The AC^0-Complexity of Visibly Pushdown Languages

Stefan Göller
School of Electrical Engineering and Computer Science, Universität Kassel, Germany

Nathan Grosshans
Independent Scholar, Paris Region, France

Abstract

We study the question of which visibly pushdown languages (VPLs) are in the complexity class AC^0 and how to effectively decide this question. Our contribution is to introduce a particular subclass of one-turn VPLs, called *intermediate VPLs*, for which the raised question is entirely unclear: to the best of our knowledge our research community is unaware of containment or non-containment in AC^0 for *any* language in our newly introduced class. Our main result states that there is an algorithm that, given a visibly pushdown automaton, correctly outputs exactly one of the following: that its language L is in AC^0, some $m \geq 2$ such that MOD_m (the words over $\{0,1\}$ having a number of 1's divisible by m) is constant-depth reducible to L (implying that L is not in AC^0), or a finite disjoint union of intermediate VPLs that L is constant-depth equivalent to. In the latter of the three cases one can moreover effectively compute $k, l \in \mathbb{N}_{>0}$ with $k \neq l$ such that the concrete intermediate VPL $L(S \to \varepsilon \mid ac^{k-1}Sb_1 \mid ac^{l-1}Sb_2)$ is constant-depth reducible to the language L. Due to their particular nature we conjecture that either all intermediate VPLs are in AC^0 or all are not. As a corollary of our main result we obtain that in case the input language is a visibly counter language our algorithm can effectively determine if it is in AC^0 – hence our main result generalizes a result by Krebs et al. stating that it is decidable if a given visibly counter language is in AC^0 (when restricted to well-matched words).

For our proofs we revisit so-called Ext-algebras (introduced by Czarnetzki et al.), which are closely related to forest algebras (introduced by Bojańczyk and Walukiewicz), and use Green's relations.

2012 ACM Subject Classification Theory of computation → Grammars and context-free languages; Theory of computation → Circuit complexity

Keywords and phrases Visibly pushdown languages, Circuit Complexity, AC0

Digital Object Identifier 10.4230/LIPIcs.STACS.2024.38

Related Version *Full Version, Extended Version*: `https://doi.org/10.48550/arXiv.2302.13116`

Funding *Stefan Göller*: The author was supported by the Agence Nationale de la Recherche grant no. ANR-17-CE40-0010.

1 Introduction

This paper studies the circuit complexity of formal word languages. It is well-known that the regular word languages are characterized as the languages recognizable by finite monoids. When restricting the finite monoids to be aperiodic Schützenberger proved that one obtains precisely the star-free regular languages [22]. In terms of logic, these correspond to the languages definable in first-order logic $\mathsf{FO}[<]$ by a result of McNaughton and Papert [23]. The more general class of regular languages expressible in $\mathsf{FO}[\mathsf{arb}]$, i.e. first-order logic with arbitrary numerical predicates, coincides with the regular languages in AC^0 [13, 16]. These can be characterized algebraically as the regular languages whose syntactic morphism is quasi-aperiodic [5]. The latter algebraic characterization also shows that it is decidable if a regular language is in AC^0.

Generalizing regular languages, input-driven languages were introduced by Mehlhorn [21]. They are described by pushdown automata whose input alphabet is partitioned into letters that are either of type call, internal, or return. Rediscovered by Alur and Madhusudan in 2004 [2] under the name of *visibly pushdown languages (VPLs)*, it was shown that they enjoy many of the desirable effective closure properties of the regular languages. For instance, the visibly pushdown languages form an effective Boolean algebra. Algebraically, VPLs were characterized by Alur et al. [1] by congruences on well-matched words of finite index. Extending upon these, Czarnetzki et al. introduced so-called Ext-algebras [9]; these involve pairs of monoids (R, O) where O is a submonoid of R^R. Being tailored towards recognizing word languages, Ext-algebras are closely connected to forest algebras, introduced by Bojańczyk and Walukiewicz [7]: in [9] it is shown that a language of well-matched words is visibly pushdown if, and only if, its syntactic Ext-algebra is finite. While context-free languages are generally in $\mathsf{LOGCFL} = \mathsf{SAC}^1$, the visibly pushdown languages, as the regular languages, are known to be in NC^1 [10]. By a famous result of Barrington [4], there already exist regular languages that are NC^1-hard.

Related work. Visibly pushdown languages (VPLs) were introduced [2] via deterministic visibly pushdown automata (DVPA for short). Inspired by forest algebras [7] the paper [9] introduces Ext-algebras. Unfortunately, the definition of Ext-algebra morphisms in [9] is incorrect in that it provably does not lead to freeness.

The regular languages that are in AC^0 were effectively characterized by Barrington et al. [5]. By an automata-theoretic approach, Krebs et al. [19] effectively characterized the visibly counter languages that are in AC^0. These are particular VPLs that are essentially accepted by visibly pushdown automata that use only one stack symbol. In his PhD thesis [20] Ludwig already considers the question of which VPLs are in AC^0. Yet, his conjectural characterization contains several serious flaws – a detailled discussion of these shortcomings can be found in Section 8 in [12].

Our contribution. We reintroduce Ext-algebras, fix the notion of Ext-algebra morphisms and define the languages they recognize. We also reintroduce the syntactic Ext-algebra of languages of well-matched words. We rigorously prove classical results like freeness and minimality of syntactic Ext-algebras with respect to recognition. We prove that a language of well-matched words is a VPL if, and only if, it is recognizable by a finite Ext-algebra. While these results essentially revisit the constructions of [9], we use Ext-algebras as a technical tool for studying the complexity of visibly pushdown languages.

Fix a visibly pushdown alphabet Σ, i.e. Σ is partitioned into Σ_{call} (call letters), Σ_{int} (internal letters), and Σ_{ret} (return letters). Denoting $\Delta(u)$ as the difference between the number of occurrences of call and return letters in $u \in \Sigma^*$, a word $w \in \Sigma^*$ is *well-matched* if $\Delta(w) = 0$ and $\Delta(u) \geq 0$ for all prefixes u of w. A *context* is a pair (u, v) such that uv is well-matched – contexts have a natural composition operation: $(u, v) \circ (u', v') = (uu', v'v)$.

A set of contexts $\mathcal{R}$ is *length-synchronous* if $|u|/|v| = |u'|/|v'|$ for all $(u, v), (u', v') \in \mathcal{R}$ with $\Delta(u), \Delta(u') > 0$ and *weakly length-synchronous* if $u = u'$ implies $|v| = |v'|$ and $v = v'$ implies $|u| = |u'|$ for all $(u, v), (u', v') \in \mathcal{R}$ with $\Delta(u), \Delta(u') > 0$. Any language L of well-matched words induces a congruence $\equiv_L$ on contexts: $(u, v) \equiv_L (u', v')$ if $xuwvy \in L \Leftrightarrow xu'wv'y \in L$ for all contexts (x, y) and all well-matched words w. We introduce the notion of quasi-counterfreeness: a language is *quasi-counterfree* if for all contexts $\sigma \in \Sigma^k \times \Sigma^l$ for arbitrary k and l at least one of the following holds: (1) there exists some $n \in \mathbb{N}$ such that $\sigma^n \equiv_L \sigma^{n+1}$ or (2) no context in $\Sigma^k \times \Sigma^l$ is $\equiv_L$-equivalent to $\sigma \circ \sigma$. Finally, we introduce our central class of *intermediate VPLs*: a VPL is *intermediate* if it is quasi-counterfree and generated by a context-free grammar containing the production $S \to_G \varepsilon$, where S is the start nonterminal and whose other productions are of the form $T \to_G uT'v$ where uv is well-matched, $u \in (\Sigma_{\text{int}}^* \Sigma_{\text{call}} \Sigma_{\text{int}}^*)^+$ and $v \in (\Sigma_{\text{int}}^* \Sigma_{\text{ret}} \Sigma_{\text{int}}^*)^+$, such that the set of contexts $\{(u,v) \mid S \Rightarrow_G^* uSv\}$ is weakly length-synchronous but not length-synchronous. Note that intermediate VPLs are particular one-turn visibly pushdown languages, that is, visibly pushdown languages that are subsets of $(\Sigma \setminus \Sigma_{\text{ret}})^*(\Sigma \setminus \Sigma_{\text{call}})^*$. As an example, for all $k, l \geq 1$ with $k \neq l$, a concrete intermediate VPL, denoted by $\mathcal{L}_{k,l}$, is the one that is generated by the context-free grammar $S \to \varepsilon \mid ac^{k-1}Sb_1 \mid ac^{l-1}Sb_2$: here a is a call letter, c is an internal letter and b_1 and b_2 are return letters.

As far as we know the techniques known to our community do not directly suffice to show whether *at all* there is some intermediate VPL that is *provably in* AC^0 *or provably not in* AC^0 – analogous remarks apply to ACC^0. Our main result states that there is an algorithm that, given a DVPA A correctly outputs exactly one of the following: $L(A) \in \mathsf{AC}^0$, some $m \geq 2$ such that MOD_m is constant-depth reducible to L (thus witnessing that $L(A) \notin \mathsf{AC}^0$), or a non-empty disjoint finite union of intermediate VPLs that $L(A)$ is constant-depth equivalent to. In the latter of the three cases one can moreover effectively compute $k, l \in \mathbb{N}_{>0}$ with $k \neq l$ such that the above-mentioned $\mathcal{L}_{k,l}$ is constant-depth reducible to $L(A)$. We conjecture that either all intermediate VPLs are in AC^0 or all are not: note that together with our main result this conjecture implies the existence of an algorithm that can effectively determine if a given visibly pushdown language is in AC^0. As a corollary of our main result we obtain that in case the input language is a visibly counter language our algorithm can effectively determine if it is in AC^0, hence our main result generalizes a result by Krebs et al. stating that it is decidable if a given visibly counter lanugage is in AC^0 (when restricted to well-matched words).

For our main result we extensively study Ext-algebras, the syntactic morphisms of VPLs, and make use of Green's relations.

Organization. Our paper is organized as follows. We introduce notation and give an overview of our main result in Section 2. In Section 3 we first recall general algebraic concepts and then revisit Ext-algebras and their correspondence to visibly pushdown languages. Section 4 introduces central notions like length-synchronicity and weak length-synchronicity for Ext-algebra morphisms and visibly pushdown languages. The proof of our main result is content of Section 5. We conclude in Section 6. Full proofs can be found in [12].

2 Preliminaries

By $\mathbb{N}$ we denote the non-negative integers and by $\mathbb{N}_{>0}$ the positive integers. For all $a \in \Gamma$ and all $w \in \Gamma^*$ we write $|w|_a$ to denote the number of occurrences of a in w. We define the languages $\text{EQUALITY} = \{w \in \{0,1\}^* : |w|_0 = |w|_1\}$ and $\text{MOD}_m = \{w \in \{0,1\}^* : m \text{ divides } |w|_1\}$ for each $m \geq 2$. A *visibly pushdown alphabet* is a finite alphabet $\Sigma = \Sigma_{call} \cup \Sigma_{int} \cup \Sigma_{ret}$, where the alphabets $\Sigma_{call}, \Sigma_{int}$, and Σ_{ret} are pairwise disjoint. The set of *well-matched words over a visibly pushdown alphabet* Σ, denoted by $\Sigma^\triangle$, is the smallest set satisfying the following: $\varepsilon \in \Sigma^\triangle$ and $c \in \Sigma^\triangle$ for all $c \in \Sigma_{\text{int}}$, $awb \in \Sigma^\triangle$ for all $w \in \Sigma^\triangle$, $a \in \Sigma_{\text{call}}$ and $b \in \Sigma_{\text{ret}}$, and $uv \in \Sigma^\triangle$ for all $u, v \in \Sigma^\triangle \setminus \{\varepsilon\}$. A well-matched word $w \in \Sigma^\triangle$ is *one-turn* if $w \in (\Sigma \setminus \Sigma_{\text{ret}})^*(\Sigma \setminus \Sigma_{\text{call}})^*$. A language $L \subseteq \Sigma^\triangle$ is *one-turn* if it contains only one-turn words. Let Σ be a visibly pushdown alphabet. We define $\Delta \colon \Sigma^* \to \mathbb{Z}$ to be the height monoid morphism such that $\Delta(w) = |w|_{\Sigma_{call}} - |w|_{\Sigma_{ret}}$ for all $w \in \Sigma^*$.

A *context* is a pair $(u, v) \in \Sigma^* \times \Sigma^*$ such that $uv \in \Sigma^\triangle$. The *composition* of two contexts $(u, v), (x, y) \in \mathsf{Con}(\Sigma)$ is defined as $(u, v) \circ (x, y) = (ux, yv)$. For $\sigma \in \mathsf{Con}(\Sigma)$ by σ^k we denote the k-fold composition $\sigma \circ \cdots \circ \sigma$. For any context $(u, v) \in \mathsf{Con}(\Sigma)$ and well-matched word $w \in \Sigma^\triangle$ we define $(u, v)w = uwv$. An equivalence relation $\equiv$ on $\mathsf{Con}(\Sigma)$ is a *congruence* if for all $\chi, \chi', \sigma, \tau \in \mathsf{Con}(\Sigma)$ we have that $\sigma \equiv \tau$ implies $\chi \circ \sigma \circ \chi' \equiv \chi \circ \tau \circ \chi'$. Given a congruence $\equiv$ over $\mathsf{Con}(\Sigma)$ we denote by $[\sigma]_\equiv$ the equivalence class of σ. Given a language of well-matched words $L \subseteq \Sigma^\triangle$ we write $\sigma \equiv_L \tau$ if for all $\chi \in \mathsf{Con}(\Sigma)$ and all $w \in \Sigma^\triangle$ we have $(\chi \circ \sigma)w \in L$ if, and only if, $(\chi \circ \tau)w \in L$. Clearly, $\equiv_L$ is a congruence.

A *context-free grammar* is a tuple $G = (V, \Sigma, P, S)$, where V is a finite set of *nonterminals*, Σ is a non-empty finite *alphabet*, $P \subseteq V \times (V \cup \Sigma)^*$ is a finite set of *productions*, and $S \in V$ is the *start nonterminal*. We write $T \rightarrow_G y$ whenever $(T, y) \in P$. The binary relation $\Rightarrow_G$ over $(V \cup \Sigma)^*$ is defined as $u \Rightarrow_G v$ if there exists a production $T \rightarrow_G y$ and $x, z \in (V \cup \Sigma)^*$ such that $u = xTz$ and $v = xyz$. By $L(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$ we denote the *language* of G where $\Rightarrow_G^*$ is the reflexive transitive closure of $\Rightarrow_G$.

In the following we introduce deterministic visibly pushdown automata, remarking that nondeterministic visibly pushdown automata are determinizable [2]. A *deterministic visibly pushdown automaton* (DVPA) is a tuple $A = (Q, \Sigma, \Gamma, \delta, q_0, F, \perp)$, where Q is a finite set of *states*, Σ is a visibly pushdown alphabet, the *input alphabet*, Γ is a finite alphabet, the *stack alphabet*, $q_0 \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, $\perp \in \Gamma$ is the *bottom-of-stack symbol*, and $\delta \colon Q \times \Sigma \times \Gamma \rightarrow Q \times \big(\{\varepsilon\} \cup \Gamma \cup (\Gamma \setminus \{\perp\})\Gamma\big)$ is the *transition function* such that for all $q \in Q, a \in \Sigma, \alpha \in \Gamma$: if $a \in \Sigma_{\text{call}}$, then $\delta(q, a, \alpha) \in Q \times (\Gamma \setminus \{\perp\})\alpha$, if $a \in \Sigma_{\text{ret}}$, then $\delta(q, a, \alpha) \in Q \times \{\varepsilon\}$, and if $a \in \Sigma_{\text{int}}$, then $\delta(q, a, \alpha) \in Q \times \{\alpha\}$. We define the *extended transition function* $\widehat{\delta} \colon Q \times \Sigma^* \times \Gamma^* \rightarrow Q \times \Gamma^*$ inductively as $\widehat{\delta}(q, \varepsilon, \beta) = (q, \beta)$ for all $q \in Q$ and $\beta \in \Gamma^*$, $\widehat{\delta}(q, w, \varepsilon) = (q, \varepsilon)$ for all $q \in Q$ and $w \in \Sigma^+$, and $\widehat{\delta}(q, aw, \alpha\beta) = \widehat{\delta}(p, w, \gamma\beta)$, where $\delta(q, a, \alpha) = (p, \gamma)$ for all $q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$, $\alpha \in \Gamma$ and $\beta \in \Gamma^*$. The *language* accepted by A is the language $L(A) = \{w \in \Sigma^* \mid \widehat{\delta}(q_0, w, \perp) \in F \times \{\perp\}\}$. We call such a language a *visibly pushdown language* (VPL). We remark that visibly pushdown languages are always subsets of $\Sigma^\triangle$.

We refer to [14] for further details on formal language theory.

2.1 Complexity and logic

We assume familiarity with standard circuit complexity, we refer to [24, 18] for an introduction to the topic. Recall the following Boolean functions: the AND-function, the OR-function, the majority function (that outputs 1 if the majority of its inputs are 1s), and the mod_m function (that outputs 1 if the number of its inputs that are 1s is divisible by m) for all $m \geq 2$.

A circuit family $(C_n)_{n\in\mathbb{N}}$ *decides* a binary language $L \subseteq \{0, 1\}^*$ if C_n is a circuit with n inputs such that $L \cap \{0, 1\}^n = \{x_1 \ldots x_n \in \{0, 1\}^n \mid C_n(x_1, \ldots, x_n) = 1\}$ for all $n \in \mathbb{N}$. In this paper, we will consider circuits deciding languages over arbitrary finite alphabets: to do this, we just consider implicitly that any language over an arbitrary finite alphabet comes with a fixed binary encoding that encodes each letter as a block of bits of fixed size. By $\leq_{\text{cd}}$ we mean *constant-depth truth table reducibility* (or just constant-depth reducibility) as introduced in [8]. Formally for two languages $K \subseteq \Gamma^*$ and $L \subseteq \Sigma^*$ for finite alphabets Σ, Γ, we write $K \leq_{\text{cd}} L$ in case there is a polynomial p, a constant $d \in \mathbb{N}$, and circuit family $(C_n)_{n\in\mathbb{N}}$ deciding L such that each circuit C_n satisfies the following: it is of depth at most d and size at most $p(n)$ and its non-input gates are either AND-labeled, OR-labeled, or so-called oracle gates, labeled by L, that are gates deciding $L \cap \Sigma^m$, where $m \leq p(n)$, such that there is no path from the output of an oracle gate to an input of another oracle gate.

We write $K =_{\mathrm{cd}} L$ if $K \leq_{\mathrm{cd}} L$ and $L \leq_{\mathrm{cd}} K$; we also say that K and L are *constant-depth equivalent*. We say a language L is *hard* for a complexity class C (or just C-*hard*) if $L' \leq_{\mathrm{cd}} L$ for all $L' \in \mathsf{C}$. We say L is C-*complete* if L is C-hard and $L \in \mathsf{C}$. The following complexity classes are relevant in this paper: AC^0 is the class of all languages decided by circuit families with NOT gates, AND, OR gates of unbounded fan-in, constant depth and polynomial size; ACC^0 is the class of all languages decided by circuit families with NOT gates, AND, OR and modular gates (for some fixed m) of unbounded fan-in, constant depth and polynomial size; TC^0 is the class of all languages decided by circuit families with NOT gates, AND, OR and majority gates of unbounded fan-in, constant depth and polynomial size; NC^1 is the class of all languages decided by circuit families with NOT gates, AND, OR gates of bounded fan-in, logarithmic depth and polynomial size.

We also consider the framework of first order logic over finite words. (See [17, 23] for a proper introduction to the topic.) A *numerical predicate of arity* $r \in \mathbb{N}_{>0}$ is a symbol of arity r associated to a subset of ${\mathbb{N}_{>0}}^r$. Given a class $\mathcal{C}$ of numerical predicates and a finite alphabet Σ, we call $\mathsf{FO}_\Sigma[\mathcal{C}]$-formula a first order formula over finite words using the alphabet Σ and numerical predicates from the class $\mathcal{C}$. On occasions, we might also consider $\mathsf{FO}_{\Sigma,\leftrightsquigarrow}[\mathcal{C}]$-formulas that in comparison to the previous ones can use an additional binary predicate $\leftrightsquigarrow$ and are interpreted on structures (w, M) with $w \in \Sigma^*$ and $M \subseteq [1, |w|]^2$, where everything is interpreted as for $\mathsf{FO}_\Sigma[\mathcal{C}]$-formulas on w excepted for $\leftrightsquigarrow$ that is interpreted by M. Given a class $\mathcal{C}$ of numerical predicates, by $\mathsf{FO}[\mathcal{C}]$ we denote the class of all languages over any finite alphabet Σ defined by a $\mathsf{FO}_\Sigma[\mathcal{C}]$-sentence. A classical result at the interplay of circuit complexity and logic is that $\mathsf{AC}^0 = \mathsf{FO}[\mathsf{arb}]$, where arb denotes the class of all numerical predicates (see [23, Theorem IX.2.1] or [17, Corollary 5.32]). The other numerical predicates that we will encounter in this paper are $<$, $+$ and MOD_m for all $m \in \mathbb{N}_{>0}$, where MOD_m tests if m divides the number of 1's (gathered together in the set $\mathsf{MOD} = \{\mathsf{MOD}_m \mid m > 0\}$).

2.2 Main result

The notion of length-synchronicity and weak length-synchronicity will be a central notion in our main result. In the following Σ always denotes a visibly pushdown alphabet.

▶ **Definition 1** ((Weak) Length-Synchronicity). *Let $\mathcal{R} \subseteq \mathsf{Con}(\Sigma)$ be a set of contexts. We say $\mathcal{R}$ is* length-synchronous *if $|u| / |v| = |u'| / |v'|$ for all $(u, v), (u', v') \in \mathcal{R}$ with $\Delta(u), \Delta(u') > 0$; we say $\mathcal{R}$ is* weakly length-synchronous *if $u = u'$ implies $|v| = |v'|$ and $v = v'$ implies $|u| = |u'|$ for all $(u, v), (u', v') \in \mathcal{R}$ with $\Delta(u), \Delta(u') > 0$.*

Note that a set of contexts $\mathcal{R}$ is weakly length-synchronous if $\mathcal{R}$ is length-synchronous. Indeed, if, say $(u, v), (u, v') \in \mathcal{R}$, where $|v| \neq |v'|$ and $\Delta(u) > 0$, then $|u|, |v|, |v'| > 0$ and so the quotients $\frac{|u|}{|v|}$ and $\frac{|u|}{|v'|}$ are distinct, thus violating length-synchronicity of $\mathcal{R}$.

▶ **Definition 2** (Quasi-Counterfree). *A VPL $L \subseteq \Sigma^{\triangle}$ is* quasi-counterfree *if for all $\sigma = (u, v) \in \mathsf{Con}(\Sigma)$ at least one of the following holds: (1) there exists some $n \in \mathbb{N}$ such that $\sigma^n \equiv_L \sigma^{n+1}$ or (2) for all $\tau \in \Sigma^{|u|} \times \Sigma^{|v|} \cap \mathsf{Con}(\Sigma)$ we have $\tau \not\equiv_L \sigma \circ \sigma$.*

We will later show that quasi-counterfreeness of a VPL $L \subseteq \Sigma^{\triangle}$ is equivalent to the condition that there is no $k, l \in \mathbb{N}$ such that there is a subset of $\mathsf{Con}(\Sigma) \cap \Sigma^k \times \Sigma^l$ that forms a non-trivial group when considering the associated equivalence classes with respect to $\equiv_L$.

▶ **Example 3.** Consider the visibly pushdown alphabet Σ, where $\Sigma_{\mathrm{call}} = \{a\}$, $\Sigma_{\mathrm{int}} = \{c\}$ and $\Sigma_{\mathrm{ret}} = \{b_1, b_2\}$. For all $k, l \in \mathbb{N}_{>0}$ satisfying $k \neq l$, consider the language $\mathcal{L}_{k,l}$ generated by the context-free grammar $S \to ac^{k-1}Sb_1 \mid ac^{l-1}Sb_2 \mid \varepsilon$. We have that the set of contexts

$\{(u,v) \in \mathsf{Con}(\Sigma) \mid S \Rightarrow_G^* uSv\}$ is weakly length-synchronous since both the relation and its reverse is a partial function – however, it is not length-synchronous. It is also not hard to see that $\mathcal{L}_{k,l}$ is quasi-counterfree. Indeed, given $(u,v) \in \mathsf{Con}(\Sigma)$, if u contains the letter b_1 or b_2, or v contains the letter a or the letter c along with the letter b_1 or b_2, we have that $(\chi \circ (u,v))w \notin \mathcal{L}_{k,l}$ for all $\chi \in \mathsf{Con}(\Sigma)$ and all $w \in \Sigma^{\triangle}$, so $(u,v)^2 \equiv_{\mathcal{L}_{k,l}} (u,v)^3$. If u and v happen to contain the letter c and only this letter, we can argue similarly. In the cases remaining, u contains only letters from $\{a,c\}$ and v letters from $\{b_1,b_2\}$.

We say a context-free grammar $G = (V,\Sigma,P,S)$ is *vertically visibly pushdown* if the underlying alphabet Σ is a visibly pushdown alphabet, $S \to_G \varepsilon$, and all other productions of G are of the form $T \to_G uT'v$, where $uv \in \Sigma^{\triangle}$ is one-turn such that $u \in (\Sigma_{\mathrm{int}}^*\Sigma_{\mathrm{call}}\Sigma_{\mathrm{int}}^*)^+$ and $v \in (\Sigma_{\mathrm{int}}^*\Sigma_{\mathrm{ret}}\Sigma_{\mathrm{int}}^*)^+$. Note that each grammar generating $\mathcal{L}_{k,l}$ in Example 3 is vertically visibly pushdown. Note that languages generated by vertically visibly pushdown grammars are obviously one-turn VPLs.

▶ **Definition 4** (Intermediate VPL). *A VPL L is intermediate if it is quasi-counterfree and $L = L(G)$ for some vertically visibly pushdown grammar G for which $\mathcal{R}(G) = \{(u,v) \in \mathsf{Con}(\Sigma) \mid S \Rightarrow_G^* uSv\}$ is weakly length-synchronous but not length-synchronous.*

We remark that every intermediate languages is in TC^0. Thus the languages $\mathcal{L}_{k,l}$ from Example 3 are all intermediate VPLs. Loosely speaking, they are the simplest intermediate VPLs. We have the following conjecture.

▶ **Conjecture 5.** *There is no intermediate VPL that is in ACC^0 or TC^0-hard.*

In fact, the authors are not even aware of any intermediate VPL that is provably not in AC^0. An indication for the inadequacy of known techniques to prove it is that the *robustness* [18] of intermediate VPLs can be proven to be constant. Further techniques, based for instance on the *switching lemma* [15] or on the *polynomial method* [6] also do not seem to be applicable.

Our main result is the following theorem.

▶ **Theorem 6.** *There is an algorithm that, given a DVPA A, correctly outputs either*

- $L(A) \in \mathsf{AC}^0$,
- $m \geq 2$ *such that* $\mathrm{MOD}_m \leq_{\mathrm{cd}} L(A)$ *(hence implying $L(A) \notin \mathsf{AC}^0$),*
- *vertically visibly pushdown grammars $G_1,\ldots,G_m$ each generating intermediate VPLs such that $L =_{\mathrm{cd}} \biguplus_{i=1}^m L(G_i)$. In this case one can moreover effectively compute $k,l \in \mathbb{N}$ with $k \neq l$ such that $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L(A)$.*

Theorem 6 and the following conjecture imply the existence of an algorithm that decides if a given visibly pushdown language is in AC^0.

▶ **Conjecture 7.** *Either all intermediate VPLs are in AC^0 or all are not.*

We refer the reader to [3] for the definition of visibly counter automata. Visibly counter automata (m-VCA) are essentially restricted visibly pushdown automata manipulating a counter which can moreover explicitly test if the current counter has a value in $[0,m-1]$ or at least m. The following corollary of Theorem 6 implies the main result of [19] when restricted to well-matched words.

▶ **Corollary 8.** *There is an algorithm that, given an m-VCA A, correctly outputs either that $L(A)$ is in AC^0 or some $m \geq 2$ such that $\mathrm{MOD}_m \leq_{\mathrm{cd}} L(A)$ (hence implying $L(A) \notin \mathsf{AC}^0$).*

3 Ext-Algebras

This section builds on [9], but identifies an inaccuracy in the definition of Ext-algebra morphisms to establish freeness.

Let $(M, \cdot, 1_M)$ be a monoid. For each $m \in M$, we shall respectively denote by left_m and right_m the *left-multiplication map* $x \mapsto m \cdot x$ and the *right-multiplication map* $x \mapsto x \cdot m$.

An Ext-*algebra* $(R, O, \cdot, \circ)$ consists of a monoid $(R, \cdot, 1_R)$ and a monoid $(O, \circ, 1_O)$ that is a submonoid of $(R^R, \circ)$ containing the maps left_r and right_r for each $r \in R$. An Ext-*algebra morphism from* Ext-*algebra* (R, O) *to* Ext-*algebra* (S, P) is a pair (φ, ψ) of monoid morphisms $\varphi\colon R \to S$ and $\psi\colon O \to P$ such that: for all $e \in O$ and $r \in R$ we have $\psi(e)(\varphi(r)) = \varphi(e(r))$ and for all $r \in R$ we have $\psi(\mathrm{left}_r) = \mathrm{left}_{\varphi(r)}$ and $\psi(\mathrm{right}_r) = \mathrm{right}_{\varphi(r)}$. When it is clear from the context, we shall write *morphism* to mean Ext-*algebra morphism*. We remark that in the above definition, φ is totally determined by ψ, because for each $r \in R$, we have $\varphi(r) = \varphi(\mathrm{left}_r(1_R)) = \psi(\mathrm{left}_r)(\varphi(1_R)) = \psi(\mathrm{left}_r)(1_S)$.

For the rest of this section, let us fix some visibly pushdown alphabet Σ. For all $(u, v) \in \mathsf{Con}(\Sigma)$, consider the function $\mathrm{ext}_{u,v}\colon \Sigma^{\triangle} \to \Sigma^*$ such that $\mathrm{ext}_{u,v}(x) = uxv$ for all $x \in \Sigma^{\triangle}$. It is not hard to prove that $\mathrm{ext}_{u,v}$ is a function from $\Sigma^{\triangle}$ to $\Sigma^{\triangle}$. Consider now the set $\mathcal{O}(\Sigma^{\triangle})$ of all functions $\mathrm{ext}_{u,v}$ for $(u, v) \in \mathsf{Con}(\Sigma)$: it is a subset of $(\Sigma^{\triangle})^{\Sigma^{\triangle}}$ closed under composition. Thus, $(\mathcal{O}(\Sigma^{\triangle}), \circ)$ is a submonoid of $((\Sigma^{\triangle})^{\Sigma^{\triangle}}, \circ)$. As for all $w \in \Sigma^{\triangle}$ we have $\mathrm{left}_w = \mathrm{ext}_{w,\varepsilon}$ and $\mathrm{right}_w = \mathrm{ext}_{\varepsilon,w}$, the set $\mathcal{O}(\Sigma^{\triangle})$ contains left_w and right_w for all $w \in \Sigma^{\triangle}$. Thus, $(\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle}), \cdot, \circ)$ is an Ext-algebra.

The following important proposition establishes freeness of $(\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle}))$.

▶ **Proposition 9.** *Let (R, O) be an* Ext-*algebra and consider two functions $\varphi\colon \Sigma_{int} \to R$ and $\psi\colon \{\mathrm{ext}_{a,b} \mid a \in \Sigma_{call}, b \in \Sigma_{ret}\} \to O$. Then there exists a unique* Ext-*algebra morphism $(\overline{\varphi}, \overline{\psi})$ from $(\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle}))$ to (R, O) satisfying $\overline{\varphi}(c) = \varphi(c)$ for each $c \in \Sigma_{int}$ and $\overline{\psi}(\mathrm{ext}_{a,b}) = \psi(\mathrm{ext}_{a,b})$ for each $a \in \Sigma_{call}, b \in \Sigma_{ret}$.*

We remark that the requirement that for all $r \in R$ we have $\psi(\mathrm{left}_r) = \mathrm{left}_{\varphi(r)}$ and $\psi(\mathrm{right}_r) = \mathrm{right}_{\varphi(r)}$ does not appear in the definition of Ext-algebra morphisms given in [9]. But this is actually problematic, because then Proposition 9 would not hold in general. A counter-example is given in [12].

A language $L \subseteq \Sigma^{\triangle}$ is *recognized* by an Ext-algebra (R, O) whenever there exists a morphism $(\varphi, \psi)\colon (\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle})) \to (R, O)$ such that $L = \varphi^{-1}(F)$ for some $F \subseteq R$. The *syntactic* Ext-*algebra congruence of a language* $L \subseteq \Sigma^{\triangle}$ is the congruence $\sim_L$ on $\Sigma^{\triangle}$ defined by $u \sim_L v$ for $u, v \in \Sigma^{\triangle}$ whenever $xuy \in L \Leftrightarrow xvy \in L$ for all $(x, y) \in \mathsf{Con}(\Sigma)$. By $[x]_{\sim_L}$ we denote the *congruence class* of $x \in \Sigma^{\triangle}$.

▶ **Example 10.** Consider the language $L = \mathcal{L}_{1,2} = L(S \to aSb_1 \mid acSb_2 \mid \varepsilon)$ from Example 3 over the visibly pushdown alphabet Γ, where $\Gamma_{\text{int}} = \{c\}$, $\Gamma_{\text{call}} = \{a\}$ and $\Gamma_{\text{ret}} = \{b_1, b_2\}$. Set $R_L = \{[acb_1]_{\sim_L}, [\varepsilon]_{\sim_L}, [c]_{\sim_L}, [cab_1]_{\sim_L}, [ab_1]_{\sim_L}\}$, and, given for all $(u, v) \in \mathsf{Con}(\Sigma)$ the function $f_{u,v} \in (R_L)^{R_L}$ satisfying $f_{u,v}([x]_{\sim_L}) = [uxv]_{\sim_L}$ for all $x \in \Sigma^{\triangle}$, set

$$O_L = \{f_{acb_1,\varepsilon}, f_{\varepsilon,\varepsilon}, f_{c,\varepsilon}, f_{\varepsilon,c}, f_{ab_1,\varepsilon}, f_{\varepsilon,ab_1}, f_{cab_1,\varepsilon}, f_{a,b_2}, f_{ca,b_2}, f_{ca,ab_1b_2}, f_{ca,b_1}, f_{a,ab_1b_2}, f_{a,b_1}\} .$$

For instance, note that $[ab_1]_{\sim_L} = [acb_2]_{\sim_L}$, that $[acb_1]_{\sim_L}$ is the zero of R_L and that $f_{acb_1,\varepsilon}$ is the zero of O_L. Then (R_L, O_L) is an Ext-algebra recognizing L thanks to the morphism $(\varphi_L, \psi_L)\colon (\Gamma^{\triangle}, \mathcal{O}(\Gamma^{\triangle})) \to (R_L, O_L)$ satisfying $\varphi_L(c) = [c]_{\sim_L}$, $\psi_L(\mathrm{ext}_{a,b_1}) = f_{a,b_1}$ and $\psi_L(\mathrm{ext}_{a,b_2}) = f_{a,b_2}$. Note that $L = \varphi_L^{-1}(\{[\varepsilon]_{\sim_L}, [ab_1]_{\sim_L}\})$. Finally, note that for instance $\psi_L(\mathrm{ext}_{ca,ab_1b_2}) = f_{ca,ab_1b_2} \neq f_{a,ab_1b_2} = \psi_L(\mathrm{ext}_{a,ab_1b_2})$ since we have $\psi_L(\mathrm{ext}_{a,b_2}) \circ \psi_L(\mathrm{ext}_{ca,ab_1b_2})([c]_{\sim_L}) = [acacab_1b_2b_2]_{\sim_L} = [ab_1]_{\sim_L}$ whereas $\psi_L(\mathrm{ext}_{a,b_2}) \circ \psi_L(\mathrm{ext}_{a,ab_1b_2})([c]_{\sim_L}) = [aacab_1b_2b_2]_{\sim_L} = [acb_1]_L$.

The following lemma is proven in several steps. For this, classical notions like sub-Ext-algebra, division and quotients are introduced.

▶ **Lemma 11.** *Let $L \subseteq \Sigma^\triangle$. The pair $(R_L, O_L) = (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle))/{\sim_L}$, where $R_L = \Sigma^\triangle/{\sim_L}$ and where $O_L = \{e' \in (R_L)^{R_L} \mid \exists \mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^\triangle) \forall x \in \Sigma^\triangle : e'([x]_{\sim_L}) = [\mathrm{ext}_{u,v}(x)]_{\sim_L}\}$ is an* Ext*-algebra. Moreover the pair (φ_L, ψ_L) of functions $\varphi_L \colon \Sigma^\triangle \to R_L$ and $\psi_L \colon \mathcal{O}(\Sigma^\triangle) \to O_L$ satisfying $\varphi_L(x) = [x]_{\sim_L}$ for all $x \in \Sigma^\triangle$ and $\psi(\mathrm{ext}_{u,v})([x]_{\sim_L}) = [\mathrm{ext}_{u,v}(x)]_{\sim_L}$ for all $\mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^\triangle)$ and $x \in \Sigma^\triangle$ is an* Ext*-algebra morphism. We have $L = \varphi_L^{-1}(\varphi_L(L))$.*

We call (R_L, O_L) the *syntactic* Ext*-algebra* of L along with its unique associated morphism $(\varphi_L, \psi_L) \colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R_L, O_L)$, the *syntactic* Ext*-algebra morphism of* L. (An example of each of these is already given in Example 10.)

We say that an Ext-algebra (R, O) is *finite* whenever R is finite (which is the case if and only if O is finite). The following theorem establishes the equivalence between visibly pushdown languages and languages recognizable by finite Ext-algebras. Its proof provides effective translations from DVPAs to Ext-algebras and vice versa.

▶ **Theorem 12.** *A language $L \subseteq \Sigma^\triangle$ is a VPL if, and only if, it is recognized by a finite* Ext*-algebra.*

4 (Weak) length-synchronicity, nesting depth, and quasi-aperiodicity

For the rest of this section let us fix a visibly pushdown alphabet Σ, a finite Ext-algebra (R, O) and consider a morphism $(\varphi, \psi) \colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R, O)$.

In this section we introduce the notions of weak length-synchronicity and length-synchronicity for Ext-algebra morphisms and visibly pushdown languages. Before we do that, let us give some motivation how TC^0-hardness can be proven if the syntactic morphism maps certain $\mathrm{ext}_{u,v}, \mathrm{ext}_{u',v}$ with $|u| \neq |u'|$ to particular idempotents. For these we require the following notion of reachability.

For $F \subseteq R$ we say that an element $r \in R$ is *F-reaching* if $e(r) \in F$ for some $e \in O$. We say $e \in O$ is *F-reaching* if $e(r)$ is F-reaching for some $r \in R$. Fix any VPL L, its syntactic Ext-algebra (R_L, O_L) along with its syntactic morphism (φ_L, ψ_L). Assume there exists some idempotent $e \in O_L$ that is $\varphi_L(L)$-reaching.

We claim that if $\psi_L(\mathrm{ext}_{u,v}) = \psi_L(\mathrm{ext}_{u',v}) = e$ and $\Delta(u), \Delta(u') > 0$ for some $\mathrm{ext}_{u,v}, \mathrm{ext}_{u',v} \in \mathcal{O}(\Sigma^\triangle)$ with $|u| \neq |u'|$, then L is TC^0-hard. We remark that we must have $\Delta(u) = \Delta(u')$. Exploiting the fact that $|u| \neq |u'|$ we give a constant-depth reduction from the TC^0-complete language EQUALITY to L. As $\psi_L(\mathrm{ext}_{u,v})$ is $\varphi(L)$-reaching, we can fix some $x, y, z \in \Sigma^*$ such that $xuyvz \in L$. Given a word $w \in \{0,1\}^*$ of length $2n$ (binary words of odd length can directly be rejected) we map it to $xh(w)yv^{n\cdot(|u|+|u'|)}z$, where $h \colon \{0,1\}^* \to \Sigma^*$ is the length-multiplying morphism (i.e. $\exists l \in \mathbb{N} : h(0), h(1) \in \Sigma^l$) satisfying $h(0) = u^{|u'|}$ and $h(1) = u'^{|u|}$. We have $w \in$ EQUALITY if, and only if, $|w|_0 = |w|_1 = n$ if, and only if, $\Delta(h(w)) = n \cdot (|u| + |u'|) \cdot \Delta(u) = -n \cdot (|u| + |u'|) \cdot \Delta(v)$ if, and only if, $h(w)v^{n\cdot(|u|+|u'|)} \in \Sigma^\triangle$. Hence, since $\psi_L(\mathrm{ext}_{u^s,v^s}) = \psi_L(\mathrm{ext}_{(u')^t,v^t}) = e$ for all $s, t \geq 1$ it follows that $w \in$ EQUALITY if, and only, if $xh(w)yv^{n\cdot(|u|+|u'|)}z \in \Sigma^\triangle$ if, and only if, $xh(w)yv^{n\cdot(|u|+|u'|)}z \in L$.

Dually, one can show that L is TC^0-hard in case $\psi_L(\mathrm{ext}_{u,v}) = \psi_L(\mathrm{ext}_{u,v'}) = e$ and $\Delta(u) > 0$ for some $\mathrm{ext}_{u,v}, \mathrm{ext}_{u,v'} \in \mathcal{O}(\Sigma^\triangle)$ with $|v| \neq |v'|$.

The following definition of weak length-synchronicity captures the situation when such idempotents do not exist – it adapts the notion of weak length-synchronicity of sets of contexts, given in Definition 1, to morphisms and VPLs, respectively.

The morphism $(\varphi, \psi)\colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R, O)$ is F-weakly-length-synchronous (where $F \subseteq R$) if for all F-reaching idempotents $e \in O$ the set of contexts $\mathcal{R}_e := \{(u,v) \in \mathsf{Con}(\Sigma) \mid \psi(\mathrm{ext}_{u,v}) = e\}$ is weakly length-synchronous. We call a VPL $L \subseteq \Sigma^\triangle$ *weakly length-synchronous* if its syntactic morphism (φ_L, ψ_L) is $\varphi_L(L)$-weakly-length-synchronous.

Instead of considering those pairs (u, v) such that $\mathrm{ext}_{u,v}$ is being mapped to an F-reaching idempotent, the following characterization of weak length-synchronicity consider pairs (u, v) such that $\mathrm{ext}_{u,v}$ is being mapped to an element that behaves neutrally with respect to right multiplication with an F-reaching element that is not necessarily idempotent.

▶ **Proposition 13.** *For all $F \subseteq R$ we have that (φ, ψ) is F-weakly-length-synchronous if, and only if, for all F-reaching $e \in O$ the set of contexts $\mathcal{U}_e := \{(u,v) \in \mathsf{Con}(\Sigma) \mid e \circ \psi(\mathrm{ext}_{u,v}) = e\}$ is weakly length-synchrononous.*

One can prove that each $\mathrm{ext}_{u,v}$ has a unique *stair factorization* $\mathrm{ext}_{u,v} = \mathrm{ext}_{x_1,y_1} \circ \mathrm{ext}_{a_1,b_1} \circ \cdots \circ \mathrm{ext}_{x_{h-1},y_{h-1}} \circ \mathrm{ext}_{a_{h-1},b_{h-1}} \circ \mathrm{ext}_{x_h,y_h}$ satisfying $h \geq 1$, $x_i, y_i \in \Sigma^\triangle$ for all $i \in [1,h]$, $a_i \in \Sigma_{\text{call}}$, and $b_i \in \Sigma_{\text{ret}}$ for all $i \in [1, h-1]$. We refer to the x_i and y_i as *hills* of the stair factorization. From the following proposition it follows that weak length-synchronicity of a VPL L implies that for all $\varphi_L(L)$-reaching $e \in O_L$ and all $(u,v) \in \mathcal{U}_e$, the stair factorization of $\mathrm{ext}_{u,v}$ has small hills of constant size.

▶ **Proposition 14.** *There is a constant $n \in \mathbb{N}$ such that for all $F \subseteq R$, all F-reaching $e \in O$, and all $(u,v) \in \mathcal{U}_e$, if (φ, ψ) is F-weakly-length-synchronous, then the stair factorization $\mathrm{ext}_{u,v} = \mathrm{ext}_{x_1,y_1} \circ \mathrm{ext}_{a_1,b_1} \circ \cdots \circ \mathrm{ext}_{x_{h-1},y_{h-1}} \circ \mathrm{ext}_{a_{h-1},b_{h-1}} \circ \mathrm{ext}_{x_h,y_h}$ satisfies $|x_i|, |y_i| \leq n$ for all $i \in [1,h]$.*

As above, the following definition adapts the notion of length-synchronicity of sets of contexts, given in Definition 1, to Ext-algebra morphisms and VPLs, respectively.

The morphism $(\varphi, \psi)\colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R, O)$ is F-length-synchronous (where $F \subseteq R$) if for all F-reaching idempotents $e \in O$ the set of contexts $\mathcal{R}_e = \{(u,v) \in \mathsf{Con}(\Sigma) \mid \psi(\mathrm{ext}_{u,v}) = e\}$ is length-synchronous. We call a VPL $L \subseteq \Sigma^\triangle$ *length-synchronous* if its syntactic morphism (φ_L, ψ_L) is $\varphi_L(L)$-length-synchronous.

Consider our running example $\mathcal{L}_{1,2} = L(S \to aSb_1 \mid acSb_2 \mid \varepsilon)$. Recall that the monoid $O_{\mathcal{L}_{1,2}}$ of the syntactic Ext-algebra $(R_{\mathcal{L}_{1,2}}, O_{\mathcal{L}_{1,2}})$ and syntactic morphism $(\varphi_{\mathcal{L}_{1,2}}, \psi_{\mathcal{L}_{1,2}})$ of $\mathcal{L}_{1,2}$, given in Example 10, has the idempotents $f_{\varepsilon,\varepsilon}$, $f_{acb_1,\varepsilon}$ and f_{a,b_1}. Also recall that $\varphi_{\mathcal{L}_{1,2}}(\mathcal{L}_{1,2}) = \{[\varepsilon]_{\sim_{\mathcal{L}_{1,2}}}, [ab_1]_{\sim_{\mathcal{L}_{1,2}}}\}$. Since $\psi^{-1}_{\mathcal{L}_{1,2}}(f_{\varepsilon,\varepsilon}) = \{\mathrm{ext}_{\varepsilon,\varepsilon}\}$ and $f_{acb_1,\varepsilon}$ is a zero we have that $O_{\mathcal{L}_{1,2}}$'s only idempotent that is $\{[\varepsilon]_{\sim_{\mathcal{L}_{1,2}}}, [ab_1]_{\sim_{\mathcal{L}_{1,2}}}\}$-reaching and whose pre-image under $\psi_{\mathcal{L}_{1,2}}$ contains at least one $\mathrm{ext}_{u,v}$ with $\Delta(u) > 0$ is the idempotent f_{a,b_1}. However, both ext_{a,b_1} and ext_{ac,b_2}, where $\Delta(a) = \Delta(ac) = 1 > 0$, are sent to the idempotent $f_{a,b_1} = f_{a,b_2} \circ f_{c,\varepsilon}$. Since $|a|/|b_1| = 1 \neq 2 = |ac|/|b_2|$, we have that $\mathcal{L}_{1,2}$ is not length-synchronous. On the other hand, note that if any $\mathrm{ext}_{u,v}$ and $\mathrm{ext}_{u',v}$ (resp. $\mathrm{ext}_{u,v}$ and $\mathrm{ext}_{u,v'}$) are sent to f_{a,b_1} then $u = u'$ and thus $|u| = |u'|$ (resp. $v = v'$ and thus $|v| = |v'|$). Hence, $\mathcal{L}_{1,2}$ is weakly length-synchronous.

The two following propositions characterize length-synchronicity of Ext-algebra morphisms and of the set of contexts $\mathcal{U}_e$, which will be of particular importance when approximating the matching relation of a length-synchronous VPL in terms of $\mathsf{FO}[+]$. This will be an important ingredient to proving that VPLs that both are length-synchronous and have a quasi-aperiodic syntactic morphism (a notion to be defined below) are in $\mathsf{FO}[+]$ and thus in AC^0.

▶ **Proposition 15.** *For all $F \subseteq R$ we have that (φ, ψ) is F-length-synchronous if, and only if, for all F-reaching $e \in O$ the set of contexts $\mathcal{U}_e = \{(u,v) \in \mathsf{Con}(\Sigma) \mid e \circ \psi(\mathrm{ext}_{u,v}) = e\}$ is length-synchronous.*

▶ **Proposition 16.** *Let $F \subseteq R$ and assume (φ, ψ) is F-weakly-length-synchronous. Then for all F-reaching $e \in O$ the following two statements are equivalent.*

1. *$\mathcal{U}_e = \{(u, v) \in \mathsf{Con}(\Sigma) \mid e \circ \psi(\mathrm{ext}_{u,v}) = e\}$ is length-synchronous.*
2. *There exist $\alpha \in \mathbb{Q}_{>0}$, $\beta \in \mathbb{N}$, $\gamma \in \mathbb{N}_{>0}$ such that for all $(u, v) \in \mathcal{U}_e$ with $\Delta(u) > 0$ we have: (a) $\frac{|u|}{|v|} = \alpha$, (b) for all $u', v' \in \Sigma^+$ with u' prefix of u and v' suffix of v such that $\frac{|u'|}{|v'|} = \alpha$, we have that $-\Delta(v') - \beta \leq \Delta(u') \leq -\Delta(v') + \beta$, (c) for all factors $u' \in \Sigma^*$ of u such that $|u'| = \gamma$, we have $\Delta(u') \geq 1$, and (d) for all factors $v' \in \Sigma^*$ of v such that $|v'| = \gamma$, we have $\Delta(v') \leq -1$.*

The nesting depth of visibly pushdown languages. Another central notion is the nesting depth of well-matched words, which is the Horton-Strahler number [11] of the underlying trees. The *nesting depth* of well-matched words is inductively defined as follows: $\mathrm{nd}(\varepsilon) = 0$; $\mathrm{nd}(c) = 0$ for all $c \in \Sigma_{\mathrm{int}}$; $\mathrm{nd}(uv) = \max\{\mathrm{nd}(u), \mathrm{nd}(v)\}$ for all $u \in \Sigma_{\mathrm{call}}\Sigma^{\triangle}\Sigma_{\mathrm{ret}} \cup \Sigma_{\mathrm{int}}$ and $v \in \Sigma^{\triangle} \setminus \{\varepsilon\}$; $\mathrm{nd}(awb) = \mathrm{nd}(w) + 1$ if $w = uv$ for some $u, v \in \Sigma^{\triangle}$ with $\mathrm{nd}(w) = \mathrm{nd}(u) = \mathrm{nd}(v)$ and $\mathrm{nd}(w)$ otherwise, for all $a \in \Sigma_{\mathrm{call}}$, $b \in \Sigma_{\mathrm{ret}}$ and $w \in \Sigma^{\triangle}$.

An important property of weakly length-synchronous VPLs is that their words have bounded nesting depth. Assume any weakly length-synchronous VPL $L \subseteq \Sigma^{\triangle}$. Let n be the constant of Proposition 14. One can prove that if there exists $w \in L$ with $\mathrm{nd}(w) > d = n+1$, then there exists a factorization $w = uv = \mathrm{ext}_{u,v}(\varepsilon)$ such that for stair factorization $\mathrm{ext}_{u,v} = \mathrm{ext}_{x_1,y_1} \circ \mathrm{ext}_{a_1,b_1} \circ \cdots \circ \mathrm{ext}_{x_{h-1},y_{h-1}} \circ \mathrm{ext}_{a_{h-1},b_{h-1}} \circ \mathrm{ext}_{x_h,y_h}$ we must have $\max\{|x_i|, |y_i| : i \in [1, h]\} > n$, clearly contradicting Proposition 14. We obtain the following proposition.

▶ **Proposition 17.** *For each weakly length-synchronous VPL $L \subseteq \Sigma^{\triangle}$ there exists a constant $d \in \mathbb{N}$ such that $L \subseteq \{w \in \Sigma^{\triangle} \mid \mathrm{nd}(w) \leq d\}$.*

Quasi-aperiodicity. Towards characterizing the circuit complexity of visibly pushdown languages the notion of quasi-aperiodicity has already been defined for visibly pushdown languages in [20]. Let $(\varphi, \psi) \colon (\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle})) \to (R, O)$ for a visibly pushdown alphabet Σ and a finite Ext-algebra (R, O). We define $\mathcal{O}(\Sigma^{\triangle})^{k,l} = \{\mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^{\triangle}) : |u| = k, |v| = l\}$ for all $k, l \in \mathbb{N}$. We say (φ, ψ) is *quasi-aperiodic* if all semigroups contained in the set $\psi(\mathcal{O}(\Sigma^{\triangle})^{k,l})$ are aperiodic for all $k, l \in \mathbb{N}$.

5 Proof of the main theorem

The following proposition implies that the syntactic Ext-algebra and the syntactic morphism of a given visibly pushdown language L is computable and that it is decidable if L is quasi-aperiodic, length-synchronous, and weakly length-synchronous, respectively.

▶ **Proposition 18.** *The following computability and decidability results hold:*

1. *Given a DVPA A, one can effectively compute the syntactic* Ext*-algebra of $L = L(A)$, its syntactic morphism (φ_L, ψ_L) and $\varphi_L(L)$.*
2. *Given a morphism $(\varphi, \psi) \colon (\Sigma^{\triangle}, \mathcal{O}(\Sigma^{\triangle})) \to (R, O)$ for a visibly pushdown alphabet Σ and a finite* Ext*-algebra (R, O), all of the following are decidable for (φ, ψ): (a) Quasi-aperiodicity: in case (φ, ψ) is not quasi-aperiodic, one can effectively compute $k, l \in \mathbb{N}$ such that $\psi(\mathcal{O}(\Sigma^{\triangle})^{k,l})$ is not aperiodic; (b) F-length-synchronicity for a given $F \subseteq R$: in case (φ, ψ) is not F-length-synchronous, one can effectively compute a quadruple $(k, l, k', l') \in \mathbb{N}^4_{>0}$ such that there exist $uv, u'v' \in \Sigma^{\triangle}$ and some F-reaching idempotent $e \in O$ such that $\psi(\mathrm{ext}_{u,v}) = \psi(\mathrm{ext}_{u',v'}) = e$, $\Delta(u) > 0$, $\Delta(u') > 0$, $k = |u|, l = |v|, k' = |u'|$, $l' = |v'|$, and $\frac{k}{l} \neq \frac{k'}{l'}$; (c) F-weakly-length-synchronicity for a given $F \subseteq R$.*

Proof outline for Theorem 6. Towards proving our main result (Theorem 6), given a DVPA A, where $L = L(A)$ is a VPL over a visibly pushdown alphabet Σ, we apply Proposition 18 and compute its syntactic Ext-algebra (R_L, O_L) along with its syntactic morphism (φ_L, ψ_L) and the subset $\varphi_L(L)$. Then the following effective case distinction implies Theorem 6.

1. If L is not weakly length-synchronous, then L is TC^0-hard and hence not in AC^0 (Proposition 19 in Section 5.1). We output some $m > 1$ since $\mathrm{MOD}_m \leq_{\mathrm{cd}} \mathrm{EQUALITY} \leq_{\mathrm{cd}} L$.
2. If L is not quasi-aperiodic, then one can compute some $m \geq 2$ such that $\mathrm{MOD}_m \leq_{\mathrm{cd}} L$ (Proposition 20 in Section 5.1).
3. If L is length-synchronous and (φ_L, ψ_L) is quasi-aperiodic, then $L \in \mathsf{AC}^0$ (Theorem 22 in Section 5.2).
4. If L that is weakly length-synchronous, not length-synchronous, and its syntactic morphism (φ_L, ψ_L) is quasi-aperiodic, one can compute vertically visibly pushdown grammars $G_1, \ldots, G_m$ generating intermediate VPLs such that $L =_{\mathrm{cd}} \biguplus_{i=1}^{m} L(G_i)$ (Theorem 29 in Section 5.3). Already if L is weakly length-synchronous but not length-synchronous, one can compute $k, l \in \mathbb{N}_{>0}$ with $k \neq l$ such that $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L$ (Proposition 30 in Section 5.3). ◀

5.1 Lower bounds

The following lower bound has already been given in Section 4.

▶ **Proposition 19.** *If L is not weakly length-synchronous, then L is TC^0-hard.*

The following proposition has essentially already been shown in [20, Proposition 135], yet with some inaccuracies (we refer to Section 8 in [12]) that we fix. The idea is again a standard reduction from the word problem of non-trivial cyclic subgroups of $\psi_L(\mathcal{O}(\Sigma)^{k,l})$, in case the latter set contains a non-trivial group.

▶ **Proposition 20.** *If L is not quasi-aperiodic, then one can compute some $m \geq 2$ such that $\mathrm{MOD}_m \leq_{\mathrm{cd}} L$.*

As final lower bound result we prove a stronger lower bound, namely when the syntactic morphism not only is not quasi-aperiodic but the syntactic Ext-algebra is not solvable. We say the Ext-algebra (R, O) is *solvable* if all subsets of R or O that are groups (under the multiplication of R, resp. of O) are solvable. It is worth mentioning that one can prove that if $(\varphi, \psi)\colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R, O)$ is quasi-aperiodic, then (R, O) is solvable.

Our proof that L is NC^1-hard (and thus TC^0-hard) when (R_L, O_L) is not solvable can essentially be reduced to the case for words [4], by showing that already $\psi_L(\mathcal{O}(\Sigma^\triangle)^{k,l})$ contains such a non-solvable group for some fixed $k, l \geq 0$.

▶ **Proposition 21.** *If (R_L, O_L) is not solvable, then L is NC^1-hard and thus not in AC^0.*

5.2 In AC^0: Length-synchronous and quasi-aperiodic

In this section we concern ourselves with the following result.

▶ **Theorem 22.** *If L is length-synchronous and (φ_L, ψ_L) is quasi-aperiodic, then L is in $\mathsf{FO}[+]$ and thus in AC^0.*

For the rest of this section let us fix a VPL L, its syntactic Ext-algebra (R_L, O_L), and its syntactic morphism $(\varphi_L, \psi_L)\colon (\Sigma^\triangle, \mathcal{O}(\Sigma^\triangle)) \to (R_L, O_L)$. We first introduce suitably padded word languages mimicking the evaluation problem of the monoid R_L and the monoid O_L, respectively.

For all $k \in \mathbb{N}$, we define $\mathcal{O}(\Sigma^{\triangle})^{k,*} = \{\mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^{\triangle}) : |u| = k\}$ and $\mathcal{O}(\Sigma^{\triangle})^{*,k} = \{\mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^{\triangle}) : |v| = k\}$. We also define $\mathcal{O}(\Sigma^{\triangle})_{\uparrow} = \{\mathrm{ext}_{u,v} \in \mathcal{O}(\Sigma^{\triangle}) \mid \Delta(u) > 0\}$ and finally for all $k \in \mathbb{N}$, we define $\mathcal{O}(\Sigma^{\triangle})^{k,*}_{\uparrow} = \mathcal{O}(\Sigma^{\triangle})^{k,*} \cap \mathcal{O}(\Sigma^{\triangle})_{\uparrow}$ and $\mathcal{O}(\Sigma^{\triangle})^{*,k}_{\uparrow} = \mathcal{O}(\Sigma^{\triangle})^{*,k} \cap \mathcal{O}(\Sigma^{\triangle})_{\uparrow}$. Consider the alphabets $\Gamma_{\varphi_L} = \varphi_L(\Sigma^{\triangle} \setminus \{\varepsilon\}) \cup \{\$\}$ and $\Gamma_{\psi_L} = \psi_L\big(\mathcal{O}(\Sigma^{\triangle})_{\uparrow}\big) \cup \{\$\}$ for a letter $\$ \notin R_L \cup O_L$. We also define $\mathcal{V}_{\varphi_L} = \{\$^k s \mid k \in \mathbb{N}, s \in \varphi_L(\Sigma^{k+1})\}^*$ and $\mathcal{V}_{\psi_L} = \big\{\$^k f \mid k \in \mathbb{N}, f \in \psi_L\big(\mathcal{O}(\Sigma^{\triangle})^{k+1,*}_{\uparrow}\big)\big\}^*$. The following lemma holds irrespective of whether the syntactic morphism (φ_L, ψ_L) of L is quasi-aperiodic or not.

▶ **Lemma 23.** *$\mathcal{V}_{\varphi_L}$, $\mathcal{V}_{\psi_L}$ are regular languages whose syntactic morphisms are quasi-aperiodic.*

Define the φ_L-evaluation morphism $\mathrm{eval}_{\varphi_L} \colon \Gamma^*_{\varphi_L} \to R_L$ by $\mathrm{eval}_{\varphi_L}(s) = s$ for all $s \in \varphi_L(\Sigma^{\triangle} \setminus \{\varepsilon\})$ and $\mathrm{eval}_{\varphi_L}(\$) = 1_R$. Similarly, let the morphism $\mathrm{eval}_{\psi_L} : \Gamma^*_{\psi_L} \to O_L$ be defined as $\mathrm{eval}_{\psi_L}(f) = f$ for all $f \in \psi_L\big(\mathcal{O}(\Sigma^{\triangle})_{\uparrow}\big)$ and $\mathrm{eval}_{\psi_L}(\$) = 1_{O_L}$. Finally, for all $r \in R_L$, we set $\mathcal{E}_{\varphi_L,r} = \mathcal{V}_{\varphi_L} \cap \mathrm{eval}^{-1}_{\varphi_L}(r)$ and for all $e \in O_L$, $\mathcal{E}_{\psi_L,e} = \mathcal{V}_{\psi_L} \cap \mathrm{eval}^{-1}_{\psi_L}(e)$.

The following proposition states that the respective evaluation languages $\mathcal{E}_{\varphi_L,r}$ and $\mathcal{E}_{\psi_L,e}$ are all quasi-aperiodic if the syntactic morphism (φ_L, ψ_L) of L is and L is additionally length-synchronous.

▶ **Proposition 24.** *Let L be a VPL for which (φ_L, ψ_L) is quasi-aperiodic. Then $\mathcal{E}_{\varphi_L,r}$ is a regular language whose syntactic morphism is quasi-aperiodic for all $r \in R_L$. If L is length-synchronous, then $\mathcal{E}_{\psi_L,e}$ is a regular language whose syntactic morphism is quasi-aperiodic for all $e \in O_L$.*

The following remark states that the length-synchronicity condition in the second point of Proposition 24 is important. In fact it shows that weak length-synchronicity is not sufficient.

▶ Remark 25. For the second point of Proposition 24 it is generally not sufficient to assume that L is weakly length-synchronous. Indeed, the VPL K generated by the grammar with rules $S \to aSb_1 \mid acTb_2 \mid \varepsilon$ and $T \to aTb_1 \mid acSb_2$ using S as start symbol is not length-synchronous (but weakly length-synchronous) and has a quasi-aperiodic syntactic morphism. However, for the syntactic Ext-algebra (R_K, O_K) and the syntactic morphism (φ_K, ψ_K) of K, one can prove that that there exists $e \in O_K$ such that $\mathcal{E}_{\psi_K,e}$ is a regular language whose syntactic morphism is not quasi-aperiodic. Details can be found in [12].

Approximate matchings generalize the classical matching relation on well-matched words with respect to our VPL L in the sense that they are subsets of the matching relation but must equal the matching relation on all those words that are in L. Approximate matchings in the context of visibly pushdown languages were introduced by Ludwig [20].

For any word $w \in \Sigma^*$, we say that two positions $i, j \in [1, |w|]$ in w are *matched* whenever $i < j$, $w_i \in \Sigma_{\mathrm{call}}$, $w_j \in \Sigma_{\mathrm{ret}}$ and $w_{i+1} \cdots w_{j-1} \in \Sigma^{\triangle}$; we also say that i is *matched to* j in w. Given a word $w \in \Sigma^{\triangle}$, we denote by $M^{\triangle}(w)$ its *matching relation* (or *matching*), that is the relation $\{(i,j) \in [1,|w|]^2 \mid i \text{ is matched to } j \text{ in } w\}$. An *approximate matching relative to* $L \subseteq \Sigma^{\triangle}$ is a function $M \colon \Sigma^* \to {\mathbb{N}_{>0}}^2$ such that $M(w) = M^{\triangle}(w)$ for all $w \in L$ and $M(w) \subseteq M^{\triangle}(w)$ for all $w \in \Sigma^* \setminus L$.

The next lemma is an important tool for defining an approximate matching relation.

▶ **Lemma 26.** *Assume that (φ_L, ψ_L) is weakly length-synchronous. Let $e \in O_L$ be $\varphi_L(L)$-reaching and let $\mathcal{U}_e = \{(u,v) \in \mathsf{Con}(\Sigma) \mid e \circ \psi_L(\mathrm{ext}_{u,v}) = e\}$ be length-synchronous. Then there exists an $\mathsf{FO}_\Sigma[+]$-formula $\pi_e(x, x', y', y)$ such that for all $w \in \Sigma^+$ and $i, i', j', j \in [1, |w|], i \le i' < j' \le j$ the following holds:*

- *if $w \models \pi_e(i, i', j', j)$, then $w_i \cdots w_{i'} w_{j'} \cdots w_j \in \Sigma^{\triangle}$ and*
- *if $w_i \cdots w_{i'} w_{j'} \cdots w_j \in \Sigma^{\triangle}$ and $(w_i \ldots w_{i'}, w_{j'} \ldots w_j) \in \mathcal{R}_e$, then $w \models \pi_e(i, i', j', j)$.*

The proof of Lemma 26 takes several steps. The formula π_e expresses the characterization of length-synchronicity given by Proposition 16 via an $\mathsf{FO}_\Sigma[+]$-formula. To realize these we make use of aperiodicity of the following languages $L_{p,q}$. For each $p \in \mathbb{N}$ let $\Gamma_p = \{a_{-p}, \ldots, a_{-1}, a_0, a_1, \ldots, a_p\}$ and let $\Delta_p \colon \Gamma_p^* \to \mathbb{Z}$ be the morphism satisfying $\Delta_p(a_h) = h$ for all $a_h \in \Gamma_p$. Let $L_{p,q} = \{w \in \Gamma_p^* \mid \Delta_p(w) = 0 \wedge \forall i \in [1, |w|], -q \leq \Delta_p(w_1 \cdots w_i) \leq q\}$. One can prove that this language is recognized by a finite aperiodic monoid. This implies, by a theorem by McNaughton and Papert (see [23, Theorem VI.1.1]), that there exists an $\mathsf{FO}_{\Gamma_{n+d}}[<]$-sentence defining $L_{p,q}$. These ingredients are central for expressing the characterization of length-synchronicity given by Proposition 16 via an $\mathsf{FO}_\Sigma[+]$-formula.

With the help of the predicates π_e provided by Lemma 26 one can build an $\mathsf{FO}_\Sigma[+]$-definable approximate matching relative to any length-synchronous visibly pushdown language. The proof is by induction on the nesting depth.

▶ **Proposition 27.** *If $L \subseteq \Sigma^\triangle$ is length-synchronous, then there exists an $\mathsf{FO}_\Sigma[+]$-formula $\eta(x, y)$ such that $M \colon \Sigma^* \to {\mathbb{N}_{>0}}^2$ defined by $M(w) = \{(i, j) \in [1, |w|]^2 \mid w \models \eta(i, j)\}$ for all $w \in \Sigma^*$ is an approximate matching relative to L.*

The following proposition states that a VPL L is definable by some $\mathsf{FO}_{\Sigma,\leftrightsquigarrow}[+]$ sentence in case L has bounded nesting depth, the evaluation languages $\mathcal{E}_{\varphi_L,r}$ and $\mathcal{E}_{\psi_L,e}$ are all quasi-aperiodic, and any approximate matching is present as built-in predicate.

▶ **Proposition 28.** *Assume a VPL L has bounded nesting depth and $\mathcal{E}_{\varphi_L,r}$ and $\mathcal{E}_{\psi_L,e}$ are regular languages whose syntactic morphisms are quasi-aperiodic for all $r \in R_L$ and for all $e \in O_L$. Then there exists an $\mathsf{FO}_{\Sigma,\leftrightsquigarrow}[+]$-sentence η such that for all approximate matchings M relative to L, we have $w \in L$ if, and only if, $(w, M(w)) \models \eta$ for all $w \in \Sigma^*$.*

Proof (Sketch). By hypothesis, there exists $d_L \in \mathbb{N}$ bounding the nesting depth of the words in L. By hypothesis also, for each $r \in R_L$, the language $\mathcal{E}_{\varphi_L,r}$ is regular and its syntactic morphism is quasi-aperiodic. This implies, by [23, Theorem VI.4.1], that for each $r \in R_L$, there exists an $\mathsf{FO}_{\Gamma_{\varphi_L}}[<, \mathsf{MOD}]$-sentence $\nu_{\varphi_L,r}$ defining $\mathcal{E}_{\varphi_L,r}$. Finally, by hypothesis, for each $e \in O_L$, the language $\mathcal{E}_{\psi_L,e}$ is regular and its syntactic morphism is quasi-aperiodic. Analogously, for each $e \in O_L$, there exists an $\mathsf{FO}_{\Gamma_{\psi_L}}[<, \mathsf{MOD}]$-sentence $\nu_{\psi_L,e}$ defining $\mathcal{E}_{\psi_L,e}$.

To build the $\mathsf{FO}_{\Sigma,\leftrightsquigarrow}[+]$-sentence η, we build $\mathsf{FO}_{\Sigma,\leftrightsquigarrow}[+]$-formulas $\eta^\uparrow_{d,r}(x, y)$ and $\eta_{d,r}(x, y)$ for all $d \in [0, d_L]$ and all $r \in R_L$. They will have the following properties for all $w \in \Sigma^\triangle$ and all $i, j \in [1, |w|]$, where $M^\triangle$ is the full matching relation: (1) if i is matched to j in w, then $(w, M^\triangle(w)) \models \eta^\uparrow_{d,r}(i, j)$ if, and only if, $\mathrm{nd}(w_i \cdots w_j) \leq d$ and $\varphi_L(w_i \cdots w_j) = r$ and (2) if $w_i \cdots w_j \in \Sigma^\triangle$, then $(w, M^\triangle(w)) \models \eta_{d,r}(i, j)$ if, and only if, $\mathrm{nd}(w_i \cdots w_j) \leq d$ and $\varphi_L(w_i \cdots w_j) = r$. It is not difficult to construct a formula $N_d(x, y)$ (that also accesses the full matching relation) such that for all $w \in \Sigma^*$ and all infixes $w_i \ldots w_j \in \Sigma^\triangle$ we have $w \models N_d(i, j)$ if, and only if, $\mathrm{nd}(w_i \ldots w_j) \leq d$. Let the formula E be $\forall x(x \neq x)$ if $\varepsilon \in L$ and $\bot = \exists x(x \neq x)$ otherwise. Observe that $w \models \forall x(x \neq x)$ if, and only if, $w = \varepsilon$. Letting $\leftrightsquigarrow$ being interpreted over any approximate matching relation, the formula η is then defined as the conjunction of $\forall z \exists t\big((\Sigma_{\text{call}}(z) \to z \leftrightsquigarrow t) \wedge (\Sigma_{\text{ret}}(z) \to t \leftrightsquigarrow z)\big)$ and $E \vee \exists x \exists y\big(\neg\exists x'(x' < x) \wedge \neg\exists y'(y < y') \wedge \bigvee_{r \in \varphi_L(L)} \eta_{d_L,r}(x, y)\big)$.

Let us give some intuition on how to build $\eta^\uparrow_{d,r}(x, y)$ and $\eta_{d,r}(x, y)$ for all $d \in \mathbb{N}$ and $r \in R_L$. The construction is by induction on d. Let $r \in R_L$. We define $\eta^\uparrow_{0,r}(x, y) = \bot$. We define $\eta_{0,r}$ as $\eta_{0,r}(x, y) = \neg N_1(x, y) \wedge \tau_0(\nu_{\varphi_L,r})$, where the translation τ_0 is defined as follows: $\tau_0(z < z') = z < z'$, $\tau_0(s(z)) = \bigvee_{c \in \varphi_L^{-1}(s) \cap \Sigma_{\text{int}}} c(z)$ for all $s \in \varphi_L(\Sigma^\triangle \setminus \{\varepsilon\})$, $\tau_0(\mathsf{MOD}_m(z)) = \exists t(z - x + 1 = t \cdot m)$ for all $m \in \mathbb{N}_{>0}$, $\tau_0(\$(z)) = \bot$, $\tau_0(\rho_1(\boldsymbol{z}_1) \wedge \rho_2(\boldsymbol{z}_2)) = \tau_0(\rho_1(\boldsymbol{z}_1)) \wedge \tau_0(\rho_1(\boldsymbol{z}_2))$, $\tau_0(\neg\rho(\boldsymbol{z})) = \neg\tau_0(\rho(\boldsymbol{z}))$, and $\tau_0(\exists z \rho(z, \boldsymbol{z})) = \exists z\big(x \leq z \leq y \wedge \tau_0(\rho(z, \boldsymbol{z}))\big)$

Now let $d > 0$. Define the formula $A(x,y,z) = \exists x' \exists y' (x \leq x' \leq z < y' \leq y \wedge x' \leftrightsquigarrow y')$ that expresses that for all $w \in \Sigma^\triangle$ and $i,j,k \in [1,|w|]$ satisfying $w_i \cdots w_j \in \Sigma^\triangle$, we have $(w, M^\triangle(w)) \models A(i,j,k)$ if, and only if, $i \leq k < j$ and $\Delta(w_i \cdots w_k) > 0$. Let us first define $\eta_{d,r}$ when assuming that we have already defined $\eta^\uparrow_{d,r}$. Note that in case $\mathrm{nd}(u) \leq d$, then one can factorize u as $u = u_1 \cdots u_m$ such that $u_i \in \Sigma^+_{\mathrm{int}} \cup \Sigma_{\mathrm{call}} \Sigma^\triangle \Sigma_{\mathrm{ret}}$ and $\mathrm{nd}(u_i) \leq d$ for all $i \in [1,m]$. Using this observation we define $\eta_{d,r}(x,y) = \neg N_{d+1}(x,y) \wedge \tau_1(\nu_{\varphi_L,r})$, where the translation τ_1 agrees with the above translation τ_0 (where, as expected, occurrences of τ_0 are replaced by τ_1) except for the following kinds of subformulas: $\tau_1(\$(z)) = A(x,y,z)$ and $\tau_1(s(z)) = \neg A(x,y,z) \wedge \Big(\bigvee_{c \in \varphi_L^{-1}(s) \cap \Sigma_{\mathrm{int}}} c(z) \vee \exists t \big(x \leq t \leq y \wedge t \leftrightsquigarrow z \wedge \eta^\uparrow_{d,s}(t,z)\big)\Big)$.

Similarly, in the definition of $\eta^\uparrow_{d,r}$ we make use of our sentences $\nu_{\psi_L,e}$ for the evaluation languages $\mathcal{E}_{\psi_L,e}$ for all $e \in O_L$. For these however we make use of an auxiliary formula U such that for all $w \in \Sigma^\triangle$ and $i,i',k \in [1,|w|]$ we have $(w, M^\triangle(w)) \models U(i,i',k)$ if, and only if, $i \leq k \leq i'$ and k is matched with some position larger than i', thus expressing that for some positions j, j' the position k is an "upward stair position" in the stair factorization of $\mathrm{ext}_{u_i \cdots u_{i'}, u_{j'} \cdots u_j} = \mathrm{ext}_{x_1,y_1} \circ \cdots \mathrm{ext}_{a_{h-1},b_{h-1}} \circ \mathrm{ext}_{x_h,y_h}$: more precisely k is one of the positions $\{i + |x_1 a_1| - 1, i + |x_1 a_1 x_2 a_2| - 1, \ldots, i + |x_1 a_1 \ldots x_{h-1} a_{h-1}| - 1\}$. These positions k will be precisely the ones where the predicate \$ does not hold in a suitable translation of $\nu_{\psi_L,e}$. ◀

Proof of Theorem 22. Proposition 24 implies that $\mathcal{E}_{\varphi_L,r}$ and $\mathcal{E}_{\psi_L,e}$ are regular languages whose respective syntactic morphisms are quasi-aperiodic for all $r \in R_L$ and all $e \in O_L$, respectively. Thus, the first two conditions of Proposition 28 are satisfied. Moreover, Proposition 27 provides a first-order definable approximate matching relation relative to L, being a predicate assumed by Proposition 28. Finally, Proposition 28 implies Theorem 22. ◀

5.3 The intermediate case

The following theorem effectively characterizes the remaining case, namely those VPLs that are weakly length-synchronous but not length-synchronous and whose syntactic morphism is quasi-aperiodic: such VPLs are shown to be constant-depth equivalent to a non-empty disjoint union of intermediate languages.

▶ **Theorem 29.** *If a VPL L is weakly length-synchronous, not length-synchronous, and its syntactic morphism (φ_L, ψ_L) is quasi-aperiodic, one can compute vertically visibly pushdown grammars $G_1, \ldots, G_m$ generating intermediate VPLs such that $L =_{\mathrm{cd}} \biguplus_{i=1}^m L(G_i)$.*

Let $L \subseteq \Sigma^\triangle$ be a weakly length-synchronous VPL that is not length-synchronous, and whose syntactic morphism (φ_L, ψ_L) is quasi-aperiodic. By Proposition 18 one can compute its syntactic Ext-algebra (R_L, O_L), (φ_L, ψ_L) and $\varphi_L(L)$ from (a given DVPA for) L. For all $\varphi_L(L)$-reaching $e \in O_L$ and some fresh internal letter $\# \notin \Sigma$ let $M_e = \{u\#v \mid \Delta(u) > 0, (u,v) \in \mathcal{U}_e\}$, which can be shown to be a computable VPL.

The set $\mathcal{Z} = \{e \in O_L \mid e \text{ is } \varphi_L(L)\text{-reaching and } \mathcal{U}_e \text{ is not length-synchronous}\}$ can be proven to be computable. Observe that as L is not length-synchronous by assumption, we have $\mathcal{Z} \neq \emptyset$ (Proposition 15). Next make use of Lemma 14 stating that the hills in the stair factorization of any $\mathrm{ext}_{u,v} \in \psi_L^{-1}(e)$ are constantly bounded for all $\varphi_L(L)$-reaching $e \in O_L$. This gives rise to computable intermediate languages N_e such that $N_e =_{\mathrm{cd}} M_e$ for all $e \in \mathcal{Z}$. Letting $L_f = \{u\#v \mid \psi(\mathrm{ext}_{u,v}) = f\}$ for all $f \in O_L$, it is standard to show $L_f \leq_{\mathrm{cd}} L$ for all $\varphi_L(L)$-reaching $f \in O_L$. Finally, one proves $M_e \leq_{\mathrm{cd}} \biguplus_{f \in O_L \text{ is } \varphi_L(L)\text{-reaching}} L_f$ for all $e \in \mathcal{Z}$ and $L \leq_{\mathrm{cd}} \biguplus_{e \in \mathcal{Z}} M_e$ thus establishing $L =_{\mathrm{cd}} \biguplus_{e \in \mathcal{Z}} N_e$. The proof of $L \leq_{\mathrm{cd}} \biguplus_{e \in \mathcal{Z}} M_e$ is the technically most demanding and is an adaption of the proof of Proposition 28: alas, one cannot assume presence of the evaluation $\mathsf{FO}_{\Gamma_{\psi_L}}[<, \mathsf{MOD}]$-sentence $\nu_{\psi_L,e}$ for each $e \in O_L$ since $\mathcal{E}_{\psi_L,e}$ can possibly have a non-quasi-aperiodic syntactic morphism by Remark 25. Yet, one can realize evaluation via access to the oracle $\biguplus_{e\in\mathcal{Z}} M_e$.

The following proposition implies the computability of $k, l \in \mathbb{N}_{>0}$ such that $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L$ already when a VPL L is weakly length-synchronous but not length-synchronous.

▶ **Proposition 30.** *If a VPL L is weakly length-synchronous but not length-synchronous, one can compute $k, l \in \mathbb{N}_{>0}$ with $k \neq l$ such that $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L$.*

Proof. Let $L \subseteq \Sigma^\triangle$ be a weakly length-synchronous VPL that is not length-synchronous. According to Point 2 (b) of Proposition 18 one can compute a quadruple $(k_0, l_0, k_0', l_0') \in \mathbb{N}_{>0}^4$ for which there exist $\mathrm{ext}_{u,v}, \mathrm{ext}_{u',v'} \in \mathcal{O}(\Sigma^\triangle)$ such that $|u| = k_0$, $|v| = l_0$, $|u'| = k_0'$, $|v'| = l_0'$, $\psi_L(\mathrm{ext}_{u,v}) = \psi_L(\mathrm{ext}_{u',v'})$ is a $\varphi_L(L)$-reaching idempotent, $\Delta(u), \Delta(u') > 0$, and $\frac{k_0}{l_0} = \frac{|u|}{|v|} \neq \frac{|u'|}{|v'|} = \frac{k_0'}{l_0'}$. We can compute such $\mathrm{ext}_{u,v}$ and $\mathrm{ext}_{u',v'}$ by just doing an exhaustive search. This enables us to assume without loss of generality that $\Delta(u) = \Delta(u')$: indeed, in case $\Delta(u) \neq \Delta(u')$, we can consider $\mathrm{ext}_{u_1,v_1} = \mathrm{ext}_{u^{\Delta(u')},v^{\Delta(u')}}$ and $\mathrm{ext}_{u_2,v_2} = \mathrm{ext}_{(u')^{\Delta(u)},(v')^{\Delta(u)}}$.

Let us now define Green's relations on O_L. Let us consider two elements x, y of O_L. We write $x \leq_{\mathfrak{J}} y$ whenever there are elements e, f of O_L such that $x = e \circ y \circ f$. We write $x \mathrel{\mathfrak{J}} y$ if $x \leq_{\mathfrak{J}} y$ and $y \leq_{\mathfrak{J}} x$. We write $x <_{\mathfrak{J}} y$ if $x \leq_{\mathfrak{J}} y$ and $x \not\mathrel{\mathfrak{J}} y$. We write $x \leq_{\mathfrak{R}} y$ whenever there is an element e of O_L such that $x = y \circ e$. We write $x \mathrel{\mathfrak{R}} y$ if $x \leq_{\mathfrak{R}} y$ and $y \leq_{\mathfrak{R}} x$. We write $x \leq_{\mathfrak{L}} y$ whenever there is an element e of O_L such that $x = e \circ y$. We write $x \mathrel{\mathfrak{L}} y$ if $x \leq_{\mathfrak{L}} y$ and $y \leq_{\mathfrak{L}} x$. We write $x \mathrel{\mathfrak{H}} y$ if $x \mathrel{\mathfrak{R}} y$ and $x \mathrel{\mathfrak{L}} y$.

Observe that because $\Delta(u) = \Delta(u')$, we have that $uv' \in \Sigma^\triangle$ and $u'v \in \Sigma^\triangle$, so that we can consider the elements $\mathrm{ext}_{uuu,vv'v} = \mathrm{ext}_{u,v} \circ \mathrm{ext}_{u,v'} \circ \mathrm{ext}_{u,v}$ and $\mathrm{ext}_{uu'u,vvv} = \mathrm{ext}_{u,v} \circ \mathrm{ext}_{u',v} \circ \mathrm{ext}_{u,v}$ in $\mathcal{O}(\Sigma^\triangle)$. These elements satisfy $\psi_L(\mathrm{ext}_{uuu,vv'v}) \leq_{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$ and $\psi_L(\mathrm{ext}_{uu'u,vvv}) \leq_{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$. We claim that we actually have $\psi_L(\mathrm{ext}_{uuu,vv'v}) <_{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$ and $\psi_L(\mathrm{ext}_{uu'u,vvv}) <_{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$. Indeed, assume we had $\psi_L(\mathrm{ext}_{uu'u,vvv}) \mathrel{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$. Set $x = \psi_L(\mathrm{ext}_{u,v})$ and $y = \psi_L(\mathrm{ext}_{u',v})$. Since it would hold that $x \circ y \circ x \leq_{\mathfrak{R}} x$ and $x \circ y \circ x \mathrel{\mathfrak{J}} x$, we would have $x \circ y \circ x \mathrel{\mathfrak{R}} x$ and dually, since it would hold that $x \circ y \circ x \leq_{\mathfrak{L}} x$ and $x \circ y \circ x \mathrel{\mathfrak{J}} x$, we would have $x \circ y \circ x \mathrel{\mathfrak{L}} x$. Therefore, we would have $x \circ y \circ x \mathrel{\mathfrak{H}} x$. As x is an idempotent, its $\mathfrak{H}$-class is a group, hence for $\omega \in \mathbb{N}_{>0}$ the idempotent power of O_L, we would have $(x \circ y \circ x)^\omega = x^\omega = x$ (as the only idempotent element in a group is the identity). This would finally entail that $\psi_L(\mathrm{ext}_{(uu'u)^\omega,(vvv)^\omega}) = \psi_L(\mathrm{ext}_{(uuu)^\omega,(vvv)^\omega})$ is a $\varphi_L(L)$-reaching idempotent and $\Delta((uu'u)^\omega) = \Delta((uuu)^\omega) > 0$ but $|(uu'u)^\omega| \neq |(uuu)^\omega|$, a contradiction to the fact that (φ_L, ψ_L) is $\varphi_L(L)$-weakly-length-synchronous. Symmetrically, we can prove that if we had $\psi_L(\mathrm{ext}_{uuu,vv'v}) \mathrel{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$, this would contradict the fact that (φ_L, ψ_L) is $\varphi_L(L)$-weakly-length-synchronous.

Here we only treat the case when $|v| = |v'|$ and refer to [12] for the full proof of the other cases. We prove that there exist $k, l \in \mathbb{N}_{>0}, k \neq l$ such that $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L_{\psi_L(\mathrm{ext}_{u,v})}$, so that since $L_{\psi_L(\mathrm{ext}_{u,v})} \leq_{\mathrm{cd}} L$ (as already mentioned in Section 5.3 this is standard) and by transitivity of $\leq_{\mathrm{cd}}$ we have $\mathcal{L}_{k,l} \leq_{\mathrm{cd}} L$. In that case, we necessarily have $|u| \neq |u'|$. Then, we can exploit the fact that matching u^3 with $vv'v$ or $uu'u$ with v^3 makes us fall down to a smaller $\mathfrak{J}$-class to reduce $\mathcal{L}_{3|u|,2|u|+|u'|}$ to $L_{\psi_L(\mathrm{ext}_{u,v})}$. The constant-depth reduction works as follows on input $w \in \Sigma^*$: (i) check if $w = xy$ with $x \in (ac^{3|u|-1} + ac^{2|u|+|u'|-1})^*$ and $y \in (b_1 + b_2)^*$, reject otherwise; (ii) build x' by sending $ac^{3|u|-1}$ to u^3, $ac^{2|u|+|u'|-1}$ to $uu'u$ and y' by sending b_1 to v^3 and b_2 to $vv'v$; (iii) accept whenever $x'\#y' \in L_{\psi_L(\mathrm{ext}_{u,v})}$. This forms a valid reduction. Indeed, take a word $w = xy$ with $x \in (ac^{3|u|-1} + ac^{2|u|+|u'|-1})^n$ for $n \in \mathbb{N}$ and $y \in (b_1 + b_2)^m$ for $m \in \mathbb{N}$ and consider $x'\#y'$ produced by the reduction with $x' \in (u^3 + uu'u)^n$ and $y' \in (v^3 + vv'v)^m$. If $w \in \mathcal{L}_{3|u|,2|u|+|u'|}$, then it easily follows that

$x'\#y' \in L_{\psi_L(\mathrm{ext}_{u,v})}$. Otherwise, if $w \notin \mathcal{L}_{3|u|,2|u|+|u'|}$, then either $n \neq m$ and thus $x'y'$ is not well-matched because $\Delta(x') = n \cdot 3 \cdot \Delta(u)$ and $\Delta(y') = m \cdot 3 \cdot \Delta(v)$, or $n = m$ and thus $x'y'$ is well-matched, so $\mathrm{ext}_{x',y'} = \mathrm{ext}_{z'_1,t'_1} \circ \cdots \circ \mathrm{ext}_{z'_n,t'_n}$ with $z'_1, \ldots, z'_n \in \{u^3, uu'u\}$ and $t'_1, \ldots, t'_n \in \{v^3, vv'v\}$ such that there exists $i \in [1,n]$ satisfying $\mathrm{ext}_{z'_i,t'_i} \in \{\mathrm{ext}_{u^3,vv'v}, \mathrm{ext}_{uu'u,v^3}\}$, thereby implying $\psi_L(\mathrm{ext}_{x',y'}) \leq_{\mathfrak{J}} \psi_L(\mathrm{ext}_{z'_i,t'_i}) <_{\mathfrak{J}} \psi_L(\mathrm{ext}_{u,v})$. Hence, our algorithm outputs the pair $(k,l) = (3k_0, 2k_0 + k'_0)$. ◀

6 Conclusion

In this paper we have studied the question of which visibly pushdown languages lie in the complexity class AC^0. We have introduced the notions of length-synchronicity, weak length-synchronicity and quasi-counterfreeness. We have introduced intermediate VPLs: these are quasi-counterfree VPLs generated by context-free grammars G involving the production $S \to_G \varepsilon$ for the start nonterminal S and whose further productions are all of the form $T \to_G uT'v$, where uv is well-matched, $u \in (\Sigma_{\mathrm{int}}^*\Sigma_{\mathrm{call}}\Sigma_{\mathrm{int}}^*)^+$, $v \in (\Sigma_{\mathrm{int}}^*\Sigma_{\mathrm{ret}}\Sigma_{\mathrm{int}}^*)^+$, and the set of contexts $\{(u,v) \in \mathsf{Con}(\Sigma) \mid S \Rightarrow_G^* uSv\}$ is weakly length-synchronous but not length-synchronous. To the best of our knowledge it is unclear whether at all there is an intermediate VPL that is provably in AC^0 (even in ACC^0) or provably not in AC^0. We conjecture that none of the intermediate VPLs are in ACC^0 nor TC^0-hard. Our main result states that there is an algorithm that, given a visibly pushdown language L, outputs if L surely lies in AC^0, surely does not lie in AC^0 (by providing some $m > 1$ such that MOD_m is constant-depth reducible to L), or outputs a disjoint finite union of intermediate VPLs that L is constant-depth equivalant to. In the latter of the three cases one can moreover compute distinct $k, l \in \mathbb{N}_{>0}$ such that already $\mathcal{L}_{k,l} = L(S \to \varepsilon \mid ac^{k-1}Sb_1 \mid ac^{l-1}Sb_2)$ is constant-depth reducible to L. We conjecture that due to the particular nature of intermediate VPLs, either all of them are in AC^0 or all are not: this conjecture together with our main result indeed implies that there is an algorithm that decides if a given visibly pushdown language is in AC^0. As main tools we carefully revisited Ext-algebras, introduced by Czarnetzki et al. [9], being closely related to forest algebras, introduced by Bojańczyk and Walukiewicz [7]. For the reductions from $\mathcal{L}_{k,l}$ we made use of Green's relations.

Natural questions arise. Is there any *concrete* intermediate VPL that is provably in ACC^0, provably not in AC^0, or hard for TC^0? Another exciting question is whether one can effectively compute those visibly pushdown languages that lie in the complexity class TC^0. Is there a $\mathsf{TC}^0/\mathsf{NC}^1$ complexity dichotomy? For these questions new techniques seem to be necessary. In this context it is already interesting to mention there is an NC^1-complete visibly pushdown language whose syntactic Ext-algebra is aperiodic. Another exciting question is to give an algebraic characterization of the visibly counter languages.

References

1 Rajeev Alur, Viraj Kumar, P. Madhusudan, and Mahesh Viswanathan. Congruences for visibly pushdown languages. In Luís Caires, Giuseppe F. Italiano, Luís Monteiro, Catuscia Palamidessi, and Moti Yung, editors, *Automata, Languages and Programming, 32nd International Colloquium, ICALP 2005, Lisbon, Portugal, July 11-15, 2005, Proceedings*, volume 3580 of *Lecture Notes in Computer Science*, pages 1102–1114. Springer, 2005. doi:10.1007/11523468_89.

2 Rajeev Alur and P. Madhusudan. Visibly pushdown languages. In László Babai, editor, *Proceedings of the 36th Annual ACM Symposium on Theory of Computing, Chicago, IL, USA, June 13-16, 2004*, pages 202–211. ACM, 2004. doi:10.1145/1007352.1007390.

3 Vince Bárány, Christof Löding, and Olivier Serre. Regularity problems for visibly pushdown languages. In Bruno Durand and Wolfgang Thomas, editors, *STACS 2006, 23rd Annual Symposium on Theoretical Aspects of Computer Science, Marseille, France, February 23-25, 2006, Proceedings*, volume 3884 of *Lecture Notes in Computer Science*, pages 420–431. Springer, 2006. doi:10.1007/11672142_34.

4 David A. Mix Barrington. Bounded-Width Polynomial-Size Branching Programs Recognize Exactly Those Languages in NC^1. *J. Comput. Syst. Sci.*, 38(1):150–164, 1989. doi:10.1016/0022-0000(89)90037-8.

5 David A. Mix Barrington, Kevin J. Compton, Howard Straubing, and Denis Thérien. Regular languages in nc^1. *J. Comput. Syst. Sci.*, 44(3):478–499, 1992. doi:10.1016/0022-0000(92)90014-A.

6 Richard Beigel. The polynomial method in circuit complexity. In *Proceedings of the Eigth Annual Structure in Complexity Theory Conference, San Diego, CA, USA, May 18-21, 1993*, pages 82–95. IEEE Computer Society, 1993. doi:10.1109/SCT.1993.336538.

7 Mikolaj Bojanczyk and Igor Walukiewicz. Forest algebras. In Jörg Flum, Erich Grädel, and Thomas Wilke, editors, *Logic and Automata: History and Perspectives [in Honor of Wolfgang Thomas]*, volume 2 of *Texts in Logic and Games*, pages 107–132. Amsterdam University Press, 2008.

8 Ashok K. Chandra, Larry J. Stockmeyer, and Uzi Vishkin. Constant depth reducibility. *SIAM J. Comput.*, 13(2):423–439, 1984. doi:10.1137/0213028.

9 Silke Czarnetzki, Andreas Krebs, and Klaus-Jörn Lange. Visibly pushdown languages and free profinite algebras. *CoRR*, abs/1810.12731, 2018. arXiv:1810.12731.

10 Patrick W. Dymond. Input-driven languages are in log n depth. *Inf. Process. Lett.*, 26(5):247–250, 1988. doi:10.1016/0020-0190(88)90148-2.

11 Javier Esparza, Michael Luttenberger, and Maximilian Schlund. A brief history of strahler numbers. In Adrian-Horia Dediu, Carlos Martín-Vide, José Luis Sierra-Rodríguez, and Bianca Truthe, editors, *Language and Automata Theory and Applications - 8th International Conference, LATA 2014, Madrid, Spain, March 10-14, 2014. Proceedings*, volume 8370 of *Lecture Notes in Computer Science*, pages 1–13. Springer, 2014. doi:10.1007/978-3-319-04921-2_1.

12 Stefan Göller and Nathan Grosshans. The ac^0-complexity of visibly pushdown languages. *CoRR*, abs/2302.13116, 2023. doi:10.48550/ARXIV.2302.13116.

13 Yuri Gurevich and Harry R. Lewis. A logic for constant-depth circuits. *Inf. Control.*, 61(1):65–74, 1984. doi:10.1016/S0019-9958(84)80062-5.

14 Michael A. Harrison. *Introduction to Formal Language Theory*. Addison-Wesley, 1978.

15 Johan Håstad. Almost optimal lower bounds for small depth circuits. In Juris Hartmanis, editor, *Proceedings of the 18th Annual ACM Symposium on Theory of Computing, May 28-30, 1986, Berkeley, California, USA*, pages 6–20. ACM, 1986. doi:10.1145/12130.12132.

16 Neil Immerman. Languages that capture complexity classes. *SIAM J. Comput.*, 16(4):760–778, 1987. doi:10.1137/0216051.

17 Neil Immerman. *Descriptive complexity*. Graduate texts in computer science. Springer, 1999. doi:10.1007/978-1-4612-0539-5.

18 Stasys Jukna. *Boolean Function Complexity - Advances and Frontiers*, volume 27 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-24508-4.

19 Andreas Krebs, Klaus-Jörn Lange, and Michael Ludwig. Visibly counter languages and constant depth circuits. In Ernst W. Mayr and Nicolas Ollinger, editors, *32nd International Symposium on Theoretical Aspects of Computer Science, STACS 2015, March 4-7, 2015, Garching, Germany*, volume 30 of *LIPIcs*, pages 594–607. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2015. doi:10.4230/LIPIcs.STACS.2015.594.

20 Michael Ludwig. *Tree-Structured Problems and Parallel Computation*. PhD thesis, University of Tübingen, Germany, 2019. URL: https://publikationen.uni-tuebingen.de/xmlui/handle/10900/85960/.

21 Kurt Mehlhorn. Pebbling Moutain Ranges and its Application of DCFL-Recognition. In J. W. de Bakker and Jan van Leeuwen, editors, *Automata, Languages and Programming, 7th Colloquium, Noordweijkerhout, The Netherlands, July 14-18, 1980, Proceedings*, volume 85 of *Lecture Notes in Computer Science*, pages 422–435. Springer, 1980. doi:10.1007/3-540-10003-2_89.

22 Marcel Paul Schützenberger. On finite monoids having only trivial subgroups. *Inf. Control.*, 8(2):190–194, 1965. doi:10.1016/S0019-9958(65)90108-7.

23 Howard Straubing. *Finite Automata, Formal Logic, and Circuit Complexity*. Birkhäuser Boston, 1994. doi:10.1007/978-1-4612-0289-9.

24 Heribert Vollmer. *Introduction to Circuit Complexity - A Uniform Approach*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 1999. doi:10.1007/978-3-662-03927-4.