}{2} \rfloor - i - 1$. If $|C(u)|$ is odd, then $2d(u,r) + 2i + 3 \leq |C(u)|$, which contradicts the choice of $C(u)$. Therefore, $|C(u)|$ is even and $2d(u,r) + 2 + 2i \leq |C(u)|$. However, $|C| \leq 2d(u,r) + 2i + 2$. Thus $|C| = |C(u)|$ and therefore C a min cycle, by Lemma 2. Furthermore, C is directed by Lemma 3. This is a contradiction because then xy is in a directed shortest path, and must be part of T and not an edge in X_0. Hence $d(u,r) \geq \lfloor \frac{|C(u)|}{2} \rfloor - i$, as desired. ◀

We present one last technical lemma in this section. We remark that this lemma is of critical value in our analysis and, we believe, may be of importance in achieving future improvements.

▶ **Lemma 9.** *If uvw is a directed path in H and $\deg_H(v) = 2$, then $|S_H(v)| \geq \frac{\alpha}{2(|C(v)|-3)}$. If, in addition, uvw is a directed path of down-edges in T, then $|S_H(v)| \geq |T(w)|$.*

Proof. Consider the strategy change of u selling uv and buying uw. As $\deg_H(v) = 2$, the only vertices which become farther from u are those of $S_H(v)$. They all become farther from u by 1. All the vertices in $S_u(w)$ conversely, become closer to u by 1. Therefore, by the equilibrium conditions, we must have $|S_H(v)| \geq |S_u(w)|$, otherwise u has an incentive to switch strategies. Further, if uvw is a path of down-edges in T, then because u is the ancestor of w and T is a shortest path tree, $T(w) \subseteq S_u(w)$. Thus $|S_H(v)| \geq |T(w)|$, as desired.

Now consider a second strategy change in which u deletes uv. There are two sets of vertices whose distance to u increases. Shortest paths from u to $S_H(v)$ go from using uv to using $C(v) \setminus uv$; note that as $\deg_H(v) = 2$ it must be that case that $uv, vw \in C(v)$. The distance of these vertices from u then increase by $|C(v)| - 2$. Meanwhile, paths from u to $S_u(w)$ go from using uvw to using at most $C(v) \setminus \{uv, vw\}$, an increase in distance of at most $|C(v)| - 4$. No other vertices have increased distance to u. Therefore u's cost as a result changes by at most

$$(|C(v)| - 2) \cdot |S_H(v)| + (|C(v)| - 4) \cdot |S_u(w)| - \alpha \leq (2|C(v)| - 6) \cdot |S_H(v)| - \alpha$$

Here the inequality holds as $|S_H(v)| \geq |S_u(w)|$. This change must non-negative by the equilibrium condition. This implies $|S_H(v)| \geq \frac{\alpha}{2(|C(v)|-3)}$, as desired. ◀

3 A Class of Deviation Strategies

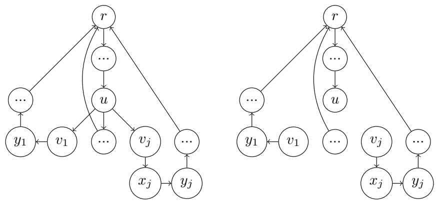
We now present a new class of deviation strategies. Specifically, in Section 3.1 we study three related strategies that a vertex in the biconnected component may ponder. Concretely, we derive bounds on the change in cost to the vertex resulting from using these strategy changes. In particular, we will use the fact that, at a Nash equilibrium, these cost changes must be non-negative. Then, in Section 3.2, we apply these deviation strategies to make three substantial observations concerning vertices that buy multiple edges of specific types in H.

3.1 Three Deviation Strategies

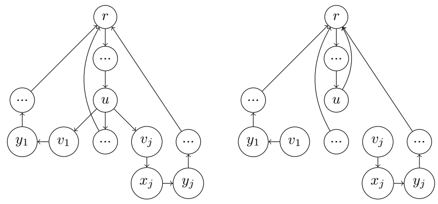
As stated, we now introduce a class of deviation strategies. The first, shown in Figure 2a, involves a vertex u selling edges of X_2. Lemma 7 guarantees that the graph is still connected without these edges. Now u's cost will change: it saves α for each edge sold, but its distance to many vertices may increase by varying amounts, requiring us to bound the new total distance to all vertices. The second deviation, depicted in Figure 2b, is very similar to the first. The only change is that u buys the edge ur. This reduces the saving by α for the extra edge bought, but it also reduces the bound on the increased distance to the other vertices considerably. Finally, the third deviation, pictured in Figure 2c, is slightly different than the second. The bound given is weaker, because u may sell edges of X_2^+, rather than just X_2. Selling the edge to u's parent reduces some of the savings from the previous strategies.

▶ **Lemma 10.** *If $\alpha > 2n$, u buys $uv_1 \in X_{i_1}, ..., uv_j \in X_{i_j}$ for $j \geq 1$, $i_k \leq 2$, for all $k, 1 \leq k \leq j$, $u = u_0, u_1, ...u_{d(u,r)-1}, u_{d(u,r)} = r$ is the path from u to r in T, then the change of u selling $uv_1, ..., uv_j$ results in a change of cost of at most*

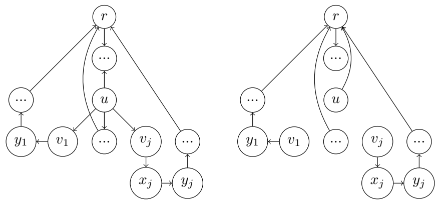
$$d(u,r) \cdot n - \sum_{l=0}^{d(u,r)-1} 2 \cdot |T(u_l)| - j \cdot \alpha + \sum_{k=1}^{j} (2i_k + 2d(u,r)) \cdot |T(uv_{i_k})| \tag{3}$$



(a) Application of Lemma 10. u sells its edges to v_1,..., v_j.



(b) Application of Corollary 11. u sells its edges to v_1,..., v_j and buys edge ur.



(c) Application of Corollary 12. u sells its edges to v_1,..., v_j and buys edge ur.

Figure 2 The three main strategy changes for vertex u.

Proof. We commence by bounding $D(u)$ before the changes, by comparing it to $D(r)$. By the triangle inequality, u's distance to any vertex v is at most $d(r,u) + d(r,v)$ because there is a path from u to r and a path from r to v. This is the bound we use for vertices in $T(r) \setminus T(u_{d(u,r)-1})$. For other vertices this is obviously not the shortest path. For instance, for any vertex $v \in T(u)$, $d(u,v) = d(r,v) - d(r,u)$. Furthermore, for l in $1 \leq l \leq d(u,r)$, the length of u's shortest path to $v \in |T(u_l) \setminus T(u_{l-1})|$ differs from r's by $(d(u,r) - 2l)$. Therefore we have the bound:

$$D(u) \leq D(r) - d(u,r) \cdot |T(u)| - \sum_{l=1}^{d(u,r)} (d(u,r) - 2l) \cdot |T(u_l) \setminus T(u_{l-1})|$$

Now $D(r) \leq D(u)$. Moreover, if we rearrange, we can more concisely write this as

$$0 \leq d(u,r) \cdot n - \sum_{l=0}^{d(u,r)-1} 2 \cdot |T(u_l)|$$

To complete the proof, consider what happens when u sells $uv_1, ..., uv_j$. First, u obviously saves $j\alpha$ from it edge costs. Second, by Lemma 7, we know that all $w \in T(uv_i)$ have a path of length $\leq 2i_k + d(w,r)$ to r that does not contain u. This leads to an increase in distance from u to w of at most

$$(d(u,r) + 2i_k + d(w,r)) - (d(w,r) - d(u,r)) = 2d(u,r) + 2i_k$$

This yields our desired bound on the change of cost for u:

$$0 \leq d(u,r) \cdot n - \sum_{l=0}^{d(u,r)-1} 2 \cdot |T(u_l)| - j \cdot \alpha + \sum_{k=1}^{j} (2i_k + 2d(u,r)) \cdot |T(uv_{i_k})|$$ ◀

▶ **Corollary 11.** *If $\alpha > 2n$, u buys $uv_1 \in X_{i_1}, ..., uv_j \in X_{i_j}$ for $j \geq 1$, $i_k \leq 2$, for all $k, 1 \leq k \leq j$, $u = u_0, u_1, ...u_{d(u,r)-1}, u_{d(u,r)} = r$ is the path from u to r in T, then the strategy change of u selling $uv_1, ..., uv_j$ and buying ur results in a change of cost of at most*

$$n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| - (j-1) \cdot \alpha + \sum_{k=1}^{j} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})|$$

where the second term is removed if $\frac{d(u,r)}{2}$ is not integral.

Proof. This strategy change is identical that of to Lemma 10 except u also buys ur. This means that the new path from u to r is now length 1, not length $d(u,r)$. Hence we replace $d(u,r) \cdot n$ with n in Equation (3). There are still vertices which are as close or closer to u than to r. The vertices of $|T_{u_{\frac{d(u,r)}{2}}}|$, which only exists if $\frac{d(u,r)}{2}$ is integral, are at-least as close to u as to r, therefore u does not use the edge ur on the path to any of the vertices in $|T_{u_{\frac{d(u,r)}{2}}}|$, thus we subtract $|T_{u_{\frac{d(u,r)}{2}}}|$ from Equation (3). For the remaining sets $T(u_l)$ for $l < \frac{d(u,r)}{2}$, each time l decreases by 1, the vertices in the set are 1 closer to u and 1 farther from r.

Finally, the last two terms of Equation (3) are affected as well. We now buy an additional edge, so we add α, resulting in the $(j-1) \cdot \alpha$ term. Furthermore, as previously stated. The path from u to r now has length 1, meaning for all $w \in T(uv_i)$ the increase in distance from u to w is at most

$$(2d(u,r) + 2i_k) - (d(u,r) - 1) = 1 + d(u,r) + 2i_k$$

This yields our desired bound on the change of cost for u:

$$n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l<\frac{d(u,r)}{2}} 2 \cdot |T(u_l)| - (j-1) \cdot \alpha + \sum_{k=1}^{j} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})|$$ ◀

▶ **Corollary 12.** *If $\alpha > 2n$, u buys $uv_1 \in X^+_{i_1}, ..., uv_j \in X^+_{i_j}$ for $j \geq 1$, $i_k \leq 2$, for all $k, 1 \leq k \leq j$, then the strategy change of u selling $uv_1, ..., uv_j$ and buying ur results in a change of cost of at most*

$$n - (j-1)\alpha - (d(u,r) + 1) \cdot |T(u)| + \sum_{k=1}^{j} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})|$$

Proof. This strategy change is nearly identical to Corollary 11, except now u is potentially selling the edge to its parent in T. We are no longer certain that u is closer to its former parent than r is, nor any vertices along the original path from u to r. Thus we lose the savings of all $T(u_l)$ except for $l = 0$, as $T(u_0) = T(u)$. ◀

3.2 Observations arising from the Deviation Strategies

We now apply these three deviation strategies to make three observations concerning vertices that buy multiple edges of specific types in H. In turn these observations will be instrumental in proving the main result in Section 4. The first observation simply states that no vertex can buy two edges in X^+_1.

▶ **Observation 13.** *No vertex $u \in H$ buys two edges $uv_1, uv_2 \in X^+_1$.*

Proof. Suppose $u \in H$ buys $uv_1, uv_2 \in X^+_1$. Observe r cannot buy an edge in X^+_1, as this would imply the existence of a short cycle, contradicting Corollary 5; therefore $u \neq r$. Hence, Lemma 6 implies $|T(u)| \leq \frac{n}{2}$. We now apply Corollary 12 to the strategy change where u sells $uv_1, uv_2 \in X^+_1$ and buys ur. The change in cost for u is at most

$$\begin{aligned}
& n - \alpha - (d(u,r) + 1) \cdot |T(u)| + \sum_{k=1}^{2} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})| \\
= \quad & n - \alpha - (d(r,u) + 1) \cdot (|T(u)| - |T(uv_1)| - |T(uv_2)|) + 2\left(|T(uv_1)| + |T(uv_2)|\right) \\
\leq \quad & n - \alpha + 2 \cdot |T(u)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad n - \alpha + 2\frac{n}{2} \\
&\leq \quad 2n - \alpha \\
&< \quad 0
\end{aligned}
$$

This contradicts the Nash equilibrium conditions. ◀

Next we observe that no vertex can buy three edges in X_2^+. Furthermore any vertex that buys two edges in X_2^+ must be the parent of a child at the root of a large subtree in T.

▶ **Observation 14.** *No vertex $u \in H$ buys three edges $uv_1, uv_2, uv_3 \in X_2^+$, and if u buys $uv_1, uv_2 \in X_2^+$, then $|T(uv_1) \cup T(uv_2)| > \frac{n}{4}$.*

Proof. Suppose $u \in H$ buys $uv_1, uv_2, uv_3 \in X_2^+$. Again, r cannot buy edges in X_2, as this would contradict Corollary 5; so $u \neq r$. Thus Lemma 6 implies $|T(u)| \leq \frac{n}{2}$. Now we apply Corollary 12 to the strategy change where v sells $vv_1, vv_2, vv_3 \in X_2^+$ and buys vr. The change in cost for v is:

$$
\begin{aligned}
&n - 2\alpha - (d(u,r)+1) \cdot |T(u)| + \sum_{k=1}^{2} (2i_k + d(u,r)+1) \cdot |T(uv_{i_k})| \\
&\quad = \quad n - 2\alpha - (d(r,u)+1) \cdot (|T(u)| - |T(uv_1)| - |T(uv_2)| - |T(uv_3)|) \\
&\qquad\quad +4\,(|T(uv_1)| + |T(uv_2)| + |T(uv_3)|) \\
&\quad \leq \quad n - 2\alpha + 4 \cdot |T(u)| \\
&\quad \leq \quad n - 2\alpha + 4\frac{n}{2} \\
&\quad \leq \quad 3n - 2\alpha \\
&\quad < \quad 0
\end{aligned}
$$

This contradicts the Nash equilibrium conditions. Now suppose $u \in H$ buys $uv_1, uv_2 \in X_2$. We can apply Corollary 12 to the strategy change where v sells $uv_1, uv_2 \in X_2^+$ and buys ur. The change in cost for u is at most

$$
\begin{aligned}
&n - \alpha - (d(u,r)+1) \cdot (|T(v)| - |T(uv_1)| - |T(uv_2)|) + 4 \cdot (|T(uv_1)| + |T(uv_2)|) \\
&\leq n - \alpha + 4 \cdot |T(uv_1) \cup T(uv_2)| \\
&< 4 \cdot |T(uv_1) \cup T(uv_2)| - n
\end{aligned}
$$

This is non-negative by the Nash equilibrium conditions. Therefore $|T(vv_1) \cup T(vv_2)| > \frac{n}{4}$, as desired. ◀

Finally, we observe some properties that follow when a vertex buys two edges in X_2.

▶ **Observation 15.** *If u buys two edges $uv_1, uv_2 \in X_2$, then $d(r,u) \geq 3$. If, in addition, $|T(uv_1) \cup T(uv_2)| > \frac{n}{4}$ then $d(r,u) = 3$.*

Proof. By Corollary 5, r cannot buy edges of X_2, therefore $u \neq r$. Further, by Lemma 6, $|T(u)| \leq \frac{n}{2}$.

We now apply Lemma 10 with the strategy change where u sells uv_1, uv_2. Its cost changes by at most

$$
\begin{aligned}
&d(r,u) \cdot n - 2\alpha - 2d(r,u) \cdot |T(u)| + (2d(r,u)+4) \cdot |T(uv_1) \cup T(uv_2)| \\
&\leq d(r,u) \cdot n - 2\alpha + 4 \cdot |T(uv_1) \cup T(uv_2)| \\
&\leq d(r,u) \cdot n - 2\alpha + 4\frac{n}{2}
\end{aligned}
$$

$$
\begin{aligned}
&< d(r,u) \cdot n - 2(2n) + 2n \\
&= (d(r,u) - 2) \cdot n
\end{aligned}
$$

If $d(r,u) \leq 2$ then this cost change is negative, a contradiction. Thus $d(r,u) \geq 3$.

Now suppose $|T(uv_1) \cup T(uv_2)| > \frac{n}{4}$. Without loss of generality, let $|T(uv_1)| \geq |T(uv_2)|$. Therefore $|T(uv_1)| > \frac{n}{8}$, meaning $T(uv_1) = T(v_1)$. As $uv_1 \in X_2$, we know v_1 buys an edge in X_0 or $X_1 \setminus X_0$. Suppose v_1 buys $v_1y \in X_0$. By Corollary 11, v selling v_1y and buying v_1r changes v_1's cost by

$$
\begin{aligned}
&\leq n - (d(v_1,r)+1) \cdot |T(v_1)| - (d(v_1,r)-1) \cdot |T(u) \setminus T(v_1)| \\
&\leq n - 2 \cdot |T(v_1)| - (d(v,r)-1) \cdot |T(u)| \\
&< n - 2\frac{n}{8} - (d(v_1,r)-1)\frac{n}{4} \\
&= n - d(v_1,r)\frac{n}{4} \\
&= n - (d(u,r)+1)\frac{n}{4} \\
&\leq n - (3+1)\frac{n}{4} \\
&= 0
\end{aligned}
$$

This contradicts the equilibrium condition, therefore v_1 buys $v_1w \in X_1 \setminus X_0$. w buys $wy \in X_0$. By Corollary 11, w selling wy and buying wr changes w's cost by

$$
\begin{aligned}
&\leq n - (d(w,r)+1)|T(w)| - (d(w,r)-1)|T(v_1) \setminus T(w)| - (d(w,r)-3)|T(u) \setminus T(v_1)| \\
&\leq n - 2 \cdot |T(v_1)| - (d(w,r)-3) \cdot |T(u)| \\
&< n - 2\frac{n}{8} - (d(w,r)-3)\frac{n}{4} \\
&= \frac{3}{2}n - d(w,r)\frac{n}{4} \\
&= \frac{3}{2}n - (d(u,r)+2)\frac{n}{4} \\
&= n - d(u,r)\frac{n}{4}
\end{aligned}
$$

If $d(r,u) \geq 4$, this leads to a contradiction. Thus $d(r,u) = 3$, as desired. ◀

4 The Tree Conjecture holds for $\alpha > 2n$

We now have all the resources necessary to prove the tree conjecture holds if $\alpha > 2n$. To do this, we prove two final lemmas which together give the main result. The first of these lemmas investigates the path to the root r from a vertex $u \in H$ that buys an out-edge.

▶ **Lemma 16.** *Let $\alpha > 2n$. If u_0 buys $u_0v \in X_0$ and $u_0, u_1, ...u_{d(u,r)-1}, u_{d(u,r)} = r$ is the path from u_0 to r in T then at most one of u_0, u_1, u_2 has $\deg_H = 2$*

Proof. By Observation 13, no vertex buys two edges in X_1^+. Thus u_0 cannot buy u_0u_1 as well as u_0v. Therefore u_1 buys u_1u_0. Similarly, by Observation 13, u_1 cannot buy u_1u_2 as well as u_1u_0. Therefore u_2 buys u_2u_1.

Now if $\deg_H(u_0) = 2$ then $S_H(u_0) = T(u_0)$, because u_1 is u_0's parent and $v \notin T(u_0)$, thus $T(u_0) \cap H = \{u_0\}$. Similarly, if $\deg_H(u_1) = 2$ then $S_H(u_1) = T(u_1) \setminus T(u_0)$, and if $\deg_H(u_2) = 2$ then $S_H(u_2) = T(u_2) \setminus T(u_1)$.

Next by Lemma 8, we know that $d(r, u_2) \geq \lfloor \frac{|C(u_2)|}{2} \rfloor - 2$, that $d(r, u_1) \geq \lfloor \frac{|C(u_1)|}{2} \rfloor - 1$ and that $d(r, u_0) \geq \lfloor \frac{|C(u_0)|}{2} \rfloor$.

Furthermore, if $\deg_H(u_1) = 2$ then, by Lemma 9, $|S_H(u_1)| \geq |T(u_0)|$. If $\deg_H(u_2) = 2$ then, by Lemma 3, u_2u_1 is in a directed cycle, which means u_3 buys u_3u_2. Thus, by Lemma 9, we have $|S_H(u_2)| \geq |T(u_1)|$

We proceed by case analysis. First, assume that $\deg_H(u_2) = \deg_H(u_1) = 2$. Then, by Lemma 9, we have $|S_H(u_1)| \geq \frac{\alpha}{2(|C(u_1)|-3)}$ and $|S_H(u_2)| \geq |T(u_1)|$. Thus $|T(u_2)| = |S_H(u_2)| + |T(u_1)| \geq 2|T(u_1)| \geq 2|S_H(u_1)| \geq \frac{2\alpha}{2(|C(u_1)|-3)}$. Now, by Lemma 6, $|T(u_2)| \leq \frac{n}{2}$. Therefore $\frac{n}{2} \geq \frac{2\alpha}{2(|C(u_1)|-3)}$. Rearranging gives $|C(u_1)| \geq 8$. By Observation 15, $d(u_1, r) \geq 3$.

Applying Corollary 11 when u_1 sells u_1u_0 and buys u_1r, the change in cost to u_1 is at most

$$
\begin{aligned}
& n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| - (j-1)\cdot\alpha + \sum_{k=1}^{j} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})| \\
&= n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| + (d(u_1,r) + 1 + 2) \cdot |T(u_1u_0)| \\
&= n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| + (d(u_1,r) + 3) \cdot |T(u_0)| \\
&\leq n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| + (2d(u_1,r)) \cdot |T(u_0)| \\
&\leq n - 2|T(u_1)| - (d(u_1,r) - 1) \cdot |T(u_2)| + (2d(u_1,r)) \cdot |T(u_0)| \\
&= n - (d(u_1,r) + 1) \cdot |T(u_1)| - (d(u_1,r) - 1) \cdot (|T(u_2) \setminus |T(u_1)|) + (2d(u_1,r)) \cdot |T(u_0)| \\
&= n - (d(r,u_1) + 1) \cdot |T(u_1)| - (d(u_1,r) - 1) \cdot |S_H(u_2)| + (d(u_1,r) + 3) \cdot |T(u_0)| \\
&\leq n - (d(r,u_1) + 1) \cdot |T(u_1)| - (d(u_1,r) - 1) \cdot |T(u_1)| + (2d(u_1,r)) \cdot |T(u_0)| \\
&= n - 2(d(r,u_1)) \cdot |T(u_1)| + 2(d(u_1,r)) \cdot |T(u_0)| \\
&= n - 2(d(r,u_1)) \cdot |T(u_1) \setminus T(u_0)| \\
&= n - 2(d(r,u_1)) \cdot |S_H(u_1)| \\
&\leq n - 2(d(r,u_1)) \cdot \tfrac{\alpha}{2(|C(u_1)|-3)} \\
&\leq n - 2(\lfloor \tfrac{|C(u_1)|}{2} \rfloor - 1) \cdot \tfrac{\alpha}{2(|C(u_1)|-3)} \\
&\leq n - (|C(u_1)| - 3) \cdot \tfrac{\alpha}{2(|C(u_1)|-3)} \\
&= n - \tfrac{\alpha}{2} \\
&< 0
\end{aligned}
$$

Therefore it cannot be that $\deg_H(u_2) = \deg_H(u_1) = 2$.

Second, suppose $\deg_H(u_0) = \deg_H(u_2) = 2$. By Lemma 9, we have $|S_H(u_0)| \geq \frac{\alpha}{2(|C(u_1)|-3)}$ and $|S_H(u_2)| \geq |T(u_1)|$. Because $u_0v \in X_0$, it follows that $T(v) \not\subseteq T(u_0)$. Thus $T(u_0v) = \emptyset$. Applying Corollary 11 when u_0 sells u_0v and buys u_0r, the change in cost to u_0 is at most

$$
\begin{aligned}
& n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| - (j-1)\cdot\alpha + \sum_{k=1}^{j} (2i_k + d(u,r) + 1) \cdot |T(uv_{i_k})| \\
&= n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l < \frac{d(u,r)}{2}} 2 \cdot |T(u_l)| \\
&\leq n - 2|T(u_0)| - 2|T(u_1)| - (d(u_0,r) - 3) \cdot |T(u_2)|
\end{aligned}
$$

$$
\begin{aligned}
&= n - 2|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_1)| - (d(u_0,r)-3)\cdot|T(u_2)\setminus T(u_1)| \\
&= n - 2|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_1)| - (d(u_0,r)-3)\cdot|S_H(u_2)| \\
&\le n - 2|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_1)| - (d(u_0,r)-3)\cdot|T(u_1)| \\
&= n - 2|T(u_0)| - (2d(u_0,r)-4)\cdot|T(u_1)| \\
&\le n - (2d(u_0,r)-2)|T(u_0)| \\
&= n - (2d(u_0,r)-2)|S_H(u_0)| \\
&\le n - (2d(r,u_0)-2)\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&\le n - (2\lfloor\tfrac{|C(u_0)|}{2}\rfloor-2)\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&\le n - (|C(u_0)|-3)\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&= n - \tfrac{\alpha}{2} \\
&< 0
\end{aligned}
$$

Therefore it is not true that $\deg_H(u_2) = \deg_H(u_0) = 2$.

Third, suppose $\deg_H(u_0) = \deg_H(u_1) = 2$. By Lemma 9, we have $|S_H(u_0)| \ge \frac{\alpha}{2(|C(u_1)|-3)}$ and $|S_H(u_1)| \ge |T(u_0)|$. Because $u_0v \in X_0$, it follows that $T(v) \not\subseteq T(u_0)$. Thus $T(u_0v) = \emptyset$. Applying Corollary 11 when u_0 sells u_0v and buys u_0r, the change in cost to u_0 is at most

$$
\begin{aligned}
&n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l<\frac{d(u,r)}{2}} 2\cdot|T(u_l)| - (j-1)\cdot\alpha + \sum_{k=1}^{j}(2i_k + d(u,r)+1)\cdot|T(uv_{i_k})| \\
&= n - |T_{u_{\frac{d(u,r)}{2}}}| - \sum_{l<\frac{d(u,r)}{2}} 2\cdot|T(u_l)| \\
&\le n - 2|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_1)| \\
&= n - (d(u_0,r)+1)\cdot|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_1)\setminus T(u_0)| \\
&= n - (d(u_0,r)+1)\cdot|T(u_0)| - (d(u_0,r)-1)\cdot|S_H(u_1)| \\
&\le n - (d(u_0,r)+1)\cdot|T(u_0)| - (d(u_0,r)-1)\cdot|T(u_0)| \\
&\le n - (2d(u_0,r))\cdot|T(u_0)| \\
&= n - (2d(u_0,r))\cdot|S_H(u_0)| \\
&\le n - (2d(r,u_0))\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&\le n - (2\lfloor\tfrac{|C(u_0)|}{2}\rfloor)\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&\le n - (|C(u_0)|-3)\cdot\tfrac{\alpha}{2(|C(u_0)|-3)} \\
&= n - \tfrac{\alpha}{2} \\
&< 0
\end{aligned}
$$

Therefore it is not true that $\deg_H(u_1) = \deg_H(u_0) = 2$. Thus, at most one of u_0, u_1, u_2 has $\deg_H = 2$, as desired. ◀

We now apply a counting argument to upper bound the number of vertices in H that buy two edges in X_2.

▶ **Lemma 17.** *The number of vertices $v \in H$ that buy two edges in X_2 is $< \deg_H(r)$.*

Proof. The only vertices that can buy two edges e_1, e_2 in X_2 are those with $|T(e_1) \cup T(e_2)| > \frac{n}{4}$ by Observation 14. All of these vertices are distance 3 from r, by Observation 15. The common ancestor in T of any pair of these vertices must be r, otherwise there is a vertex

$u \neq r$ with $|T(u)| > \frac{n}{2}$, contradicting Lemma 6. Furthermore, r cannot buy any edge e such that $T(e)$ contains such a vertex v, otherwise r could sell e and buy rv, reducing its distance to $T(v)$ by 2 and increasing its distance to $T(e) \setminus T(v)$ by ≤ 2. This gives a net increase in cost of $\leq 2|T(v)| - 2|T(e) \setminus T(v)| \leq 4|T(v)| - 2|T(e)| < 4\frac{n}{4} - 2\frac{n}{2} = 0$, a contradiction. Therefore the number of vertices which buy two edges in X_2 is $\leq deg^-_H(r)$. By Lemma 3, we must have $deg^+_H(r) \geq 1$, implying $deg^-_H(r) < deg_H(r)$ and completing the proof. ◀

Putting everything together we can now prove the main result.

Proof of Theorem 1. Here we use combine all to prove that if G is a Nash equilibrium graph for the network creation game (n, α) and $\alpha > 2n$, G is a tree.

This is a proof by contradiction based on the assumption that there exists a biconnected component H in G containing n_H vertices. Within H, the sum of degrees of all vertices in H equals $2(n_H - 1) + 2|X_0|$. That is twice the number of edges in the spanning tree on H induced by T plus twice the number of out-edges in H.

Let's instead count the degrees in the following way:

$$
\begin{aligned}
&\sum_{i=2}^{n_H} \sum_{v \in H : \deg_H(v) = i} i \\
&= 2n_H + \sum_{i=3}^{n_H} \sum_{v \in H : \deg_H(v) = i} i - 2 \\
&= 2n_H + \deg_H(r) - 2 + \sum_{i=3}^{n_H} \sum_{v \in H - r : \deg_H(v) = i} i - 2 \\
&\geq 2n_H + \deg_H(r) - 2 + 2|X_0| - \sum_{i=2}^{|X_2|} \sum_{v \in H : v \text{ is in } i \text{ paths of } X_2 \text{ edges}} i - 1 \quad \text{[by Lemma 16]} \\
&= 2n_H + \deg_H(r) - 2 + 2|X_0| - \sum_{v \in H : v \text{ is in } 2 \text{ paths of } X_2 \text{ edges}} 1 \qquad \text{[by Observation 14]} \\
&> 2n_H + \deg_H(r) - 2 + 2|X_0| - \deg_H(r) \qquad \text{[by Lemma 17]} \\
&= 2(n_H - 1) + 2|X_0|
\end{aligned}
$$

This is a contradiction, implying that H does not exist for $\alpha > 2n$. ◀

5 Conclusion

In this paper we proved the revised tree conjecture holds for $\alpha > 2n$. Moreover, we have reached a natural limit in the quest to settle this conjecture. Specifically, for $\alpha \in (n - 3, 2n]$, the range in which the conjecture is unsettled, we are no longer certain that directed cycles must be present in non-tree equilibrium graphs. To close this remaining gap, we believe min-cycles still have an important role to play but, to allow their usage, more precise analyses will be necessary. A potentially useful intermediate step would be to determine conditions that allow for an undirected min-cycle.

References

1 Susanne Albers, Stefan Eilts, Eyal Even-Dar, Yishay Mansour, and Liam Roditty. On nash equilibria for a network creation game. *ACM Transactions on Economics and Computation*, 2(1):1–27, 2014. doi:10.1145/2560767.

2 N. Alon, E. Demaine, M. Hajiaghayi, and T. Leighton. Basic network creation games. *SIAM Journal on Discrete Mathematics*, 27(2):656–668, 2013. doi:10.1137/090771478.

3 C. Àlvarez and A. Messegué. Network creation games: Structure vs anarchy. arXiv:1706.09132.

4 Davide Bilò, Luciano Gualà, Stefano Leucci, and Guido Proietti. Locality-based network creation games. *ACM Transactions on Parallel Computing*, 3(1):1–26, 2016. doi:10.1145/2938426.

5 Davide Bilò, Luciano Gualà, Stefano Leucci, and Guido Proietti. Network creation games with traceroute-based strategies. *Algorithms*, 14(2):35, 2021. doi:10.3390/a14020035.

6 Davide Bilò and Pascal Lenzner. On the tree conjecture for the network creation game. *Theory of Computing Systems*, 64(3):422–443, 2019. doi:10.1007/s00224-019-09945-9.

7 A. Chauhan, P. Lenzner, A. Melnichenko, and L. Molitor. Selfish network creation with non-uniform edge cost. In *Proceedings of 10th Symposium on Algorithmic Game Theory (SAGT)*, pages 160–172, 2017. doi:10.1007/978-3-319-66700-3_13.

8 A. Cord-Landwehr and P. Lenzner. Network creation games: Think global - act local. In *Proceedings of 40th Symposium on Mathematical Foundations of Computer Science (MFCS)*, pages 248–260, 2015. doi:10.1007/978-3-662-48054-0_21.

9 E. Demaine, M. Hajiaghayi, H. Mahini, and M. Zadimoghaddam. The price of anarchy in cooperative network creation games. *ACM Transactions on Economics and Computation*, 8(2), 2012.

10 J. Dippel and A. Vetta. An improved bound for the tree conjecture in network creation games. In *Proceedings of 15th Symposium on Algorithmic Game Theory (SAGT)*, pages 241–257, 2022.

11 Alex Fabrikant, Ankur Luthra, Elitza Maneva, Christos H. Papadimitriou, and Scott Shenker. On a network creation game. *Proceedings of the twenty-second annual symposium on Principles of distributed computing*, 2003. doi:10.1145/872035.872088.

12 P. Lenzner. *On Selfish Network Creation*. PhD thesis, Humboldt-Universität zu Berlin, 2014.

13 Akaki Mamageishvili, Matúš Mihalák, and Dominik Müller. Tree nash equilibria in the network creation game. *Internet Mathematics*, 11(4–5):472–486, 2015. doi:10.1080/15427951.2015.1016248.

14 Matúš Mihalák and Jan Christoph Schlegel. The price of anarchy in network creation games is (mostly) constant. *Theory of Computing Systems*, 53(1):53–72, 2013. doi:10.1007/s00224-013-9459-y.

15 C. Papadimitriou. Algorithms, games, and the internet. In *Proceedings of the 33rd ACM Symposium on the Theory of Computing (STOC)*, pages 749–753, 2001.

16 Q. Wang. On tree equilibria in max-distance network creation games. In *Proceedings of 15th Symposium on Algorithmic Game Theory (SAGT)*, pages 293–310, 2022.

17 Carme Àlvarez and Arnau Messegué Buisan. On the poa conjecture: Trees versus biconnected components. *SIAM Journal on Discrete Mathematics*, 37(2):1030–1052, 2023. doi:10.1137/21m1466426.