# GAPP: Generic Aggregation of Polynomial Protocols

Chaya Ganesh<sup>1</sup>, Sikhar Patranabis<sup>2</sup>, Shubh Prakash<sup>3</sup>, and Nitin Singh<sup>4</sup>

1,3Indian Institute of Science 2,4IBM Research India

#### Abstract

We propose a generic framework called GAPP for aggregation of polynomial protocols. This allows proving  $n$  instances of a polynomial protocol using a single *aggregate* proof that has  $O(\log n)$  size, and can be verified using  $O(\log^2 n)$  operations. The satisfiability of several univariate polynomial identities over a domain is reduced to the satisfiability of a *single* bivariate polynomial identity over a related domain, where the bivariate polynomials interpolate a batch of univariate polynomials over the domain. We construct an information-theoretic protocol for proving the satisfiability of the bivariate polynomial identity, which is then compiled using any bivariate polynomial commitment scheme (PCS) to yield an argument of knowledge for the aggregation relation. GAPP can be applied to several popular SNARKs over bilinear groups that are modeled as polynomial protocols in a black-box way.

We present a new bivariate polynomial commitment scheme, **bPCLB**, with succinct verification that yields an efficient instantiation of GAPP. In addition, the prover only performs sublinear cryptographic operations in the evaluation proof. Towards constructing bPCLB, we show a new folding technique that we call *Lagrangian folding*. The bivariate PCS **bPCLB** and the Lagrangian folding scheme are of independent interest. We implement bPCLB and experimentally validate the practical efficiency of our GAPP instantiation. For the popular PLONK proof system, we achieve  $25-30\%$  faster proof generation than the naïve baseline of generating n separate PLONK proofs. Compared to all existing aggregation schemes that incur additional prover overheads on top of the baseline, we achieve significantly more efficient proving, while retaining succinct verification.

We demonstrate the versatility of our GAPP framework by outlining applications of practical interest: tuple lookups that significantly outperform existing lookup arguments in terms of prover overheads; and proofs for non-uniform computation with "à la carte" prover cost.

### 1 Introduction

Proof systems have a rich history in the theory of computation (notably in works such as [\[GMW86,](#page-41-0) For 87,  $BGG+90$ . At the same time, proof systems are also fundamental building blocks in several cryptographic constructions such as public-key encryption [\[NY90\]](#page-42-0), digital signatures [\[CS97\]](#page-40-1), anonymous credentials [\[CL01\]](#page-40-2), secure multi-party computation [\[GMW87\]](#page-41-1), and more recently, in modern systems like ZCash [\[BCG](#page-39-1)+14] and Monero [\[NMT\]](#page-42-1).

Succinct Arguments. Zero-knowledge (ZK) proofs [\[GMR89,](#page-41-2) [GMW86\]](#page-41-0) allow a prover to convince a verifier about the truth of a statement without revealing anything beyond this. Consider an NP relation R that defines the language  $\mathcal L$  of all statements x for which there exists a witness w so that  $\mathcal{R}(x, w) = 1$ . In a zero-knowledge proof for R, the goal is for a prover who knows a witness w to convince a verifier that  $x \in \mathcal{L}$  without revealing any additional information about w. When considering proofs that are only *computationally sound* (called argument systems) the communication complexity can be smaller than the length of the witness [\[BCC88\]](#page-39-2), and are called succinct arguments [\[Kil92,](#page-41-3) [Mic94\]](#page-42-2).

Succinct Non-interactive ARguments of Knowledge (SNARKs) enable one to prove the integrity of a computation such that the proof size and the verifier's work to check the proof do not scale with the size of the computation. Zero-knowledge variants (zkSNARKs) additionally guarantee that the proof hides all private inputs involved in the computation. zkSNARKs are a fundamental building block in modern cryptographic systems that crucially need small proofs and efficient verification. These have been constructed in several works [\[Gro10,](#page-41-4) [Lip12,](#page-42-3) [BCCT12,](#page-39-3) [BCI](#page-39-4)+13, [GGPR13,](#page-41-5) [PHGR13,](#page-42-4) [BCG](#page-39-5)+13, [Lip13,](#page-42-5) [BCTV14\]](#page-39-6).

Polynomial Protocols. A design methodology underlying several recent constructions of efficient SNARKs is the following. First, an unconditionally secure idealized protocol is obtained for some NP complete language, which is then compiled into a computationally sound argument via a cryptographic compiler. Polynomial protocols, and related notions of Polynomial Interactive Oracle Proofs (PIOPs) and Algebraic Holographic Proofs (AHPs) offer a mathematically elegant framework for constructing secure idealized protocols. Informally, the prover in this idealized setting is restricted to sending low-degree polynomial oracles to the verifier, who infers the membership of the statement in the NP complete language by checking certain identities on the polynomials provided by the prover. In the compiled cryptographic argument, the polynomial oracles are realized via a polynomial commitment scheme (PCS) that allows the prover to send a short commitment to a polynomial and then open evaluations in a verifiable way. The zkSNARK typically inherits the complexity of the PCS in proof size and prover/verifier complexity. PCSs (and the compiled zkSNARKs) are either in the Structured Reference String (SRS) model or in idealized models (like ROM,GGM,AGM) or both.

Polynomial protocols have several useful applications. Many popular SNARKs [\[CHM](#page-40-3)+20, [GWC19\]](#page-41-6) are modeled as polynomial protocols. They have also been used to construct lookup arguments  $[BCG<sup>+</sup>18, GW20, EFG22, CFF<sup>+</sup>24]$  $[BCG<sup>+</sup>18, GW20, EFG22, CFF<sup>+</sup>24]$ , which are particularly useful in constructing efficient SNARKs by moving "SNARK-unfriendly" operations into lookups over pre-computed tables. Additionally, arguments for several other useful relations such as permutations etc. have efficient realizations based on polynomial protocols.

**Our Work.** We propose *generic aggregation of polynomial protocols* where n instances of a polynomial protocol can be proved via a single *aggregate* proof that has  $O(\log n)$  size, and can be verified using  $O(\log^2 n)$  operations (with only  $O(\log n)$  of these being cryptographic operations). Our proof aggregation applies in a black-box manner to several popular SNARKs over bilinear groups that are modeled as polynomial protocols. Along the way, we obtain new lookup arguments over tuples that, together with our proof aggregation, enable efficient proofs of non-uniform computation with "à la carte" prover cost.

#### 1.1 Our Contributions

We expand on our results below.

GAPP. We propose a generic technique for aggregating polynomial protocols, that we call GAPP. Our framework allows proving n instances of a polynomial protocol using a single  $O(\log n)$ -sized aggregate proof, which can be succinctly verified. Broadly, we reduce the satisfiability of several univariate polynomial identities over a domain to the satisfiability of a single bivariate polynomial identity over a related domain, where the bivariate polynomials interpolate a batch of univariate polynomials over the domain. We construct an information-theoretic protocol for proving the satisfiability of the bivariate polynomial identity, which is then be compiled using any bivariate polynomial commitment scheme (PCS) to yield an argument of knowledge for the aggregation relation. We refer to Section [3](#page-14-0) for a formal exposition.

The use of a bivariate polynomial in Lagrange basis to capture a batch of univariate constraints has appeared in Caulk  $[ZBK^+22]$  $[ZBK^+22]$ , Sublonk  $[CGG^+24]$  $[CGG^+24]$ , and Pianist  $[LXZ^+24]$  $[LXZ^+24]$ . However, in all of these prior works, the techniques are presented for very specific polynomial protocols, namely, for multi-unity proof aggregation in [\[ZBK](#page-43-0)+22, [CGG](#page-40-6)+24], and PLONK PIOP in [\[LXZ](#page-42-6)+24]. As a first contribution, we present a framework that generalizes these approaches to arbitrary polynomial protocols. Our framework can be viewed as a natural analogue of representing a batch of arithmetic constraints via a polynomial identity over interpolated polynomials, which is a key step in several SNARK constructions. Unlike prior works  $[ZBK^+22, CGG^+24, LXZ^+24]$  $[ZBK^+22, CGG^+24, LXZ^+24]$ , our framework is described in a manner that is agnostic to the specific bivariate PCS used. This abstraction highlights the role of the bivariate PCS as the key primitive determining the efficiency of aggregated proof generation.

Bivariate PCS. It turns out that instantiating the GAPP framework with existing bivariate PCS (such as those based on AFG  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$ , Dory  $[Lee21]$  and KZG variants  $[ZBK^+22,$  $[ZBK^+22,$  $LXZ+24$  $LXZ+24$ ) incurs large overheads with respect to the size of the public parameters and proof generation (see Section [1.3](#page-6-0) for a more detailed discussion). To this end, we present a new bivariate PCS, which we call bPCLB. At a high level, bPCLB is an analogue of AFG  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$ in the Lagrange basis. In comparison to KZG-based bivariate PCS  $[ZBK^+22, LXZ^+24]$  $[ZBK^+22, LXZ^+24]$  $[ZBK^+22, LXZ^+24]$  $[ZBK^+22, LXZ^+24]$ , which incur  $O(mn)$ -sized public parameters and  $O(mn)$  cryptographic operations to generate evaluation proofs for bivariate polynomials of degree  $(n, m)$ , the corresponding costs for **bPCLB** are  $O(m + n)$ . While this is similar to the overheads incurred by AFG-based bivariate PCS  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$  $[AFG^+16, BMM^+21]$ in isolation, the fact that bPCLB operates directly over Lagrangian components of the bivariate polynomial makes it more suitable for instantiating the GAPP framework than AFG based as well as Dory-based bivariate PCS [\[Lee21\]](#page-42-7), which work over the monomial representation of polynomials (again, see Section [1.3](#page-6-0) for a more detailed discussion). To achieve succinct verification in bPCLB, we require a new folding technique that we call *Lagrangian folding*, and a novel application of the sumcheck protocol. The bivariate PCS bPCLB and the Lagrangian folding scheme are of independent interest. See Section [4](#page-20-0) for the technical details.

We present a concrete instantiation of the GAPP framework using **bPCLB**. Unlike the aggregation frameworks from [\[GMN22,](#page-41-8) [ABST23,](#page-38-1) [YZRM24,](#page-43-1) [LXZ](#page-42-6)+24] that are tailored to specific protocols (such as Groth16 or PLONK), our scheme is generally applicable to all polynomial protocols, and features a universal setup that can be reused across different polynomial IOPs. We implement this scheme and experimentally validate its practical efficiency in several applications, as we discuss below.

Applications. While the above instantiation of GAPP can be applied to any polynomial protocol, we showcase certain applications of practical interest, as outlined below. See Section [5](#page-28-0) for a more detailed exposition.

*Proof Aggregation.* We use the above instantiation of GAPP with PLONK PIOP to obtain a simpler and modular proof aggregation for PLONK with universal setup. In terms of proof generation, our scheme outperforms the popular aggregation scheme aPlonk [\[ABST23\]](#page-38-1) currently used in validity rollups for the Tezos blockchain. Our scheme also supports more efficient proof generation than the recently introduced scheme from  $[LXZ+24]$  $[LXZ+24]$  as well as proof aggregation approaches based on incrementally verifiable computation (IVC) [\[KST22,](#page-42-8) [KS22,](#page-42-9) [BC23,](#page-39-8) [KS24\]](#page-42-10). Some other advantages of our scheme are:

- $\bullet$  Unlike [\[LXZ](#page-42-6)<sup>+</sup>24], we avoid the need for customized KZG commitments, which are incompatible with existing PLONK circuits based on publicly available universal SRS generated using powers of tau ceremony [\[pot\]](#page-43-2).
- Unlike IVC-based aggregation frameworks [\[KST22,](#page-42-8) [KS22,](#page-42-9) [BC23,](#page-39-8) [KS24\]](#page-42-10) that make non-blackbox use of cryptographic primitives (e.g., hash functions, bilinear groups etc.) and are therefore difficult to instantiate in a plug-and-play manner, we use cryptohraphic primitives in a fully black-box manner.

We benchmark our scheme against the naïve baseline of generating  $n$  separate PLONK proofs in Figure [1.](#page-4-0) We achieve 25-30% faster proof generation than the baseline. In comparison with [\[ABST23,](#page-38-1)  $LXZ+24$  $LXZ+24$ , [KST22,](#page-42-8) [KS22,](#page-42-9) [BC23,](#page-39-8) KS24, all of which incur additional prover overheads on top of the baseline, we support significantly more efficient proving, while retaining succinct verification. Our advantage in proof generation over the baseline stems from the fact that we avoid computing evaluation proofs for the "quotient" polynomials for each instance, and instead only compute an evaluation proof for the bivariate polynomial aggregating them. The cryptographic operations incurred by evaluation proof using our bivariate PCS bPCLB are almost the same as that for one instance of univariate evaluation proof.

Tuple lookup. We present a new lookup protocol for tuples based on our bivariate PCS bPCLB. For integers k, m, n, the vector of m-tuples  $\mathbf{A} = (\mathbf{a}_0, \ldots, \mathbf{a}_{n-1})$  is a subvector of  $\mathbf{T} = (\mathbf{t}_0, \ldots, \mathbf{t}_{k-1}),$ if for all  $i \in [n]$ , there exists  $j \in [k]$  such that  $\mathbf{a}_i = \mathbf{t}_j$ . Typically, we wish to check that **A** is a subvector of  $T$  given commitments to  $A$  and  $T$  under a suitable commitment scheme. When  $m = 1$ , this is a regular lookup argument. For most of the  $m = 1$  schemes, the case of  $m > 1$  can be obtained by committing vectors of  $m$ -tuples and then using a random linear combination over the field vectors to reduce the subvector relation over tuples to the one over field vectors. However, this straightforward approach causes both the commitment size and verification complexity to scale as  $O(m)$ . Recent works  $[\text{CGG}^+24, \text{D}XNT23]$  achieve argument size and verification cost independent

<span id="page-4-0"></span>

Figure 1: Comparative benchmarks for aggregation of PLONK proofs between the naïve baseline of generating individual proofs, the aPlonk scheme from [\[ABST23\]](#page-38-1) and our work. We conservatively estimate a 10% overhead for aPlonk over the na¨ıve baseline, based on performance reported in [\[ABST23\]](#page-38-1). Our verification time is similar to aPlonk, and proof size scales as  $\approx 2.8 \log n$  KB for BLS12-381 curve. Individual circuits are of size 214. The comparisons were run using all six cores on a desktop class machine with 32GB RAM.

<span id="page-4-1"></span>

Scheme	Setup	UР	$t_V$	$\pi$
Naïve	O(t)	$O(m+t)$	O(m)	O(m)
$[CGG+24]$	O(mt)	O(mt)	O(1)	O(1)
[DXNT23]	O(mt)	O(mt)	O(1)	O(1)
Our Work	$O(m+t)$	$O(m+t)$	$O(\log t)$	$O(\log t)$

Table 1: Comparison of arguments of knowledge for tuple lookups. Here  $m$  denotes the tuple size,  $n$  denotes the size of the subvector, while k denotes the size of the parent table/vector and  $t = \max(n, k)$ . We denote prover cost by  $t_P$ , verification cost by  $t_V$  and argument size by  $|\pi|$ . We only report cryptographic operations for  $t_P$  and  $t_V$ .

of m, but incur a multiplicative overhead  $(O(mn))$  in the size of the public parameters and the cryptographic operations required by the prover.

We present an approach for tuple lookup based on **bPCLB** where the size of public parameters is  $O(m+n)$ , proof generation requires  $O(m+n)$  cryptographic operations, and the argument size and verification cost are logarithmic. We construct protocols for lookup (the vectors are committed) and committed index lookup (both the vectors and the positions are committed). We compare our tuple lookup with [\[CGG](#page-40-6)+24] and [\[DXNT23\]](#page-40-8) in Table [1.](#page-4-1)

 $\hat{A}$  la carte Prover.  $\hat{A}$  la carte prover cost profile refers to the prover's complexity being proportional only to the size (sum of sizes) of circuit(s) of the operations invoked by a program execution (and independent of the size of circuits corresponding to non-invoked instructions). We combine our aggregation techniques with the above tuple lookup to achieve an "à la carte" proof system for non-uniform computations captured in PLONK constraints. Our scheme makes purely black-box use of cryptoprimitives, and serves as a practical alternative to non-uniform IVC schemes [\[KS22,](#page-42-9) [BC23,](#page-39-8) [KS24\]](#page-42-10).

Our scheme achieves faster proof generation than existing approaches that make black-box use of cryptographic primitives, such as  $[DXNT23, CGG<sup>+</sup>24]$  $[DXNT23, CGG<sup>+</sup>24]$  $[DXNT23, CGG<sup>+</sup>24]$ . In both the prior works, non-uniform proof generation involves two key steps (i) prover uses the tuple lookup argument to lookup "sub-circuits" (modelled as tuples) involved in the computation, from a pre-defined table of such sub-circuits and later (ii) proves the correctness of the circuit assembled from the looked up sub-circuits. The first step incurs  $O(mn)$  cryptographic operations, while the second step invokes Plonk/Marlin prover on the  $O(mn)$ -sized assembled circuit. Following a similar approach, we first use tuple lookup to commit to circuit polynomials for each step (potentially in an input defined manner), and then provide an aggregated proof for all the steps. As noted earlier, our tuple lookup is substantially faster. Further, our proof aggregation is  $25 - 30\%$  more efficient than the monolithic proof for  $O(mn)$  sized circuit.

We reiterate that our focus is only to showcase the versatility of the GAPP framework and its efficient instantiation from our bivariate PCS bPCLB. While à la carte prover for non-uniform computation can also be used to obtain proofs of machine execution (zkVM), where current instruction determines the invoked computation, we leave detailed, end-to-end optimized construction of zkVM using our methods and its comparison with existing approaches as future work.

#### 1.2 Related Work

Proof Aggregation. Popular proof aggregation schemes include SnarkPack [\[GMN22\]](#page-41-8) for aggregating Groth16 proofs, aPlonk [\[ABST23\]](#page-38-1) for aggregating PLONK [\[GWC19\]](#page-41-6) proofs, and aHyper-Proofs [\[YZRM24\]](#page-43-1) which is a multivariate counterpart to aPlonk and aggregates HyperPlonk [\[CBBZ23\]](#page-40-9) proofs. Hyperproofs  $[SCP+22]$  $[SCP+22]$  provides Merkle-like proofs based on polynomial commitments which can be efficiently aggregated. [\[GMN22,](#page-41-8) [ABST23,](#page-38-1) [YZRM24\]](#page-43-1) rely on multi-polynomial commitments and inner pairing-product arguments from  $[BMM+21]$  $[BMM+21]$  to fold pairing checks for n proof verifications into one pairing check over multi-commitments. The PIOP based schemes [\[ABST23,](#page-38-1) [YZRM24\]](#page-43-1) subsequently verify the polynomial identities inside a *meta* arithmetic circuit, whose correctness is proved using a separate SNARK proof. These existing aggregation frameworks require relationspecific setup for each instance due to their dependence on the arithmetic circuit defining the aggregation relation. These approaches are also tailored to specific protocols, such as Groth16 and PLONK. In contrast, our methods apply more generally and avoid the need for relation-specific setup.

The recent work of  $[{\rm L}XZ^+24]$  uses bivariate polynomials in Lagrange basis to capture a batch of univariate constraints, albeit towards a completely different goal of distributing a SNARK prover. Their distributed protocol also requires proof aggregation as a key technique. As noted earlier, their techniques are very specific to PLONK and crucially rely on customized KZG commitments, which are incompatible with existing PLONK circuits based on publicly available universal SRS generated using powers of tau ceremony [\[pot\]](#page-43-2). Our proof aggregation mechanism applies generally to any polynomial protocol, and avoids the need for such customized KZG commitments.

**Lookup Arguments.** Early lookup arguments such as Arya  $[BCG^+18]$  $[BCG^+18]$  and Plookup  $[GW20]$ were constructed as means for "custom gates" in SNARKs. These schemes incur proving cost of  $O(n + k)$  cryptographic operations where n and k are the sizes of the table and the subvector, respectively. Recent works  $[BCG+18, GW20, Hab22, ZBK+22, PK22, ZGK+22, GK22, EFG22,$  $CFF+24$  $CFF+24$  have introduced substantial improvements. The works of  $[EFG22, CFF+24]$  $[EFG22, CFF+24]$  $[EFG22, CFF+24]$  based on

"cached quotients" incur proving cost of  $O(k)$ , substantially improving over earlier works when  $k \ll n$ . Moreover, recent work [\[DGP](#page-40-10)+24] also obtains committed index lookups via an efficient reduction to un-indexed lookups. In the context of lookups over m-tuples, all of these works focus on  $m = 1$ . The naïve approach of constructing m-tuple lookup for  $m > 1$  using these works incurs  $O(m)$  multiplicative overhead in commitment/argument size and verification complexity.

Recent works  $[\text{CGG}^+24, \text{D} \text{X} \text{N} \text{T} 23]$  avoid the  $O(m)$  multiplicative overhead by extending CQ [\[EFG22\]](#page-40-4) and Plookup [\[GW20\]](#page-41-7) PIOPs with additional algebraic constraints to atomically associate each m-tuple in a table  $\bf{T}$  consisting of n such tuples with m distinct positions in a flattened table  $\bf{T}$ of size  $mn$ . An m tuple is now interpreted as a set of contiguous positions (equivalently, a segment) in  $[CGG+24]$  $[CGG+24]$ , or as a coset of a subgroup of order n in  $[DXNT23]$ . While both the works achieve argument size and verification independent of m, they both require a setup of size  $O(mn)$  and proof generation involving  $O(mn)$  cryptographic operations. In contrast, we propose a lookup argument for m-tuples with  $O(m + n)$  setup,  $O(m + n)$  cryptographic operations for proof generation, and logarithmic argument size and verification cost.

A la carte Prover. Proving correct machine computation involves proving the state transition function determined at each step by the specific instruction type. This typically incurs large prover costs due to the use of a universal circuit to capture any instruction supported by the machine. General purpose SNARKs for such applications require large universal circuits that encompass all possible execution paths, incurring similar proof generation overheads. the prover cost should only depend on the actual execution path (i.e., the clause that is actually executed), such that the cost only grows with the sizes of circuits corresponding to the the operations invoked by the program execution. Such "`a la carte" prover cost profile when proving machine executions is enabled by recent works on non-uniform IVC [\[KS22,](#page-42-9) [BC23,](#page-39-8) [KS24\]](#page-42-10). However, these works make use of non-black-box use of cryptographic objects such as hash functions, groups etc, which limits their portability. Certain recent works [\[AST24,](#page-38-2) [STW24\]](#page-43-5) avoid non-uniform verification of the CPU state transition by proposing an approach based on lookup arguments.

Verifying non-uniform relations while only making black-box use of cryptography has been explored in MuxProofs  $[DXNT23]$  and SubPlonk  $[CGG+24]$  $[CGG+24]$ . The authors of  $[DXNT23, CGG+24]$  $[DXNT23, CGG+24]$  $[DXNT23, CGG+24]$ leverage SNARKs with updatable setup to verifiably obtain computation commitments (similar to relation-specific public parameters) for the active sub-circuit determined by the inputs. The correctness of the committed active sub-circuit is then proved by a specific SNARK, namely Marlin [\[CHM](#page-40-3)+20] in [\[DXNT23\]](#page-40-8) and PLONK in [\[CGG](#page-40-6)+24]. Our proposed scheme also makes black-box use of cryptoprimitives, and achieves faster proof generation than  $[DXNT23, CGG<sup>+</sup>24]$  $[DXNT23, CGG<sup>+</sup>24]$  $[DXNT23, CGG<sup>+</sup>24]$ .

For disjunctive NP relations, [\[GGHAK22,](#page-41-11) [GHAKS23\]](#page-41-12) describe stacking protocols for efficiently handling disjunctions by compiling IOPs and Sigma protocol composition. This approach incurs prover cost that depends only on the size of the clause executed and is independent of the total number of clauses. Other recent works such as [\[YHH](#page-43-6)<sup>+23]</sup> have used the MPC-in-the-head paradigm to construct efficient protocols for disjunctions and batched disjunctions.

#### <span id="page-6-0"></span>1.3 Technical overview

In this section, we present an overview of our core techniques. Throughout, we use [a] for  $a \in \mathbb{N}$ to denote the set  $[0, a - 1]$ . A polynomial protocol involves proving that a polynomial identity holds over some subset  $\nabla$  of  $\nabla$ , where the polynomials involved in the identity are committed using some polynomial commitment scheme PC. Typically, the subset  $V$  is taken to be the set of  $m<sup>th</sup>$  roots of unity for some  $m \in \mathbb{Z}$ . Formally, the goal is to prove a polynomial identity of the form  $G(p_0(Y), \ldots, p_{\ell-1}(Y)) = 0$  vanishes over V, for some multivariate polynomial G, given a set

of commitments  $(C_0, \ldots, C_{\ell-1})$ , where each  $C_j$  is a commitment to the polynomial  $p_j(Y)$  under the polynomial commitment scheme (PCS) PC. Without loss of generality, we assume that, for some  $k \in [\ell],$  the commitments  $(C_0, \ldots, C_{k-1})$  are honestly generated (i.e., trusted by the verifier), while the remaining commitments  $(C_k, \ldots, C_{\ell-1})$  are adversarially generated. For example, proof generation in PLONK [\[GWC19\]](#page-41-6) involves showing that a polynomial

$$
G_{\mathbf{p}}(q_M(Y), q_L(Y), q_R(Y), q_O(Y), q_C(Y), a(Y), b(Y), c(Y))
$$

vanishes over V, where  $G_p(q_M, q_L, q_R, q_O, q_C, a, b, c) = q_Mab + q_La + q_Rb + q_Oc + q_C$ . Here, commitments to the circuit polynomials  $(q_M, q_L, q_R, q_O, q_C)$  are trusted (being outputs of one-time preprocessing), while commitments to the witness polynomials  $(a, b, c)$ , generated by the prover, may be malicious.

Aggregating Polynomial Protocols. Now consider a scenario where a prover wishes to prove n homogeneous polynomial identities

$$
G(p_{i,0}(Y),\ldots,p_{i,\ell-1}(Y)) = 0 \text{ mod } Z_{\mathbb{V}}(Y) \quad \forall i \in [n],
$$
\n
$$
(1)
$$

given a set of commitments  $(C_{i,0},...,C_{i,\ell-1})$ , where  $C_{i,j}$  is a commitment to  $p_{i,j}(Y)$  under the PCS PC. Note that the naïve approach would be to run  $n$  instances of any polynomial protocol compatible with PC. This approach involves proving a statement of size  $O(n\ell)$ , and results in an argument size of  $O(n\ell |\pi_{PC}|)$  (where  $|\pi_{PC}|$  denotes the opening size of PC).

Aggregation using Bivariate Polynomials. First, to reduce the statement size from  $n \times \ell$ to  $\ell$ , we create a single commitment to a vector of polynomials  $(p_{i,j})_{i\in[0,n-1]}$  for a given  $j\in[\ell]$ . Let  $\mathbb{H} = \{1, \omega, \ldots, \omega^{n-1}\}\)$  be the subgroup consisting of the  $n^{th}$  roots of unity in  $\mathbb{F}$ , and let  $\mu_i^{\mathbb{H}}$  $_{i}^{\mathbb{H}}(X)$ be the corresponding Lagrange polynomial for each  $i \in [n]$ . We say that the *packed* polynomial corresponding to the vector of univariate polynomials  $(p_{i,j})_{i\in[n]}$  is the bivariate polynomial  $P_j(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}$  $\mathbb{H}^1(X)p_{i,j}(Y)$ . For this packing scheme, the satisfiability of a set of n univariate polynomial identities over V reduces to showing that the bivariate polynomial  $Q(X, Y) =$  $G(P_0(X, Y), \ldots, P_{\ell-1}(X, Y))$  vanishes over  $\mathbb{H} \times \mathbb{V}$ . This is analogous to compressing n arithmetic constraints into an equivalent polynomial constraint over interpolated polynomials in several SNARK constructions. See Lemma [3.1](#page-14-1) for a precise exposition.

Now the goal is to prove knowledge of bivariate polynomials  $(P_0, \ldots, P_{\ell-1})$  corresponding to commitments  $(C_0, \ldots, C_{\ell-1})$  s.t.  $G(p_{i,0}(Y), \ldots, p_{i,\ell-1}(Y))$  vanishes over  $\mathbb {V}$  for each  $i \in [n]$ , where for each  $j \in [\ell], p_{0,j}(Y), \ldots, p_{n-1,j}(Y)$  are the uniquely determined univariate components of  $P_j(X, Y)$ with respect to the polynomial basis  $(\mu_0^{\mathbb{H}})$  $\mathbb{H}(X), \ldots, \mu_{n-1}^{\mathbb{H}}(X)$ . In a real application, for some  $k \in [\ell],$ the commitments  $(C_0, \ldots, C_{k-1})$  would be honestly generated (i.e., trusted by the verifier), while the remaining commitments  $(C_k, \ldots, C_{\ell-1})$  would serve as aggregated commitments to the prover's witness, and thus could be adversarially generated.

The GAPP Relation. Let bPC be a bivariate PCS with public parameter pp. We informally define the relation  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  as follows:

**Definition 1.1** (GAPP (informal)). Let  $C = (C_0, \ldots, C_{\ell-1})$  be a vector of commitments, and let  $\mathbf{w} = (P_0(X, Y), \ldots, P_{\ell-1}(X, Y))$  be a vector of bivariate polynomials. We say that  $(\mathbf{C}, \mathbf{w}) \in$  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  if:

1. For each  $j \in [\ell], P_i(X, Y)$  opens the commitment  $C_j$ .

2. For each  $i \in [n]$ ,  $G(P_0(\omega^i, Y), \ldots, P_{\ell-1}(\omega^i, Y)) = 0$  mod  $Z_{\mathbb{V}}(Y)$ , where  $\omega$  is the canonical primitive n<sup>th</sup> root of unity in  $\mathbb{F}$ , and where  $P_j(\omega^i, Y)$  is the i<sup>th</sup> univariate polynomial  $p_{i,j}(Y)$ "packed" into  $P_i$ .

In Section [3,](#page-14-0) we formally define a more generalized version of this relation that allows capturing polynomial protocols where the same polynomial appears with different parameterizations. See Definition [3.1](#page-15-0) for the details.

**Argument of Knowledge for GAPP.** To construct an argument of knowledge for  $\mathcal{R}_{pp,G,n,m}^{GAPP}$ , we design: (i) an information-theoretic protocol, where the prover's messages are restricted to be low-degree (univariate and bivariate) polynomials, and (ii) a novel bivariate PCS to compile this information-theoretic protocol into an argument of knowledge for  $\mathcal{R}_{pp,G,n,m}^{GAPP}$  with succinct verification and an efficient prover.

The Information-Theoretic Protocol. The information-theoretic protocol requires the prover to show that  $Q(X, Y) = G(P_0(X, Y), \ldots, P_{\ell-1}(X, Y))$  vanishes over  $\mathbb{H} \times \mathbb{V}$ . At a high level, the prover produces low-degree polynomials  $Q(X, Y)$  and  $H(X, Y)$  such that  $Q(X, Y) - Z_{\mathbb{V}}(Y)H(X, Y)$  vanishes over H, to which the verifier has oracle access. The verifier probabilistically checks the identity  $Q(X, y) - Z_{\mathbb{V}}(y)H(X, y) = 0$  mod  $Z_{\mathbb{H}}(X)$  for  $y \leftarrow \mathbb{F}$ . Concretely, the prover produces univariate polynomials  $q(X) = Q(X, y)$ ,  $h(X) = H(X, y)$ , and  $u(X)$ , and the verifier probabilistically checks  $q(X) - Z_{\mathbb{V}}(y)h(X) = u(X)Z_{\mathbb{H}}(X)$  by querying the oracles for  $q(X), h(X)$  and  $u(X)$  at a random point. Finally, to establish that  $q(X) = Q(X, y)$  and  $h(X) = H(X, y)$ , the verifier samples  $x \leftarrow \mathbb{F}$ , queries the polynomials Q and H at  $(x, y)$ , queries the polynomials q and h at x, and checks that  $Q(x, y) = q(x)$  and  $H(x, y) = h(x)$ .

Bivariate Polynomial Commitment. The above information-theoretic protocol can be compiled into an argument of knowledge for  $\mathcal{R}_{pp,G,n,m}^{GAPP}$  with succinct verification using any univariate PCS and any bivariate PCS, such that both schemes support succinct verification of evaluation proofs (see Section [3.2](#page-15-1) for a detailed treatment). Our key innovation is a bivariate PCS that, together with the KZG PCS for univariate polynomials, yields a concretely efficient argument of knowledge based on the above information-theoretic protocol while achieving *efficient prover* and minimizing the size of the public parameters.

Our starting point is the two-tiered commitment scheme based on a bilinear pairing  $e : \mathbb{G}_1 \times$  $\mathbb{G}_2 \to \mathbb{G}_T$  from [\[AFG](#page-38-0)<sup>+</sup>16, [BMM](#page-40-7)<sup>+</sup>21, [GMN22\]](#page-41-8) that computes a commitment  $C_P$  to a bivariate polynomial  $P(X,Y)$  as follows: (i) represent  $P(X,Y)$  as  $\sum_{i=0}^{n-1} p_i(Y)X^i$ , (ii) compute commitments  $C_i \in \mathbb{G}_1$  to polynomials  $p_i(Y)$  for all  $i \in [n]$ , and (iii) compute the commitment  $C_P$  to the vector  $\mathbf{C} = (C_0, \ldots, C_{n-1})$  as  $C_P = \sum_{i=0}^{n-1} e(C_i, w_i)$ , where  $\mathbf{w} = (w_0, \ldots, w_{n-1}) \in \mathbb{G}_2^n$  is a commitment key. However, this scheme is not amenable to the above information-theoretic protocol, as we elaborate below.

 $\sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}$ Recall that our approach requires computing a commitment to a packed polynomial  $P(X, Y) =$  $\mathbb{H}(X)p_i(Y)$ . Further, assume that a commitment to each  $p_i(Y)$  is available as part of some pre-processing step (such a scenario, in fact, arises in our application of GAPP for proving nonuniform computation in Section [5\)](#page-28-0). To use the commitment scheme from [\[BMM](#page-40-7)+21], one needs to express the polynomial  $P(X,Y)$  as  $\sum_{i=0}^{n-1} p_i'(Y) X^i$ , which is equivalent to computing the monomial coefficients  $\{p_i'(Y)\}$  given the Lagrange coefficients  $\{p_i(Y)\}\$ . This requires FFT over the polynomial ring  $\mathbb{F}[Y]$ , which incurs  $\Omega(mn \log n)$  field operations. Moreover, the prover needs to commit to each  $p'_{i}(Y)$  and then compute the pairing product, which additionally incurs  $\Omega(mn)$  group operations and n pairings. The overall commitment cost is  $\Omega(mn \log n)\mathbb{F} + \Omega(mn)\mathbb{G} + n\mathbb{P}$ .

In Section [4.1,](#page-20-1) we present a novel bivariate PCS that totally avoids the overheads for FFT and the associated group operations by computing the pairing product directly over commitments to the Lagrange coefficients  $\{p_i(Y)\}\$  of the bivariate polynomial  $P(X, Y)$ . This significantly reduces the the commitment cost to just  $n$  pairings. We now present an overview of our bivariate PCS in the rest of this section.

**Our Bivariate PCS.** Similar to the approach in  $[BMM<sup>+</sup>21]$  $[BMM<sup>+</sup>21]$ , we use KZG PCS to create commitments  $\{C_i\}$  to the Lagrange coefficients  $\{p_i(Y)\}\$  of the bivariate polynomial  $P(X, Y)$ , and then compute  $C_P = \sum_{i=0}^{n-1} e(C_i, w_i)$ , where  $\mathbf{w} = (w_0, \dots, w_{n-1}) \in \mathbb{G}_2^n$  is a commitment key. Our core technical innovation is in achieving a logarithmic size opening for this commitment. To illustrate the challenges thereof, we recall the techniques used in prior works [\[BMM](#page-40-7)+21, [GMN22\]](#page-41-8).

To open the bivariate polynomial  $P(X, Y)$  with commitment  $C_P$  (computed using the monomial basis), the existing approaches in  $[BMM<sup>+</sup>21, GMN22]$  $[BMM<sup>+</sup>21, GMN22]$  $[BMM<sup>+</sup>21, GMN22]$  generalize the "split and fold" technique used in the inner product protocols in  $[BCC^+16, BBB^+18]$  $[BCC^+16, BBB^+18]$  $[BCC^+16, BBB^+18]$  $[BCC^+16, BBB^+18]$  and in the compressed sigma protocol framework of [\[AC20\]](#page-38-4). In more detail, to open the polynomial commitment to a value v at  $(x, y)$ , the prover first computes the commitment  $C_p$  to the univariate polynomial  $P(x, Y)$  and sends it to the verifier. Since  $P(x,Y) = \sum_{i=0}^{n-1} p_i(Y) x^i$ , the homomorphic property of the KZG commitment scheme implies that  $C_p = \sum_{i=0}^{n-1} x^i C_i$  is a commitment to  $P(x, Y)$ . Next, the prover opens the univariate polynomial  $P(x, Y)$  to the value v at  $Y = y$ . The key step in the protocol is for the prover to convince the verifier that the commitment  $C_p$  to the univariate polynomial is consistent with the commitment  $C_P$  to the bivariate polynomial. In other words, the prover proves knowledge of the vector  $C_0, \ldots, C_{n-1} \in \mathbb{G}_1^n$  such that:

<span id="page-9-1"></span>
$$
C_P = \sum_{i=0}^{n-1} e(C_i, w_i) \bigwedge C_p = \sum_{i=0}^{n-1} x^i C_i \tag{2}
$$

In the above,  $C_P$  can be considered a commitment to  $\mathbf{C} = (C_0, \ldots, C_{n-1})$  under the commitment key  $\mathbf{w} = (w_0, \dots, w_{n-1})$  $\mathbf{w} = (w_0, \dots, w_{n-1})$  $\mathbf{w} = (w_0, \dots, w_{n-1})$  which is *doubly homomorphic*<sup>1</sup>, whereas  $C_p$  can be considered a *linear form* given by the key  $(1, x, \ldots, x^{n-1})$ .

Note that the relation in Equation [2](#page-9-1) can be proved using the split and fold technique of [\[BCC](#page-39-9)+16,  $BBB+18$  $BBB+18$ , [AC20\]](#page-38-4). However, naïvely using the techniques from these works results in a linear time verifier. This is due to the fact that verifier is required to compute the "folded" commitment key and linear form in each round. To address this, the prior works [\[BMM](#page-40-7)+21, [GMN22\]](#page-41-8) consider commitment keys  $w_i, i \in [n]$  with monomial structure, i.e.  $w_i = \tau^i \cdot g_2 = [\tau_i^i]_2$  for some trapdoor  $\tau \leftarrow$ **F.** The linear form already inherits the monomial structure  $(1, x, ..., x^{n-1})$  from the representation of the bivariate polynomial  $P(X, Y)$  in the monomial basis consisting of powers of X. The key observation made about such structured commitment keys is that the folding can be delegated to the prover, and the verifier can efficiently (in  $O(\log n)$ ) time) check the correctness of the final commitment key and the linear form.

A New Folding Technique. At a high level, we need to prove a relation similar to the one in Equation [2,](#page-9-1) except that  $C_p = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}$  $\mathbb{H}^1(x)C_i$  since, in our case,  $\{C_i\}$  are commitments to the Lagrange coefficients  $\{p_i(Y)\}\$  of the bivariate polynomial  $P(X, Y)$ . Since the commitment key  $(\mu_i^{\mathbb{H}})$  $\mathbb{H}^1(x))_{i\in [N]}$  does not have an obvious tensor (e.g., monomial) structure, the verifier cannot delegate the folding to the prover, which can be verified in logarithmic time, as in prior works [\[BMM](#page-40-7)+21, [GMN22\]](#page-41-8). To address this, we introduce a new folding technique, which allows a verifier to check

<span id="page-9-0"></span> $1$ <sup>1</sup>The commitment is homomorphic both in commitment key and message

the folding of commitment keys structured as Lagrange basis polynomials (instead of monomial basis) with logarithmic effort.

Our key insight here is that while Lagrange polynomials has no obvious structure in the monomial basis, they can be succinctly described in the Lagrange basis, where their coefficients correspond to standard unit vectors. This allows us to efficiently fold in the Lagrange basis, and later recover the final folded polynomial in the monomial basis by applying the inverse-FFT transformation to the final folded polynomial in the Lagrange basis. The protocol for checking correctness of the inverse-FFT transformation uses a novel application of multivariate sum-check protocol [\[LFKN92\]](#page-42-12) over a boolean hypercube. We defer the details to Section [4.2.](#page-24-0)

### 2 Preliminaries

We present preliminary background material in this section.

**Notations.** We use [n] to denote the set of integers  $\{0, \ldots, n-1\}$  and F to denote a prime field of order p. We denote by  $\lambda$  a security parameter. We use negl to denote a negligible function: for any integer  $c > 0$ , there exists  $n \in \mathbb{N}$ , such that  $\forall x > n$ , negl $(x) \leq 1/x^c$ . We assume a bilinear group generator BG which on input  $\lambda$  outputs parameters for the protocols. Specifically  $BG(1^{\lambda})$  outputs  $(\mathbb{F}, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2, g_t)$  where:  $\mathbb{F} = \mathbb{F}_p$  is a prime field of super-polynomial size in  $\lambda$ , with  $p = \lambda^{\omega(1)}$ ;  $\mathbb{G}_1, \mathbb{G}_2$  and  $\mathbb{G}_T$  are groups of order p, and e is an efficiently computable non-degenerate bilinear pairing  $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ ; Generators  $g_1, g_2$  are uniformly chosen from  $\mathbb{G}_1$  and  $\mathbb{G}_2$  respectively and  $g_t = e(g_1, g_2)$ . We write groups  $\mathbb{G}_1$  and  $\mathbb{G}_2$  additively, and use the shorthand notation  $[x]_1$  and  $[x]_2$  to denote group elements  $x \cdot g_1$  and  $x \cdot g_2$  respectively for  $x \in \mathbb{F}$ . We implicitly assume that all the setup algorithms for the protocols invoke BG to generate descriptions of groups and fields over which the protocol is instantiated. We will als use sets  $\mathbb{F}, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$  to specify the type of operations, where additionally, we have  $\mathbb P$  to denote pairings and  $\mathbb M$  to denote multiexponentiation.

Lagrange Polynomials. At different points, we use groups  $\mathbb{H}, \mathbb{V}$  and  $\mathbb{K}$  generated by primitive  $n^{th}$ ,  $m^{th}$  and  $k^{th}$  roots of unity. We use  $\{\mu_i^{\mathbb{H}}\}$  $\{ \mu_i^{\mathbb{H}}(X) \}_{i=0}^{n-1},$   $\{\mu_i^{\mathbb{V}}$  $\widetilde{\Psi}(X)$   $\}_{i=0}^{m-1}$  and  $\{\mu_i^{\mathbb{K}}\}$  $_{i}^{\mathbb{K}}(X)\}_{i=0}^{k-1}$  as the Lagrange polynomials for sets H, V and K respecitvely. We use  $Z_{\mathbb{H}}(X)$ ,  $Z_{\mathbb{V}}(X)$  and  $Z_{\mathbb{K}}(X)$  to denote the vanishing polynomials of the respective sets. We will generally use  $\omega$  as the  $n^{th}$  primitive root of unity and  $\nu$  as the primitive  $m^{th}$  root of unity.

#### <span id="page-10-0"></span>2.1 Succinct Argument of Knowledge

Let R be a NP-relation and L be the corresponding NP-language, where  $\mathcal{L} = \{x : \exists w \text{ such that }$  $(x, w) \in \mathcal{R}$ . Here, a prover P aims to convince a verifier V that  $x \in \mathcal{L}$  by proving that it knows a witness w for a public statement x such that  $(x, w) \in \mathcal{R}$ . An interactive argument of knowledge for a relation R consists of a PPT algorithm Setup that takes as input the security parameter  $\lambda$ , and outputs the public parameters **pp**, and a pair of interactive PPT algorithms  $\langle \mathcal{P}, \mathcal{V} \rangle$ , where  $\mathcal{P}$  takes as input (pp, x, w) and V takes as input (pp, x). An interactive argument of knowledge  $\langle \mathcal{P}, \mathcal{V} \rangle$  must satisfy completeness and knowledge soundness.

**Definition 2.1** (Completeness). For all security parameter  $\lambda \in \mathbb{N}$  and statement x and witness w such that  $(x, w) \in \mathcal{R}$ , we have

$$
\Pr\left(b=1\,:\,\frac{\mathsf{pp} \leftarrow \mathsf{Setup}(1^{\lambda})}{b \leftarrow \langle \mathcal{P}(w), \mathcal{V} \rangle(\mathsf{pp},x) } \right) = 1.
$$

**Definition 2.2** (Knowledge Soundness). For any PPT malicious prover  $\mathcal{P}^* = (\mathcal{P}_1^*, \mathcal{P}_2^*)$ , there exists a PPT algorithm  $\mathcal E$  such that the following probability is negligible:

$$
\Pr\left(\begin{matrix}\mathsf{pp} \leftarrow \mathsf{Setup}(1^\lambda)\\ b = 1 \wedge & (x, \mathsf{st}) \leftarrow \mathcal{P}_1^*(1^\lambda, \mathsf{pp})\\ (x, w) \not\in \mathcal{R} \, \, : \, \, b \leftarrow \langle \mathcal{P}_2^*(\mathsf{st}), \mathcal{V} \rangle(\mathsf{pp}, x) \\ w \leftarrow \mathcal{E}^{\mathcal{P}_2^*}(\mathsf{pp}, x)\end{matrix}\right) \right).
$$

A succinct argument of knowledge  $\langle \mathcal{P}, \mathcal{V} \rangle$  for a relation R, must satisfy completeness and knowledge soundness and additionally be *succinct*, that is, the communication complexity between prover and verifier, as well as the verification complexity is bounded by  $\text{poly}(\lambda, \log |w|)$ .

in Appendix [2.1.](#page-10-0)

#### <span id="page-11-0"></span>2.2 Polynomial Commitment Scheme

A polynomial commitment scheme (PCS) introduced in [\[KZG10\]](#page-42-13) allows a prover to open evaluations of the committed polynomial succinctly. A PCS over  $\mathbb F$  is a tuple  $PC = (Setup, Com, Open,Eval)$ where:

- pp  $\leftarrow$  Setup $(1^{\lambda}, n, \{D_i\}_{i \in [n]})$ . On input security parameter  $\lambda$ , number of variables n and upper bounds  $D_i \in \mathbb{N}$  on the degree of each variable  $X_i$  for a *n*-variate polynomial, Setup generates public parameters pp.
- $(C, \tilde{c}) \leftarrow \textsf{Com}(pp, f(X_1, \dots, X_n), d_1, \dots, d_n)$ . On input the public parameters pp, and a nvariate polynomial  $f(X_1, \dots, X_n) \in \mathbb{F}[X_1, \dots, X_n]$  with degree at most  $\deg(X_i) = d_i \leq D_i$ for all  $i$ , Com outputs a commitment to the polynomial  $C$ , and additionally an opening hint ˜c.
- $b \leftarrow \text{Open}(pp, f(X_1, \dots, X_n), d_1, \dots, d_n, C, \tilde{c})$ . On input the public parameters pp, the commitment C and the opening hint  $\tilde{\mathbf{c}}$ , a polynomial  $f(X_1, \dots, X_n)$  with  $d_i \leq D_i$ , Open outputs a bit indicating accept or reject.
- $\bullet \, b \leftarrow \text{Eval}(pp, C, (d_1, \ldots, d_n), (x_1, \ldots, x_n), v; f(X_1, \cdots, X_n)).$  A public coin interactive protocol  $\langle P_{eval}(f(X_1,\dots,X_n)), V_{eval} \rangle$  (pp,  $C, (d_1,\dots,d_n), (x_1,\dots,x_n), v$ ) between a PPT prover and a PPT verifier. The parties have as common input public parameters  $pp$ , commitment  $C$ , degree d, evaluation point x, and claimed evaluation v. The prover has, in addition, the opening  $f(X_1, \dots, X_n)$  of C, with  $\deg(X_i) \leq d_i$ . At the end of the protocol, the verifier outputs 1 indicating accepting the proof that  $f(x_1, \ldots, x_n) = v$ , or outputs 0 indicating rejecting the proof.

A polynomial commitment scheme must satisfy completeness, binding and extractability.

**Definition 2.3** (Completeness). For all polynomials  $f(X_1, \dots, X_n) \in \mathbb{F}[X_1, \dots, X_n]$  with degree  $\deg(X_i) = d_i \leq D_i$ , for all  $(x_1, \ldots, x_n) \in \mathbb{F}^n$ ,

$$
\Pr\left(b=1: \begin{matrix} \text{pp} \leftarrow \text{Setup}(1^{\lambda}, n, \{D_i\}_{i \in [n]}) \\ (C, \tilde{\mathbf{c}}) \leftarrow \text{Com}(\text{pp}, f(X_1, \cdots, X_n), d_1, \ldots, d_n) \\ v \leftarrow f(x) \\ b \leftarrow \text{Eval}(\text{pp}, C, (d_1, \ldots, d_n), (x_1, \ldots, x_n), v; f(X_1, \cdots, X_n)) \end{matrix}\right) = 1.
$$

**Definition 2.4** (Binding). A polynomial commitment scheme PC is binding if for all PPT A, the following probability is negligible in  $\lambda$ :

$$
\Pr\left(\begin{matrix}\mathsf{Open}(\mathsf{pp}, f_0, \mathbf{d_0}, C, \mathbf{\tilde{c}_0})=1 \wedge & \mathsf{pp} \leftarrow \mathsf{Setup}(1^\lambda, n, \{D_i\}_{i \in [n]}) \\ \mathsf{Open}(\mathsf{pp}, f_1, \mathbf{d_1}, C, \mathbf{\tilde{c}_1})=1 \wedge : & (C, f_0, f_1, \mathbf{\tilde{c}_0}, \mathbf{\tilde{c}_1}, \mathbf{d_0}, \mathbf{d_1}) \leftarrow \mathcal{A}(\mathsf{pp}) \\ f_0 \neq f_1\end{matrix}\right).
$$

**Definition 2.5** (Knowledge Soundness). For any PPT adversary  $A = (A_1, A_2)$ , there exists a PPT algorithm  $\mathcal E$  such that the following probability is negligible in  $\lambda$ :

$$
\Pr\left(\begin{matrix} b=1 \wedge& \text{pt} \leftarrow \text{Setup}(1^{\lambda},n,\{D_i\}_{i \in [n]}) \\ b=1 \wedge& (C,\mathbf{d},\mathbf{x},v,\mathbf{st}) \leftarrow \mathcal{A}_1(\mathsf{pp}) \\ \mathcal{R}_{\mathsf{Eval}}(\mathsf{pp},C,\mathbf{x},v;\tilde{f},\tilde{\mathbf{c}})=0 & \begin{matrix} (\tilde{f},\tilde{\mathbf{c}}) \leftarrow \mathcal{E}^{\mathcal{A}_2}(\mathsf{pp},C,d) \\ (\tilde{f},\tilde{\mathbf{c}}) \leftarrow \mathcal{E}^{\mathcal{A}_2}(\mathsf{pp},C,d) \\ b \leftarrow \langle \mathcal{A}_2(\mathsf{st}),V_{\mathsf{eval}} \rangle(\mathsf{pp},C,\mathbf{d},\mathbf{x},v) \end{matrix} \right) \end{matrix}.
$$

where the relation  $\mathcal{R}_{\text{Eval}}$  is defined as follows:

$$
\mathcal{R}_{\mathsf{Eval}} = \{ ((\mathsf{pp}, C \in \mathbb{G}, \mathbf{x} \in \mathbb{F}^n, v \in \mathbb{F}); (f(X_1, \cdots, X_n), \mathbf{\tilde{c}})) : \\ (\mathsf{Open}(\mathsf{pp}, f, \mathbf{d}, C, \mathbf{\tilde{c}}_0) = 1) \land v = f(\mathbf{x}) \}
$$

We denote by Prove, Verify, the non-interactive prover and verifier algorithms obtained by applying FS to the Eval public-coin interactive protocol, giving a non-interactive PCS scheme  $(pp \leftarrow \mathsf{Setup}(1^{\lambda}, n, d), C \leftarrow \mathsf{Com}(pp, f(\mathbf{X})), (v, \pi) \leftarrow \mathsf{Prove}(pp, f(\mathbf{X}), x), b \leftarrow \mathsf{Verify}(pp, C, v, x, \pi).$ 

<span id="page-12-0"></span>**Definition 2.6** (Knowledge Soundness for Non-Interactive PCS). For any PPT adversary A, there exists a PPT algorithm  $\mathcal E$  such that the following probability is negligible in  $\lambda$ :

$$
\Pr\left(\begin{matrix} \mathbf{p} & \mathbf{p} \leftarrow \mathsf{Setup}(1^\lambda,n,\{D_i\}_{i\in[n]})\\ b=1 \wedge & (C,\mathbf{d},\mathbf{x},v,\pi) \leftarrow \mathcal{A}^{\mathsf{RO}}(\mathbf{p}\mathbf{p})\\ \mathcal{R}_{\mathsf{Eval}}(\mathbf{p}\mathbf{p},C,\mathbf{x},v;\tilde{f},\tilde{\mathbf{c}})=0 & (\tilde{f},\tilde{\mathbf{c}}) \leftarrow \mathcal{E}^{\mathcal{A},\mathsf{RO}}(\mathbf{p}\mathbf{p},C,d)\\ b \leftarrow \mathsf{Verify}(\mathbf{p}\mathbf{p},C,\mathbf{d},\mathbf{x},v,\pi)\end{matrix}\right)\right).
$$

where the relation  $\mathcal{R}_{\text{Eval}}$  is defined as follows:

$$
\mathcal{R}_{\mathsf{Eval}} = \{ ((\mathsf{pp}, C \in \mathbb{G}, \mathbf{x} \in \mathbb{F}^n, v \in \mathbb{F}); (f(X_1, \cdots, X_n), \mathbf{\tilde{c}})) : \\ (\mathsf{Open}(\mathsf{pp}, f, \mathbf{d}, C, \mathbf{\tilde{c}_0}) = 1) \land v = f(\mathbf{x}) \}
$$

**Definition 2.7** (Succinctness). We require the commitments and the evaluation proofs to be of size independent of the degree of the polynomial, that is the scheme is proof succinct if  $|C|$  is  $poly(\lambda)$ ,  $|\pi|$  is poly( $\lambda$ ) where  $\pi$  is the transcript obtained by applying FS to Eval. Additionally, the scheme is verifier succinct if Eval runs in time  $\text{poly}(\lambda) \cdot \log(d)$  for the verifier.

**Fiat-Shamir.** An interactive protocol is *public-coin* if the verifier's messages are uniformly random strings. Public-coin protocols can be transformed into non-interactive arguments in the Random Oracle Model (ROM) by using the Fiat-Shamir (FS) [\[FS87\]](#page-40-11) heuristic to derive the verifier's messages as the output of a Random Oracle. All protocols in this work are public-coin interactive protocols in the structured reference string (SRS) model where both the parties have access to a SRS, that are then compiled into non-interactive arguments using FS. We denote by Prove, Verify, the non-interactive prover and verifier algorithms obtained by applying FS to the Eval publiccoin interactive protocol, giving a non-interactive PCS scheme (pp  $\leftarrow$  Setup $(1^{\lambda}, n, d), (C, \tilde{\mathbf{c}}) \leftarrow$ Com(pp,  $f(\mathbf{X})$ ),  $(v, \pi) \leftarrow$  Prove(pp,  $f(\mathbf{X}), x$ ),  $b \leftarrow$  Verify(pp,  $C, v, x, \pi$ ).

KZG PCS. The KZG univariate PCS was introduced in [\[KZG10\]](#page-42-13). We denote the KZG scheme by the tuple of PPT algorithms (KZG.Setup,KZG.Commit, KZG.Prove, KZG.Verify) as defined below.

**Definition 2.8** (KZG PCS). Let  $(\mathbb{F}, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2, g_t)$  be output of bilinear group generator  $\mathsf{B}\mathsf{G}(1^{\lambda}).$ 

• KZG.Setup on input  $(1^{\lambda}, d)$ , where d is the degree bound, outputs

$$
\mathsf{srs} = (\{ [\tau]_1, \ldots, [\tau^d]_1 \}, \{ [\tau]_2, \ldots, [\tau^d]_2 \}
$$

- KZG.Commit on input (srs,  $p(X)$ ), where  $p(X) \in \mathbb{F}_{\le d}[X]$ , outputs  $C = [p(\tau)]_1$
- KZG.Prove on input (srs,  $p(X), \alpha$ ), where  $p(X) \in \mathbb{F}_{\le d}[X]$  and  $\alpha \in \mathbb{F}$ , outputs  $(v, \pi)$  such that  $v = p(\alpha)$  and  $\pi = [q(\tau)]_1$ , for  $q(X) = \frac{p(X) - p(\alpha)}{X - \alpha}$
- KZG.Verify on input (srs,  $C, v, \alpha, \pi$ ), outputs 1 if the following equation holds, and 0 otherwise:  $e(C - v[1]_1 + \alpha \pi, [1]_2) \stackrel{?}{=} e(\pi, [\tau]_2)$

#### 2.3 Polynomial Protocols

A modular approach for designing efficient succinct arguments consists of two steps: (i) constructing an information theoretic protocol in an idealized model, (ii) compiling the information-theoretic protocol via a cryptographic compiler to obtain an argument system. Informally, the prover and the verifier interact where the prover provides oracle access to a set of polynomials, and the verifier accepts or rejects by checking certain identities over the polynomials output by the prover and possibly public polynomials known to the verifier. Such a polynomial protocol is compiled into a succinct argument of knowledge by realizing the polynomial oracles using a *polynomial commitment* scheme. A polynomial commitment scheme allows a prover to commit to polynomials, and later verifiably open evaluations at chosen points by giving evaluation proofs. This enables the verifier to probabilistically check polynomial identities at random points of F. Many recent constructions of zkSNARKs [\[BFS20,](#page-39-10) [CHM](#page-40-3)+20, [GWC19\]](#page-41-6) follow this approach where the information theoretic object is a polynomial protocol and the cryptographic compiler is a polynomial commitment scheme.

#### 2.4 Models and Assumptions

Algebraic Group Model. We analyze the security of our protocols in the Algebraic Group Model (AGM) introduced in [\[FKL18\]](#page-40-12). An adversary  $A$  is called *algebraic* if every group element output by  $A$  is accompanied by a representation of that group element in terms of all the group elements that  $A$  has seen so far (input and output). In the AGM, an adversary  $A$  is restricted to be algebraic, which in our SRS-based protocol means a PPT algorithm satisfying the following: Given  $\textsf{sts} = (\textsf{sts}_1, \textsf{sts}_2)$ , whenever A outputs an element  $A \in \mathbb{G}_i, i \in 1, 2$ , it is accompanied by its representation, i.e, A also outputs a vector **v** over  $\mathbb{F}$  such that  $A = \langle \mathbf{v}, \mathsf{srs}_i \rangle$ .

Definition 2.9 (q-DLOG Assumption). The q-DLOG assumption with respect to G holds if for all  $\lambda$  and for all PPT  $\mathcal{A}$ , we have:

$$
\Pr\left[\begin{matrix} \tau = \tau'\\ \tau' \leftarrow \mathcal{A}(1^\lambda,\mathsf{pp}) \end{matrix}\right. : \begin{matrix} (\mathbb{F},\mathbb{G}_1,\mathbb{G}_2,\mathbb{G}_T,e,g_1,g_2,g_t) \leftarrow \mathsf{BG}(1^\lambda), \, \tau \leftarrow \mathbb{F} \\ \mathsf{pp} := (g_1^\tau,g_1^{\tau^2},\ldots,g_1^{\tau^q},g_2^\tau,g_2^{\tau^2},\ldots,g_2^{\tau^q}) \end{matrix}\right] \leq \mathsf{negl}(\lambda)
$$

### <span id="page-14-0"></span>3 Generic Aggregation of Polynomial Protocols

In this section, we present our GAPP framework for generic aggregation of polynomial protocols. Let  $G(X_0, \ldots, X_{\ell-1}) \in \mathbb{F}[X_0, \ldots, X_{\ell-1}]$  be an  $\ell$ -variate polynomial. As outlined in the overview, we consider a scenario where a prover wishes to prove a set of  $n$  polynomial identities of the form

$$
G(p_{i,0}(Y),\ldots,p_{i,\ell-1}(Y)) = 0 \text{ mod } Z_{\mathbb{V}}(Y) \quad \forall i \in [n],
$$
\n
$$
(3)
$$

given a set of commitments  $(C_{i,0},\ldots,C_{i,\ell-1})$ , where for each  $(i,j) \in [n] \times [\ell], C_{i,j}$  is a commitment to the polynomial  $p_{i,j}(Y) \in \mathbb{F}[Y]$  under a polynomial commitment scheme PC.

#### 3.1 Aggregation using Bivariate Polynomials

Packed Polynomials. Let bPC be a generic bivariate polynomial commitment scheme. Let  $\mathbb{H} = \{1, \omega, \ldots, \omega^{n-1}\}\$ be the subgroup consisting of the  $n^{\hat{t}h}$  roots of unity in  $\mathbb{F}$ , and let  $\mu_i^{\mathbb{H}}$  $_{i}^{\mathbb{H}}(X)$ be the corresponding Lagrange polynomial for each  $i \in [n]$ . We say that the *packed* polynomial corresponding to the vector of univariate polynomials  $(p_{i,j}(Y))_{i\in[n]}$  is the bivariate polynomial

<span id="page-14-2"></span>
$$
P_j(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) p_{i,j}(Y).
$$
 (4)

It turns out that, for the above packing scheme, the satisfiability of a set of  $n$  univariate polynomial identities over V reduces to a single bivariate polynomial identity over the packed polynomials over the domain  $\mathbb{H} \times \mathbb{V}$ , given by

$$
Q(X,Y) = G(P_0(X,Y), \ldots, P_{\ell-1}(X,Y))
$$
 vanishes over  $\mathbb{H} \times \mathbb{V}$ .

We capture this formally using the following lemma.

<span id="page-14-1"></span>**Lemma 3.1** (Packing Lemma). Let  $m, n, \ell \in \mathbb{N}$  be positive integers. Let  $\mathbb{V} = \langle \nu \rangle$  and  $\mathbb{H} = \langle \omega \rangle$  be the subgroups generated by primitive  $m^{th}$  and  $n^{th}$  roots of unity in  $\mathbb F$  respectively. Let  $G(X_0, \ldots, X_{\ell-1}) \in$  $\mathbb{F}[X_0,\ldots,X_{\ell-1}]$  be an  $\ell$ -variate polynomial, and let  $p_{i,j}(Y)$  be a univariate polynomial in  $\mathbb{F}[Y]$  for each  $(i, j) \in [n] \times [\ell]$ . Let  $P_i(X, Y)$  denote the packed (bivariate) polynomial corresponding to the vector of univariate polynomials  $(p_{0,j}(Y), \ldots, p_{n-1,j}(Y))$  as in Equation [4.](#page-14-2) Then the univariate polynomial  $G(p_{i,0}(Y), \ldots, p_{i,\ell-1}(Y))$  vanishes over V for all  $i \in [n]$  if and only if the bivariate polynomial

$$
Q(X, Y) = G(P_0(X, Y), \dots, P_{\ell-1}(X, Y))
$$

vanishes over the set  $\mathbb{H} \times \mathbb{V}$ .

*Proof.* We first state some identities. Recall that  $\mathbb{H} = \{1, \omega, \dots, \omega^{n-1}\}\$ is the subgroup consisting of the  $n^{th}$  roots of unity in F. It follows from Equation [4](#page-14-2) that for each  $i \in [N]$  and each  $j \in [\ell]$ , we have

$$
P_j(\omega^i, Y) = \sum_{i'=0}^{n-1} \mu_{i'}(\omega^i) p_{i',j}(Y) = p_{i,j}(Y)
$$

which follows from the facts that: (i)  $\mu_i(\omega^i) = 1$  for each  $i \in [N]$ , and (ii)  $\mu_{i'}(\omega^i) = 0$  for each  $i, i' \in [N]$  such that  $i \neq i'$ . Hence, for all  $i \in [n]$ :

<span id="page-14-3"></span>
$$
Q(\omega^{i}, Y) = G(p_{i,0}(Y), \dots, p_{i,\ell-1}(Y))
$$
\n(5)

We now prove the "if" part of the statement of Lemma [3.1.](#page-14-1) Suppose that  $Q(X, Y) = G(P_0(X, Y), \ldots, P_{\ell-1}(X, Y))$ vanishes over  $\mathbb{H} \times \mathbb{V}$ . This implies that the following must be true for each  $i \in [n]: Q(\omega^i, Y) =$ 0 mod  $Z_{\mathbb{V}}(Y)$ . By Equation [5,](#page-14-3) we have that for all  $i \in [n]$ , the following univariate polynomial holds

 $Q(\omega^i, Y) = 0 \text{ mod } Z_{\mathbb{V}}(Y) \implies G(p_{i,0}(Y), \ldots, p_{i,\ell-1}(Y)) = 0 \text{ mod } Z_{\mathbb{V}}(Y)$ 

as desired. We now prove the "only if" part of the statement of Lemma [4.1.](#page-28-1) Suppose that for all  $i \in [n]: G(p_{i,0}(Y), \ldots, p_{i,\ell-1}(Y)) = 0 \text{ mod } Z_{\mathbb{V}}(Y)$ . By equation [5,](#page-14-3) for all  $i \in [n]: Q(\omega^i, Y) =$ 0 mod  $Z_{\mathbb{V}}(Y)$ . But this precisely implies that  $Q(X, Y)$  vanishes over K, as desired. This completes the proof of Lemma [3.1.](#page-14-1)  $\Box$ 

**The GAPP Relation.** In the GAPP relation defined in the overview, we considered  $\ell$  polynomial commitments for an  $\ell$ -variate form G, intuitively, binding each polynomial to a distinct variable of G. Often we need to associate the *same* commitment with more than one variable in  $G$ , specifically when a polynomial appears in the identity with different parameterizations. Looking ahead, in the application of GAPP to aggregate PLONK proofs, the form of polynomial identity is given by Equation [\(19\)](#page-29-0), in which the polynomial z appears with parameterizations as  $z(Y)$  and  $z(\nu Y)$  for  $\nu \in \mathbb{F}$ .

Formally, we consider commitments  $(C_0, \ldots, C_{r-1})$  to  $r \leq \ell$  polynomials  $(P_0, \ldots, P_{r-1})$ , each of which is potentially bound to several variables in G with different parameterizations. In general, we consider s parameterization polynomials  $h_i(Y)$  for  $i \in [s]$ , and the maps  $\kappa : [\ell] \to [r]$  and  $\theta : [\ell] \to [s]$ . For each  $i \in [\ell]$ , we define  $K_i(X, Y) = P_{\kappa(i)}(X, h_{\theta(i)}(Y))$  for  $i \in [\ell]$ , where  $K_i$  specifies the  $i^{\text{th}}$  input to G.

Let  $bPC$  be a bivariate polynomial commitment scheme as before with commitment space  $C$ . Given  $pp \leftarrow bPC.Setup(1^{\lambda}, (d_x, d_y))$ , we define the relation  $\mathcal{R}_{pp,G,n,m}^{GAPP}$  for degree bounds  $(n, m) \le$  $(d_x, d_y)$  as follows:

<span id="page-15-0"></span>**Definition 3.1** (GAPP Relation). Let  $C = (C_0, \ldots, C_{r-1}) \in C^r$  be a vector of commitments,  $\mathbf{w}_0 = (P_0(X, Y), \ldots, P_{r-1}(X, Y)) \in (\mathbb{F}[X, Y])^r$  be a vector of bivariate polynomials, and let  $\mathbf{w}_1 =$  $(\tilde{\mathbf{c}}_0, \ldots, \tilde{\mathbf{c}}_{r-1})$  be a vector of opening hints. Additionally, let  $\mathbf{h} = (h_0(Y), \ldots, h_{s-1}(Y))$  be a vector of parameterization polynomials, maps  $\kappa : [\ell] \to [r]$  and  $\theta : [\ell] \to [s]$  assigning witness polynomials with parameterization for variables in G. We say that  $(\mathbf{x}, \mathbf{w}) \in \mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$  for  $\mathbf{x} = (\kappa, \theta, \mathbf{h}, \mathbf{C})$  and  $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1)$  if:

- 1. For each  $j \in [r]$ , bPC.Open(pp,  $P_i(X, Y)$ ,  $(d_x, d_y)$ ,  $C_i$ ,  $\tilde{\mathbf{c}}_i$ ) = 1.
- 2. For each  $i \in [n], G(K_0(\omega^i, Y), \ldots, K_{\ell-1}(\omega^i, Y)) = 0 \mod Z_{\mathbb{V}}(Y)$ , where  $\omega$  is the canonical primitive n<sup>th</sup> root of unity in  $\mathbb F$  and where for each  $j \in [\ell]$ , we have  $K_j(X,Y) = P_{\kappa(j)}(X, h_{\theta(j)}(Y))$ .

**Remark 3.1.** Typically,  $\mathcal{R}_{pp,G,n,m}^{GAPP}$  will be invoked on  $(C_0,\ldots,C_{r-1})$  where, without loss of generality, for some  $k \in [r]$ ,  $(C_0, \ldots, C_{k-1})$  are honestly generated (i.e., trusted/supplied by the verifier), while the remaining commitments correspond to the prover's witness.

#### <span id="page-15-1"></span>3.2 Argument of Knowledge for the GAPP Relation

In this subsection, we present a argument of knowledge for  $\mathcal{R}_{pp,G,n,m}^{\text{GAPP}}$  given any univariate and bivariate polynomial commitment schemes. This argument of knowledge relies on certain algebraic observations which we state next. Recall that the formal definition of  $\mathcal{R}_{pp,G,n,m}^{\mathsf{GAPP}}$  involves proving that the bivariate polynomial  $Q(X, Y) = G(P_0(X, Y), \ldots, P_{\ell-1}(X, Y))$  vanishes over  $\mathbb{H} \times \mathbb{V}$ . We use the following algebraic criterion to show that a bivariate polynomial Q vanishes over the domain  $\mathbb{H} \times \mathbb{V}$ .

<span id="page-16-0"></span>**Lemma 3.2.** Let  $m, n, \ell \in \mathbb{N}$  be positive integers and let V and  $\mathbb{H}$  as before be the subgroups consisting of the m<sup>th</sup> and n<sup>th</sup> roots of unity in  $\mathbb{F}$ , respectively. A polynomial  $Q \in \mathbb{F}[X, Y]$  vanishes over  $\mathbb{H} \times \mathbb{V}$  if and only there exists polynomial  $H \in \mathbb{F}[X, Y]$  with  $\deg_X(H) < n$  such that  $Q(X, Y)$  –  $Z_{\mathbb{V}}(Y)H(X,Y) = 0 \text{ mod } Z_{\mathbb{H}}(X)$ . Moreover, the polynomial H is explicitly described as:

$$
H(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) \frac{Q(\omega^i, Y)}{Z_{\mathbb{V}}(Y)}
$$
(6)

*Proof.* First, assume that Q vanishes on  $\mathbb{H} \times \mathbb{V}$ . Then for  $i \in [n]$ ,  $Q(\omega^i, Y)$  vanishes on V and hence is divisible by  $Z_V(Y)$ . Thus all the univariate components of H are indeed polynomials. Substituting  $X = \omega^k$  in expressions for polynomials Q and H we see that  $Q(\omega^k, Y) = Z_{\mathbb{V}}(Y)H(\omega^k, Y)$ . Thus, by factor theorem,  $(X - \omega^k)$  divides  $Q(X, Y) - Z_{\mathbb{V}}(Y)H(X, Y)$  for all  $k \in [n]$ . Since these factors are relatively prime we have  $Z_{\mathbb{H}}(X) = \prod_{k=0}^{n-1} (X - \omega^k)$  divides  $Q(X,Y) - Z_{\mathbb{V}}(Y)H(X,Y)$  which proves the claim. The other direction is trivial, as existence of H satisfying  $Q(X, Y) - Z_{\mathbb{V}}(Y)H(X, Y) =$ 0 mod  $Z_{\mathbb{H}}(X)$  implies Q vanishes over  $\mathbb{H} \times \mathbb{V}$ .  $\Box$ 

**Argument of Knowledge for**  $\mathcal{R}_{pp,G,n,m}^{\text{GAPP}}$ . We now describe an argument of knowledge for the relation  $\mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$ . Let uPC and bPC be any univariate and bivariate polynomial commitment schemes as defined in Section [2.2.](#page-11-0) The argument of knowledge presented below is an interactive public-coin protocol (i.e., all of the verifier's messages are uniformly random strings). It can be made non-interactive using the standard FS transform. We describe how the interactive protocol works below. For a succinct description of the protocol, see Figure [2.](#page-17-0)

Setup and Inputs. The setup phase of the protocol generates the following public parameters for uPC and bPC:  $pp_{\text{uPC}} \leftarrow$  uPC.Setup $(1^{\lambda}, d_x)$  and  $pp_{\text{bPC}} \leftarrow$  bPC.Setup $(1^{\lambda}, (d_x, d_y))$ . The public input (common to both the prover  $P$  and the verifier  $V$ ) consists of:

- A vector of bivariate commitments  $\mathbf{C} = (C_0, \dots, C_{r-1}) \in \mathcal{C}^r$ .
- A vector of parameterization polynomials  $\mathbf{h} = (h_0(Y), \ldots, h_{s-1}(Y)).$
- The maps  $\kappa : [\ell] \to [r]$  and  $\theta : [\ell] \to [s]$ .

The (honest) prover  $P$  additionally inputs its witness ( $\mathbf{w}_0$ ,  $\mathbf{w}_1$ ) where

$$
\mathbf{w}_0 = (P_0(X, Y), \dots, P_{r-1}(X, Y)) \in (\mathbb{F}[X, Y])^r, \quad \mathbf{w}_1 = (\mathbf{\tilde{c}}_0, \dots, \mathbf{\tilde{c}}_{r-1})
$$

such that  $(\mathbf{x}, \mathbf{w}) \in \mathcal{R}_{\text{pp}, G, n,m}^{\text{GAPP}}$  for  $\mathbf{x} = (\kappa, \theta, \mathbf{h}, \mathbf{C})$  and  $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1)$ . We define the following auxiliary polynomials:  $K_j(X,Y) = P_{\kappa(j)}(X, h_{\theta(j)}(Y))$  for  $j \in [\ell]$  and  $Q(X,Y) = G(K_0(X,Y), \ldots, K_{\ell-1}(X,Y)).$ 

The Interactive Protocol. Given the public parameters ( $pp_{\text{uPC}}$ ,  $pp_{\text{bPC}}$ ) and the inputs as described above,  $P$  and  $V$  engage in an interactive protocol that proceeds as follows:

**Round-1:** In the first round,  $P$  computes the polynomial  $H(X, Y)$  according to Lemma [3.2,](#page-16-0) and sends a commitment  $C_H$  to the polynomial  $H(X, Y)$  under the bivariate PCS bPC. To check that  $Q(X, Y) = Z_{\mathbb{V}}(Y)H(X, Y) \mod Z_{\mathbb{H}}(X)$ , V sends a random challenge  $y \leftarrow \mathbb{F}$ , and asks P to prove that  $Z_{\mathbb{H}}(X)$  divides the univariate polynomial  $Q(X, y) - Z_{\mathbb{V}}(y)H(X, y)$ .

<span id="page-17-0"></span>• Setup: Setup generates the following public parameters:

$$
\mathsf{pp}_{\mathsf{uPC}} \leftarrow \mathsf{uPC}.\mathsf{Setup}(1^{\lambda},d_x), \quad \mathsf{pp}_{\mathsf{bPC}} \leftarrow \mathsf{bPC}.\mathsf{Setup}(1^{\lambda},(d_x,d_y)).
$$

- Common Input: C =  $(C_0, ..., C_{r-1}) \in \mathcal{C}^r$ , h =  $(h_0(Y), ..., h_{s-1}(Y))$ , maps  $\kappa : [\ell] \to [r]$  and  $\theta : [\ell] \to [s]$
- **Prover's Input:**  $(\mathbf{w}_0, \mathbf{w}_1)$  where  $\mathbf{w}_0 = (P_0(X, Y), \dots, P_{r-1}(X, Y)) \in (\mathbb{F}[X, Y])^r$ , and  $\mathbf{w}_1 = (\mathbf{\tilde{c}}_0, \dots, \mathbf{\tilde{c}}_{r-1}),$ such that  $(\mathbf{x}, \mathbf{w}) \in \mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$  for  $\mathbf{x} = (\kappa, \theta, \mathbf{h}, \mathbf{C})$  and  $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1)$ .
- Additional Notations:  $K_j(X, Y) = P_{\kappa(j)}(X, h_{\theta(j)}(Y))$  for  $j \in [\ell]$  and

 $Q(X, Y) = G(K_0(X, Y), \ldots, K_{\ell-1}(X, Y))$ 

- Round 1:  $P$  commits to the polynomial  $H(X, Y)$  computed as in Lemma [3.2.](#page-16-0)
- 1. P sends commitment  $C_H \leftarrow \text{bPC}.\text{Com}(pp_{\text{bPC}}, H)$ .
- 2. The verifier  $\mathcal V$  sends  $y \leftarrow \mathbb F$ .
- Round 2:  $P$  commits to an auxiliary univariate polynomial  $u(X)$ .
- 1. P computes  $u(X) = (Q(X, y) Z_{\mathbb{V}}(y)H(X, y))/Z_{\mathbb{H}}(X)$ .
- 2. P sends the univariate commitment  $C_u \leftarrow \text{uPC.Com}(pp_{\text{uPC}}, u)$ .
- 3.  $\mathcal V$  sends  $x \leftarrow \mathbb F$ .

• Round 3:  $V$  checks:  $G(K_0(X, y),..., K_{\ell-1}(X, y)) = Z_V(y)h_y(X) + u(X)Z_{\mathbb{H}}(X)$ .

- 1. P computes:  $(\tilde{h}, {\{\tilde{k}_j\}}_{j \in [\ell]}, \tilde{u})$  where  $\tilde{h} = H(x, y), \tilde{u} = u(x),$  and  $\tilde{k}_j = K_j(x, y)$  for each  $j \in [\ell].$
- 2. P computes
	- $\pi_h \leftarrow \mathsf{bPC}.\mathsf{Prove}(pp_{\mathsf{bPC}}, H(X, Y), (x, y)).$
	- $\pi_u \leftarrow$  uPC.Prove(pp<sub>uPC</sub>,  $u(X)$ , x).
	- $-\pi_j \leftarrow \text{bPC.Prove}(pp_{bPC}, P_{\kappa(j)}(X, Y), (x, h_{\theta(j)}(y)))$  for each  $j \in [\ell].$
- 3. P sends  $((\tilde{h}, \pi_h), {\tilde{k}_j}, \pi_j\}_{j\in[\ell],} (\tilde{u}, \pi_u)).$
- 4.  $\nu$  performs the following verification checks:
	- $-b_h = bPC.Verify(pp_{bPC}, C_H, (n, m), (x, y), \tilde{h}, \pi_h).$
	- $b_u = uPC.Verify(pp_{uPC}, C_u, n, x, \tilde{u}, \pi_u).$
	- $b_j \leftarrow \mathsf{bPC}.\mathsf{Verify}(\mathsf{pp}_{\mathsf{bPC}}, C_{\kappa(j)},(n,m),(x,h_{\theta(j)}(y)),\tilde{k}_j,\pi_j)$  for each  $j \in [\ell].$
- 5. V also checks that  $G(\tilde{k}_j, \ldots, \tilde{k}_{\ell-1}) = Z_{\mathbb{V}}(y) \tilde{h} + \tilde{u} Z_{\mathbb{H}}(x)$ .
- 6. If all of the above verification checks pass,  $\mathcal V$  accepts. Otherwise, it rejects.

Figure 2: Argument of Knowledge for the extended relation  $\mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$ .

**Round-2:** P computes  $u(X) = (Q(X, y) - Z_{\mathbb{Y}}(y)H(X, y))/Z_{\mathbb{H}}(X)$ , and sends a commitment  $C_u$  to  $u(X)$  under the univariate PCS uPC. Note that P computes  $Q(X, y)$  as  $G(K_0(X, y), \ldots, K_{\ell-1}(X, y))$ without explicitly computing the bivariate polynomial  $Q$ . At this point,  $\mathcal V$  wishes to check

$$
Q(X, y) = G(K_0(X, y), \dots, K_{\ell-1}(X, y)) = Z_{\mathbb{V}}(y)H(X, y) + u(X)Z_{\mathbb{H}}(X)
$$

To check this,  $\mathcal V$  sends a second random challenge  $x \leftarrow \mathbb F$ , and asks  $\mathcal P$  to send a set of polynomial evaluations  $(\tilde{h}, \tilde{u}, {\{\tilde{k}_j\}}_{j \in [\ell]})$  where  $\tilde{h} = H(x, y), \tilde{u} = u(x)$ , and for each  $j \in [\ell], \tilde{k}_j = K_j(x, y) =$  $P_{\kappa(j)}(x, h_{\theta(j)}(y)).$ 

**Round-3:** P sends the above polynomial evaluations to  $V$ , along with the corresponding evaluation proofs, computed as:

•  $\pi_h \leftarrow \text{bPC.Prove}(pp_{\text{bPC}}, H(X, Y), (x, y))$ 

- $\pi_u \leftarrow$  uPC.Prove(pp<sub>uPC</sub>,  $u(X), x$ ).
- $\pi_j \leftarrow \text{bPC.Prove}(pp_{\text{bPC}}, P_{\kappa(j)}(X, Y), (x, h_{\theta(j)}(y)))$  for each  $j \in [\ell]$ .

**Final Verification Checks:**  $V$  verifies the evaluation proofs sent by  $P$  as:

- $b_h =$  bPC.Verify(pp<sub>bPC</sub>,  $C_H$ ,  $(n, m)$ ,  $(x, y)$ ,  $\tilde{h}$ ,  $\pi_h$ ).
- $b_u =$  uPC.Verify(pp<sub>uPC</sub>,  $C_u$ ,  $n, x, \tilde{u}, \pi_u$ ).
- $b_j = \text{bPC.Verify}(\text{pp}_{\text{bPC}}, C_{\kappa(j)}, (n, m), (x, h_{\theta(j)}(y)), \tilde{k}_j, \pi_j)$  for each  $j \in [\ell]$ .

V also verifies that the following relation holds with respect to the evaluations sent by  $\mathcal{P}$ :  $G(\tilde{k}_0,\ldots,\tilde{k}_{\ell-1}) =$  $Z_{\mathbb{V}}(y)\tilde{h} + \tilde{u}Z_{\mathbb{H}}(x)$ . If all of these verification checks pass, V accepts. Otherwise, it rejects.

<span id="page-18-0"></span>Theorem 3.1. Assuming that uPC and bPC are polynomial commitment schemes as defined in Section [2.2,](#page-11-0) the above protocol is a succinct argument of knowledge for the relation  $\mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$  in Definition [3.1.](#page-15-0)

Proof. We argue both knowledge-soundness and succinctness for the protocol in Figure [2](#page-17-0) below.

Knowledge-Soundness. Consider a PPT cheating prover A that interacts with an honest verifier  $V$  to produce an accepting transcript of the form

$$
(\mathbf{C} = (C_0, \dots, C_{r-1}), (x, y), C_H, C_u, (\tilde{h}, \pi_h), (\tilde{u}, \pi_u), \{(\tilde{p}_j, \pi_j)\}_{j \in [\ell]})
$$

where  $x, y \leftarrow \mathbb{F}$ . Let  $\mathcal{E}_{\mu}$  and  $\mathcal{E}_{\text{bPC}}$  be the PPT extractors for  $\mu$ PC and bPC, respectively, as per Definition [2.6.](#page-12-0) We construct an extractor  $\mathcal{E}_{\mathsf{GAPP}}$  with oracle access to A as follows:

•  $\mathcal{E}_{\text{GAPP}}$  uses its oracle access to  $\mathcal{A}$  to extract the following:

$$
- (P_j(X, Y), \tilde{c}_j) \leftarrow \mathcal{E}_{bPC}^{\mathcal{A}}(C_j, (n, m), (x, y), \tilde{p}_j, \pi_j) \text{ for each } j \in [r].
$$
  
- 
$$
(H(X, Y), \tilde{c}_H) \leftarrow \mathcal{E}_{bPC}^{\mathcal{A}}(C_H, (n, m), (x, y), \tilde{h}, \pi_h).
$$
  
- 
$$
(u(X), \tilde{c}_u) \leftarrow \mathcal{E}_{uPC}^{\mathcal{A}}(C_u, n, x, \tilde{u}, \pi_u).
$$

•  $\mathcal{E}_{\text{GAPP}}$  uses the vector of parameterization polynomials  $\mathbf{h} = (h_0(Y), \ldots, h_{s-1}(Y)),$  and the maps  $\kappa : [\ell] \to [r]$  and  $\theta : [\ell] \to [s]$  to compute for each  $j \in [\ell]$ 

$$
K_j = P_{\kappa(j)}(X, h_{\theta(j)}(Y)).
$$

- $\mathcal{E}_{\text{GAPP}}$  outputs  $\bot$  if any of the following hold:
	- Any of the above extractions fail.
	- $-$  For some  $j \in [\ell], \widetilde{p}_j \neq K_j(x, y) = P_{\kappa(j)}(x, h_{\theta(j)}(y)),$  or  $h \neq H(x, y)$ , or  $\widetilde{u} \neq u(x)$ .
	- For some  $j \in [r]$ , bPC.Open(pp,  $P_j(X, Y)$ ,  $(d_x, d_y)$ ,  $C_j$ ,  $\tilde{\mathbf{c}}_j$ ) = 0.
- Otherwise,  $\mathcal{E}$  outputs  $(\mathbf{w}_0, \mathbf{w}_1)$  where

$$
\mathbf{w}_0 = (P_0(X, Y), \dots, P_{r-1}(X, Y)), \quad \mathbf{w}_1 = (\mathbf{\tilde{c}}_0, \dots, \mathbf{\tilde{c}}_{r-1})
$$

First of all, assuming the knowledge-soundness of uPC and bPC,  $\mathcal{E}_{GAPP}$  outputs  $\perp$  with negligible probability for any PPT adversary  $A$ . Indeed, if this is not the case for some PPT adversary  $A$ , then one can use  $A$  to construct a PPT adversary  $A'$  that breaks knowledge-soundness of either uPC or bPC with non-negligible probability.

Now, assuming that  $\mathcal{E}_{\mathsf{GAPP}}$  does not output  $\bot$ , we argue that we must have  $(\mathbf{C},(\mathbf{w}_0,\mathbf{w}_1)) \in$  $\mathcal{R}_{\textsf{pp},G,n,m}^{\textsf{GAPP}},$  except with negligible probability. To see this, observe the following:

• Since the above transcript passes all verification checks by an honest  $V$ , we must have

$$
G(\tilde{p}_j,\ldots,\tilde{p}_{\ell-1})=Z_{\mathbb{V}}(y)\tilde{h}+\tilde{u}Z_{\mathbb{H}}(x)
$$

This follows immediately from the description of the protocol in Figure [2,](#page-17-0) since the verifier would reject otherwise.

• Further, since  $\mathcal{E}_{\mathsf{GAPP}}$  did not output  $\bot$ , we must have

$$
\widetilde{p}_j = K_j(x, y) = P_{\kappa(j)}(x, h_{\theta(j)}(y)) \quad \forall j \in [\ell], \quad h = H(x, y), \quad \widetilde{u} = u(x).
$$

• Since  $x \leftarrow \mathbb{F}$  and all of the commitments  $(\{C_j\}_{j\in[r]}, C_H, C_u)$  were produced by A before the honest  $V$  sent across the challenge x, by the Schwartz-Zippel lemma, the following must be true (except with probability  $(r+2)/|\mathbb{F}| = \mathsf{negl}(\lambda)$ ):

$$
Q(X, y) = G(p_0(X, y), \dots, p_{\ell-1}(X, y)) = Z_{\mathbb{V}}(y)H(X, y) + u(X)Z_{\mathbb{H}}(X)
$$

which is turn implies

$$
Q(X, y) - Z_{\mathbb{V}}(y)H(X, y) = 0 \text{ mod } Z_{\mathbb{H}}(X)
$$

• Finally, since  $y \leftarrow F$  and the commitments  $(\{C_j\}_{j\in[r]}, C_H)$  were produced by A before the honest  $V$  sent across the challenge y, by the Schwartz-Zippel lemma, the following must again be true (except with probability  $(r+1)/|\mathbb{F}| = \mathsf{negl}(\lambda)$ ):

$$
Q(X,Y) - Z_{\mathbb{V}}(Y)H(X,Y) = 0 \text{ mod } Z_{\mathbb{H}}(X)
$$

This in turn implies that, except with negligible probability, we must have  $(C,(\mathbf{w}_0,\mathbf{w}_1)) \in$  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  by Lemma [3.2.](#page-16-0)

Succinctness. The succinctness of the protocol in Figure [2](#page-17-0) follows immediately from the succinctness of uPC and bPC. Concretely, to argue proof-succinctness of the protocol, it suffices to observe that:

- $\bullet$   $|C_j| = \text{poly}(\lambda)$  and  $|\pi_j| = \text{poly}(\lambda)$  for each  $j \in [r]$ .
- $|C_H|$  = poly( $\lambda$ ) and  $|\pi_h|$  = poly( $\lambda$ ).
- $|C_u|$  = poly( $\lambda$ ) and  $|\pi_u|$  = poly( $\lambda$ ).

The first two requirements are satisfied assuming the proof-succinctness of bPC, while the third requirement is satisfied assuming the proof-succinctness of uPC. Finally, verifier-succinctness of the protocol follows immediately from the verifier-succinctness of uPC and bPC.

This completes the proof of Theorem [3.1.](#page-18-0)

 $\Box$ 

### <span id="page-20-0"></span>4 Efficient Instantiation of GAPP

We now describe an efficient instantiation of the generic GAPP protocol over the bilinear group BG, using our new bivariate polynomial commitment scheme bPCLB. Like the earlier works  $[BMM^+21]$  $[BMM^+21]$ , our construction also relies on techniques from Inner Product Arguments(IPA) [\[BCC](#page-39-9)+16, [BBB](#page-38-3)+18, [BMM](#page-40-7)+21] and the closely related Compressed Sigma Protocols(CSP) [\[AC20\]](#page-38-4). We describe these techniques as they apply to the bilinear group BG, while a more general exposition may be found in Appendix [A.](#page-43-7)

Inner Product Arguments, Compressed Sigma Protocols. Let CM denote the commitment scheme with key space  $\mathbb{G}_2^n$ , message space  $\mathbb{G}_1^n$  and commitment space as  $\mathbb{G}_T$ , where commitment to  $\mathbf{u} \in \mathbb{G}_1^n$  under the key  $\mathbf{v} \in \mathbb{G}_2^n$  is given by  $C_u = \text{CM}(\mathbf{v}, \mathbf{u}) = \sum_{i=0}^{n-1} e(u_i, v_i)$ . For simplicity, we have considered a *non-hiding* commitment here, though the modification to a hiding commitment is straightforward. Similarly, for  $\mathbf{a} \in \mathbb{F}^n$ , we call the map  $L_a: \mathbb{G}_1^n \to \mathbb{G}_1$  defined by  $\mathbf{u} \mapsto \sum_{i=0}^{n-1} a_i u_i$  as the linear form on  $\mathbb{G}_1^n$ , defined by the vector **a**. We also conveniently denote the commitment CM and the linear form as inner products  $\langle v, u \rangle_{\otimes}$  and  $\langle a, u \rangle$  respectively. The CSPs and IPAs provide elegant arguments of knowledge for proving linear forms over committed vectors. In particular, they provide  $O(\log n)$  size argument for the following relation over  $(v, a, C_P, C_p; C)$  given by  $\langle v, C \rangle_{\otimes} =$  $C_P$  and  $\langle \mathbf{a}, \mathbf{C} \rangle = C_p$ .

#### <span id="page-20-1"></span>4.1 Bivariate PCS in Lagrange Basis: bPCLB

Our bivariate polynomial commitment scheme in Lagrange Basis bPCLB is described in Figure [3.](#page-21-0) We also describe the key steps here.

Setup. The setup for bPCLB for degree bound  $(d_x, d_y)$  consists of KZG setup (uPC.pk, uPC.ck) =  $(([\tau^i]_1)_{i=0}^{d_y}, [\tau]_2)$  for  $\tau \leftarrow \mathbb{F}$ , which forms the *inner commitment scheme*. For *outer commitment*, the setup generates  $\mathbf{v} = (\begin{bmatrix} \beta^i \end{bmatrix}_2)_{i=0}^{d_x}$  for  $\beta \leftarrow \mathbb{F}$ . The setup outputs  $pk = (uPC,pk,v)$  and  $ck =$  $(uPC.ck, [\beta]_1).$ 

In [\[BMM](#page-40-7)<sup>+</sup>21, [GMN22\]](#page-41-8), the authors note that the outer commitment  $\mathbf{C} \mapsto \sum_{i=0}^{n} e(C_i, [\beta^i]_2)$ is not binding, as  $[\beta]_1$  in the verification key can be used to construct a collision; in particular the vectors  $([\beta]_1, [0]_1)$  and  $([0]_1, [1]_1)$  yield the same commitment. To ensure binding, in [\[BMM](#page-40-7)+21] encode only even powers of  $\beta$  in the vector **v**, while in [\[GMN22\]](#page-41-8) they also commit the vector using an independently sampled key in  $\mathbb{G}_2^n$ . However, the authors in aPlonk [\[ABST23\]](#page-38-1) observed that these mitigation strategies are not required when using KZG as the inner commitment scheme with an independently sampled setup trapdoor  $\tau$ . They define precise requirements for the inner commitment scheme which ensure the overall commitment is binding as inner product binding and inner product extractable and show that KZG PCS satisfies them. Thus, in this work, we do not constrain the commitment key v in any way.

Commitment. To commit to polynomial  $P \in \mathbb{F}[X, Y]$  with  $deg_X(P) < n$  and  $deg_Y(P) < m$ , the prover first writes P in Lagrange basis as

$$
P(X, Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) p_i(Y)
$$

Then, the prover computes inner commitments  $(C_i, \tilde{c}_i) = uPC$ .Com(uPC.pk,  $m, p_i$ ) for  $i \in [n]$ . Finally, the prover computes the outer commitment  $(C_P, \tilde{C}) = CM(v, C)$  with  $C_P = \sum_{i=0}^{n-1} e(C_i, v_i)$  <span id="page-21-0"></span>Let  $BG = (F, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, g_1, g_2, g_t)$ , be a bilinear group with efficiently computable non-degenerate bilinear pairing  $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ . Let uPC denote the KZG univariate PCS with commitments in  $\mathbb{G}_1$ . bPCLB consists of PPT algorithms (bPCLB.Setup, bPCLB.Com, bPCLB.Eval, bPCLB.Open) defined below.

- bPCLB.Setup( $1^{\lambda}$ ,  $d_x$ ,  $d_y$ ) takes degree bounds  $d_x$  and  $d_y$  in variables X and Y respectively as inputs. It outputs as follows:
	- 1. (uPC.pk, uPC.ck) = KZG.Setup $(1^{\lambda}, d_y)$ .

2. 
$$
\mathbf{v} = (\begin{bmatrix} \beta^i \end{bmatrix}_2)_{i \in [d_x]}
$$
 for  $\beta \leftarrow \mathbb{F}$ .

- 3.  $\mathbb{H} = \{1, \omega, \ldots, \omega^{d_x 1}\}\$ , where  $\omega \in \mathbb{F}$  is primitive  $d_x^{th}$  root of unity.
- 4.  $pk = (uPC.pk, v), ck = (uPC.ck, [\beta]_1).$

5. Output (pk, ck)

- bPCLB.Com(pk,  $F,(n,m)$ ) takes proving key pk and polynomial  $F \in \mathbb{F}[X,Y]$  with  $\deg_X(F) < n$  and  $\deg_Y(F) <$  $\overline{m}$ . The algorithm outputs commitment  $C_F$  as follows:
	- 1. Compute  $(C_i, \tilde{c}_i) \leftarrow$  uPC.Com(uPC.pk,  $F(\omega_n^i, Y), m$ ) for  $i \in [n]$ . Here  $\omega_n \in \mathbb{H}$  is a primitive  $n^{th}$  root of unity. We assume that  $n | d_x$ .
	- 2. Compute  $C_F = \mathsf{CM}_e(\mathbf{v}, \mathbf{C}) = \sum_{i=0}^{n-1} e(C_i, v_i)$ . Here  $\mathsf{CM}_e$  denotes inner product commitment given by bilinear operator  $v_i \otimes C_i = e(C_i, v_i)$ .
	- 3. Output  $(C_F, \widetilde{\mathbf{c}})$  where  $\widetilde{\mathbf{c}} = (\widetilde{c}_0, \ldots, \widetilde{c}_{n-1}).$
- bPCLB.Eval is an interactive protocol between  $\mathcal{P}(\mathsf{pk}, F, \tilde{\mathsf{c}},(n, m), (x, y), v)$  and  $\mathcal{V}(\mathsf{ck}, C_F, (n, m), (x, y), v)$ .
	- Round 1: Prover commits to univariate restriction.
		- 1. P computes  $f(Y) = F(x, Y)$ , commitment  $C_f = \mathsf{uPC}.$ Com( $\mathsf{uPC}.$ pk, f).
		- 2. P computes  $\pi \leftarrow$  uPC.Prove(uPC.pk, f, y).
		- 3. P sends the commitment  $C_f$  and opening proof  $\pi$ .
	- Round 2: Verifier checks consistency of univariate restriction.
		- 1. P and V run a CSP argument of knowledge  $\pi_{\text{csp}}$  to prove knowledge of  $\mathbf{w} = (w_0, \dots, w_{n-1}) \in \mathbb{G}_1^n$  for the relation:

$$
\sum_{i=0}^{n-1} e(w_i, v_i) = C_F \wedge \sum_{i=0}^{n-1} \mu_i(x) \cdot w_i = C_f \tag{7}
$$

2. V outputs 1 if the  $\pi_{\text{csn}}$  verifier accepts and if

uPC.Verify(uPC.ck, 
$$
C_f
$$
,  $\pi$ ,  $y$ ,  $v$ ) = 1



as the commitment to P. Since, we are considering non-hiding outer commitment, we may assume that  $\tilde{\mathbf{C}} = [0]_1$ .

Succinct Evaluation Proof. As in  $[BMM+21]$  $[BMM+21]$ , the evaluation proof of a bivariate polynomial  $P(X, Y)$  at the point  $(x, y)$  proceeds in two steps: In the first step, the prover commits to the univariate polynomial  $p(Y) = P(x, Y)$ . Subsequently, the prover uses the univariate PCS to prove that  $p(y) = v$ , where v is the claimed evaluation. Next, the prover to shows that the commitment  $C_p$  to the polynomial  $p(Y)$  is consistent with the commitment  $C_P$  to  $P(X, Y)$  by proving knowledge of the commitments  $(C_0, \ldots, C_{n-1}) \in \mathbb{G}_1^n$  to the univariate components of P satisfying:

<span id="page-21-1"></span>
$$
\sum_{i=0}^{n-1} e(C_i, v_i) = C_P \quad \wedge \quad \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(x) \cdot C_i = C_p \tag{8}
$$

The above relation can be proved using compressed sigma protocols. We now describe the concrete CSP for Equation [8,](#page-21-1) which we denote by  $\pi_{\text{csp}}$ . In the description below, "boxes" highlight those computations which are delegated by the verifier to obtain succinct verification. Their correctness is succinctly verified in the final round.

Interactive Protocol  $\pi_{\textsf{csp}}$ . Let  $P_0 = C_P$ ,  $v_0 = C_p$ ,  $\mathbf{w}_0 = (C_0, \ldots, C_{n-1})$ ,  $\mathsf{ck}_0 = \mathbf{v} = (\begin{bmatrix} 1 \end{bmatrix}_2, \begin{bmatrix} \beta \end{bmatrix}_2, \ldots, \begin{bmatrix} \beta^{n-1} \end{bmatrix}_2)$ ,  $\mathbf{a}_0 = (\mu_0^{\mathbb{H}}$  $u_0^{\mathbb{H}}(x), \ldots, \mu_{n-1}^{\mathbb{H}}(x)$ . The prover and verifier now interact in  $\ell$  rounds where  $\ell = \log(n)$ . In round  $i \in [\ell]$ , the prover  $(\mathcal{P})$  and verifier  $(\mathcal{V})$  proceed as follows:

- P computes: Prover splits the vectors  $\mathbf{w}_i$ , ck<sub>i</sub> and  $\mathbf{a}_i$  of size  $n/2^i$  into two vectors of size  $n/2^{i+1}$ each by splitting them in the middle.

$$
(\mathbf{w}_i^{(L)}, \mathbf{w}_i^{(R)}) \leftarrow \text{split}(\mathbf{w}_i),
$$

$$
\boxed{(\mathsf{ck}_i^{(L)}, \mathsf{ck}_i^{(R)}) \leftarrow \text{split}(\mathsf{ck}_i), (\mathbf{a}_i^{(L)}, \mathbf{a}_i^{(R)}) \leftarrow \text{split}(\mathbf{a}_i)}
$$

- $P$  computes:  $A_i = \langle c k_i^{(L)}, w_i^{(R)} \rangle$  $\left\langle k^{(R)}\right\rangle_{\otimes},\, A'_i=\langle$ ck $^{(R)}_i\,,\, {\bf w}_i^{(L)}$  $\left\langle L^{(L)}\right\rangle _{\otimes},\, u_{i}=\langle\mathbf{a}_{i}^{(L)}% \boldsymbol{\hat{\beta}}_{i}^{(L)}\rangle_{\otimes}\,,\, u_{i}=\langle\mathbf{a}_{i}^{(L)}% \boldsymbol{\hat{\beta}}_{i}^{(L)}\rangle_{\otimes}\,,\, u_{i}=\langle\mathbf{a}_{i}^{(L)}% \boldsymbol{\hat{\beta}}_{i}^{(L)}\rangle_{\otimes}\,,\, u_{i}=\langle\mathbf{a}_{i}^{(L)}% \boldsymbol{\hat{\beta}}_{i}^{(L)}\rangle_{\otimes}\,,\, u_{i}=\langle\mathbf{a}_{i}^{(L)}% \boldsymbol{\hat$  $\mathbf{w}_i^{(L)}\,,\,\mathbf{w}_i^{(R)}$  $\langle R \rangle$ <sub>i</sub>,  $u'_i = \langle \mathbf{a}_i^{(R)} \rangle$  $\mathbf{w}_i^{(R)}\,,\,\mathbf{w}_i^{(L)}$  $\binom{L}{i}$ . It sends  $A_i, A'_i, u_i$  and  $u'_i$  to  $V$ .
- The verifier sends a challenge  $c_i \leftarrow \mathbb{F}$ .
- The prover computes folded vectors  $w_{i+1}$ ,  $ck_{i+1}$  and  $a_{i+1}$ , as follows:

$$
\mathbf{w}_{i+1} = \mathbf{w}_i^{(L)} + c_i^{-1} \cdot \mathbf{w}_i^{(R)},
$$

$$
c\mathbf{k}_{i+1} = c\mathbf{k}_i^{(L)} + c_i \cdot c\mathbf{k}_i^{(R)}, \mathbf{a}_{i+1} = \mathbf{a}_i^{(L)} + c_i \cdot \mathbf{a}_i^{(R)}
$$

- P and V compute:  $P_{i+1} = c_i \cdot A'_i + P_i + c_i^{-1} \cdot A_i, v_{i+1} = c_i u'_i + v_i + c_i^{-1} u_i.$ 

Final Check. After  $\ell$  iterations as above, the prover sends size 1 vectors  $ck_{\ell}, a_{\ell}$  and  $w_{\ell}$  to the verifier. The verifier checks  $e(\mathbf{w}_{\ell}, \mathbf{c} \mathbf{k}_{\ell}) = P_{\ell}$  and  $\mathbf{a}_{\ell} \cdot \mathbf{w}_{\ell} = v_{\ell}$ . Note that we treat size 1 vectors as scalars here.

Verifier Checks Folding of Inner Product Commitment Key. As noted in prior works [\[BMM](#page-40-7)+21, Lee 21, the final folded commitment key  $\mathsf{ck}_\ell$  is a KZG commitment the polynomial  $g(X) = \prod_{i=0}^{\ell-1} (1+\ell)^{-1}$  $c_iX^{2^{\ell-1-i}}$  in the group  $\mathbb{G}_2$  with v as the commitment key. To check this, the verifier requests the prover for an evaluation proof at a point  $r \leftarrow \mathbb{F}$ . Subsequently, the verifier checks the evaluation proof using  $ck_{\ell}$  as the commitment to g, and  $g(r)$  as the evaluation which it computes itself in  $O(\log n)$  time.

Verifier Checks Folding of Linear Form. The commitment key for the linear form is

$$
\mathbf{a}_0 = (\mu_0^{\mathbb{H}}(x), \dots, \mu_{n-1}^{\mathbb{H}}(x))
$$

where the polynomials  $\{\mu_i^{\mathbb{H}}\}$  ${}_{i}^{\mathbb{H}}\}_{i=0}^{n-1}$  are the Lagrange basis polynomials for the subgroup  $\mathbb{H} = \langle \omega_n \rangle$ generated by primitive  $n^{th}$  root of unity. To verify folding of such structured keys succinctly, we introduce a new technique which we call Lagrangian Folding. Looking ahead, the prover shows that  $p(x) = a_\ell$  for a polynomial  $p \in \mathbb{F}[X]$ , where the verifier "knows" coefficients of p in Lagrange basis.

Applying the Lagrangian folding technique yields an evaluation proof for bPCLB of size  $O(\log n)$ , with verification costing  $O(\log^2 n)$  F-operations and  $O(\log n)$  group operations (see Lemma [4.4\)](#page-27-0).

We now describe the Lagrangian folding technique, where we first consider folding vectors of polynomials.

**Definition 4.1.** Let  $n = 2^{\ell}$  for  $\ell \geq 1$  and let  $\mathbf{A} = (a_0(X), \ldots, a_{n-1}(X))$  be a vector of polynomials in  $\mathbb{F}[X]$ . For a scalar  $c \in \mathbb{F}$ , we define  $\mathsf{Fold}(\mathbf{A}, c)$  as a vector  $\mathbf{A}'$  of polynomials of length  $n/2$ , where  $\mathbf{A}'[i] = \mathbf{A}[i] + c \cdot \mathbf{A}[n/2 + i]$  for  $i \in [n/2]$ . For  $\mathbf{A} = (a(X))$  of size 1, we define  $\mathsf{Fold}(\mathbf{A}) = a(X)$ .

Our next definition captures successive folding of keys to obtain the final commitment key.

**Definition 4.2.** For a vector of polynomials  $A = (a_0(X), \ldots, a_{n-1}(X)) \in \mathbb{F}[X]^n$  with  $n = 2^{\ell}$ ,  $\ell > 1$ , and scalar vector  $\mathbf{c} = (c_0, \ldots, c_{\ell-1}) \in \mathbb{F}^{\ell}$ , we define FullFold $(\mathbf{A}, \mathbf{c})$  to be the polynomial FullFold(Fold( $A, c_0$ ),  $(c_1, \ldots, c_{\ell-1})$ ). For  $\ell = 1$ , we define FullFold( $A, c$ ) = Fold( $A, c_0$ ) where  $c =$  $(c_0)$ .

It is readily observed that if  $(a_0(x), \ldots, a_{n-1}(x))$  is the initial commitment key for some polynomials  $a_0(X), \ldots, a_{n-1}(X)$ , the final commitment key after folding challenges  $c_0, \ldots, c_{\ell-1}$  for  $\ell = \log n$  is given by  $p(x)$ , where  $p(X) = \text{FullFold}(\mathbf{A}, (c_0, \ldots, c_{\ell-1}))$  with  $\mathbf{A} = (a_0(X), \ldots, a_{n-1}(X)).$ The logarithmic verification of prior works is based on the fact that for  $\mathbf{A} = (1, X, \dots, X^{n-1})$ , the polynomial FullFold $(\mathbf{A}, (c_0, \ldots, c_{\ell-1})) = \prod_{i=0}^{\ell-1} (1 + c_i \cdot X^{2^{\ell-1-i}})$  can be evaluated at any point  $x \in \mathbb{F}$ in  $O(\log n)$  time. Our key idea is to fold the vector  $\mathbf{A} = (\mu_0^{\mathbb{H}})$  $\mathbb{H}(X), \ldots, \mu_{n-1}^{\mathbb{H}}(X))$  in Lagrange basis, where the polynomial representations are *n*-dimensional standard unit vectors in  $\mathbb{F}^n$ . The resulting (folded) vector  $\tilde{p}$  will be the coefficients of the folded polynomial p in Lagrange basis. We then show that the vector  $\tilde{p}$  corresponds to evaluations of a multilinear polynomial  $f_c$  over the  $\ell$ -dimensional boolean hypercube  $\mathbb{B}^{\ell}$  for  $\ell = \log n$ . The polynomial  $f_{c}$  is given by Equation [\(9\)](#page-23-0), which can be evaluated at any point on the hypercube in logarithmic cost. Now, the evaluation  $p(x)$  can be obtained as  $\tilde{\mathbf{p}} \cdot \mathbf{W} \cdot (1, x, \dots, x^{n-1})^T$ , where **W** is the inverse-FTT matrix. We cast the last check as an instance of sum-check protocol of [\[LFKN92\]](#page-42-12).

We start with Lemma [4.1](#page-23-1) below:

<span id="page-23-1"></span>**Lemma 4.1** (Lagrangian Folding). Let  $\omega$  be a primitive  $n^{th}$  root of unity in  $\mathbb{F}$  and  $\mathbb{H} = \{1, \omega, \ldots, \omega^{n-1}\}.$ Let  $\mathbf{A} = (\mu_0^{\mathbb{H}})$  $\mathbb{H}^1_0(X), \ldots, \mu_{n-1}^{\mathbb{H}}(X))$  be the vector of Lagrange basis polynomials for the set  $\mathbb{H}$ . Then for  $\ell = \log n$ , FullFold $(A, c_0, \ldots, c_{\ell-1})$  is the unique polynomial  $p(X) \in \mathbb{F}_{\leq n}[X]$  such that  $p(\omega^k) =$  $f_{\mathbf{c}}(k_0,\ldots,k_{\ell-1})$  where  $k_0,\ldots,k_{\ell-1}$  is the binary decomposition of k and  $f_{\mathbf{c}}(X_0,\ldots,X_{\ell-1})$  is the multilinear polynomial given by:

<span id="page-23-0"></span>
$$
f_{\mathbf{c}}(X_0, \dots, X_{\ell-1}) = \prod_{i=0}^{\ell-1} \left( 1 + (c_{\ell-1-i} - 1) \cdot X_i \right) \tag{9}
$$

*Proof.* To a polynomial  $f \in \mathbb{F}_{\leq n}[X]$ , associate an "equivalent" multilinear polynomial  $\tilde{f} \in \mathbb{F}[X_0, \ldots, X_{\ell-1}]$ , satisfying  $f(\omega^i) = \tilde{f}(i)$  for all i; where as before we identify  $i \in [n]$ , with its binary decomposition in  $\mathbb{B}^{\ell}$ . Thus, a Lagrange basis polynomial  $\mu_i^{\mathbb{H}}$  $\mathbb{H}^1(X)$  over  $\mathbb H$  is equivalent to the corresponding Lagrange polynomial  $eq_i^{\ell}(X_0,\ldots,X_{\ell-1})$  over the boolean hypercube  $\mathbb{B}^{\ell}$ , which evaluates to 1 precisely at i viewed as point in  $\mathbb{B}^{\ell}$ . It is easy to see that the folding operations preserve the above equivalence between polynomials in  $\mathbb{F}_{\leq n}[X]$  and multilinear polynomials in  $\mathbb{F}[X_0, \ldots, X_{\ell-1}]$ . Consider the folding of polynomial vector  $\mathbf{A} = (\mathsf{eq}^\ell_0, \dots, \mathsf{eq}^\ell_{n-1})$  using the challenge  $c_0$ . Let  $\mathbf{B} = (b_0, \dots, b_{n/2-1})$  be the resulting vector of polynomials. We see that for all  $i \in [n/2]$ ,

$$
b_i(X_0, \ldots, X_{\ell-1}) = \mathsf{eq}_i^{\ell}(X_0, \ldots, X_{\ell-1}) + c_0 \cdot \mathsf{eq}^{\ell}_{n/2+i}(X_0, \ldots, X_{\ell-1})
$$

Noticing that for all  $i \in [n/2]$ , the most significant bit of i and  $n/2 + i$  are 0 and 1 respectively, we can write the above equation by substituting the known coefficients of  $X_{\ell-1}$  as:

$$
b_i(X_0, \dots, X_{\ell-1})
$$
  
=  $(1 - X_{\ell-1}) \operatorname{eq}_i^{\ell-1}(X_0, \dots, X_{\ell-2}) + c_0 \cdot X_{\ell-1} \operatorname{eq}_i^{\ell-1}(X_0, \dots, X_{\ell-2})$   
=  $(1 + X_{\ell-1}(c_0 - 1)) \operatorname{eq}_i^{\ell-1}(X_0, \dots, X_{\ell-2})$  (10)

Thus, the folded vector **B** is simply the Lagrange polynomials over the boolean hypercube  $\mathbb{B}^{\ell-1}$ , scaled by the polynomial  $(1 + X_{\ell-1}(c_0 - 1))$ . Now, an easy induction shows that the final folded polynomial is the polynomial  $f_c$  as in Equation [9.](#page-23-0) This polynomial is "equivalent" to the univariate polynomial obtained by folding the vector  $(\mu_0^{\mathbb{H}})$  $\mathbb{H}^1_0(X),\ldots,\mu_{n-1}^{\mathbb{H}}(X))$  of Lagrange polynomials over  $\mathbb{H},$ which is what we wanted to prove.  $\Box$ 

#### <span id="page-24-0"></span>4.2 Sumcheck for Lagrangian Folding

Let  $n, \ell$  be as defined previously and  $\phi = \omega^{-1}$ . Here we present a sum-check based argument for checking that given  $z \in \mathbb{F}$ ,  $v \in \mathbb{F}$ ,  $\mathbf{c} = (c_0, \ldots, c_{\ell-1}) \in \mathbb{F}^{\ell}$  and the implied multilinear polynomial  $f_{\mathbf{c}}(X_0, \ldots, X_{\ell-1}) = \prod_{i=0}^{\ell-1} (1 + (c_{\ell-1-i} - 1) \cdot X_i)$ , the following holds:

<span id="page-24-1"></span>
$$
n \cdot v = \left(f_{\mathbf{c}}(0), \dots, f_{\mathbf{c}}(n-1)\right) \cdot \mathbf{W} \cdot \left(1, z, \dots, z^{n-1}\right)^{T}
$$
\n(11)

where **W** is the scaled inverse-FFT matrix defined by  $\mathbf{W}(i, j) = \phi^{ij}$ . We note that to verify the folding of linear form in a CSP, v will be set to  $a_{\ell}$ , the final folded key for linear form. To this end, we can define 2l-variate multilinear extension W of the matrix **W** such that  $W(i, j) = \phi^{ij}$ The integers i and j are interchangeably viewed as points on the hypercube  $\mathbb{B}^{\ell}$  and as elements of [n]. Similarly, let  $f_z$  be a multilinear polynomial in  $\ell$  variables given by  $f_z(\mathbf{X}) = \prod_{k=0}^{\ell-1} (1 +$  $X_k(z^{2^k}-1)$ , which satisfies  $f_z(i) = z^i$  for all  $i \in [n]$ . Then, the identity in [\(11\)](#page-24-1) can be written as  $\sum_{j\in[n]} f_{\mathbf{c}}(j) \sum_{i\in[n]} W(i,j) f_z(i) = n \cdot v.$  Further, defining  $F(\mathbf{X}, \mathbf{Y}) = f_{\mathbf{c}}(\mathbf{Y}) \cdot W(\mathbf{X}, \mathbf{Y}) \cdot f_z(\mathbf{X})$ , where  $\mathbf{X} = (X_0, \ldots, X_{\ell-1})$  and  $\mathbf{Y} = (Y_0, \ldots, Y_{\ell-1}),$  Equation [\(11\)](#page-24-1) is equivalent to  $\sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{B}^{2\ell}} F(\mathbf{x}, \mathbf{y}) =$  $n \cdot v$ . The preceding summation can be checked using the sum-check protocol [\[LFKN92\]](#page-42-12), but it would incur prover complexity of  $\widetilde{O}(n^2)$ , which is prohibitive for large values of n. Therefore, we consider an alternate multivariate extension of the (scaled) inverse-FFT function which incurs a proving cost of only  $O(n \log n (\log \log n)^2)$ . Our eventual function F will no longer be *multilinear*, but multivariate with degree in some variables upto  $\log n$ .

Reducing Sum-check Cost. Instead of the multilinear extension of the inverse-FFT function from  $\mathbb{B}^{2\ell} \to \mathbb{F}$ , we will use the following multivariate polynomial, which agrees with the inverse-FFT function over the hypercube  $\mathbb{B}^{2\ell}$ . Define  $2\ell$ -variate polynomial  $\widetilde{W}$  as:

<span id="page-24-3"></span>
$$
\widetilde{W}(\mathbf{X}, \mathbf{Y}) = \prod_{k=0}^{\ell-1} \left( 1 + X_k \left( \prod_{t=0}^{\ell-1} \left( 1 + Y_t (\phi^{2^k \cdot 2^t} - 1) \right) - 1 \right) \right) \tag{12}
$$

We observe that the above polynomial has degree 1 in variables in  $X$  while it has degree  $\ell$  in variables in **Y**. Using routine calculation (see Lemma [4.2](#page-24-2) below) it can be verified that  $\widetilde{W}(x, y) = \phi^{xy}$  for  $(x, y) \in \mathbb{B}^{2\ell}$ .

<span id="page-24-2"></span>**Lemma 4.2.** For  $(x_0, \ldots, x_{\ell-1}) \in \mathbb{B}^{\ell}$ ,  $(y_0, \ldots, y_{\ell-1}) \in \mathbb{B}^{\ell}$ , we have

$$
\widetilde{W}(x_0, \dots, x_{\ell-1}, y_0, \dots, y_{\ell-1}) = \phi^{xy}
$$
  
where  $x = \sum_{i=0}^{\ell-1} 2^i x_i$  and  $y = \sum_{i=0}^{\ell-1} 2^i y_i$ .

*Proof.* Let  $(x_0, \ldots, x_{\ell-1}), (y_0, \ldots, y_{\ell-1})$  be arbitrary but fixed vectors in  $\mathbb{B}^{\ell}$ . For  $y_t \in \{0, 1\}$ , note that  $1 + y_t(\phi^{2^k \cdot 2^t} - 1) = \phi^{2^k \cdot (y_t 2^t)} = (\phi^{2^t y_t})^{2^k}$ . Thus, we have for all  $k \in [\ell],$ 

$$
\prod_{t=0}^{\ell-1} (1 + y_t(\phi^{2^k \cdot 2^t} - 1)) = \prod_{t=0}^{\ell-1} (\phi^{2^t y_t})^{2^k} = (\phi^{\sum_{t=0}^{\ell-1} 2^t y_t})^{2^k} = \phi^{y \cdot 2^k}
$$

where  $y = \sum_{t=0}^{\ell-1} 2^t y_t$ . Now, we can write  $\widetilde{W}(x, y)$  as:

$$
\widetilde{W}(x,y) = \prod_{k=0}^{\ell-1} \left( 1 + x_k \left( \phi^{y \cdot 2^k} - 1 \right) \right) = \prod_{k=0}^{\ell-1} \left( \phi^y \right)^{x_k 2^k} = (\phi^y)^{\sum_{k=0}^{\ell-1} 2^k x_k} = \phi^{xy}
$$

where  $x = \sum_{k=0}^{\ell-1} 2^k x_k$  as required.

We now use the formulation of inverse-FFT function given by Equation [\(12\)](#page-24-3), for the final sum-check protocol. Thus, we rewrite the sum-check identity (Equation [11\)](#page-24-1) as:

$$
\sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{B}^{2\ell}} \widetilde{F}(\mathbf{x}, \mathbf{y}) = v \tag{13}
$$

where  $\widetilde{F}(\mathbf{X}, \mathbf{Y}) = f_{\mathbf{c}}(\mathbf{Y}) \cdot \widetilde{W}(\mathbf{X}, \mathbf{Y}) \cdot f_z(\mathbf{X})$ . We recall the outline of sum-check protocol of [\[LFKN92\]](#page-42-12) below for the polynomial  $\widetilde{F}$  where we assume that the order in which the variables are bound to random challenges is  $X_0, \ldots, X_{\ell-1}$  followed by  $Y_0, \ldots, Y_{\ell-1}$ .

- For  $i \in [\ell]$ , the  $(i + 1)^{th}$  message from the prover is the polynomial  $h_i(X_i)$  computed as:

$$
h_i(X_i) = \sum \widetilde{F}(r_0, \dots, r_{i-1}, X_i, x_{i+1}, \dots, x_{\ell-1}, y_0, \dots, y_{\ell-1})
$$
\n(14)

where the summation runs over  $(x_{i+1},...,x_{\ell-1},y_0,...,y_{\ell-1}) \in \mathbb{B}^{2\ell-i-1}$  and  $r_0,...,r_{i-1}$  are the challenges sent by the verifier in the previous rounds.

- After all the variables in **X** are bound to challenges  $\mathbf{r} = (r_0, \ldots, r_{\ell-1})$ , as  $(\ell+1+i)^{th}$  message for  $i \in [\ell]$ , the prover sends the polynomial  $h'_i(Y_i)$  computed as:

$$
h'_{i}(Y_{i}) = \sum \widetilde{F}(r_{0}, \ldots, r_{\ell-1}, r'_{0}, \ldots, r'_{i-1}, Y_{i}, y_{i+1}, \ldots, y_{\ell-1})
$$
\n(15)

where summation is over  $(y_{i+1},...,y_{\ell-1}) \in \mathbb{B}^{\ell-i-1}$  and  $r_0,...,r_{\ell-1}, r'_0,...,r'_{i-1}$  are the challenges sent by the verifier in previous rounds.

For notational convenience, we define  $g_k(\mathbf{Y}) = \prod_{t=0}^{\ell-1} (1 + Y_t(\phi^{2^k \cdot 2^t} - 1)) - 1$  for all  $k \in [\ell]$ . Thus,  $\widetilde{\mathbf{W}}(\mathbf{X}, \mathbf{Y}) = \prod_{k=0}^{\ell-1} (1 + X_k g_k(\mathbf{Y}))$ . We employ techniques from [\[XZZ](#page-43-8)+19] and [\[CBBZ23\]](#page-40-9) to optimize the sum-check through effective pre-computation and leveraging multiplicative structures in  $\overline{F}$ .

**Lemma 4.3.** The polynomials  $h_i$ ,  $i \in [\ell]$  can be computed by the prover in  $O(n \log n)$  F-operations, while the polynomials  $h'_i$ ,  $i \in [\ell]$  incur a total cost of  $O(n \log n (\log \log n)^2)$  F-operations.

*Proof.* Consider the computation of  $h_i(X_i)$  for  $i \in [\ell]$ . We write  $\widetilde{W}$  as:

$$
\widetilde{W}(\mathbf{X}, \mathbf{Y}) = \prod_{k=0}^{\ell-1} (1 + X_k g_k(\mathbf{Y})) \text{ for } g_k(\mathbf{Y}) = \left( \prod_{t=0}^{\ell-1} (1 + Y_t (\phi^{2^k \cdot 2^t} - 1)) - 1 \right)
$$

 $\Box$ 

Then, we have  $\sum_{(\mathbf{x}, \mathbf{y}) \in \mathbb{B}^{2\ell}} F(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{y}} f_{\mathbf{c}}(\mathbf{y}) \sum_{\mathbf{x}} W(\mathbf{x}, \mathbf{y}) f_{z}(\mathbf{x})$ . Now leveraging the multiplicative structure of W and  $f_z$  we can re-organize the summation to write  $h_i(X_i) = p_0(\mathbf{r}_i) \sum_{\mathbf{y}} f_{\mathbf{c}}(\mathbf{y})$ .  $p_1(\mathbf{r}_i, \mathbf{y}) \cdot p_2(i, \mathbf{y}) \cdot q(X_i, \mathbf{y})$ , where  $p_0(\mathbf{r}_i) = \prod_{k=0}^{i-1} (1 + r_k(z^{2^k} - 1))$  and

$$
p_1(\mathbf{r}_i, \mathbf{y}) = \prod_{k=0}^{i-1} (1 + r_k g_k(\mathbf{y})), \, q(X_i, \mathbf{y}) = (1 + X_i g_i(\mathbf{y}))(1 + X_i(z^{2^i} - 1)),
$$
\n
$$
p_2(i, \mathbf{y}) = \sum_{\mathbf{x}_i \in \mathbb{B}^{\ell-i-1}} \prod_{k=i+1}^{\ell-1} \left( (1 + x_k g_k(\mathbf{y}))(1 + x_k(z^{2^k} - 1)) \right)
$$

In the preceding expressions we use  $\mathbf{r}_i$  and  $\mathbf{x}_i$  to denote tuples  $(r_0, \ldots, r_{i-1})$  and  $(x_{i+1}, \ldots, x_{\ell-1})$ respectively. Next, in the expression for  $p_2(\mathbf{r}_i, \mathbf{y})$  we replace each quadratic factor of the form  $(1+x_k g_k(y))(1+x_k(z^{2^k}-1))$  with the linear term  $1+x_k(z^{2^k}+z^{2^k} g_k(y)-1)$  using the fact that  $x_k^2 = x_k$  for all k. Now, exchanging the sum and the product we get:

<span id="page-26-0"></span>
$$
p_2(i, \mathbf{y}) = \prod_{k=i+1}^{\ell-1} \left( \sum_{x_k \in \{0,1\}} \left( 1 + x_k (z^{2^k} + z^{2^k} g_k(\mathbf{y}) - 1) \right) \right)
$$
  
= 
$$
\prod_{k=i+1}^{\ell-1} (1 + z^{2^k} + z^{2^k} g_k(\mathbf{y}))
$$
(16)

It is clear that  $h_i(\alpha)$  can be computed for any  $\alpha \in \mathbb{F}$  in  $O(n)$  field operations, given pre-computed tables  $T(\mathbf{y}) = f_c(\mathbf{y}), A_i(\mathbf{y}) = p_1(\mathbf{r}_i, \mathbf{y})$  and  $B_i(\mathbf{y}) = p_2(i, \mathbf{y})$  for  $\mathbf{y} \in \mathbb{B}^{\ell}$ . Using expressions for  $p_1(\mathbf{r}_i, \mathbf{y})$  and  $p_2(i, \mathbf{y})$  (Equation [16\)](#page-26-0), we also see that tables  $A_{i+1}(\cdot)$  and  $B_{i+1}(\cdot)$  can be computed from tables  $A_i$  and  $B_i$  respectively in  $O(n)$  F-operations, given access to evaluations  $g_i(\mathbf{y})$  for  $i \in [\ell], \mathbf{y} \in \mathbb{B}^{\ell}$ . To interpolate  $h_i(X_i)$ , one computes  $h_i(\alpha)$  for  $\alpha \in \{0,1,2\}$ . The computation of  $h_i(X_i)$  for all  $i \in [\ell]$  takes  $O(n \log n)$  F-operations by initially computing  $T(\cdot, \cdot), g_i(\cdot), i \in [\ell]$  over the boolean hypercube  $\mathbb{B}^{\ell}$  in  $O(n \log n)$  time  $(O(n)$  for each of the log n tables) and then updating in  $O(n)$  cost for each of the log n rounds.

Now, consider computing the polynomial  $h'_i(Y_i)$ , where the variables in **X** have been bound to random values  $r_0, \ldots, r_{\ell-1}$  and variables  $Y_0, \ldots, Y_{i-1}$  are bound to  $\mathbf{r}' = (r'_0, \ldots, r'_{i-1})$ . For brevity, let **r**,  $\mathbf{r}'_i$  and  $\mathbf{y}_i$  denote tuples  $(r_0, \ldots, r_{\ell-1})$ ,  $(r'_0, \ldots, r'_{i-1})$  and  $(y_{i+1}, \ldots, y_{\ell-1})$  respectively. In this case, we write

$$
h_i'(Y_i) = f_z(\mathbf{r}) \sum_{\mathbf{y}_i \in \mathbb{B}^{\ell-i-1}} f_c(\mathbf{r}_i', Y_i, \mathbf{y}_i) \cdot \prod_{k=0}^{\ell-1} \left(1 + r_k g_k(\mathbf{r}_i', Y_i, \mathbf{y}_i)\right)
$$
(17)

To compute the polynomial  $h'_i(Y_i)$ , the prover maintains pre-computed tables  $T_i(\mathbf{y}) = f_c(\mathbf{r}'_i, \mathbf{y})$ ,  $A_{ik}(\mathbf{y}) = g_k(\mathbf{r}'_i, \mathbf{y}), k \in [\ell]$  where  $\mathbf{y} \in \mathbb{B}^{\ell-i}$ . Initializing the tables for  $i = 0$  costs  $O(n \log n)$ operations, while updates from iteration i to iteration  $(i+1)$  cost  $O(n \log n/2^i)$  operations using the update rule below (illustrated for the table  $T_i$ , with other multi-linear functions updated similarly).

$$
T_{i+1}(\mathbf{y}) = f_c(\mathbf{r}'_{i+1}, \mathbf{y}) = f_c(\mathbf{r}'_i, r'_i, \mathbf{y})
$$
  
=  $(1 - r'_i) f_c(\mathbf{r}'_i, 0, \mathbf{y}) + r'_i f_c(\mathbf{r}'_i, 1, \mathbf{y})$   
=  $(1 - r'_i) T_i(0, \mathbf{y}) + r'_i T_i(1, \mathbf{y})$  (18)

Thus the total cost of initializing and updating the tables costs  $O(n \log n)$  across  $\log n$  iterations. It remains to determine the cost of computing  $h_i'(Y_i)$  given access to the pre-computed tables. We

follow the approach in [\[CBBZ23\]](#page-40-9). For a fixed  $y_i \in \mathbb{B}^{\ell-i}$ , the linear polynomial  $f_c(\mathbf{r}'_i, Y_i, \mathbf{y}_i)$  can be inferred from its evaluations at 0 and 1, which are  $T_i(0, y_i)$  and  $T_i(1, y_i)$  respectively. Similarly, the linear polynomials  $1 + r_k g_k(\mathbf{r}'_i, Y_i, \mathbf{y}_i)$  for  $k \in [\ell]$  can be computed by querying the tables  $A_{ik}, k \in [\ell]$ . Finally, the  $\ell + 1$  linear polynomials can be multiplied recursively using FFT in  $O(\ell \log^2 \ell)$  $O(\ell \log^2 \ell)$  $O(\ell \log^2 \ell)$  F-operations<sup>2</sup>. The polynomial  $h'_i(Y_i)$  is obtained by adding such polynomials for all  $y_i \in \mathbb{B}^{\ell-i}$  which costs  $O(n\ell \log^2 \ell/2^i)$  for iteration i. Summing over all iterations, and substituting  $\ell = \log n$ , the prover incurs  $O(n \log n (\log \log n)^2)$  F-operations.  $\Box$ 

**Reducing Communication in Sum-Check to**  $O(\log n)$ . The preceding sum-check leads to argument size of  $O(\log^2 n)$ , as the latter  $\log n$  rounds involve message polynomials of degree  $\log n$  each. However, we can reduce argument size to  $O(\log n)$  by having the prover send a KZG commitment to the log n degree polynomial. In the final round, the verifier can request for evaluations of the committed polynomials at the verification points. This modification results in argument size of  $O(\log n)$ and verification complexity of  $O(\log^2 n)$  F-operations (to evaluate  $\widetilde{F}$  at a  $(r_0, \ldots, r_{\ell-1}, r'_0, \ldots, r'_{\ell-1})$ ) and  $O(\log n)$  group operations. One could use data parallel SNARK such as Virgo [\[ZXZS20\]](#page-43-9) to model inverse-FFT as a GKR-friendly circuit with depth  $O(\log n)$ . However, it leads to argument size of  $O(\log^2 n)$ .

Parameters for bPCLB. We now state the parameters attained by the bivariate polynomial commitment scheme bPCLB described in Definition [3,](#page-21-0) and those by the argument for relation  $\mathcal{R}_{\textsf{pp},G,n,m}^{\textsf{GAPP}}$  in Figure [2](#page-17-0) by using bPCLB as the bivariate commitment scheme in conjunction with the KZG univariate commitment scheme.

<span id="page-27-0"></span>Lemma 4.4. Assuming that the q-DLOG assumption holds for the bilinear group generator BG, the scheme bPCLB  $=$  (Setup, Com, Open, Prove, Verify) *obtained by applying Fiat-Shamir heuristic to the* interactive procedure bPCLB. Eval in Figure [3](#page-21-0) is a polynomial commitment scheme for polynomials in  $\mathbb{F}[X, Y]$  in the algebraic group model (AGM) and achieves following efficiency parameters:

$$
\begin{array}{ll} |\pi^{\mathsf{bPCLB}}| &= 2 \log n \mathbb{G}_T + 4 \log n \mathbb{G}_1 + 6 \log n \mathbb{F} \\ \mathsf{t}_\mathsf{C}^\mathsf{bPCLB} &= mn \, \mathbb{M} + n \, \mathbb{P} + mn \, \mathbb{F} \\ \mathsf{t}_\mathsf{P}^\mathsf{bPCLB} &= O(m+n) \, \mathbb{M} + O(n) \, \mathbb{P} + O(n \log n (\log \log n)^2) \, \mathbb{F} \\ \mathsf{t}_\mathsf{V}^\mathsf{bPCLB} &= 2 \log n \, \mathbb{G}_T + 4 \log n \, \mathbb{G}_1 + O(1) \, \mathbb{P} + O(\log^2 n) \, \mathbb{F} \end{array}
$$

In the above, n and m denote the degree bounds on variables X and Y respectively, while  $|\pi|$ , t<sub>C</sub>,  $t_P$ ,  $t_V$  denote the proof size, commit time, prover time and verifier time respectively. We exclude the time to evaluate F at the evaluation point in the prover time  $t_P$ .

*Proof.* Assume that for all nodes p at height d labelled with the statement  $C_p$ , the extractor outputs vector  $(w_0^p)$  $_{0}^{p}(Y), \ldots, w_{h}^{p}$  $_{h-1}^{p}(Y)$ ) such that  $\sum_{i=0}^{h-1} e([w_i^p])$  $\binom{p}{i}$  $\binom{p}{i}$ , ck $\binom{p}{i}$  =  $C_p$ , where we have  $h = 2^d$  and  $ck_p$  is the folded key ck corresponding to challenges determined by node p. It can be seen that for a node q at height  $d+1$ , the vector  $(w_0^q)$  $_{0}^{q}(Y),\ldots,w_{h}^{q}$  $_{h'-1}^{q}(Y)$ ), with  $h' = 2^{d+1}$ such that  $\sum_{i=0}^{h'-1} e([w_i^q$  $\binom{q}{i}$ ,  $\mathsf{ck}^q[i]) = C_q$  can be obtained by considering child nodes of  $q$  for three distinct challenges  $c_{\ell-1-d}$ . This follows from the usual extraction in CSPs, noticing that underlying representation can also be linearly combined. All that we need to do is to argue the case  $d = 0$ , for which we invoke the AGM assumption as follows. Consider a node in the tree at depth  $\ell$ (and thus height 0), after all CSP challenges have been specified. Let the node correspond to the statement  $(C_\ell, w_\ell, ck_\ell)$  where  $C_\ell$  is the folded commitment computed by the verifier,  $w_\ell$  and  $ck_\ell$  are

<span id="page-27-1"></span><sup>&</sup>lt;sup>2</sup>In practice, for small values of  $\ell$ , Karatsuba's multiplication gives better concrete efficiency

the folded witness and commitment key output by  $A$ . Since  $A$  is algebraic, it outputs polynomials  $w(Y)$ ,  $\bar{g}(X)$  of degree at most  $d_y$  and  $d_x$  respectively such that  $\mathbf{w}_{\ell} = [w(\tau)]_1$ ,  $\mathsf{ck}_{\ell} = [\bar{g}(\beta)]_2$ . Let z be the subsequent evaluation challenge at depth  $\ell+1$ , with  $\pi$  as the KZG evaluation proof output by A. Again, A outputs  $q(X)$  such that  $\pi = [q(\beta)]_2$ . Let  $g(X)$  be the polynomial  $\prod_{i=0}^{\ell-1} (1+c_i X^{2^{\ell-1-i}})$ . For an accepting transcript, we must have:  $e(\mathbf{w}_{\ell}, \mathbf{c}\mathbf{k}_{\ell}) = C_{\ell}, e([\beta - z]_1, \pi) = e([1]_1, \mathbf{c}\mathbf{k}_{\ell} - [g(z)]_2)$ . By q-DLOG assumption, the second pairing equality implies  $q(X)(X-z) = \bar{g}(X) - g(z)$ , otherwise  $\beta$  can be obtained by factoring the non-zero difference polynomial. Then, with overwhelming probability,  $\overline{g}(X) = g(X)$ , as z was sampled independent of  $\overline{g}$  and g. Now, from first pairing equality we have  $e([w(\tau)]_1, [g(\beta)]_2) = C_{\ell}$ . The extractor outputs  $w(Y)$ , thus proving the base case of induction.

We state the following theorem for the protocol in Figure [2,](#page-17-0) instantiated with the KZG as the univariate and bPCLBas the bivariate polynomial commitment schemes respectively in the bilinear group.

<span id="page-28-1"></span>Theorem 4.1. Assuming q-DLOG assumption holds for the bilinear group generator BG, there exists an argument of knowledge for the relation  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  in the algebraic group model. Moreover, the setup  $pp \leftarrow$  Setup $(\lambda, d_x, d_y)$  is universal for all relations  $\mathcal{R}_{pp,G,n,m}^{GAPP}$  satisfying  $n \leq d_x$ ,  $m \cdot$  $deg(G) \leq d_v$ . The protocol satisfies the following efficiency parameters:

$$
\begin{array}{ll} |\pi^{\mathsf{GAPP}}|&=(\ell+2)\,\mathbb{F}+|\pi^{\mathsf{bPCLB}}(n,m)|\\ \mathsf{t}^{\mathsf{GAPP}}_{\mathsf{V}}&=n\log(\deg(G))\cdot\mathsf{FFT}(m\deg(G))\,\mathbb{F}+nm\deg(G)\,\mathbb{M}+\mathsf{t}^{\mathsf{bPCLB}}_{\mathsf{P}}(n,m)\\ \mathsf{t}^{\mathsf{GAPP}}_{\mathsf{V}}&=\ell\,\mathbb{G}_T+||G||\,\mathbb{F}+\mathsf{t}^{\mathsf{bPCLB}}_{\mathsf{V}}(n,m)\end{array}
$$

In the above  $||G||$  denotes the size of arithmetic circuit computing  $G, \ell$  denotes number of variables in G, while  $\pi$ , tp and t<sub>V</sub> denote proof size, prover complexity and verifier complexity respectively.

Proof. The proof follows from the proof of Theorem [3.1,](#page-18-0) the security of our **bPCLB** commitment scheme proved in Lemma [4.4,](#page-27-0) and the security of the KZG commitment scheme.  $\Box$ 

# <span id="page-28-0"></span>5 Applications

In this section, we present certain illustrative applications of the argument for the relation  $\mathcal{R}_{pp,G,n,m}^{GAPP}$ , and the bivariate polynomial commitment scheme bPCLB.

#### <span id="page-28-2"></span>5.1 Proof Aggregation

Proof aggregation is a natural application of GAPP framework introduced in this paper. For simplicity, we describe proof aggregation for  $n$  identical PLONK circuits. This is a common setting in rollup applications, with each circuit verifying a batch of transactions. We also benchmark our scheme against the naive approach of generating  $n$  separate proofs. The circuit for a relation in PLONK PIOP is described by the vector

$$
((q_M(Y), q_L(Y), q_R(Y), q_O(Y), q_C(Y), S_a(Y), S_b(Y), S_c(Y))) \in (\mathbb{F}[Y])^8
$$

where the first five polynomials represent the gate constraints, while the latter three represent the "wiring" constraints. The commitments to above polynomials are known to the verifier. As part of the proof, the prover commits to "witness" polynomials  $a(Y)$ ,  $b(Y)$  and  $c(Y)$  (we ignore public inputs for simplicity, the parts of witness may be opened orthogonal to the aggregation). Following the verifier's challenges after prover commits to the witness, a PLONK proof essentially requires prover to show that the polynomial  $q(Y)$  as defined by

<span id="page-29-0"></span>
$$
q(Y) \equiv (q_M(Y)a(Y)b(Y) + q_L(Y)a(Y) + q_R(Y)b(Y) + q_O(Y)c(Y) + q_C(Y)) + \alpha((a(Y) + \beta Y + \gamma)(b(Y) + \beta k_1 Y + \gamma)(c(Y) + \beta k_2 Y + \gamma)z(Y)) - \alpha((a(Y) + \beta S_a(Y) + \gamma)(b(Y) + \beta S_b(Y) + \gamma)(c(Y) + \beta S_c(Y) + \gamma)z(\nu Y)) + \alpha^2 \mu_0^{\mathbb{H}}(Y)(z(Y) - 1)
$$
(19)

vanishes over the domain V consisting of m roots of unity  $\{1, \nu, \ldots, \nu^{m-1}\}\$ . Here m denotes the number of gates in the circuit, while  $\alpha$ ,  $\beta$  and  $\gamma$  denote uniform challenges from the verifier. The polynomials  $a(Y)$ ,  $b(Y)$ ,  $c(Y)$  and  $z(Y)$  are supplied by the prover. Let  $(a_i(Y), b_i(Y), c_i(Y), z_i(Y))_{i \in [n]}$ be the prover's polynomials corresponding to each of the  $n$  proofs. Define multivariate polynomial G as below:

$$
G(X_0, X_1, \ldots, X_{12}, X_{13}) :=
$$
  
\n
$$
X_0 X_8 X_9 + X_1 X_8 + X_2 X_9 + X_3 X_{10} + X_4
$$
  
\n
$$
+ \alpha ((X_8 + \beta X_{13} + \gamma) \cdot (X_9 + \beta k_1 X_{13} + \gamma) \cdot (X_{10} + \beta k_2 X_{13} + \gamma) \cdot X_{11})
$$
  
\n
$$
- \alpha ((X_8 + \beta X_5 + \gamma) \cdot (X_9 + \beta X_6 + \gamma) \cdot (X_{10} + \beta X_7 + \gamma) \cdot X_{12})
$$
  
\n
$$
+ \alpha^2 \mu_0^{\Psi} \cdot (X_{12} - 1)
$$
\n(20)

The aggregate proof requires the prover to show that for all  $i \in [n]$ , the polynomial

<span id="page-29-1"></span>
$$
G(q_M, q_L, q_R, q_O, q_C, S_a, S_b, S_c, a_i, b_i, c_i, z_i, \overline{z}_i, Y)
$$

vanishes over V for  $\overline{z_i}(Y) = z_i(\nu Y)$ . Now, by Lemma [3.1,](#page-14-1) above is equivalent to proving that the polynomial

$$
Q(X,Y) = G(q_M, q_L, q_R, q_O, q_C, S_a, S_b, S_c, A, B, C, Z, \overline{Z}, Y)
$$
\n(21)

where  $\overline{Z}(X, Y) = Z(X, \nu Y)$  vanishes over the domain  $\mathbb{H} \times \mathbb{V}$ . The aggregate proof using GAPP framework is described in Figure [4,](#page-30-0) where we omit details of PLONK argument which can be found in [\[GWC19\]](#page-41-6). Broadly, in the first two messages, the prover commits to packed polynomials  $A(X, Y), B(X, Y), C(X, Y), Z(X, Y),$  corresponding to univariate polynomials  $a_i(Y), b_i(Y), c_i(Y)$ ,  $z_i(Y)$  for  $i \in [n]$ . We note that for polynomials which are constant across n instances (circuit polynomials, Y , etc), their packed polynomial is identical to the univariate polynomial. The prover and verifier now invoke the protocol in Figure [2](#page-17-0) to prove that  $(\kappa, \theta, \mathbf{h}, \chi)$  is a valid statement in  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  where

$$
\chi = (\chi_M, \chi_L, \chi_R, \chi_O, \chi_C, \chi_a, \chi_b, \chi_c, \chi_A, \chi_B, \chi_C, \chi_Z, \bot)
$$

is a vector of commitments to polynomials, with the first eight corresponding to circuit polynomials, and last four to the prover polynomials. We set the final commitment to polynomial Y as  $\perp$ , as it can be succinctly evaluated. Next, we set  $\mathbf{h} = (Y, \nu Y)$  and set maps  $\kappa$  and  $\theta$  to ensure that variable  $X_{11}$  is bound to  $Z(X, Y)$  while variable  $X_{12}$  is bound to  $Z(X, \nu Y)$ . Above protocol is trivially extended to the case, when different circuit polynomials are used in different instances. Let  $\mathcal{C} = (\mathcal{C}_i)_{i=0}^{n-1}$  as a family of circuits, each of which is represented using at most m PLONK constraints. We define the langugage  $\mathcal{R}_{\text{nn}}^{\text{agg}}$  $\Pr_{\mathsf{pp},\mathcal{C}}^{\mathsf{agg}}$  and say that  $\mathbf{w} = (\mathbf{w}_0,\ldots,\mathbf{w}_{n-1}) \in \mathcal{R}_{\mathsf{pp},\mathcal{C}}^{\mathsf{agg}}$  if  $\mathcal{C}_i(\mathbf{w}_i) = 1$ for all  $i \in [n]$ .

Lemma 5.1. Assuming that q-DLOG holds for bilinear group generator BG there exists an argument of knowledge for the langugage  $\mathcal{R}_{\text{nn}}^{\text{agg}}$  $_{\text{pp,C}}^{\text{agg}}$  in the algerbraic group model. For the case of identical

<span id="page-30-0"></span>• Setup: Setup generates the following public parameters:

 $\mathsf{pp}_{\mathsf{uPC}} \leftarrow \mathsf{uPC}.\mathsf{Setup}(1^\lambda,d_x), \quad \mathsf{pp}_{\mathsf{bPC}} \leftarrow \mathsf{bPC}.\mathsf{Setup}(1^\lambda,(d_x,d_y)).$ 

- Common Input: Commitments  $[q_M(Y)]_1$ ,  $[q_L(Y)]_1$ ,  $[q_R(Y)]_1$ ,  $[q_O(Y)]_1$ ,  $[q_C(Y)]_1$ ,  $[S_a(Y)]_1$ ,  $[S_b(Y)]_1$ ,  $[S_c(Y)]_1$  to the circuit polynomials.
- **Prover's Input:** Witness polynomials  $(a_i(Y), b_i(Y), c_i(Y))$  for  $i \in [n]$ .
- Round 1: The prover  $P$  commits to aggregated witness.
	- 1. P computes packed polynomials  $A(X, Y)$ ,  $B(X, Y)$  and  $C(X, Y)$  as:

$$
A = \sum_{i \in [n]} \mu_i(X)a_i(Y), B = \sum_{i \in [n]} \mu_i(X)b_i(Y), C = \sum_{i \in [n]} \mu_i(X)c_i(Y)
$$

- 2. P computes commitment  $[A]_{\text{bv}} = \text{bPCLB}.\text{Com}(p_{\text{bpc}}, A)$  and similarly commitments  $[B]_{\text{bv}}$   $[C]_{\text{bv}}$  for polynomials B, C respectively.
- 3. P sends  $[A]_{\text{bv}}, [B]_{\text{bv}}$  and  $[C]_{\text{bv}}$ .
- 4. V sends  $\beta, \gamma \leftarrow \mathbb{F}$ .
- Round 2: Prover commits to auxiliary aggregated witness.
	- 1. P computes the polynomials  $z_i(Y)$ ,  $i \in [n]$  as in [\[GWC19\]](#page-41-6).
	- 2. P computes packed polynomial  $Z(X,Y) = \sum_{i \in [n]} \mu_i(X) z_i(Y)$ .
	- 3. P computes commitment  $[Z]_{\text{bv}} = \text{bPCLB}.Com(\text{pp}_{\text{bPC}}, Z)$ .
	- 4.  $\mathcal{P}$  sends  $[Z]_{\text{bv}}$
	- 5.  $\mathcal V$  sends  $\alpha \leftarrow \mathbb F$ .
- Round 3: Prover and Verifier execute the GAPP argument.
	- 1. P and V set  $G(q_M, q_L, q_R, q_O, q_C, S_a, S_b, S_c, a, b, c, z, \bar{z})$  as in Equation [\(20\)](#page-29-1).
	- 2. P and V define statement  $\mathbf{x} = (\kappa, \theta, \mathbf{h}, \chi)$  as in Section [5.1.](#page-28-2)
	- 3. P and V execute argument of knowledge (Figure [2\)](#page-17-0) for  $\mathbf{x} \in \mathcal{R}_{\text{pp},G,n,m}^{\text{GAPP}}$ .
	- 4. V accepts if the above argument accepts.



circuits, the protocol in Figure [4](#page-30-0) is an argument of knowledge for the sub-language of  $\mathcal{R}_{\text{nn}}^{\text{agg}}$  $_{\mathsf{pp},\mathcal{C}}^{\mathsf{agg}}$  consisting of identical circuits, with following efficiency parameters:

$$
\begin{array}{ll} |\pi^{\text{agg}}| &= |\pi^{\text{GAPP}}(n, m)|\\ \mathbf{t}^{\text{agg}}_{\text{P}} &= \mathbf{t}^{\text{GAPP}}_{\text{P}}(n, m) \\ \mathbf{t}^{\text{agg}}_{\text{V}} &= \mathbf{t}^{\text{GAPP}}_{\text{V}}(n, m) + 12 \, \mathbb{G}_T + 6 \, \mathbb{F} \end{array}
$$

**Comparison with baseline.** Note that computing the aggregated polynomial  $A, B, C, Z$  incur effort almost equivalent to computing and committing to the same polynomials when generating  $n$  separate proofs. Computing the univariate components of the  $H$  polynomial involves computing the form G over the univariate polynomials for each  $i \in [n]$ , and committing to them for computing commitment to H. Again, this effort is identical to computing and committing to the individual quotient polynomials for the  $n$  proof instances. However, when computing separate proofs, the prover additionally needs to compute opening proofs for the quotient polynomial, which incurs additional  $O(m)$  cryptographic operations per proof. Existing aggregation schemes such as aPlonk [\[ABST23\]](#page-38-1) incur further costs of evaluating all the polynomials and committing to their

evaluations. By contrast, GAPP protocol only requires opening proofs for the committed bivariate polynomials at a random point  $(x, y)$  which incurs only  $O(m + n)$  additional cryptographic operations, instead of  $O(mn)$ . Thus, for large enough n, proof aggregation achived by our scheme is almost 25-30% faster than naive proof generation. Moreover, we see that our scheme supports highly parallel implementation as the univariate components of the bivariate polynomials can be computed and committed in parallel. More experimental results appear in Figure [1.](#page-4-0)

#### <span id="page-31-0"></span>5.2 Lookup Protocol for Tuples

Lookup (or subvector) arguments are an important building block in modern zkSNARKs. For integers  $k, m, n \in \mathbb{Z}$ , we say that the vector of tuples  $\mathbf{A} = (\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \in (\mathbb{F}^m)^n$  is a subvector of the vector  $\mathbf{T} = (\mathbf{t}_0, \ldots, \mathbf{t}_{k-1}) \in (\mathbb{F}^m)^k$ , if for all  $i \in [n]$ , there exists  $j \in [k]$  such that  $\mathbf{a}_i = \mathbf{t}_j$ . We denote such vectors by  $A \preceq T$ . Typically, we wish to check the predicate  $A \preceq T$  given commitments to vectors A and T under a suitable commitment scheme.

**Tuple Lookup Using bPCLB.** Let  $\mathbb{K}, \mathbb{H}$  and  $\mathbb{V}$  denote the subgroups generated by primitive  $k^{th}$ ,  $n^{th}$  and  $m^{th}$  roots of unity in F respectively. Let  $\{\mu_i^{\mathbb{K}}\}$  $\mathbb{K}_{i}$ <sub>i</sub> $\in$ <sub>[k]</sub> and  $\{ \mu_{i}^{\mathbb{H}}$  $_{i}^{\mathbb{H}}\}_{i\in[n]}$  denote the Lagrange basis polynomials for the subgroups K and H respectively. To commit to a vector  $\mathbf{T} = (\mathbf{t}_0, \ldots, \mathbf{t}_{k-1}),$ we canonically associate m-tuple  $\mathbf{t}_i$  with the polynomial  $t_i(Y) = \mathsf{Enc}_{\mathbb{V}}(\mathbf{t}_i) \in \mathbb{F}_{\leq m}[Y]$ . Similarly, the components of the vector  $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_{n-1})$  are associated with polynomials  $a_i(Y) = \mathsf{Enc}_{\mathbb{H}}(\mathbf{a}_i) \in$  $\mathbb{F}_{\leq m}[Y]$ . The commitment to **T** and **A**. We encode **T** and **A** as interpolating bivariate polynomials over domains  $\mathbb{K} \times \mathbb{V}$  and  $\mathbb{H} \times \mathbb{V}$  respectively as below:

$$
\begin{aligned} \textsf{Enc}_{\mathbb{K}\times \mathbb{V}}(\mathbf{T})=T(X,Y)=\sum_{i=0}^{n-1}\mu_i^{\mathbb{K}}(X)t_i(Y),\\ \textsf{Enc}_{\mathbb{H}\times \mathbb{V}}(\mathbf{A})=A(X,Y)=\sum_{i=0}^{k-1}\mu_i^{\mathbb{H}}(X)a_i(Y) \end{aligned}
$$

We now define the relations we want to check. The first is a lookup relation that proves that one committed vector is a subvector of the other committed vector without identifying the positions where the former occurs in the latter. The second variant also commits to positions where the first vector occurs in the second vector.

**Definition 5.1** (Lookup). For  $pp \leftarrow$  bPCLB.Setup $(1^{\lambda}, d_x, d_y)$ , and integers  $k, m, n$  with  $\max(k, n)$   $<$  $d_x, m < d_y$ , we define the relation  $\mathcal{R}_{\mathsf{pp},k,m,n}^{\mathsf{lookup}}$  consisting of tuples  $(\mathbf{x}, \mathbf{w})$  where  $\mathbf{x} = (C_A, C_T) \in \mathbb{G}_T^2$ and  $\mathbf{w} = (\mathbf{A}, \mathbf{T}, \tilde{\mathbf{c}}_a, \tilde{\mathbf{c}}_t)$  such that

- $\bullet$  A  $\preceq$  T,
- bPCLB.Open(pp,  $C_A$ ,  $(n, d_y)$ ,  $A$ ,  $\tilde{\mathbf{c}}_a$ ) = 1 and
- bPCLB.Open(pp,  $C_T$ ,  $(k, d_y)$ ,  $T$ ,  $\tilde{\mathbf{c}}_t$ ) = 1.

 $Here A(X,Y) = \mathsf{Enc}_{\mathbb{H} \times \mathbb{V}}(\mathbf{A}) \text{ and } T(X,Y) = \mathsf{Enc}_{\mathbb{K} \times \mathbb{V}}(\mathbf{T}).$ 

 $\bf{Definition \ 5.2}$  (Indexed Lookup).  $\mathit{For}$   $\mathsf{pp}_{\mathsf{bPC}} \leftarrow \mathsf{bPCLB}.\mathsf{Setup}(1^\lambda,d_x,d_y),$   $\mathsf{pp}_{\mathsf{uPC}} \leftarrow \mathsf{KZG}.\mathsf{Setup}(1^\lambda,d_x)$ and integers  $k, m, n$  with  $\max(k, n) < d_x$ ,  $m < d_y$ , we define the relation  $\mathcal{R}_{\mathsf{pp},k,m,n}^{\mathsf{clockup}}$  consisting of tuples  $(\mathbf{x}, \mathbf{w})$  where  $\mathbf{x} = (C_A, C_T, C_u) \in \mathbb{G}_T^2 \times \mathbb{G}_1$  and  $\mathbf{w} = (\mathbf{A}, \mathbf{T}, \mathbf{u}, \tilde{\mathbf{c}}_a, \tilde{\mathbf{c}}_t, \tilde{\mathbf{c}}_u)$  such that

•  $\mathbf{A}[i] = \mathbf{T}[\mathbf{u}[i]]$  for all  $i \in [n]$ ,

- bPCLB.Open(pp<sub>bPC</sub>,  $C_A$ ,  $(n, d_y)$ ,  $A, \tilde{\mathbf{c}}_a$ ) = 1,
- bPCLB.Open(pp<sub>bPC</sub>,  $C_T$ ,  $(k, d_y)$ ,  $T$ ,  $\tilde{\mathbf{c}}_t$ ) = 1 and
- uPC.Open $(pp_{\text{uPC}}, C_u, n, u, \tilde{\mathbf{c}}_u) = 1.$

Here  $A(X,Y) = \mathsf{Enc}_{\mathbb{H} \times \mathbb{V}}(\mathbf{A}), T(X,Y) = \mathsf{Enc}_{\mathbb{K} \times \mathbb{V}}(\mathbf{T})$  and  $u(X) = \mathsf{Enc}_{\mathbb{H}}(\mathbf{u}).$ 

We present an approach based on bPCLB, where the prover incurs only  $O(m+n)$  cryptographic operations, as opposed to  $O(mn)$  in [\[DXNT23,](#page-40-8) [CGG](#page-40-6)<sup>+</sup>24]. Our verification is logarithmic, instead of constant in the prior works. We present our approach for the indexed lookup variant (the case of un-indexed lookup follows similarly).

**Our Approach.** Let  $A(X, Y), T(X, Y)$  and  $u(X)$  be polynomials encoding the vectors **A**, **T** and u over respective domains. For  $\gamma \in \mathbb{F}$ , let  $\mathbf{A}_{\gamma} = (a_0(\gamma), \dots, a_{n-1}(\gamma))$  and  $\mathbf{T}_{\gamma} = (t_0(\gamma), \dots, t_{k-1}(\gamma))$ be vectors obtained by evaluating the constituent polynomials at  $Y = \gamma$ . Now, we note that by Schwartz-Zippel Lemma, for a random  $\gamma \leftarrow \mathbb{F}$ , except with negligible probability  $(kn/|\mathbb{F}|)$  it holds that:

$$
\mathbf{A}_{\gamma}[i] = \mathbf{T}_{\gamma}[\mathbf{u}[i]] \forall i \in [n] \Longleftrightarrow \mathbf{A}[i] = \mathbf{T}[\mathbf{u}[i]] \forall i \in [n]
$$

Building on the above observation, and noticing that polynomials  $A(X, \gamma)$  and  $T(X, \gamma)$  encode the vectors  $\mathbf{A}_{\gamma}$  and  $\mathbf{T}_{\gamma}$  over domains H and K respectively, the verifier sends  $\gamma \leftarrow \mathbb{F}$  to the prover, who responds with KZG commitments  $C_a$  and  $C_t$  to the polynomials  $A_\gamma(X) = A(X, \gamma)$  and  $T_{\gamma}(X) = T(X, \gamma)$ . This reduces the case for general m to that for  $m = 1$ . The verifier can now check that  $(C_a, C_t, C_u)$  is a valid statement in  $\mathcal{R}_{\mathsf{pp}_{\mathsf{uPC}},k,n,1}$  using prior work such as [\[DGP](#page-40-10)+24][Lemma 3].

Additionally, the verifier needs to to check consistency of commitments  $C_t$  and  $C_a$  with polynomials T and A respectively. To do so, the verifier sends another challenge  $x \leftarrow \mathbb{F}$  and executes an argument to check: (i)  $A(x) = A(x, \gamma)$ , (ii)  $T(x) = T(x, \gamma)$ . Both the conditions are checked by requesting openings to the univariate and bivariate polynomials at x and  $(x, \gamma)$  respectively. The complete protocol appears in Figure [5.](#page-33-0)

**Lemma 5.2.** Assuming that q-DLOG is hard for the bilienar group generator BG and  $(\mathcal{P}_{sv}, \mathcal{V}_{sv})$  is an argument of knowledge for the relation  $\mathcal{R}_{\mathsf{pp},k,m,n}^{\mathsf{clockup}}$  for  $m=1$  under the q-DLOG and algebraic group model (AGM), the protocol in Figure [5](#page-33-0) is an argument of knowledge for the relation  $\mathcal{R}_{\text{on }k\,m}^{\text{clockup}}$  $\mathsf{pp},\!k,\!m,\!n$ in the algebraic group model. It satisfies following efficiency parameters:

$$
\begin{array}{ll} |\pi^{\text{lookup}}(k,m,n)|&=|\pi^{\text{sv}}|+|\pi^{\text{bPCLB}}(t,m)|\\ \mathbf{t}^{\text{lookup}}_{\text{P}}&=\mathbf{t}^{\text{bPCLB}}_{\text{V}}(t,m)+\mathbf{t}^{\text{sv}}_{\text{V}}(n,k)+O(t)\,\mathbb{M}+O(mt)\,\mathbb{F}\\ \mathbf{t}^{\text{lookup}}_{\text{V}}&=\mathbf{t}^{\text{bPCLB}}_{\text{V}}(t,m)+\mathbf{t}^{\text{sv}}_{\text{V}}(n,k) \end{array}
$$

where  $t = \max(n, k)$ , where the superscript (sv) indicates the complexities of the subvector argument.

### 5.3 "A la-carte" Proof Systems

In this section, we present an approach based on GAPP framework, using only black-box cryptography that further improves over  $[DXNT23, CGG^+24]$  $[DXNT23, CGG^+24]$  $[DXNT23, CGG^+24]$  in terms of efficiency, while being agnostic to the choice of polynomial IOP.

<span id="page-33-0"></span>• Setup: Setup generates the following public parameters:

$$
\mathsf{pp}_{\mathsf{uPC}} \leftarrow \mathsf{KZG}.\mathsf{Setup}(1^\lambda,d_x), \quad \mathsf{pp}_{\mathsf{bPC}} \leftarrow \mathsf{bPCLB}.\mathsf{Setup}(1^\lambda,(d_x,d_y)).
$$

- Common Input: Commitments  $(C_A, C_T, C_u) \in \mathbb{G}_T^2 \times \mathbb{G}_1$  and integers  $k \leq d_x$ ,  $n \leq d_x$  and  $m \leq d_y$ .
- Prover's Input: Vectors  $A = (a_0, \ldots, a_{n-1}) \in (\mathbb{F}^m)^n$ ,  $T = (t_0, \ldots, t_{k-1}) \in (\mathbb{F}^m)^k$ ,  $u \in \mathbb{F}^n$ , opening hints  $\tilde{\mathbf{c}}_a, \tilde{\mathbf{c}}_t$  and  $\tilde{\mathbf{c}}_u$ . Polynomials  $A(X, Y), T(X, Y)$  and  $u(X)$  encoding vectors **A**, **T** and **u** respectively.
- Round 1: Prover commits to univariate polynomials.
	- 1.  $\mathcal V$  sends  $\gamma \leftarrow \mathbb F$ .
	- 2. P computes vectors  $\mathbf{A}_{\gamma} = (a_i(\gamma))_{i \in [n]}$ ,  $\mathbf{T}_{\gamma} = (t_i(\gamma))_{i \in [k]}$  and polynomials  $A_{\gamma}(X)$  and  $T_{\gamma}(X)$  interpolating  $\mathbf{A}_{\gamma}$  and  $\mathbf{T}_{\gamma}$  on H and K respectively.
	- 3. P sends commitment  $C_a$  and  $C_t$  to polynomials  $A_\gamma(X) = A(X, \gamma)$  and  $T_\gamma(X) = T(X, \gamma)$  respectively.
	- 4.  $\mathcal V$  sends  $x \leftarrow \mathbb F$ .
- Round 2: Prover sends evaluations.
- 1. P sends evaluations  $\tilde{a} = A_{\gamma}(x), \tilde{t} = T_{\gamma}(x)$ .
- 2.  $\mathcal V$  sends  $r \leftarrow \mathbb F$ .
- Round 3: Prover proves evaluations.
- 1. P computes proofs

$$
\pi \leftarrow \mathsf{KZG}.\mathsf{Prove}(\mathsf{pp}_{\mathsf{uPC}}, T_{\gamma}(X) + rA_{\gamma}(X), x),
$$
  

$$
\pi' \leftarrow \mathsf{bPCLB}.\mathsf{Prove}(\mathsf{pp}_{\mathsf{bPC}}, T(X, Y) + rA(X, Y), (x, \gamma))
$$

- 2. P sends  $\pi$  and  $\pi'$  to  $\mathcal{V}$ .
- Round 4:  $V$  checks evaluations and subvector argument.
- 1.  $V$  computes:

$$
b \leftarrow \text{KZG}.\text{Verify}(pp_{\text{uPC}}, C_t + r \cdot C_a, k + n, x, \tilde{t} + r\tilde{a}, \pi)
$$
  

$$
b' \leftarrow \text{bPCLB}.\text{Verify}(pp_{\text{bPC}}, C_T + r \cdot C_A, (k + n, m), (x, \gamma), \tilde{t} + r\tilde{a}, \pi')
$$

2.  $P$  and  $V$  execute subvector argument

$$
b'' \leftarrow \langle \mathcal{P}_{\mathsf{sv}}(C_a, C_t, C_u; (\mathbf{A}_{\gamma}, \mathbf{T}_{\gamma}, \mathbf{u}, *)), \mathcal{V}_{\mathsf{sv}}(C_a, C_t, C_u) \rangle
$$

to check  $\mathbf{A}_{\gamma} \preceq \mathbf{T}_{\gamma}$ .

3. V accepts if  $b = b' = b'' = 1$ , otherwise it rejects.

Figure 5: Argument of Knowledge for the relation  $\mathcal{R}_{\mathsf{pp},k,m,n}^{\mathsf{lookup}}$ .

Model for Non-Uniform Computation. Our description of non-uniform computation is closely related to that in non-uniform IVC schemes such as  $[KS22]$  and prior work  $[CGG+24]$  $[CGG+24]$ . We represent our computations as Plonk constraints, though our scheme can be instantiated with other intermediate representations. Let  $\mathcal{F} = \{F_0, \ldots, F_{k-1}\}\$ be a family of k efficiently computable functions, which can be expressed using at most m plonk constraints. We assume that for all  $i \in [k]$ , the circuit  $F_i$  takes s inputs given by  $\mathbf{u}_i \in \mathbb{F}^s$ , and outputs  $\mathbf{v}_i \in \mathbb{F}^s$  possibly taking a non-deterministic input  $\mathbf{w}_i$  of arbitrary size. The language  $\mathcal{L}^{\text{nuc}}_{\mathcal{F},\sigma,n,m}$  expressing *n*-step execution of the non-uniform computation F with wiring permutation  $\sigma : [(2s+1)n] \to [(2s+1)n]$  consists of statements of the form  $(\mathbf{w}_{\text{io}}, \mathbf{w}_{\text{int}})$  where

$$
\mathbf{w}_{io} = (\tau_0||\mathbf{u}_0||\mathbf{v}_0||\cdots||\tau_{n-1}||\mathbf{u}_{n-1}||\mathbf{v}_{n-1}), \mathbf{w}_{int} = (\mathbf{w}_0||\cdots||\mathbf{w}_{n-1})
$$

such that  $\mathbf{v}_i = F_{\tau_i}(\mathbf{u}_i, \mathbf{w}_i)$  for all  $i \in [n]$ , and  $\mathbf{w}_{i}$   $[j] = \mathbf{w}_{i}$   $[\sigma(j)]$  for all  $j \in [(2s+1)n]$ . We call wires  $(\tau_0, \ldots, \tau_{n-1})$  as the *activation* wires, which activate a particular function from the family at each step. The wires in vector  $\mathbf{w}_{io}$  are assumed to be *interface* wires, which may be arbitrarily shorted using the permutation  $\sigma$ , whereas the wires in  $w_{int}$  are exclusive to a specific step of computation. In typical applications, we expect activation wire  $\tau_{i+1}$  to be computed as part of the output  $\mathbf{v}_i$  (which is shorted to  $\tau_{i+1}$  using  $\sigma$ ). In the non-uniform IVC scheme [\[KS22\]](#page-42-9), the activation  $\tau_{i+1}$  is specified as  $\tau_{i+1} = \varphi(\tau_i || \mathbf{u}_i || \mathbf{w}_i)$  for some efficiently computable function  $\varphi$ . In our setting, we assume  $\varphi$  is implemented as part of each circuit  $F_i$ . We additionally allow modeling "global" structure among interface wires using the permutation  $\sigma$ .

**PLONK Based Instantiation.** We now instantiate our scheme for proving  $n$ -step non-uniform computation as described above, using the PLONK PIOP. For all  $j \in [k]$ , let  $F_j$  be given by following vector of PLONK circuit polynomials:

$$
\left(q_M^j(Y), q_L^j(Y), q_R^j(Y), q_O^j(Y), q_C^j(Y), S_a^j(Y), S_b^j(Y), S_c^j(Y)\right) \in (\mathbb{F}[Y])^8
$$

Let  $T_M$ ,  $T_L$ ,  $T_R$ ,  $T_O$ ,  $T_C$ ,  $T_a$ ,  $T_b$  and  $T_c$  denote the tables (vectors) of polynomials  $(q_M^j(Y))_{j \in [k]},$  $\ldots, (S_c^j(Y))_{j \in [k]}$  respectively. Let  $\mathcal{T}_M, \ldots, \mathcal{T}_c$  denote the (trusted) commitments to the tables  $T_M$ ,  $\ldots$ ,  $T_a$  obtained as bPCLB commitments of packed polynomials below (see Section [5.2\)](#page-31-0)

$$
T_M(X,Y) = \sum_{j=0}^{k-1} \mu^{\mathbb{K}}(X) q_M^j(Y), \dots, T_c(X,Y) = \sum_{j=0}^{k-1} \mu^{\mathbb{K}}(X) S_c^j(Y)
$$

As before, let  $a_i(Y), b_i(Y)$  and  $c_i(Y)$  be polynomials that interpolate the witness (left, right and output wires) for each of the m constraints in the circuit  $F_{\tau_i}$ . To make the scheme concrete, we assume that  $m > 2s + 1$  ans all the interface wires for each step, namely  $\tau_i$ ,  $\mathbf{u}_i$  and  $\mathbf{v}_i$  appear as the first  $2s + 1$  left wires. Thus, we have for all  $i \in [q]$ :  $a_i(\nu^{0}) = \tau_i$ ,  $a_i(\nu^{j+1}) = \mathbf{u}_i[j]$  and  $a_i(\nu^{s+1+j}) = \mathbf{v}_i[j]$  for  $j \in [s]$ . We now present the argument of knowledge for the language  $\mathcal{L}^{\text{nuc}}_{\mathcal{F},\sigma,n,m}$ . In the sketch below, we assume  $\sigma$  to have a simple structure which ensures wire  $\tau_{i+1}, \mathbf{v}_i[0]$  have the same value for  $i \in [n-1]$ , i.e, the activation wire in a step is computed as the first output wire of the previous step. The case of general  $\sigma$  is presented in Section [5.4.](#page-36-0)

Setup and Inputs. The setup phase of the protocol generates the public parameters for polynomial commitment schemes uPC and bPC:

$$
\mathsf{pp}_{\mathsf{uPC}} \leftarrow \mathsf{KZG}.\mathsf{Setup}(1^\lambda,d_x), \quad \mathsf{pp}_{\mathsf{bPC}} \leftarrow \mathsf{bPCLB}.\mathsf{Setup}(1^\lambda,(d_x,d_y))
$$

It also generates commitments  $\mathbf{C}_{\mathcal{F}} = (\mathcal{T}_M, \mathcal{T}_L, \dots, \mathcal{T}_c)$  to the functions in the family  $\mathcal F$  as defined earlier. The prover's input consists of  $(\mathbf{w}_{io}, \mathbf{w}_{int}) \in \mathcal{L}^{\text{nuc}}_{\mathcal{F}, \sigma, n,m}$ .

Interactive Protocol. The interactive protocol between the prover  $(\mathcal{P})$  and honest verifier  $(\mathcal{V})$ proceeds as:

• Prover Commits to Witness and Circuit Polynomials: The prover computes commitments  $[A]_{\text{bv}}, [B]_{\text{bv}}$ and  $[C]_{\text{bv}}$  to packed witness polynomials  $A(X, Y), B(X, Y)$  and  $C(X, Y)$  as described previously. The prover also computes packed polynomials corresponding to circuit polynomials activated at each step. Specifically, the prover computes polynomials

$$
q_M(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) q_M^{\tau_i}(Y), q_L(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) q_L^{\tau_i}(Y),
$$
  
\n
$$
q_R(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) q_R^{\tau_i}(Y), q_O(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) q_O^{\tau_i}(Y),
$$
  
\n
$$
q_C(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) q_C^{\tau_i}(Y), S_a(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) S_a^{\tau_i}(Y),
$$
  
\n
$$
S_b(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) S_b^{\tau_i}(Y), S_c(X,Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) S_c^{\tau_i}(Y)
$$

- The prover sends commitments  $[A]_{\text{bv}}, [B]_{\text{bv}}, [C]_{\text{bv}}, [q_M]_{\text{bv}}, \ldots, [S_c]_{\text{bv}}$ .
- Verifier checks correctness of circuit polynomials: The verifier checks that the correct polynomials corresponding to vector  $(\tau_0, \ldots, \tau_{n-1})$  have been looked up from respective tables. The prover starts by sending commitment  $C_{\tau}$  to a polynomial  $\tau(X)$  which interpolates the vector  $(\tau_0, \ldots, \tau_{n-1})$  over  $\mathbb{H}$ .
- The verifier sends a challenge  $\chi \leftarrow \mathbb{F}$  to batch the lookup proofs.
- Verifier computes:

$$
C_a = (1, \chi, \dots, \chi^7) \cdot ([q_M]_{\text{bv}}, [q_L]_{\text{bv}}, \dots, [S_c]_{\text{bv}})
$$
  

$$
C_t = (1, \chi, \dots, \chi^7) \cdot (\mathcal{T}_M, \mathcal{T}_L, \mathcal{T}_R, \mathcal{T}_O, \mathcal{T}_C, \mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c)
$$

**•** Prover and Verifier execute an argument of knowledge to show  $(C_a, C_t, C_\tau) \in \mathcal{R}_{\text{pp},k,m,n}^{\text{clockup}}$ . Moreover to establish the correctness of the polynomial  $\tau(X)$ , the prover and verifer execute an argument to show the following:

$$
\mu_0^{\mathbb{V}}(Y)(A(X,Y) - \tau(X)) = 0
$$
 over  $\mathbb{H} \times \mathbb{V}$ 

Note that the above implies that  $\tau(\omega^i) = A(\omega^i, \nu^0) = a_i(\nu^0) = \tau_i$ , and thus  $\tau(X)$  correctly interpolates the purported vector  $(\tau_0, \ldots, \tau_{n-1})$  committed in  $[A]_{\text{bv}}$ .

• Verifier checks the interface wiring: Checking that  $\tau_{i+1} = \mathbf{v}_i[0]$  for all  $i \in [n-1]$  is equivalent to ensuring  $A(\omega^{i+1}, \nu^{0}) = A(\omega^{i}, \nu^{s+1})$  for all  $i \in [n-2]$ . To this end, the prover and the verifier define the polynomial

$$
Q'(X,Y) = (X - \omega^{n-1})\mu_0^{\mathbb{V}}(Y)(A(\omega X, Y) - A(X, \nu^{s+1}Y))
$$

and run an argument of knowledge to show that  $Q'(X,Y) = 0$  over  $\mathbb{H} \times \mathbb{V}$ .

• Prover and Verifier aggregate proofs: The protocol now proceeds similar to the one for aggregation of PLONK proofs in Section [5.1.](#page-28-2) Verifier starts by sending challenges  $\beta, \gamma \leftarrow \mathbb{F}$ .

• Prover computes polynomial  $z_i(Y)$  for each  $i \in [n]$  according to the PLONK protocol treating

$$
\left(q_M^{\tau_i}(Y), q_L^{\tau_i}(Y), q_R^{\tau_i}(Y), q_O^{\tau_i}(Y), q_C^{\tau_i}(Y), S_a^{\tau_i}(Y), S_b^{\tau_i}(Y), S_c^{\tau_i}(Y)\right)
$$

as the circuit polynomials for the step i.

- Prover sends commitment  $[Z]_{\text{bv}}$  to the aggregated polynomial  $Z(X, Y) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}$  $_{i}^{\mathbb{H}}(X)z_{i}(Y).$
- Verifier sends  $\alpha \leftarrow \mathbb{F}$ .
- Prover and Verifier execute an argument of knowledge to check that  $(\kappa, \theta, \mathbf{h}, \mathbf{C})$  is a valid statement in  $\mathcal{R}_{\mathsf{pp},G,n,m}^{\mathsf{GAPP}}$  by setting  $\mathbf{C} = ( [ q_M ]_{\text{bv}}, [ q_L ]_{\text{bv}}, [ q_B ]_{\text{bv}}, [ q_C ]_{\text{bv}}, [ g_\alpha ]_{\text{bv}}, [ S_a ]_{\text{bv}}, [ S_b ]_{\text{bv}}$  $[S_c]_{\rm bv}, [A]_{\rm bv}, [B]_{\rm bv}, [C]_{\rm bv}, [Z]_{\rm bv}), G$  as the multivariate polynomial in Equation [\(20\)](#page-29-1), h,  $\kappa$ and  $\theta$  as in the argument in Section [5.1.](#page-28-2)

**Lemma 5.3.** Assuming that  $q$ -DLOG is hard for the bilinear group generator BG, the above protocol is an argument of knowledge for the langugage  $\mathcal{L}^{\text{nuc}}_{\mathcal{F},\sigma,n,m}$  in the algebraic group model with following efficiency parameters:

$$
\begin{array}{lll} |\pi| = & |\pi^{\text{agg}}(n, m)| + |\pi^{\text{lookup}}(k, n, m)| \\ t_P = & t_P^{\text{agg}}(n, m) + t_P^{\text{lookup}}(k, n, m) + O(mn) \mathbb{F} + O(mn) \mathbb{M} \\ t_V = & t_V^{\text{agg}}(n, m) + t_V^{\text{lookup}}(k, n, m) \end{array}
$$

*Proof.* The proof essentially follows from the properties of arguments of knowledge for  $\mathcal{R}_{pp,G,n,m}^{\text{GAPP}}$ and  $\mathcal{R}_{\mathsf{pp},k,m,n}^{\mathsf{clockup}}$ .  $\Box$ 

#### <span id="page-36-0"></span>5.4 Supporting General Wiring Constraints

Given a bivariate polynomial  $A(X, Y)$  with  $\deg_X(A) < n$  and  $\deg_Y(m) < m$ , where we assume that A encodes a set of wires  $\mathbf{a} = (a_0, \dots, a_{nm-1})$  such that  $A(\omega^i, \nu^j) = a_{mi+j}$ . Given a permutation  $\sigma : [nm] \to [nm]$ , we present a bivariate PIOP, that checks  $a_{\sigma(i)} = a_i$  for all  $i \in [nm]$ . We define the following polynomials for the identity permutation and  $\sigma$ :

$$
S_{\text{id}}(X,Y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (mi+j)\mu_i^{\mathbb{H}}(X) \cdot \mu_j^{\mathbb{V}}(Y),
$$
  

$$
S_{\sigma}(X,Y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \sigma(mi+j)\mu_i^{\mathbb{H}}(X) \cdot \mu_j^{\mathbb{V}}(Y)
$$
 (22)

Our PIOP relies on the following observation.

<span id="page-36-1"></span>**Lemma 5.4.** Let  $\mathbf{a} = (a_0, \ldots, a_{nm-1})$  be a vector of length nm interpolated by the polynomial A over the domain  $\mathbb{H}\times\mathbb{V}$  as above. Let  $\sigma : [nm] \to [nm]$  be a permutation. Then, the vector **a** satisfies  $a_i = a_{\sigma(i)}$  for all  $i \in [mn]$  if and only if with high probability over the choice  $\beta, \gamma \leftarrow \mathbb{F}$ , there exists polynomials  $P(X, Y), U(X, Y) \in \mathbb{F}[X, Y]$  and  $Q(X), R(X) \in \mathbb{F}[X]$  such that the following hold:

$$
P(X, Y)(\gamma + \beta S_{\sigma}(X, Y) + A(X, Y)) - (\gamma + \beta S_{\text{id}}(X, Y) + A(X, Y)) \equiv 0 \text{ over } \mathbb{H} \times \mathbb{V}
$$
  
\n
$$
U(X, \nu Y)(\mu_0^{\mathbb{V}}(Y)Q(X) - \mu_0^{\mathbb{V}}(Y) + 1) - U(X, Y)P(X, Y) \equiv 0 \text{ over } \mathbb{H} \times \mathbb{V}
$$
  
\n
$$
\mu_0^{\mathbb{H}}(X)(R(X) - 1) \equiv 0 \text{ over } \mathbb{H}
$$
  
\n
$$
R(\omega X) - R(X)Q(X) \equiv 0 \text{ over } \mathbb{H}
$$

*Proof.* First, let us assume that the identities are true for some polynomials  $P, U, Q$  and  $R$ , given the pre-specified polynomials A,  $S_{id}$  and  $S_{\sigma}$  and uniformly sampled challenges  $\beta, \gamma \leftarrow \mathbb{F}$ . Now, putting  $X = \omega^0$  in the third identity implies  $R(\omega^0) = 1$ . Putting  $X = \omega^i$  for  $i \in [n]$  in the last equation, implies  $R(\omega^{i+1}) = R(\omega^{i})Q(\omega^{i})$  and thus

<span id="page-37-0"></span>
$$
1 = \frac{R(\omega^n)}{R(\omega^0)} = Q(\omega^0) \cdot Q(\omega^1) \cdots Q(\omega^{n-1})
$$
\n(23)

Now, from the second identity, for all  $i \in [n]$ , we have by substituting  $Y = \nu^0, \ldots, \nu^{m-1}$ ,

$$
U(\omega^i, \nu) \cdot Q(\omega^i) = U(\omega^i, \nu^0) \cdot P(\omega^i, \nu^0)
$$
  

$$
U(\omega^i, \nu^2) \cdot 1 = U(\omega^i, \nu) \cdot P(\omega^i, \nu)
$$
  

$$
\vdots
$$
  

$$
U(\omega^i, \nu^m) \cdot 1 = U(\omega^i, \nu^{m-1}) \cdot P(\omega^i, \nu^{m-1})
$$

Multiplying and observing that  $U(\omega^i, \nu^j)$  terms cancel off, we are left with  $Q(\omega^i) = \prod_{j=0}^{m-1} P(\omega^i, \nu^j)$ . Now, from Equation [\(23\)](#page-37-0), we have

<span id="page-37-1"></span>
$$
\prod_{i=0}^{n-1} \prod_{j=0}^{m-1} P(\omega^i, \nu^j) = 1
$$
\n(24)

Now, from the first identity, we have by substituting  $X = \omega^i$  and  $Y = \nu^j$ ,

$$
P(\omega^i, \nu^j) = \frac{\gamma + \beta S_{\text{id}}(\omega^i, \nu^j) + A(\omega^i, \nu^j)}{\gamma + \beta S_{\sigma}(\omega^i, \nu^j) + A(\omega^i, \nu^j)}
$$

Then from Equation [\(24\)](#page-37-1), we have:

$$
\prod_{i=0}^{n-1} \prod_{j=0}^{m-1} \left( \frac{\gamma + \beta S_{\text{id}}(\omega^i, \nu^j) + A(\omega^i, \nu^j)}{\gamma + \beta S_{\sigma}(\omega^i, \nu^j) + A(\omega^i, \nu^j)} \right) = 1
$$

Since, the above holds for uniformly sampled  $\beta, \gamma \leftarrow \mathbb{F}$ , with high probability, we have the polynomial identity:

$$
\prod_{i=0}^{n-1} \prod_{j=0}^{m-1} (X + (mi+j)Y + a_{mi+j}) = \prod_{i=0}^{n-1} \prod_{j=0}^{m-1} (X + \sigma (mi+j)Y + a_{\sigma (mi+j)})
$$

, and thus  $a_{mi+j} = a_{\sigma(mi+j)}$  for all  $(i, j) \in [n] \times [m]$ . This proves one direction of the "if and only if" claim. The other direction is straightforward, where the polynomials  $P, U, Q$  and R can be constructed according to the preceding proof.  $\Box$ 

The Lemma [5.4](#page-36-1) implies a simple argument of knowledge for checking that witness encoded by polynomial  $A \in \mathbb{F}[X, Y]$  satisfies wiring constraints given by permutation  $\sigma$ . Let  $pp_{sigma} = (C_{id}, C_{\sigma})$ denote honestly generated commitments to polynomials  $S_{\mathsf{id}}, S_{\sigma}$  as define earlier. Let pp denote all the public parameters ( $pp_{\mu PC}$ ,  $pp_{bPC}$ ,  $pp_{\sigma}$ ). We define  $\mathcal{R}_{pp,\sigma}^{perm}$  to be the relation consisting of pairs  $(C_A, A)$  such that  $bPC. Open(pp, A(X, Y), (d_x, d_y), C_A, \tilde{C}_A) = 1$  and  $a_k = a_{\sigma(k)}$  for all  $k \in [nm]$ , where  $a_{mi+j} = A(\omega^i, \nu^j)$  for  $(i, j) \in [n] \times [m]$ . The protocol is outlined below:

Setup and Inputs. The setup consists of parameters **pp** as described above. The common input consists of  $(p, C_A)$  where  $C_A$  is the commitment of the prover's polynomial. The prover in addition knows the witness polynomial  $A(X, Y)$ , the underlying witness vector  $\mathbf{a} \in \mathbb{F}^{nm}$  and other publicly specified polynomials such as  $S_{\mathsf{id}}$  and  $S_{\sigma}$ .

Prover Commits Auxiliary Polynomials. The verifier initiates the protocol by sending  $\beta, \gamma \leftarrow \mathbb{F}$ . The prover constructs polynomials  $P(X, Y)$ ,  $U(X, Y)$ ,  $Q(X)$ ,  $R(X)$  in Lagrange basis as:

$$
p_i(Y) = \sum_{j=0}^{m-1} \mu_j^{\mathbb{V}}(Y) \left( \frac{\gamma + \beta(mi + j) + a_{mi+j}}{\gamma + \beta \sigma(mi + j) + a_{\sigma(mi+j)}} \right)
$$
  

$$
u_i(Y) = \sum_{j=0}^{m-1} \mu_j^{\mathbb{V}}(Y) \prod_{k=0}^{j-1} p_i(\nu^k)
$$
  

$$
Q(X) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) p_i(\mu^{m-1})
$$
  

$$
R(X) = \sum_{i=0}^{n-1} \mu_i^{\mathbb{H}}(X) \prod_{j=0}^{i-1} Q(\omega^j)
$$

The prover sends commitments to polynomials  $P, U, Q$  and R using bPCLB as the bivariate PCS and KZG as the univariate PCS.

Prover and Verifier execute Bivariate PIOP: Prover and Verifier execute an argument of knowledge for the PIOP given by identities in Lemma [5.4](#page-36-1) as in Section [3.](#page-14-0)

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# <span id="page-43-7"></span>A Inner Product Arguments, Compressed Sigma Protocols

Compressed Sigma Protocol Framework. We briefly describe the compressed sigma protocol (CSP) framework of [\[AC20\]](#page-38-4) using the abstraction of a doubly homomorphic commitment defined in [\[BMM](#page-40-7)+21]. To ease the discussion, we define a deterministic commitment scheme below, the generalization to a hiding commitment scheme is straightforward.

**Definition A.1** ( $[BMM+21]$  $[BMM+21]$ ). Let  $(CM, Setup)$  be a computationally binding commitment scheme with message space M, key space K and commitment space C. We say that  $(CM, Setup)$  is doubly homomorphic if  $(M, +)$ ,  $(K, +)$  and  $(C, +)$  are abelian groups such that for all  $ck, ck' \in K$  and  $m, m' \in \mathcal{M}$ , we have:

$$
CM(ck + ck', m + m') = CM(ck, m) + CM(ck', m) + CM(ck, m') + CM(ck', m')
$$
 (25)

Inner Product Commitment. We consider doubly homomorphic commitment schemes obtained from bilinear maps  $\otimes : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$  for groups  $\mathcal{K}, \mathcal{M}$  and  $\mathcal{C}$  with prime order p. For  $n \in \mathbb{N}$ , we can extend the map ⊗ to the inner product  $\langle , \rangle_{\otimes} : \mathcal{K}^n \times \mathcal{M}^n \to \mathcal{C}$  between  $\mathbb{Z}_p$ -vector spaces  $\mathcal{K}^n$ and  $\mathcal{M}^n$  given by  $\langle \mathbf{a}, \mathbf{b} \rangle_{\otimes} = \sum_{i=0}^{n-1} (a_i \otimes b_i)$ . We call the doubly homomorphic commitment scheme  $CM : \mathcal{K}^n \times \mathcal{M}^n \to \mathcal{C}$  defined by  $CM(ck, w) = \langle ck, w \rangle_{\otimes}$  as an *inner product commitment*. Our notion of inner product commitment is a slight specialization of the same notion in [\[BMM](#page-40-7)<sup>+</sup>21], where we eschew full generalization to more concretely illustrate our folding techniques.

**Linear Form.** For  $K = \mathbb{Z}_p$ , the scalar multiplication  $\chi \otimes w = \chi \cdot w$  is a bilinear operator from  $\mathbb{Z}_p \times \mathcal{M}$  to M. In keeping with terminology used in CSPs, we call the resulting inner product commitment  $\langle , \rangle : \mathbb{Z}_p^n \times \mathcal{M}^n \to \mathcal{M}$  given by  $\langle$ ck, w $\rangle = \sum_{i=0}^{n-1} \chi_i \cdot w_i$ , with ck =  $(\chi_0, \ldots, \chi_{n-1}) \in \mathbb{Z}_p^n$ and  $\mathbf{w} = (w_0, \dots, w_{n-1}) \in \mathcal{M}^n$  as the linear form on  $\mathcal{M}^n$ .

CSP for Inner Product Commitments Let K and M and C be additive abelian groups of prime order p. Let (Setup, CM) denote the inner product commitment with message space  $\mathcal{M}^*$ , key space  $\mathcal{K}^*$  and commitment space C given by  $CM(\mathsf{ck}, \mathbf{w}) = \langle \mathsf{ck}, \mathbf{w} \rangle_{\otimes}$  for  $|\mathsf{ck}| = |\mathbf{w}|$ . Otherwise, we define  $CM(ck, w) = \perp$ . We now define the relation one proves using CSP.

<span id="page-44-0"></span>**Definition A.2.** Let  $K, M, C$  and CM be as above. For  $n \in \mathbb{N}$ , let  $\mathcal{R}_{\textsf{CM},n}^{\textsf{CSP}}$  be the relation consisting of pairs  $(\mathbf{x}, \mathbf{w})$  where  $\mathbf{x} = (\mathsf{ck}, \mathbf{a}, C, v)$  with  $\mathsf{ck} \in \mathcal{K}^n$ ,  $\mathbf{a} \in \mathbb{Z}_p^n$ ,  $C \in \mathcal{C}$ ,  $v \in \mathcal{M}$  and  $\mathbf{w} \in \mathcal{M}^n$  satisfying  $\langle$ ck, w $\rangle_{\otimes} = C$  and  $\langle$ a, w $\rangle = v$ .

In protocol  $\pi_{\textsf{csp}}$  to prove knowledge of  $\mathbf{w}\in\mathcal{M}^n$  such that  $(\mathbf{x},\mathbf{w})\in\mathcal{R}^{\textsf{CSP}}_{\textsf{CM},n}$ , where  $\mathbf{x}=(\mathsf{ck},\mathbf{a},C,v)$ as in Definition [A.2,](#page-44-0) the prover and verifier execute the following folding step:

- The prover splits the witness vector **w** into two equal sized vectors  $\mathbf{w}_L = (w_0, \dots, w_{n/2-1})$  and  $\mathbf{w}_R = (w_{n/2}, \ldots, w_{n-1})$ . It similarly computes  $(\mathsf{ck}_L, \mathsf{ck}_R)$  and  $(\mathbf{a}_L, \mathbf{a}_R)$  by splitting commitment key ck and the linear form a respectively. It then computes "cross terms"  $A = \langle ck_L, w_R \rangle_{\otimes}$ ,  $A' = \langle c \mathsf{k}_R, \mathbf{w}_L \rangle_{\otimes}, u = \langle \mathbf{a}_L, \mathbf{w}_R \rangle$  and  $u' = \langle \mathbf{a}_R, \mathbf{w}_L \rangle$ . It sends  $A, A', u, u'$  to the verifier.
- The verifier sends a uniform challenge  $x_0 \leftarrow \mathbb{F}$ .
- The prover computes new witness  $\mathbf{w}_1 = \mathbf{w}_L + x_0^{-1} \cdot \mathbf{w}_R$ .
- Prover and Verifier compute:  $ck_1 = ck_L + x_0 \cdot ck_R$ ,  $a_1 = a_L + x_0 \cdot a_R$ ,  $C_1 = x_0 \cdot A' + C + x_0^{-1} \cdot A$ ,  $v_1 = x_0 \cdot u' + v + x_0^{-1}u.$
- Prover and Verifier recursively run the argument of knowledge for showing  $(\mathbf{x}_1, \mathbf{w}_1) \in \mathcal{R}_{CM,n/2}^{CSP}$ where  $x_1 = (ck_1, a_1, C_1, v_1)$ .

After log n rounds of the above compression step, the final commitment keys  $\mathsf{ck}_{\log n}$  and  $\mathbf{a}_{\log n}$  have size 1, in which case the prover sends the witness  $w_{\log n}$  (of size 1) for the final relation. The verifier simply checks  $\mathsf{ck}_{\log n} \otimes \mathbf{w}_{\log n} = C_{\log n}$  and  $\mathbf{a}_{\log n} \cdot \mathbf{w}_{\log n} = v_{\log n}$ . Note that we view size 1 vectors as scalars here. While the recursive folding yields logarithmic argument size, in general the verifier incurs  $O(n)$  effort, as it is required to fold the commitment keys for the commitment and the linear form.

CSP with Logarithmic Verifier. To achieve logarithmic verification in the CSP framework, prior works such as [\[BMM](#page-40-7)+21, [GMN22\]](#page-41-8) have considered structured commitment keys. In this case, the verifier delegates the folding of the commitment keys to the prover, and can efficiently check if the final commitment key is correctly computed. The prior work has considered commitment keys with monomial structure i.e,  $\mathsf{ck} = (\chi_0, \ldots, \chi_{n-1})$  with  $\chi_i = x^i$  for some  $x \leftarrow \mathbb{F}$ . Moreover, folding of structured commitment keys which are encodings of monomials  $x^i$  in a group, e.g  $x^i \cdot g$  for a group generator  $g$  can also be verified in logarithmic time. In this work, we show that folding of commitment keys structured as Lagrange basis polynomials for the subgroup consisting of n roots of unity can be verified in  $O(\log n)$  time.