A Counterexample for Hilton–Johnson's Conjecture on List–Coloring of Graphs

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Abstract

In this paper a conjecture of A. Hilton and P. Johnson on list coloring of graphs is disproved. By modifying our counterexample, we also answer some other questions concerning Hall numbers.

1 Introduction

In this paper we consider finite undirected simple graphs. An *L–list coloring*, or L–coloring for short, of a graph G is an assignment of colors to the vertices such that each vertex v receives a color from a prescribed list $L(v)$ of colors and the adjacent vertices receive distinct colors. If an L–coloring exists then the following inequality, called *Hall's condition*, holds:

$$\sum_i t(H, L, i) \geq |V(H)|,$$

where H is an arbitrary subgraph of G and $t(H, L, i)$ denotes the maximum number of independent vertices of H having the color i in their lists, and i ranges over $\cup_{v \in V(H)} L(v)$. Clearly, to see if (G, L) satisfies Hall's condition, it is sufficient to check the inequality above for all induced subgraphs H of G.

Although Hall's condition is necessary for the existence of L–coloring, it is not sufficient unless we suppose that the sizes of lists are large enough. The following definitions appeared in [4].

Definition 1. The *Hall number* of a graph G, $h(G)$, is the smallest positive integer m such that, for every list assignment L of G with $|L(v)| \geq m$, $v \in V(G)$, if (G, L) satisfies Hall's condition, then G has an L–coloring.

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Definition 2. The *Hall index* of a graph G, $h'(G)$, is defined as $h(L(G))$, where $L(G)$ is the line graph of G and is defined to be the graph whose vertex set is in one–one correspondence with $E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are.

Because it suffices to check the inequality in Hall's condition for induced subgraphs of whatever graph is under consideration, in the case of list assignments to the edges of G, to see if Hall's condition is satisfied it suffices to check those subgraphs of the line graph of G which are line graphs of subgraphs of G. Consequently, we will permit ourselves a mild notational abuse: when L is a list assignment to the edges of G, H is a subgraph of G and i is a color, we will let $t(H, L, i)$ stand for the maximum number of independent edges of H having the color i on their lists. Thus, the requirement for Hall's condition to be satisfied is that

$$\sum_i t(H, L, i) \geq |E(H)|,$$

for all subgraphs H of G, in this case.

Definition 3. The *total Hall number* of G, $h_T(G)$, is defined as $h(T(G))$, where $T(G)$ is the total graph of G and is defined as the graph whose vertex set can be put in one–one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either two adjacent vertices, two adjacent edges, or one is an edge and the other is one of its end vertices.

We use the following lemmas frequently:

Lemma A. [3] *For a graph G we have $h(G) = 1$ iff every block of G is a complete graph.*

Lemma B. [4] *For a connected non-trivial graph G we have $h'(G) = 1$ iff G is a nontrivial tree or K_3.*

Lemma C. [5] *If H is an induced subgraph of G then $h(H) \leq h(G)$.*

Hilton and Johnson posed the following conjecture:

Conjecture. [2] *The Hall index of every graph is at most* 3.

In the next section we present a counterexample for this conjecture. Indeed, we show more, namely: For every integer k there exists a graph whose Hall index is greater than k.

2 The example

Consider the graph G_k shown in Figure 1 with the following list assignment:

$$L(ab) = L(ac) = \{1, 2, \cdots, k-1, k+1\}$$

$$L(bc) = L(bb_i) = L(cc_i) = \{1, 2, \cdots, k-1, k\}; \quad 1 \leq i \leq k-1.$$

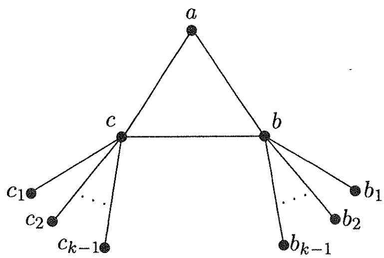

Figure 1. The graph G_k

We have $|L(e)| = k$ for all $e \in E(G)$. We claim that (G_k, L) satisfies Hall's condition but does not have any L–coloring. For G_k we have $t(G_k, L, k+1) = 1$ and $t(G_k, L, i) = 2$, for $1 \leq i \leq k$, thus $\sum_i t(G_k, L, i) = 2k+1 = |E(G_k)|$. Since for every edge e, $G_k - e$ has an L–coloring, Hall's condition holds for every proper subgraph of G_k. However, G_k does not have an L–coloring. Suppose on the contrary, ϕ is an L–coloring for G_k. Since $\deg_{G_k}(b) = k + 1$, all colors must appear in vertex b, thus $\phi(ab) = k + 1$. By a similar discussion we obtain $\phi(ac) = k + 1$, a contradiction.

As the following proposition shows, from every graph with Hall index greater than k we can construct a graph with Hall index greater than $k + 1$.

Proposition 1. *Let G be a non-trivial graph with Hall index k, $k \geq 2$, and let G^* be the graph which is obtained by joining to each vertex v of G a set S_v of k new independent vertices. Then $h'(G^*) > h'(G)$.*

Proof. Let L be a list assignment to the edges of G such that $|L(e)| \geq k - 1$, $e \in E(G)$, and (G, L) satisfies Hall's condition, but G has no L–coloring. Consider the graph G^* and assign to each of the new edges the same list of k new symbols and add one of these new symbols arbitrarily to each of the old lists on the edges of G. Denote this new list assignment by L^*. Obviously, $|L^*(e)| \geq k$, $e \in E(G^*)$. We show that (G^*, L^*) satisfies Hall's condition. Let H^* be a subgraph of G^*. We have $E(H^*) = E(H) \cup E^*$, where H is a subgraph of G and $E^* \subset E(G^*) \setminus E(G)$. Since for $e^* \in E^*$ and $e \in E(H)$, we have $L^*(e^*) \cap L(e) = \emptyset$, thus:

$$\begin{aligned} \textstyle\sum_i t(H^*, L^*, i) &\geq \textstyle\sum_r t(H, L, r) + \sum_s t(E^*, L^*, s) \\ &\geq |E(H)| + |E^*| \\ &= |E(H^*)|. \end{aligned}$$

Hence (G^*, L^*) satisfies Hall's condition. Obviously, if G^* has an L^*–coloring, then G has an L–coloring, which is a contradiction. □

One of the present authors [1] has shown that almost all graphs have Hall index greater than 2 and therefore by Proposition 1 we have infinitely many examples which disprove the conjecture. Of course, we already had infinitely many: $G_3, G_4, \cdots$. But Proposition 1 adds to the diversity of the examples.

3 Answers to some other questions

Recently, Hilton and Johnson [4], posed the following questions:

1. *Is it true that $h(G-e) \geq h(G) - 1$ for each edge e in $E(G)$?*

2. *Is it true that $h_T(G-e) \geq h_T(G) - 1$ for each edge e in $E(G)$?*

The graph G_k shown in Figure 1 also leads us to answer the above questions.

Proposition 2. *All of the following sets are unbounded:*

(a) $\{h(G) - h(G-e) |\ G$ is a simple graph$\}$.

(b) $\{h'(G) - h'(G-e) |\ G$ is a simple graph$\}$.

(c) $\{h_T(G) - h_T(G-e) |\ G$ is a simple graph$\}$.

Proof. (a) Let G_k be the graph constructed above and let $G = L(G_k)$. Note that G consists of two copies of K_{k+1}, say H_1 and H_2, which have a vertex v in common and there is an edge e which joins a vertex of $H_1 - v$ to a vertex of $H_2 - v$. Now $h(G) = h'(G_k) > k$ and since every block in $G - e$ is a complete graph, by Lemma A we have $h(G-e) = 1$.

(b) Let $G = G_k$ and $e = bc$. We have $h'(G) > k$ and since $G - e$ is a tree, by Lemma B we have $h'(G-e) = 1$.

(c) Let $G = G_k$ and $e = bc$. By Lemma C, $h_T(G) \geq h'(G) > k$. The degrees of vertices corresponding to c_i's, b_i's, and a in $T(G-e)$ are at most 4. By deleting these vertices from $T(G-e)$ we obtain a graph whose blocks are complete. Hence $h_T(G-e) \leq 5$. □

Finally, the referee informed us that M. Cropper from the University of West Virginia, also disproved the conjecture of Hilton and Johnson, by a very different means. He shows that $h'(K_{m,n}) > n-2$, for $n \geq m \geq 2$, $n \geq 3$, and that $h'(K_n) > n-2$, n odd, $n \geq 3$, and $h'(K_n) > n-3$, n even, $n \geq 4$. After submitting the first version of our paper we received a preprint of Hilton and Johnson [4] in which they show that the Petersen graph has Hall index 4.

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