Partition identities arising from involutions

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Abstract

We give a simple combinatorial proof of three identities of Warnaar. The proofs exploit involutions due to Franklin and Schur.

1 Introduction

One of the classical arguments in the combinatorial theory of partitions is Franklin's argument [1] establishing Euler's pentagonal number formula:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k+1)/2}. \tag{1}$$

This proceeds by interpreting the left side of (1) as a weighted generating function of partitions into distinct parts:

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{\lambda\in\mathcal{D}}(-1)^{n(\lambda)} q^{|\lambda|}.$$

Here $\mathcal{D}$ denotes the set of partitions with distinct parts, $|\lambda|$ is the number partitioned by λ and $n(\lambda)$ is the number of parts in λ. Franklin defines an involution σ defined on a "large" subset $\mathcal{D}' \subseteq \mathcal{D}$ with the property that $(-1)^{n(\sigma(\lambda))}q^{|\sigma(\lambda)|} = -(-1)^{n(\lambda)}q^{|\lambda|}$. Thus the sum of $(-1)^{n(\lambda)}q^{|\lambda|}$ over $\mathcal{D}'$ vanishes and Euler's formula (1) follows from noting that the sum of $(-1)^{n(\lambda)}q^{|\lambda|}$ over $\mathcal{D} - \mathcal{D}'$ is the right side of (1).

Later Schur [3] produced a proof, relying on a more complicated involution, of the Rogers-Ramanujan identities. Schur's involution later formed the basis of an explicit bijective proof due to Garsia and Milne [2] of the Rogers-Ramanujan identities.

In this paper we use Franklin's and Schur's involutions to prove bounded (polynomial rather than power series) versions of Euler's formula and the Rogers-Ramanujan identities.

Theorems 2 and 3 appear as the main theorem (Theorem 1.1) in [4]. Warnaar's proof of these results relies on an elaborate formal argument involving Bailey chains. He leaves the formula of Theorem 1 as an exercise for the reader. He also remarks that it "seems an extremely challenging problem to find a combinatorial proof of Theorem 1.1". This paper meets that challenge.

2 Franklin's involution

We adopt the standard q-series notation: for each integer $n \geq 0$ define $(a)_n = \prod_{j=0}^{n-1}(1 - aq^j)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ denote a partition, that is, a finite nonincreasing sequence of positive integers, $|\lambda| = \sum_{j=1}^{k} \lambda_j$, the number partitioned by λ, and $n(\lambda) = k$, the number of parts in λ. Let $\mathcal{D}$ denote the set of partitions having distinct parts, that is the set of λ with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. For nonempty $\lambda \in \mathcal{D}$ let $t(\lambda)$ denote the smallest part of λ and $s(\lambda)$ be the "slope" of λ, that is, the largest integer s such that $\lambda_s = \lambda_1 - s + 1 > 0$.

For $j \in \mathbf{Z}$ we define a partition $\pi_{(j)} \in \mathcal{D}$ as follows: $\pi_{(0)}$ is the empty partition, for $j > 0$, $\pi_{(j)} = (2j, 2j-1, \ldots, j+1)$ and $\pi_{(-j)} = (2j-1, 2j-2, \ldots, j)$. Then $|\pi_{(j)}| = j(3j+1)/2$ and $n(\pi_{(j)}) = |j|$.

Following Franklin [1] we define an involution σ on the set $\mathcal{D}' = \mathcal{D} - \{\pi_{(j)} : j \in \mathbf{Z}\}$ as follows:

- if $t(\lambda) \leq s(\lambda)$ remove the smallest part of λ and add 1 to each of the $t(\lambda)$ largest parts to yield $\sigma(\lambda)$;
- if $t(\lambda) > s(\lambda)$ subtract 1 from each of the $s(\lambda)$ largest parts of λ and create a new smallest part equal to $s(\lambda)$ to yield $\sigma(\lambda)$.

Then σ is an involution on $\mathcal{D}'$ and $(-1)^{n(\sigma(\lambda))}q^{|\sigma(\lambda)|} = -(-1)^{n(\lambda)}q^{|\lambda|}$.

Theorem 1 *The following identity holds for each integer $m \geq 0$:*

$$\sum_{t=0}^{\lfloor m/2 \rfloor} (-1)^t q^{t(2m-t+3)/2}(q^{t+1})_{m-2t} = \sum_{j=\lfloor -m/2 \rfloor}^{\lfloor m/2 \rfloor} (-1)^j q^{j(3j+1)/2}.$$

Proof Let $\mathcal{D}_m$ consist of the partitions in $\mathcal{D}$ with parts of size at most m. Then $\mathcal{D}_m \cap \mathcal{D}'$ is not invariant under σ. Suppose that $\lambda \in \mathcal{D}_m \cap \mathcal{D}'$ but $\sigma(\lambda) \notin \mathcal{D}_m \cap \mathcal{D}'$. In this case $\lambda_1 = m$ and $t(\lambda) \leq s(\lambda)$. Let $s = s(\lambda)$ and $t = t(\lambda) = s(\sigma(\lambda))$. Then λ contains a part $m - s + 1$ and so $m - t - 1 \geq m - s + 1 \geq t$. Were equality to hold throughout, then λ would equal $\pi_{(-t)} \notin \mathcal{D}'$. Hence $t \leq m/2$. Then $\sigma(\lambda) \in \mathcal{D}_{m,t}$, where $\mathcal{D}_{m,t}$ is the set of partitions $\lambda \in \mathcal{D}$ with largest part $m+1$, slope t and smallest part $> t$. Conversely if $\mu \in \mathcal{D}_{m,t} \cap \mathcal{D}'$, for some t, then $\sigma(\mu) \in \mathcal{D}_m$. The set $(\mathcal{D}_m \cup \bigcup_{t=1}^{\lfloor m/2 \rfloor} \mathcal{D}_{m,t}) \cap \mathcal{D}'$ is invariant under σ. It follows that

$$\sum_{\lambda \in \mathcal{D}_m \cup \bigcup_{t=1}^{\lfloor m/2 \rfloor} \mathcal{D}_{m,t}} (-1)^{n(\lambda)} q^{|\lambda|} = \sum_{j : \pi_{(j)} \in \mathcal{D}_m \cup \bigcup_{t=1}^{\lfloor m/2 \rfloor} \mathcal{D}_{m,t}} (-1)^j q^{j(3j+1)/2}. \qquad (2)$$

We now examine both sides of (2). The set $\mathcal{D}_m$ consists of all partitions in $\mathcal{D}$ with parts from $\{1, 2, \ldots, m\}$. Hence

$$\sum_{\lambda \in \mathcal{D}_m} (-1)^{n(\lambda)} q^{|\lambda|} = \prod_{j=1}^{m} (1 - q^j) = (q)_m.$$

The partitions in $\mathcal{D}_{m,t}$ must contain parts $m+1, m, m-1, \ldots, m+2-t$ and also a subset of $\{t+1, \ldots, m-t\}$. We have

$$\sum_{\lambda \in \mathcal{D}_{m,t}} (-1)^{n(\lambda)} q^{|\lambda|} = \prod_{j=m+2-t}^{m+1} (-q^j) \times \prod_{i=t+1}^{m-t} (1-q^i) = (-1)^t q^{t(2m+3-t)/2} (q^{t+1})_{m-2t}.$$

Thus

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}_m \cup \bigcup_{t=1}^{\lfloor m/2 \rfloor} \mathcal{D}_{m,t}} (-1)^{n(\lambda)} q^{|\lambda|} &= (q)_m + \sum_{t=1}^{\lfloor m/2 \rfloor} (-1)^t q^{t(2m+3-t)/2} (q^{t+1})_{m-2t} \\ &= \sum_{t=0}^{\lfloor m/2 \rfloor} (-1)^t q^{t(2m+3-t)/2} (q^{t+1})_{m-2t}. \end{aligned}$$

The partition $\pi_{(j)}$ lies in $\mathcal{D}_m$ if and only if $0 \le j \le m/2$ or $0 \ge j \ge (m-1)/2$, that is if and only if $\lfloor -m/2 \rfloor \le j \le \lfloor m/2 \rfloor$. If $j > 0$ and $\pi_{(j)} \in \mathcal{D}_{m,t}$, then $m+1 = 2j$ and $t = j$ so that $2t > m$. If $j > 0$ and $\pi_{(-j)} \in \mathcal{D}_{m,t}$, then $m+1 = 2j-1$ and $t = j$ so again $2t > m$. Hence

$$\sum_{j : \pi_{(j)} \in \mathcal{D}_m \cup \bigcup_{t=1}^{\lfloor m/2 \rfloor} \mathcal{D}_{m,t}} (-1)^j q^{j(3j+1)/2} = \sum_{j=\lfloor -m/2 \rfloor}^{\lfloor m/2 \rfloor} (-1)^j q^{j(3j+1)/2}.$$

Equating both sides of (2) gives

$$\sum_{t=0}^{\lfloor m/2 \rfloor} (-1)^t q^{t(2m+3-t)/2} (q^{t+1})_{m-2t} = \sum_{j=\lfloor -m/2 \rfloor}^{\lfloor m/2 \rfloor} (-1)^j q^{j(3j+1)/2}$$

as required. □

3 Schur's involution

Schur [3] produced a proof of the Rogers-Ramanujan identities using an involutive argument akin to Franklin's proof of Euler's formula. Let $\mathcal{R}$ denote the set of partitions in $\mathcal{D}$ having parts differing by at least 2. The first Rogers-Ramanujan identity states that

$$\sum_{\mu \in \mathcal{R}} q^{|\mu|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}.$$

Using Jacobi's triple product we see that this is equivalent to

$$\sum_{\mu \in \mathcal{R}} q^{|\mu|} = \frac{1}{(q)_\infty} \prod_{n=1}^{\infty} (1-q^{5n-3})(1-q^{5n-2})(1-q^{5n}) = \frac{1}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+1)/2}$$

and so to

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+1)/2} = (q)_\infty \sum_{\mu \in \mathcal{R}} q^{|\mu|} = \sum_{\lambda \in \mathcal{D}} \sum_{\mu \in \mathcal{R}} (-1)^{n(\lambda)} q^{|\lambda|+|\mu|}. \tag{3}$$

Hence we define

$$w((\lambda,\mu)) = (-1)^{n(\lambda)}q^{|\lambda|+|\mu|}$$

for $(\lambda,\mu) \in \mathcal{D}\times\mathcal{R}$. Let $\rho_{(j)} = (2j-1, 2j-3, \ldots, 1) \in \mathcal{R}$, and let $\mathcal{E} = \{(\pi_{(j)}, \rho_{(|j|)}) : j \in \mathbf{Z}\}$. Note that $w((\pi_{(j)}, \rho_{(|j|)})) = (-1)^j q^{j(5j+1)/2}$. Schur defined an involution τ on $(\mathcal{D}\times\mathcal{R}) - \mathcal{E}$ with the property that $w(\tau(\lambda,\mu)) = -w(\lambda,\mu)$. The formula (3) is an immediate consequence of the existence of such a τ.

We shall apply τ to the set of pairs $(\lambda,\mu) \in \mathcal{D}\times\mathcal{R}$ in which each part of λ and μ is at most m. Let $\mathcal{R}_m = \mathcal{D}_m \cap \mathcal{R}$: the set of partitions in $\mathcal{R}$ having parts of size at most m. Define

$$e_{m+2}(q) = \sum_{\mu\in\mathcal{R}_m} q^{|\mu|}.$$

The polynomials $e_{m+2}(q)$ were introduced by Schur and satisfy $e_2(q) = 1$, $e_3(q) = 1+q$ and $e_{m+2}(q) = e_{m+1}(q) + q^m e_m(q)$ for $m \geq 2$.

Theorem 2 *The following identity holds for each integer $m \geq 0$:*

$$\sum_{s=0}^{\lfloor m/2\rfloor} (-1)^s q^{s(4m-3s+5)/2}(q^{s+1})_{m-2s} e_{m-2s+2}(q) = \sum_{j=\lfloor -m/2\rfloor}^{\lfloor m/2\rfloor} (-1)^j q^{j(5j+1)/2}.$$

Proof We apply Schur's involution τ to $\mathcal{D}_m\times\mathcal{R}_m$ as best we can. For the definition of τ we follow the description of Garsia and Milne [2] who used τ to construct a bijective proof of the Rogers-Ramanujan identities.

Divide the pairs in $(\mathcal{D}\times\mathcal{R}) - \mathcal{E}$ into three disjoint classes:

- the class $\mathcal{T}$ contains those (λ,μ) with either λ or μ empty, and those with $\lambda_1 - \mu_1 \notin \{0,1\}$,
- the class $\mathcal{A}$ contains those (λ,μ) with $\lambda_1 - \mu_1 = 1$,
- the class $\mathcal{B}$ contains those (λ,μ) with $\lambda_1 - \mu_1 = 0$.

The involution τ will preserve $\mathcal{T}$ and interchange $\mathcal{A}$ and $\mathcal{B}$. It will also negate weights: if $\tau((\lambda,\mu)) = (\lambda',\mu')$ then $w((\lambda',\mu')) = -w((\lambda,\mu))$. For $(\lambda,\mu) \in \mathcal{T}$, there is a unique largest part in λ and μ; τ simply transfers this part to the other partition. Clearly τ is a weight-negating involution on $\mathcal{T}$.

We divide each of the class $\mathcal{A}$ and $\mathcal{B}$ into three subclasses. For $(\lambda,\mu) \in \mathcal{A}\cup\mathcal{B}$ we let p be the smallest part of λ, q the slope of λ and r the *2-slope* of μ, the largest integer r such that $\mu_r = \mu_1 - 2(r-1) > 0$. Then

- the class $\mathcal{A}_1$ contains those $(\lambda,\mu) \in \mathcal{A}$ with $\min(p,q,r) = p$,
- the class $\mathcal{A}_2$ contains those $(\lambda,\mu) \in \mathcal{A}$ with $\min(p,q,r) = q < p$,
- the class $\mathcal{A}_3$ contains those $(\lambda,\mu) \in \mathcal{A}$ with $\min(p,q,r) = r < \min(p,q)$,
- the class $\mathcal{B}_1$ contains those $(\lambda,\mu) \in \mathcal{B}$ with $\min(p,q,r) = p$,
- the class $\mathcal{B}_2$ contains those $(\lambda,\mu) \in \mathcal{B}$ with $\min(p,q,r) = r < p$,

- the class $\mathcal{B}_3$ contains those $(\lambda, \mu) \in \mathcal{B}$ with $\min(p, q, r) = q < \min(p, r)$.

The involution τ will interchange $\mathcal{A}_1$ with $\mathcal{B}_2$, $\mathcal{A}_2$ with $\mathcal{B}_1$ and $\mathcal{A}_3$ with $\mathcal{B}_3$.

We describe its action on each $\mathcal{A}_j$. It is then straightforward to check that $\tau : \mathcal{A}_1 \to \mathcal{B}_2$, $\tau : \mathcal{A}_2 \to \mathcal{B}_1$ and $\tau : \mathcal{A}_3 \to \mathcal{B}_3$ are all weight-negating bijections.

Let $(\lambda, \mu) \in \mathcal{A}_1$. Then we obtain $\tau((\lambda, \mu)) = (\lambda', \mu')$ by removing the smallest part p from λ and adding 1 to the p largest parts of μ.

Let $(\lambda, \mu) \in \mathcal{A}_2$. Then $\tau((\lambda, \mu)) = (\lambda', \mu)$ where $\lambda' = \sigma(\lambda)$ and σ is the Franklin involution.

Let $(\lambda, \mu) \in \mathcal{A}_3$. Then we obtain $\tau((\lambda, \mu)) = (\lambda', \mu')$ by subtracting 1 from the r largest parts of μ, then moving the largest part of λ to μ and finally adding 1 to the r largest parts of λ. That is $\lambda' = (\lambda_2 + 1, \lambda_3 + 1, \ldots, \lambda_{r+1} + 1, \lambda_{r+2}, \ldots)$ and $\mu' = (\lambda_1, \mu_1 - 1, \mu_2 - 1, \ldots, \mu_r - 1, \mu_{r+1}, \ldots)$.

Let $\mathcal{P}_m = \mathcal{D}_m \times \mathcal{R}_m$. Then

$$\sum_{(\lambda,\mu)\in\mathcal{P}_m} w((\lambda, \mu)) = (q)_m e_{m+2}(q).$$

For $1 \leq s \leq m/2$ let $\mathcal{P}_{m,s}$ denote the set of pairs $(\lambda, \mu) \in \mathcal{D} \times \mathcal{R}$ where $\lambda_1 = m + 1$, $\mu_1 = m$, λ has slope s and smallest part $> s$ and μ has 2-slope $\geq s$. The $\lambda \in \mathcal{D}$ with $\lambda_1 = m + 1$ having slope s and smallest part $> s$ must have the s parts $m+1, m, \ldots, m-s+2$, and a subset of $\{s+1, s+2, \ldots, m-s\}$. It follows that the sum of $(-1)^{n_\lambda} q^{|\lambda|}$ over these λ is

$$\prod_{j=m-s+2}^{m+1} (-q^j) \times \prod_{i=s+1}^{m-s} (1 - q^i) = (-1)^s q^{s(2m-s+3)/2} (q^{s+1})_{m-2s}.$$

The μ in $\mathcal{R}$ with $\mu_1 = m$ and having slope at least s have parts $m, m-2, \ldots, m-s+2$, together with various distinct parts $\leq m - s$ differing by at least 2. It follows that the sum of $q^{|\mu|}$ over these μ is

$$q^m q^{m-2} \cdots q^{m-2s+2} e_{m-2s+2}(q) = q^{s(m-s+1)} e_{m-2s+2}(q).$$

Hence

$$\begin{aligned} \sum_{(\lambda,\mu)\in\mathcal{P}_{m,s}} w((\lambda, \mu)) &= (-1)^s q^{s(2m-s+3)/2} (q^{s+1})_{m-2s} q^{s(m-s+1)} e_{m-2s+2}(q) \\ &= (-1)^s q^{s(4m-3s+5)/2} (q^{s+1})_{m-2s} e_{m-2s+2}(q). \end{aligned}$$

Let $\mathcal{Q}_m = \mathcal{P}_m \cup \bigcup_{s=1}^{\lfloor m/2 \rfloor} \mathcal{P}_{m,s}$. Then

$$\sum_{(\lambda,\mu)\in\mathcal{Q}_m} w((\lambda, \mu)) = \sum_{s=0}^{\lfloor m/2 \rfloor} (-1)^s q^{s(4m-3s+5)/2} (q^{s+1})_{m-2s} e_{m-2s+2}(q).$$

We claim that $\mathcal{Q}_m - \mathcal{E}$ is closed under τ. If $(\lambda, \mu) \in \mathcal{P}_m$ but $(\lambda', \mu') = \tau((\lambda, \mu)) \notin \mathcal{P}_m$ then $(\lambda, \mu) \in \mathcal{B}_1$ and so $(\lambda', \mu') \in \mathcal{A}_2$. Then $\lambda'_1 = m + 1$, $\mu'_1 = m$ and if s is the slope of λ' then all parts of λ' exceed s while the slope of μ' is at least s. Hence

$\tau((\lambda,\mu)) \in \mathcal{P}_{m,s}$. On the other hand if $(\lambda,\mu) \in \mathcal{P}_{m,s} - \mathcal{E}$, then $(\lambda,\mu) \in \mathcal{A}_2$ and so $\tau((\lambda,\mu)) \in \mathcal{P}_m$. Hence

$$\sum_{(\lambda,\mu)\in\mathcal{Q}_m-\mathcal{E}} w((\lambda,\mu)) = 0.$$

The elements of $\mathcal{Q}_m \cap \mathcal{E}$ are the $(\pi_j, \rho_{|j|})$ with $\lfloor -m/2 \rfloor \le j \le \lfloor m/2 \rfloor$. Hence

$$\begin{aligned}\sum_{(\lambda,\mu)\in\mathcal{Q}_m} w((\lambda,\mu)) &= \sum_{(\lambda,\mu)\in\mathcal{Q}_m\cap\mathcal{E}} w((\lambda,\mu)) \\ &= \sum_{j=\lfloor -m/2\rfloor}^{\lfloor m/2\rfloor} w((\pi_j,\rho_{|j|})) \\ &= \sum_{j=\lfloor -m/2\rfloor}^{\lfloor m/2\rfloor} (-1)^j q^{j(5j+1)/2}\end{aligned}$$

and the theorem follows. □

The second Rogers-Ramanujan identity states that

$$\sum_{\mu\in\mathcal{R}} q^{|\mu|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}$$

where $\mathcal{R}'$ denotes the set of $\mu \in \mathcal{R}$ with all parts at least 2. Using the Jacobi triple product, this is equivalent to

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k(5k+3)/2} = (q)_\infty \sum_{\mu\in\mathcal{R}'} q^{|\mu|} = \sum_{\lambda\in\mathcal{D}} \sum_{\mu\in\mathcal{R}'} (-1)^{n(\lambda)} q^{|\lambda|+|\mu|}. \tag{4}$$

There is also a bounded version of (4). To state it we define

$$d_{m+2}(q) = \sum_{\mu\in\mathcal{R}'_m} q^{|\mu|}$$

where $\mathcal{R}'_m = \mathcal{D}_m \cap \mathcal{R}'$ is the set of partitions in $\mathcal{R}'$ having parts of size at most m.

Theorem 3 *The following identity holds for each integer $m \ge 0$:*

$$\sum_{s=0}^{\lfloor m/2\rfloor} (-1)^s q^{s(4m-3s+5)/2} (q^{s+1})_{m-2s} d_{m-2s+2}(q) = \sum_{j=\lfloor -m/2\rfloor}^{\lfloor m/2\rfloor} (-1)^j q^{j(5j+3)/2}.$$

Proof This proof follows that of Theorem 2 *mutatis mutandis* so we do not give it in detail. We let $\rho'_{(j)} = (2j, 2j-2, \ldots, 2)$ and let $\mathcal{E}'$ be the set of pairs $(\pi_{(j)}, \rho'_{(j)})$ with $j \ge 0$ and $(\pi_{(j)}, \rho'_{(-1-j)})$ with $j < 0$. The map τ is an involution on $(\mathcal{D} \times \mathcal{R}') - \mathcal{E}'$. The proof now follows that of Theorem 2 exactly. □

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