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A characterisation of cubic parity graphs

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Abstract

A graph is $\mathbf{Z}_m$*-well-covered* if all maximal independent sets have the same cardinality modulo m. $\mathbf{Z}_m$-well-covered graphs generalise *well-covered* graphs, those in which all independent sets have the same cardinality. $\mathbf{Z}_2$-well-covered graphs are also called *parity* graphs. A characterisation of cubic well-covered graphs was given by Campbell, Ellingham and Royle. Here we extend this to a characterisation of cubic $\mathbf{Z}_m$-well-covered graphs for all integers $m \geq 2$; the most interesting case is $m = 2$, cubic parity graphs. Our main technique involves minimal non-well-covered graphs, and allows us to build our characterisation as an extension of the existing characterisation of cubic well-covered graphs.

1 Introduction

All graphs in this paper are simple and finite.

A graph is *well-covered* if all maximal independent sets have the same cardinality, and $\mathbf{Z}_m$*-well-covered*, for some $m \geq 2$, if all maximal independent sets have the same cardinality modulo m. Well-covered graphs were introduced by Plummer [11],

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and Plummer has written a useful survey paper [12]. $\mathbf{Z}_2$-well-covered graphs were introduced by Finbow and Hartnell [8, 9] under the name of *parity* graphs.

Both well-covered and $\mathbf{Z}_m$-well-covered graphs are special cases of a more general concept [4]. Given a graph whose vertices are weighted by elements of an abelian group A, we say the A-weighted graph is *well-covered* if all maximal independent sets have the same total weight. If A is a ring with identity (regarded as an additive group) then a graph is *A-well-covered* if it is well-covered when we give every vertex the weight 1 from A. Since only the additive subgroup of A generated by 1 matters here, we usually take A to be a cyclic group $\mathbf{Z}$ or $\mathbf{Z}_m$. '$\mathbf{Z}$-well-covered' just means well-covered in Plummer's original sense, and this definition of '$\mathbf{Z}_m$-well-covered' is equivalent to the previous one.

Cubic well-covered graphs were characterised by Campbell, Ellingham and Royle [2], following earlier work by Campbell and Plummer [1, 3]. In this paper we build on the characterisation of cubic well-covered graphs to obtain a characterisation of cubic $\mathbf{Z}_m$-well-covered graphs for all $m \geq 2$. Most of the work is for the case $m = 2$, so our main result is a characterisation of cubic parity graphs. This research is of interest from three different angles: the original question, the techniques used, and the results obtained.

First, characterising $\mathbf{Z}_m$-well-covered cubic graphs is a natural question. Determining whether a graph is well-covered is a co-NP-complete problem [7, 14], even for $K_{1,4}$-free graphs [6]. The same proof shows that for each cyclic group A, determining whether a graph is A-well-covered is co-NP-complete for $K_{1,4}$-free graphs. Therefore, it is unlikely that a general characterisation of A-well-covered graphs will be found, so it is of interest to find special classes of graphs where the A-well-covered graphs can be characterised. When $A = \mathbf{Z}$, two simple restrictions that allow characterisations involve the girth or the vertex degrees. When $A = \mathbf{Z}_m$, girth restrictions have been investigated [5, 9], but degree restrictions have not. (It is known that for given d and A there is a polynomial time algorithm to determine whether a graph of maximum degree at most d is A-well-covered [4], but this does not give the overall structure of the A-well-covered graphs.) Cubic graphs are the simplest interesting class of graphs with a degree restriction, so they are a natural subject for investigation.

Second, the proofs in this paper demonstrate the usefulness of the technique of 'minimal non-well-covered graphs.' This idea, developed in [4, 13, 15], and equivalent to the 'critical nongreedy hypergraphs' of [6], provides a 'minimal forbidden structure' approach to well-covered graphs. Minimal forbidden structures are an important general concept in graph theory. The most famous example is Robertson and Seymour's theory of graph minors, where many classes of graphs are characterised by forbidden minors. Another example is the characterisation of line graphs in terms of forbidden induced subgraphs. We feel that minimal non-well-covered graphs are an important tool that should be more widely used. In this paper they allow us to characterise cubic $\mathbf{Z}_m$-well-covered graphs by building on the known characterisation of cubic well-covered graphs, rather than developing this characterisation independently.

Third, the results in this paper shed some light on the difference between well-covered graphs and $\mathbf{Z}_m$-well-covered graphs. The interesting case for our results is

$m = 2$. Connected cubic well-covered graphs, with six exceptions, are made by joining together three basic building blocks (see Theorem 2.7). Since every well-covered graph is also $\mathbf{Z}_2$-well-covered, one might expect that the cubic $\mathbf{Z}_2$-well-covered graphs extend the cubic well-covered graphs by using a few more building blocks together with the known ones. However, this turns out not to be so. Cubic $\mathbf{Z}_2$-well-covered graphs seem to divide sharply into those that are well-covered and those that are not, with different structures in each case.

In Section 2 we introduce minimal non-well-covered graphs, deduce some general results using them, and apply these results to cubic graphs. This allows us to dispose of the problem of characterising cubic $\mathbf{Z}_m$-well-covered graphs for $m \geq 3$ in Theorem 2.8, and gives us some information for the case $m = 2$. In Section 3 we introduce three families of graphs and one special graph, which will turn out to be exactly the connected cubic graphs that are $\mathbf{Z}_2$-well-covered but not well-covered. In Section 4 we prove this, using a fairly lengthy case analysis. Our main results are stated as Theorem 4.3 and Corollary 4.4. In Section 5 we give some concluding remarks.

2 Minimal non-well-covered graphs

We begin by summarising some ideas from Section 2 of [4], to which we refer the reader for details and history. As in Section 1, A denotes a cyclic group $\mathbf{Z}$ or $\mathbf{Z}_m$, $m \geq 2$.

Let S be a set of vertices in a graph G. The subgraph of G induced by S is denoted $G[S]$. The *neighbourhood of* S is the set $N_G(S) = \{u : uv \in E(G) \text{ for some } v \in S\}$. The *closed neighbourhood* of S is the set $N_G[S] = S \cup N_G(S)$. We often abbreviate $N_G(S)$ to $N(S)$, $N_G[S]$ to $N[S]$, $N(\{v\})$ to $N(v)$ and $N[\{v\}]$ to $N[v]$. We say that S *dominates* another set of vertices T if $T \subseteq N[S]$. Given a subgraph H of G, we say G *encloses* H, and write $H \leq G$, if there is an independent set I of vertices of G so that $H = G - N_G[I]$. In practice we often use a different but equivalent definition: $H \leq G$ if there is an independent set I' such that H is a component of $G - N_G[I']$. The relation '$\leq$' is a partial order on graphs. We sometimes abuse notation by writing $H' \leq G$ when we mean that H' is isomorphic to some H with $H \leq G$; it should be clear when we are doing this.

The importance of this partial order is that if G is A-well-covered and $H \leq G$, then H is also A-well-covered. We use this as a tool to examine the structure of an A-well-covered graph. If H is *not* A-well-covered, then an A-well-covered graph G cannot contain any configuration that would imply that $H \leq G$. In fact, we can restrict the graphs H that we use here. H is a *minimal non-A-well-covered* graph if H is not A-well-covered, but if $J \leq H$ and $J \neq H$, then J is A-well-covered. If $A = \mathbf{Z}$, we just refer to H as *minimal non-well-covered*.

Observation 2.1 (special case of [4, Observation 2.2])**.** *A graph G is not A-well-covered if and only if there is a minimal non-A-well-covered graph H with $H \leq G$.*

This observation is more useful if we know something about the structure of minimal non-A-well-covered graphs. The following theorem provides some important information. It generalises results for $A = \mathbf{Z}$ obtained independently by Caro, Sebő and Tarsi [6] (who used a hypergraph formulation), Ramey [13], and Zverovich [15]. Here $+$ denotes join, and $\alpha(G)$ denotes the cardinality of a maximum independent set in G. Two integers p, q are *equal in* A if they have equal images under the homomorphism $\mathbf{Z} \to A$ that maps $1 \in \mathbf{Z}$ to $1 \in A$. Otherwise they are *distinct in* A. 'Equal in $\mathbf{Z}_m$' means equivalent modulo m, and 'equal in $\mathbf{Z}$' just means equal.

Theorem 2.2 (special case of [4, Theorem 2.3])**.** *A graph H is minimal non-A-well-covered if and only if there exist A-well-covered graphs $H_1, H_2, \ldots, H_k$ such that $H = H_1 + H_2 + \ldots + H_k$ and $\alpha(H_i)$ and $\alpha(H_j)$ are distinct in A for some i and j.*

Now we apply Observation 2.1 and Theorem 2.2 to prove some results for graphs of bounded degree. As usual, $\Delta(G)$ denotes the maximum degree of the graph G.

Lemma 2.3. *Suppose $\Delta(H) \leq m$, where $m \geq 2$. Then H is minimal non-$\mathbf{Z}_m$-well-covered if and only if H is minimal non-well-covered.*

Proof. If H cannot be expressed as a join, then by Theorem 2.2 H is neither minimal non-$\mathbf{Z}_m$-well-covered nor minimal non-well-covered, and the lemma holds. So, suppose that $H = H_1 + \ldots + H_k$ for some $k \geq 2$. Since $\Delta(H) \leq m$, each graph H_i must have at most m vertices. But then each H_i is $\mathbf{Z}_m$-well-covered if and only if it is ($\mathbf{Z}$-)well-covered, and moreover $\alpha(H_i)$ and $\alpha(H_j)$ are distinct in $\mathbf{Z}_m$ if and only if they are distinct in $\mathbf{Z}$. Therefore, H has a join decomposition satisfying Theorem 2.2 with $A = \mathbf{Z}_m$ if and only if it has one with $A = \mathbf{Z}$, and the lemma follows. ∎

Corollary 2.4. *Suppose $\Delta(G) \leq m$, where $m \geq 2$. Then G is $\mathbf{Z}_m$-well-covered if and only if G is well-covered.*

Proof. For any $H \leq G$, $\Delta(H) \leq m$, and so by Lemma 2.3 H is minimal non-$\mathbf{Z}_m$-well-covered if and only if it is minimal non-well-covered. The result then follows by applying Observation 2.1 with $A = \mathbf{Z}_m$ and $A = \mathbf{Z}$. ∎

Lemma 2.5. *Suppose $\Delta(H) \leq m + 1$, and H is a minimal non-well-covered graph that is $\mathbf{Z}_m$-well-covered. Then $H \cong K_{1,m+1}$.*

Proof. By Theorem 2.2 we can write $H = H_1 + \ldots + H_k$ where $k \geq 2$, each H_i is well-covered, and $\alpha(H_i) \neq \alpha(H_j)$ for some i, j. Since $\Delta(H) \leq m + 1$, each H_i has at most $m + 1$ vertices. If each H_i has no more than m vertices then H is minimal non-$\mathbf{Z}_m$-well-covered by the argument from the proof of Lemma 2.3, a contradiction. Therefore we may assume that H_1 has $m + 1$ vertices. Since $\Delta(H) \leq m + 1$, we must have $k = 2$ and H_2 must have no edges. Moreover, if H_2 has $m + 1$ vertices then $\Delta(H) \leq m + 1$ would mean that H_1 has no edges and $H \cong K_{m+1,m+1}$, which is well-covered. Therefore, H_2 has at most m vertices. Now $\alpha(H_1)$ and $\alpha(H_2)$ must be distinct in $\mathbf{Z}$ since H is minimal non-well-covered, but equal in $\mathbf{Z}_m$ since H is not minimal non-$\mathbf{Z}_m$-well-covered. Thus, one must be 1 and the other must be $m + 1$, which implies that $H \cong K_{1,m+1}$. ∎

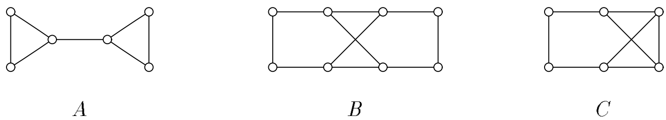


Figure 2.1

Corollary 2.6. *Suppose $\Delta(G) = m + 1$, where $m \geq 2$. If G is $\mathbf{Z}_m$-well-covered but not well-covered then $K_{1,m+1} \leq G$.*

Proof. By Observation 2.1 there is a minimal non-well-covered H with $H \leq G$. Since H is $\mathbf{Z}_m$-well-covered and $\Delta(H) \leq \Delta(G) = m + 1$, by Lemma 2.5 we have $H \cong K_{1,m+1}$. ∎

Now we apply the above results to cubic graphs. First we recall the characterisation of well-covered cubic graphs from [2]. Let A, B and C be the graphs depicted in Figure 2.1. Define a *terminal pair* to be a pair of adjacent degree two vertices. Let $\mathcal{W}$ denote the class of cubic graphs constructed as follows. Given a collection of copies of A, B and C, join every terminal pair by two edges to a terminal pair in another (possibly the same) graph, so that the result is cubic. Then we have the following characterisation (which is stated incorrectly in the survey paper [12, p. 272]).

Theorem 2.7 ([2, Theorem 6.1])**.** *Let G be a connected cubic graph. Then G is well-covered if and only if one of the following is true.*

(i) *$G \in \mathcal{W}$; or*

(ii) *G is one of six exceptional graphs: K_4, $K_{3,3}$, $K_{3,3}^*$, $C_5 \times K_2$, Q^{**} or P_{14} (see [2] for details).*

Corollary 2.4 yields the following.

Theorem 2.8. *Suppose $m \geq 3$. Then a cubic graph is $\mathbf{Z}_m$-well-covered if and only if it is well-covered. Therefore, G is a connected $\mathbf{Z}_m$-well-covered cubic graph if and only if (i) or (ii) of Theorem 2.7 holds.*

For cubic $\mathbf{Z}_2$-well-covered graphs we will make use of the following two results. The first is a special case of Corollary 2.6, and the second follows easily from Observation 2.1 and Theorem 2.2.

Lemma 2.9. *If a cubic graph G is $\mathbf{Z}_2$-well-covered but not well-covered, then $K_{1,3} \leq G$.*

Lemma 2.10. *The minimal non-$\mathbf{Z}_2$-well-covered graphs with maximum degree at most 3 are $P_3 = K_{1,2} = K_1 + 2K_1$, $L = K_1 + (K_2 \cup K_1)$, $K_{1,1,2} = K_1 + K_1 + 2K_1$, and $K_{2,3} = 2K_1 + 3K_1$. Thus, a cubic graph is $\mathbf{Z}_2$-well-covered if and only if it encloses none of these four graphs.*

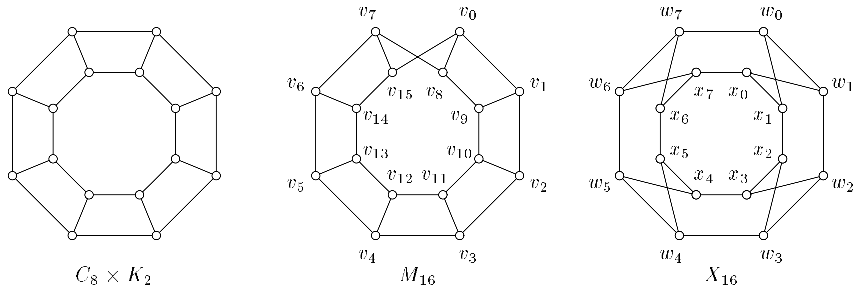


Figure 3.1

3 Families of cubic parity graphs

Every well-covered graph is also $\mathbf{Z}_m$-well-covered for all m, and in particular is $\mathbf{Z}_2$-well-covered. Therefore, the cubic $\mathbf{Z}_2$-well-covered graphs include all the cubic well-covered graphs. In this section we introduce three families of cubic graphs that (except for some small members) are $\mathbf{Z}_2$-well-covered but not well-covered. We also introduce one special graph with this property.

The first family consists of the *prisms* $C_m \times K_2$, $m \geq 3$. The second family consists of the *twisted prisms* or *Möbius ladders* M_{2m}, $m \geq 2$. M_{2m} can be defined as a cycle $v_0 v_1 v_2 \ldots v_{2m-1} v_0$ plus chords $v_i v_{i+m}$, $0 \leq i \leq m-1$. The third family consists of graphs X_{4k}, $k \geq 2$, consisting of two cycles $w_0 w_1 w_2 \ldots w_{2k-1} w_0$ and $x_0 x_1 x_2 \ldots x_{2k-1} x_0$ together with the edges $w_{2i} x_{2i+1}$ and $w_{2i+1} x_{2i}$, $0 \leq i \leq k-1$. In Figure 3.1 we show the 16-vertex members of all three families.

Theorem 3.1. (i) *The following graphs are well-covered (where we use the notation of* [2] *for graphs in* $\mathcal{W}$*):* $C_3 \times K_2 \cong -A- \in \mathcal{W}$, $C_5 \times K_2$, $M_4 \cong K_4$, $M_6 \cong K_{3,3}$, $M_8 \cong -B- \in \mathcal{W}$.

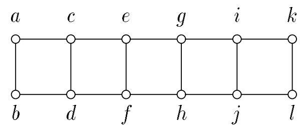
(ii) *The following graphs are* $\mathbf{Z}_2$*-well-covered but not well-covered:* $C_4 \times K_2 \cong X_8$, $C_m \times K_2$ *for* $m \geq 6$, M_{2m} *for* $m \geq 5$, *and* X_{4k} *for* $k \geq 3$.

Proof. (i) can be checked easily against Theorem 2.7.

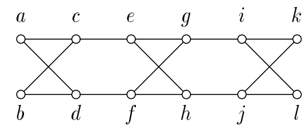
(ii) It is easy to verify that the maximal independent sets have cardinalities 2 and 4 in $C_4 \times K_2 \cong X_8$, and 3 and 5 in M_{10}, so these two graphs satisfy the theorem. We may therefore restrict our attention to the graphs with at least 12 vertices.

To prove they are $\mathbf{Z}_2$-well-covered we need only show that these graphs enclose none of the four graphs of Lemma 2.10: P_3, $L = K_1 + (K_2 \cup K_1)$, $K_{1,1,2}$ and $K_{2,3}$. However, it is clear that none of the graphs in (ii) contain induced subgraphs isomorphic to L, $K_{1,1,2}$ or $K_{2,3}$, so it suffices to show that these graphs do not enclose P_3. To prove they are not well-covered it suffices to prove that they do enclose the non-well-covered graph $K_{1,3}$.

Let G be either $C_m \times K_2$ or M_{2m} where $m \geq 6$. Then G has the subgraph shown in Figure 3.2(a), which is induced except that G possibly has edges ak, bl or al, bk.



(a)



(b)

Figure 3.2

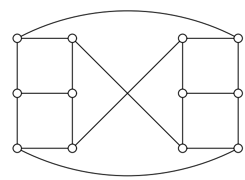


Figure 3.3: R_{12}

All P_3's in G are equivalent by an automorphism of G to either dfh or efh. Suppose that $dfh \leq G$, so that $G - N[I]$ has dfh as a component for some independent set I. Now I cannot contain c, e or g, but must contain a vertex other than f adjacent to e, which is impossible. Suppose that $efh \leq G$, so that $G - N[I]$ has efh as a component for some I. Then I cannot contain c or d, but must contain a neighbour of c other than e and a neighbour of d other than f. Therefore I contains both a and b, which is impossible. Thus, G does not enclose P_3. On the other hand, by taking $I = \{a, i\}$ we see that G does enclose a $K_{1,3}$ with edges df, ef, hf.

Now let G be X_{4k} where $k \geq 3$. Then G has the subgraph shown in Figure 3.2(b), which is induced except that G possibly has edges ak, bl. All P_3's in G are equivalent by an automorphism of G to either dfh or gfh. Suppose that $dfh \leq G$, so that dfh is a component of $G - N[I]$ for some independent set I. Now I cannot contain e, g or j. I must contain a neighbour of g other than f, so $i \in I$, but I must also contain a neighbour of j other than h, so either k or l must be in I, which is impossible. Suppose that $gfh \leq G$, so that gfh is a component of $G - N[I]$ for some independent set I. Now I cannot contain d or e. I must contain a neighbour of e other than g or h, so $c \in I$, but I must also contain a neighbour of d other than f, so either a or b must be in I, which is impossible. Thus, G does not enclose P_3. On the other hand, by taking $I = \{c, k\}$ we see that G does enclose a $K_{1,3}$ with edges df, gf, hf. ∎

There is one other cubic graph that is $\mathbf{Z}_2$-well-covered but not well-covered that we mention here. Let R_{12} be the graph of Figure 3.3. It is not difficult to show that the maximal independent sets in R_{12} have cardinalities 4 and 6.

Figure 4.1

4 Characterisation of cubic parity graphs

In this section we characterise the cubic $\mathbf{Z}_2$-well-covered graphs that are not well-covered. The two main tools are Lemmas 2.9 and 2.10. Lemma 2.9 guarantees that a $\mathbf{Z}_2$-well-covered graph that is not well-covered encloses a $K_{1,3}$. Define a *star vertex* of G to be the degree 3 vertex of some $K_{1,3}$ enclosed in G. Our main argument is a case-by-case analysis to determine the structure near a star vertex.

Theorem 4.1. *Suppose G is a connected cubic graph that is $\mathbf{Z}_2$-well-covered but not well-covered. Let v be a star vertex of G. Then either G is isomorphic to one of $C_4 \times K_2 \cong X_8$, M_{10}, or R_{12}, or else v is one of two types:*

(A) *There is an induced subgraph as shown in Figure 4.1(a) containing v in the specified position.*

(B) *There is an induced subgraph as shown in Figure 4.1(b) containing v in the specified position.*

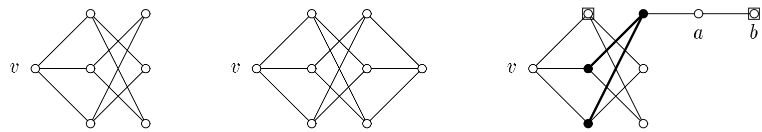
Moreover, in both cases every neighbour of v is also a star vertex of the same type as v.

Proof. First we introduce some notation. N_i denotes the set of vertices at distance i from v, and $n_i = |N_i|$. m_i denotes the number of edges with both ends in N_i, and m_{ij} denotes the number of edges with one end in N_i and the other in N_j. Since v is a star vertex, $G[N[v]]$ is isomorphic to $K_{1,3}$, so that $m_1 = 0$, i.e., the neighbours of v are pairwise nonadjacent. Moreover, there is an independent set I such that $G[N[v]]$ is a component of $G - N[I]$. We choose such an I with a minimum number of elements. $I \cap N_3$ must dominate N_2, and minimality implies that $I \subseteq N_3$.

Write $I = \{u_1, u_2, \ldots, u_p\}$. Since p is minimum, each $u_i \in I$ must have at least one *private neighbour* in N_2: a vertex of N_2 adjacent to u_i but to no other element of I. Let $P(u_i)$ denote the set of private neighbours of u_i in N_2, i.e., $P(u_i) = N_2 - \bigcup_{j \neq i} N[u_j]$. $P(u_i)$ is nonempty and disjoint from $P(u_j)$, $j \neq i$. Let $\mathcal{P} = \{P(u_1), P(u_2), \ldots, P(u_p)\}$.

Lemma 4.2. *We cannot have $P(u_i) = \{t\}$ where t has exactly one neighbour t' in N_1. We also cannot have $P(u_i) = \{s, t\}$ where s and t both have exactly one neighbour in N_1, which is the same vertex t' in each case.*

Proof. Suppose otherwise. Then, by definition of $P(u_i)$, $I' = I - \{u_i\} \cup \{t\}$ is an independent set that dominates all of N_2, apart from possibly s in the second case. It also dominates $t' \in N_1$, but no other elements of N_1. Therefore, $G - N[I']$ has



(a) (b) (c)

Figure 4.2

$G[N[v]] - t' \cong P_3$ as a component, i.e., G encloses P_3, which cannot happen since P_3 is not $\mathbf{Z}_2$-well-covered. ∎

We use the following facts throughout the rest of the proof of Theorem 4.1, often without explicit mention. Every vertex of N_2 is adjacent to a vertex of $I \subseteq N_3$, and to a vertex of N_1. Consequently, every vertex of N_2 is adjacent to at most one other vertex of N_2. We have $m_{23} \geq n_2$, $m_{12} = 6 \geq n_2$, and $m_{12} + 2m_2 + m_{23} = 6 + 2m_2 + m_{23} = 3n_2$. Consequently, $6 + 2m_2 \leq 2n_2$. It follows that $3 \leq n_2 \leq 6$. Counting edges from I to N_2 and using the fact that each $u_i \in I$ has at least one private neigbour in N_2, we obtain $n_2/3 \leq p \leq n_2$.

To first prove that v is of type (A) or (B), we divide the proof into cases identified by the value of n_2. So, our top level cases are labelled (3) to (6).

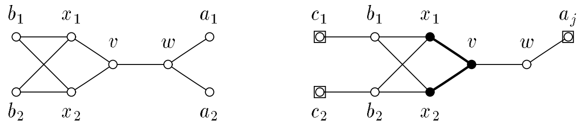
(3) Suppose that $n_2 = 3$. Since $6 + 2m_2 \leq 2n_2 = 6$, $m_2 = 0$. Consequently, $m_{23} = 3$ and every vertex of N_2 is adjacent to one vertex of N_3 and two vertices of N_1. The subgraph $G[N_1 \cup N_2]$ is bipartite and 2-regular, so it must be a 6-cycle. Thus, G contains the induced subgraph of Figure 4.2(a).

If $p = |I| = 1$ then G is the graph of Figure 4.2(b), which is the cube $C_4 \times K_2 \cong X_8$. If $p \geq 2$ then some vertex a of I has a neighbour b that does not belong to N_2. Then G encloses a P_3 as shown in Figure 4.2(c) (where it does not matter if b has neighbours in N_2), so this cannot happen. (Note that we frequently obtain a contradiction by showing that G encloses P_3, L, $K_{1,1,2}$ or $K_{2,3}$. Usually we draw a figure in which an independent set I' is indicated by open circles surrounded by squares, vertices of $N[I'] - I'$ by open circles, vertices not in $N[I']$ by solid circles, and edges of the enclosed graph by heavy lines. Occasionally a vertex whose exact status is unknown and does not matter will be indicated by a circle that is half solid and half open.)

(4) Suppose that $n_2 = 4$. Then we can write $N_2 = \{a_1, a_2, b_1, b_2\}$ where a_1 and a_2 have one neighbour in N_1 and b_1 and b_2 have two neighbours in N_1. Since b_1 and b_2 are adjacent to a vertex of $I \subseteq N_3$, the only possible edge with both ends in N_2 is a_1a_2. There are two subcases.

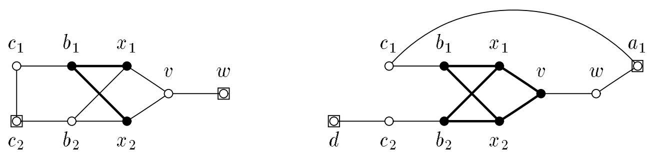
(4.1) Suppose that a_1 and a_2 have a common neighbour in N_1, which we call w. Then b_1 and b_2 are both adjacent to the other neighbours of N_1, which we call x_1 and x_2. See Figure 4.3(a).

Let c_i be the neighbour of b_i in I for $i = 1, 2$; possibly $c_1 = c_2$. If some a_j, $j = 1, 2$, is adjacent to neither c_1 nor c_2, then G encloses P_3 as shown in Figure 4.3(b). This must be the case if $c_1 = c_2$, so now we may assume that $c_1 \neq c_2$, and both a_1 and a_2 are adjacent to at least one of c_1 and c_2. Without loss of generality we may assume



(a) (b)

Figure 4.3



(a) (b)

Figure 4.4

that $a_1c_1 \in E(G)$.

If $c_1c_2 \in E(G)$, then G encloses P_3 as shown in Figure 4.4(a), so we may assume that $c_1c_2 \notin E(G)$. Since c_2 is not adjacent to w or c_1, c_2 has at least one neighbour $d \neq b_2$ that is not adjacent to a_1 (possibly $d = a_1$ or a_2). Then G encloses $K_{2,3}$ as shown in Figure 4.4(b). This concludes case (4.1).

(4.2) Suppose that a_1 and a_2 do not have a common neighbour in N_1. Let w_i be the neighbour of a_i in N_1 for $i = 1, 2$, and let x be the third vertex of N_1. Without loss of generality, we may assume that b_i is adjacent to w_i and x for $i = 1, 2$. See Figure 4.5.

Now $n_2/3 = 4/3 \leq p \leq n_2 = 4$, so $2 \leq p \leq 4$. If $p = |I| = 4$ then we must have $\mathcal{P} = \{\{a_1\}, \{a_2\}, \{b_1\}, \{b_2\}\}$, but $P(u_i) = \{a_1\}$ violates Lemma 4.2, so $p \neq 4$.

(4.2.1) Suppose that $p = 2$. Suppose b_1 and b_2 are adjacent to the same vertex $d = u_1$ of I. Then d cannot also be adjacent to a_1 or a_2, or we would have $P(u_2) = \{a_1\}$ or $\{a_2\}$, violating Lemma 4.2. Therefore a_1 and a_2 are both adjacent to the other vertex $c = u_2$ of I, and G encloses P_3 as shown in Figure 4.6(a). Thus, b_1 and b_2 have distinct neighbours in I: suppose $d_i \in I$ is adjacent to b_i, $i = 1, 2$, so that $I = \{d_1, d_2\}$. If both d_1 and d_2 are adjacent to both a_1 and a_2, then G is the graph of Figure 4.6(b), which is isomorphic to M_{10}. If one d_i, say d_1, is adjacent to both a_1 and a_2 while the other, d_2, is not, then d_2 has a neighbour $f_2 \notin \{a_1, a_2, b_2\}$, and

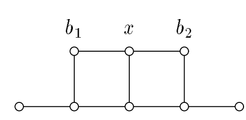


Figure 4.5

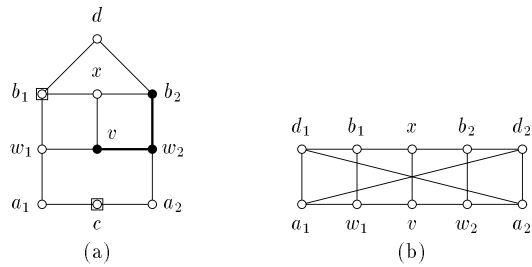


Figure 4.6

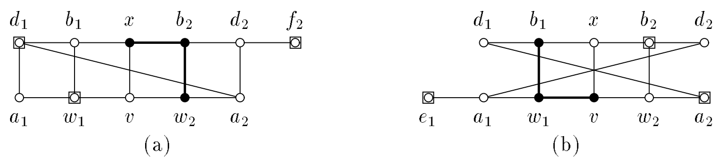


Figure 4.7

G encloses P_3 as shown in Figure 4.7(a). So, we may assume that neither d_1 nor d_2 is adjacent to both a_1 and a_2, and thus each of d_1 and d_2 is adjacent to one of a_1 or a_2.

Suppose first that $a_1d_2, a_2d_1 \in E(G)$. If a_1 and a_2 do not have a common neighbour, let $e_1 \neq w_1, d_2$ be a neighbour of a_1. Then G encloses P_3 as shown in Figure 4.7(b). Therefore a_1 and a_2 have a common neighbour e, and by a symmetric argument d_1 and d_2 have a common neighbour f. If $ef \notin E(G)$ then G encloses P_3 as shown in Figure 4.8(a). If $ef \in E(G)$ then G is the graph of Figure 4.8(b), which is isomorphic to R_{12}.

Thus, we may suppose that $a_1d_1, a_2d_2 \in E(G)$. Since $d_1, d_2 \in I$, we know that $d_1d_2 \notin E(G)$. Let $f_1 \neq a_1, b_1$ be a neighbour of d_1. If $a_1a_2 \in E(G)$ then G encloses P_3 as shown in Figure 4.9(a). Therefore, $a_1a_2 \notin E(G)$, we have the induced subgraph shown in Figure 4.9(b), and v is a star vertex of type (A) as in the statement of this theorem. This concludes case (4.2.1).

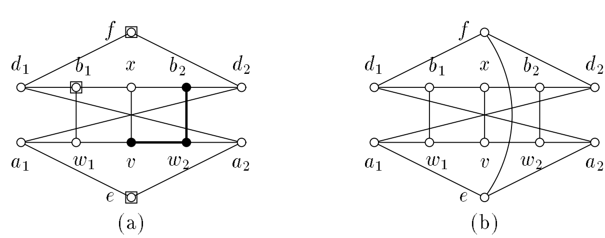


Figure 4.8

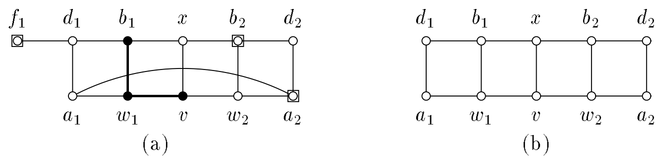


Figure 4.9

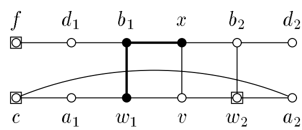


Figure 4.10

(4.2.2) Suppose that $p = 3$. By Lemma 4.2, no $P(u_i)$ can be $\{a_1\}$ or $\{a_2\}$, so the only possibility for $\mathcal{P}$ is $\{\{a_1, a_2\}, \{b_1\}, \{b_2\}\}$. Then I contains a vertex c adjacent to both a_1 and a_2, and a vertex d_i adjacent to b_i for $i = 1, 2$; there are no other adjacencies between I and N_2. Now d_1 has two neighbours other than b_1, and since neither is a_1 or a_2, at most one is adjacent to c. Thus, d_1 has a neighbour $f \neq b_1$ not adjacent to c. But then G encloses P_3 as shown in Figure 4.10. This concludes cases (4.2.2), (4.2) and (4).

(5) Suppose that $n_2 = 5$. Then one vertex c of N_2 must be adjacent to two vertices of N_1, and the remaining four vertices of N_2 must each be adjacent to one vertex of N_1. Thus, G contains the subgraph shown in Figure 4.11(a), where $d = u_1$ is the vertex of I adjacent to c, so that $c \in P(d)$. Here we may have edges between a_1, a_2, b_1, b_2.

(5.1) Suppose that $p = 2$. Write $e = u_2$. By Lemma 4.2, $|P(e)| \neq 1$, and $P(e) \neq \{a_1, a_2\}$. So, up to symmetry there are four possibilities for $P(e)$: $\{a_1, b_1\}$, $\{b_1, b_2\}$, $\{a_1, a_2, b_1\}$ and $\{a_1, b_1, b_2\}$.

(5.1.1) Suppose that $P(e) = \{a_1, b_1\}$. Then d must be adjacent to a_2 and b_2 as well as c; e may possibly be adjacent to one of a_2 or b_2 also. Thus, G has the subgraph shown in Figure 4.11(b). Let g_1 be the neighbour of b_1 other than e or x_1; possibly $g_1 = a_1$, a_2 or b_2. If $g_1 = a_2$ or b_2 then $a_2b_2 \notin E(G)$ and G encloses P_3 as shown

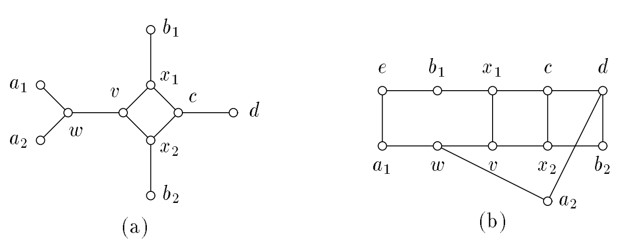


Figure 4.11

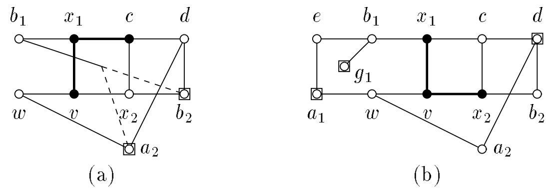


Figure 4.12

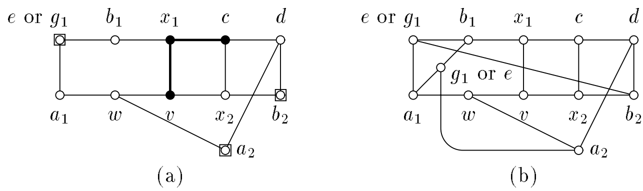


Figure 4.13

in Figure 4.12(a). So, we may assume $g_1 \neq a_2, b_2$. If $a_1g_1 \notin E(G)$ (including the case $g_1 = a_1$) then G encloses P_3 as shown in Figure 4.12(b). Therefore, g_1 is a vertex other than a_2, b_2 adjacent to a_1. If one of e or g_1 is adjacent to neither a_2 nor b_2, then G encloses P_3 as shown in Figure 4.13(a). Finally, if $a_2g_1, b_2e \in E(G)$ or $a_2e, b_2g_1 \in E(G)$, then G is the graph of Figure 4.13(b), which is isomorphic to R_{12}. This concludes case (5.1.1).

(5.1.2) Suppose that $P(e) = \{b_1, b_2\}$. Then d must be adjacent to a_1 and a_2 as well as c; e may possibly be adjacent to one of a_1 or a_2 also. Thus, G has the subgraph shown in Figure 4.14(a). To show that v is a star vertex of type (B), we must show that G does not contain the edges a_1a_2, b_1b_2, or a_ib_j where $i, j = 1, 2$. If $a_1a_2 \in E(G)$ then G encloses $K_{1,1,2}$ as shown in Figure 4.14(b). If $b_1b_2 \in E(G)$ then G encloses P_3 as shown in Figure 4.15(a). If an edge $a_ib_j \in E(G)$, then by symmetry we may assume it is a_1b_1. Then $a_1b_2 \notin E(G)$, and G encloses P_3 as shown in Figure 4.15(b). Therefore, v is a type (B) vertex as in the statement of this theorem. This concludes case (5.1.2).

(5.1.3) Suppose that $P(e) = \{a_1, a_2, b_1\}$. Then $P(d) = N(d) \cap N_2 = \{b_2, c\}$ and G is as shown in Figure 4.16(a). Let g_1 be the neighbour of b_1 other than e or x_1; possibly

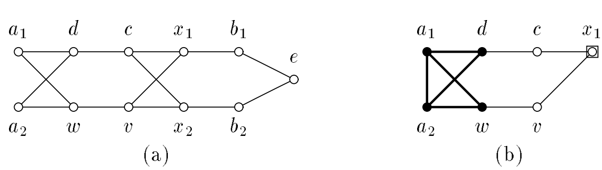


Figure 4.14

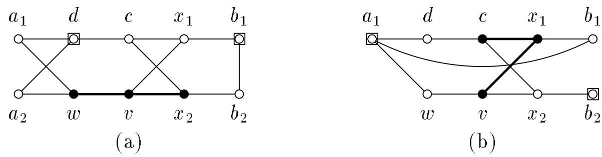


Figure 4.15

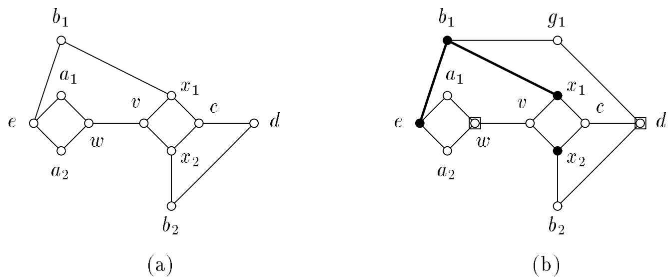


Figure 4.16

$g_1 = a_1$, a_2 or b_2. If $dg_1 \in E(G)$ (including the case $g_1 = b_2$) then G encloses a P_3 as shown in Figure 4.16(b), so $g_1 \neq b_2$ and $dg_1 \notin E(G)$. If g_1 is not adjacent to some a_i (including $g_1 = a_i$), say $a_i = a_1$, then G encloses P_3 as shown in Figure 4.17(a). Finally, if $a_1 g_1, a_2 g_1 \in E(G)$, then G encloses $K_{2,3}$ as shown in Figure 4.17(b). This concludes case (5.1.3).

(5.1.4) Suppose that $P(e) = \{a_1, b_1, b_2\}$. Then $P(d) = N(d) \cap N_2 = \{a_2, c\}$. Let f be the third neighbour of d. Then G encloses P_3, as shown in Figure 4.18(a). This concludes cases (5.1.4) and (5.1).

(5.2) Suppose that $p = 3$. By Lemma 4.2 neither $P(u_2)$ nor $P(u_3)$ has cardinality 1, and neither is equal to $\{a_1, a_2\}$. Therefore, without loss of generality we may write $u_2 = e_1$ and $u_3 = e_2$ where $P(e_i) = \{a_i, b_i\}$ for $i = 1, 2$. The only vertices of N_2 adjacent to each $u_i \in \{d, e_1, e_2\}$ are those in $P(u_i)$. Let f_1, f_2 be the neighbours

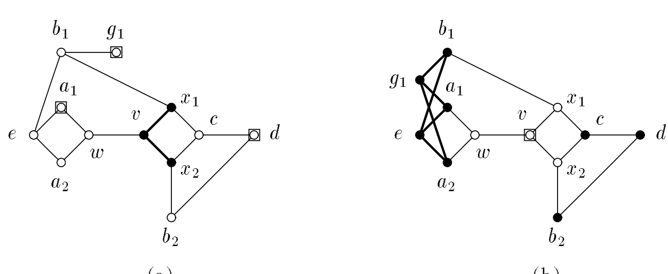


Figure 4.17

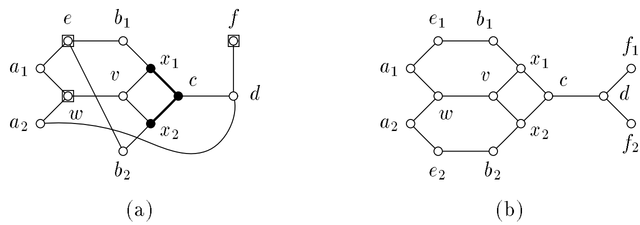


Figure 4.18

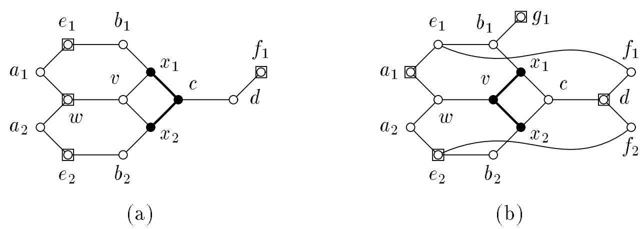


Figure 4.19

of d other than c. Thus, G has the subgraph of Figure 4.18(b). If some f_i, say f_1, is adjacent to neither e_1 nor e_2, then G encloses P_3 as shown in Figure 4.19(a). Therefore, without loss of generality we may suppose that $e_1f_1, e_2f_2 \in E(G)$.

Let g_1 be the neighbour of b_1 other than e_1 or x_1. Possibly g_1 is one of a_1, a_2, b_2, f_1, f_2. If g_1 is none of these five vertices and $a_1g_1 \notin E(G)$, or if $g_1 = a_1$, then G encloses P_3 as shown in Figure 4.19(b). If g_1 is none of the five vertices, but $a_1g_1 \in E(G)$, then G encloses P_3 as shown in Figure 4.20(a). If g_1 is one of a_2, b_2, f_1, f_2, then G encloses P_3 as shown in Figure 4.20(b): whatever the exact location of g_1, it belongs to $N[I'] - I'$. This concludes cases (5.2) and (5).

(6) Suppose that $n_2 = 6$. Then every vertex of N_2 has exactly one neighbour in N_1. It follows from Lemma 4.2 that no $u_i \in I$ has $|P(u_i)| = 1$. Therefore $|P(u_i)| = 2$ or 3 for each $u_i \in I$, and $p = |I| = 2$ or 3. If $p = 2$ there are (up to symmetry) two cases, shown in Figure 4.21(a) and (b). If $p = 3$ there is (up to symmetry) only one

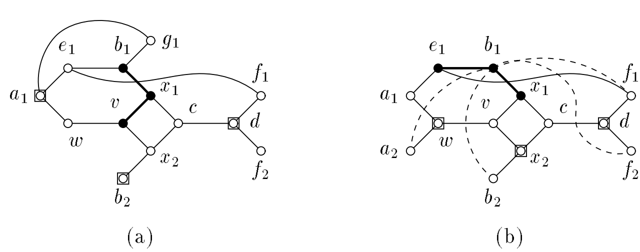


Figure 4.20

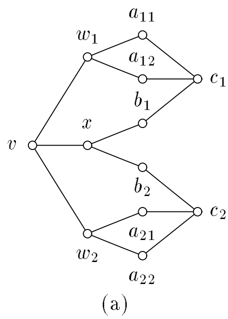


Figure 4.21

Figure 4.22

case, shown in Figure 4.21(c). A lengthy case analysis, roughly as long as cases (4) and (5) together, reveals that no star vertex can have $n_2 = 6$. The analysis focuses on the possible neighbours of vertices in N_2. We omit the details for the sake of brevity. (An expanded version of this paper with the complete argument for $n_2 = 6$ is available from the second author.)

We have now shown that every star vertex is of type (A), when $n_2 = 4$, or type (B), when $n_2 = 5$. We must now prove that every neighbour of a star vertex is also a star vertex of the same type.

Suppose v is a star vertex of type (A). We may assume that v is part of an induced subgraph labelled as in Figure 4.9(b). From $I' = \{a_1, a_2\}$ we see that x is also a star vertex with $n_2 = 4$, so x is a star vertex of type (A). Now consider w_1. Let $e_1 \neq d_1, w_1$ be the third neighbour of a_1, and $f_1 \neq a_1, b_1$ be the third neighbour of d_1. Both of these are (possibly equal) new vertices that do not appear in Figure 4.9(b). If $e_1 f_1 \notin E(G)$ (including the case $e_1 = f_1$) then G encloses a P_3 as in Figure 4.22(a). Therefore, e_1 and f_1 are distinct and adjacent. Now Figure 4.22(b) shows that w_1 is a star vertex with $n_2 = 4$, so w_1 is a star vertex of type (A). The argument for w_2 is symmetric.

Now suppose v is a star vertex of type (B). We may assume that v is part of a subgraph labelled as in Figure 4.14(a), which is induced except that e may be adjacent to one of a_1 or a_2. Let $g_i \neq d, w$ be the third neighbour of a_i, $i = 1, 2$. Suppose first that e is adjacent to some a_i; by symmetry, we may assume $a_1 e \in E(G)$. Then g_2 is a new vertex, not shown in Figure 4.14(a). If g_2 is not adjacent to some

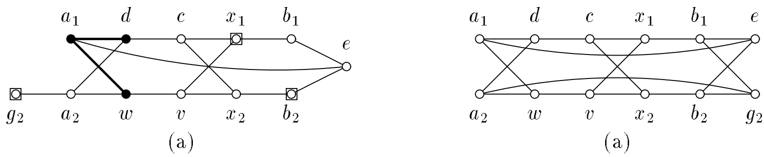


Figure 4.23

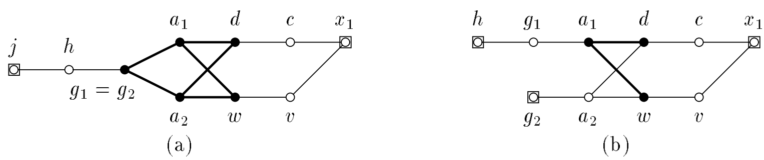


Figure 4.24

b_i, say $i = 2$ by symmetry, then G encloses P_3 as shown in Figure 4.23(a). Therefore, $b_1g_2, b_2g_2 \in E(G)$, and G is the graph of Figure 4.23(b), which is isomorphic to X_{12}, in which every vertex is a star vertex of type (B).

Suppose now that $a_1e, a_2e \notin E(G)$. We first show that w is a star vertex of type (B). Now g_1, g_2 are (possibly equal) new vertices, which do not appear in Figure 4.14(a). Let h denote a neighbour of g_1 distinct from a_1 and (if $g_1 = g_2$) also from a_2; possibly $h \in \{b_1, b_2, e, g_2\}$. Suppose that $g_1 = g_2$. Let j be any neighbour of h distinct from $g_1 = g_2$; possibly $j \in \{b_1, b_2, e, x_1, x_2\}$. Since there is no vertex adjacent to both x_1 and x_2 other than v and $j \neq v$, $jx_i \notin E(G)$ for some x_i (including the possibility that $j = x_i$); by symmetry we may assume that $x_i = x_1$. Then G encloses $K_{2,3}$ as shown in Figure 4.24(a). Therefore, $g_1 \neq g_2$. Since $h \neq v$, $hx_i \notin E(G)$ for some x_i; by symmetry we may take $x_i = x_1$. If $g_2h \notin E(G)$ (including the case $h = g_2$) then G encloses P_3 as shown in Figure 4.24(b). Therefore, $g_2h \in E(G)$. Then Figure 4.25 shows that w is a star vertex with $n_2 = 5$, i.e., a star vertex of type (B).

Now we must show that x_1 and x_2 are star vertices of type (B). Let $f_i \neq e, x_i$ denote the third neighbour of b_i for $i = 1, 2$. Then f_1, f_2 are (possibly equal) new vertices, that do not appear in Figure 4.14(a). If $f_1 \neq f_2$, then a_1 cannot be adjacent to both f_1, f_2, so $a_1f_i \notin E(G)$ for some f_i; by symmetry we may take $f_i = f_1$. Then G encloses P_3 as shown in Figure 4.26(a). Thus, $f_1 = f_2$. Then Figure 4.26(b) shows that x_1 is a star vertex with $n_2 = 5$, i.e., a star vertex of type (B). The argument for

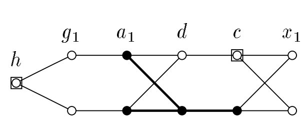
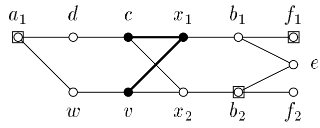


Figure 4.25



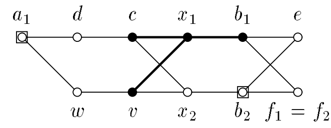


Figure 4.26

x_2 follows by symmetry.

This concludes the proof of Theorem 4.1. ∎

Theorem 4.3. *Let G be a connected cubic graph. Then G is $\mathbf{Z}_2$-well-covered but not well-covered if and only if G is one of*

(i) *R_{12} (shown in Figure 3.3),*
(ii) *$C_4 \times K_2 \cong X_8$,*
(iii) *$C_m \times K_2$ for $m \geq 6$,*
(iv) *M_{2m} for $m \geq 5$ (described in Section 3), or*
(v) *X_{4k} for $k \geq 3$ (described in Section 3).*

Proof. From Section 3, the graphs in (i) to (v) are all Z_2-well-covered but not well-covered.

Conversely, suppose that G is $\mathbf{Z}_2$-well-covered but not well-covered. Assume that G is not isomorphic to $C_4 \times K_2 \cong X_8$, M_{10}, or R_{12}. Since G is connected, by Theorem 4.1 all vertices of G are star vertices of the same type, (A) or (B). Define an edge e of G to be of *type i* (i a nonnegative integer) if e belongs to i 4-cycles of G.

Suppose all vertices of G are star vertices of type (A). Looking at Figure 4.9(b), we see that the edges of G are all type 1 or type 2. The edges of type 2 form a perfect matching. If we form a graph H whose vertices are the type 2 edges of G, with two type 2 edges adjacent if they belong to a common 4-cycle of G, then H is 2-regular, and hence is a cycle. It follows that G is isomorphic to $C_m \times K_2$ or M_{2m} for $m \geq 6$.

Now suppose that all vertices of G are star vertices of type (B). Looking at Figure 4.14(a), we see that the edges of G are all type 0 or type 1. The spanning subgraph formed by the edges of type 1 is a 2-factor whose components are all the 4-cycles of G. Each pair of opposite vertices of a 4-cycle is joined in G by a pair of type 0 edges to a pair of opposite vertices in another 4-cycle. Therefore, if we form a graph J whose vertices are the 4-cycles of G, with two 4-cycles adjacent if they are joined by a pair of edges, then J is 2-regular, and hence is a cycle. It follows that G is isomorphic to X_{4k} for $k \geq 3$. ∎

Combining this with Theorem 2.7, we obtain the following characterisation of cubic parity ($\mathbf{Z}_2$-well-covered) graphs. Note that a graph is a parity graph if and only if each of its components is a parity graph, so we may restrict our attention to connected graphs.

Corollary 4.4. *Suppose G is a connected cubic graph. Then G is $\mathbf{Z}_2$-well-covered, i.e., a parity graph, if and only if it is one of the graphs of either Theorem 2.7 or Theorem 4.3.*

5 Concluding remarks

An important point of this paper is that minimal non-well-covered graphs are a highly useful tool. Here, they enabled us to characterise cubic parity graphs by extending the characterisation of cubic well-covered graphs, rather than by starting from scratch. The original case analysis for cubic well-covered graphs took roughly 26 pages (single-spaced, with diagrams) if given in full. The case analysis in this paper takes roughly 14 pages if given in full. One would guess that a characterisation of the cubic parity graphs from scratch, which would have the well-covered and non-well-covered cases tangled together, would probably cover essentially the same cases and so take about 40 pages. Minimal non-well-covered graphs allow us to separate the well-covered and non-well-covered cases, using Lemma 2.9, and so we only had to do the 14 extra pages of the non-well-covered case, rather than the whole 40 pages.

As mentioned in Section 1, it is interesting that cubic parity graphs divide sharply into two classes, those that are well-covered and those that are not. The structures of these two classes, as described in Theorems 2.7 and 4.3, respectively, are quite different. This is surprising, because for graphs of girth 6 or more, as investigated in [5, 9], the structure of $\mathbf{Z}_m$-well-covered graphs seems to be a natural generalisation of the structure of the well-covered graphs.

In Theorem 2.7, the connected cubic well-covered graphs are, with six exceptions, constructed from the three basic well-covered building blocks A, B and C. The decomposition into building blocks is usually unique. One might reasonably have imagined that cubic $\mathbf{Z}_2$-well-covered graphs would include a few more building blocks that could be used together with A, B and C. However, the connected cubic $\mathbf{Z}_2$-well-covered non-well-covered graphs, as described in Theorem 2.7, do not seem to be made up of simple building blocks in the same way. One can decompose graphs in the families $C_m \times K_2$, M_{2m}, and X_{4k} into vertex-disjoint $\mathbf{Z}_2$-well-covered subgraphs isomorphic to $K_{1,3}$, sometimes also using one copy of P_2 or $\overline{K_2}$, but the decomposition is far from unique. The building blocks A, B and C do not appear in these graphs at all.

For well-covered graphs of bounded degree, it seems reasonable to conjecture that there will always be a decomposition into a finite family of building blocks. This is supported by the characterisation of cubic well-covered graphs, by Ramey's characterisation of well-covered graphs of maximum degree 3 in his Ph.D. thesis [13], and also by Hartnell and Plummer's characterisation of 4-connected 4-regular claw-free well-covered graphs [10].

Conjecture 5.1. *Fix $d \geq 3$. Then there is a collection $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of well-covered graphs of maximum degree at most d, and a list of rules for connecting the elements of $\mathcal{G}$ together via edges, such that with finitely many exceptions every* *well-covered graph of maximum degree at most d is obtained by joining together elements of $\mathcal{G}$ according to these rules.*

One might also conjecture that the decomposition of a graph into elements of $\mathcal{G}$ is, with finitely many exceptions, unique. For $\mathbf{Z}_m$-well-covered graphs, it is less clear whether something similar to Conjecture 5.1 is likely to hold. We expect that graphs that are minimal non-well-covered but Z_m-well-covered, such as $K_{p,km+p}$ for $p, k \geq 1$, will play an important role in the structure of graphs that are $\mathbf{Z}_m$-well-covered but not well-covered, and this is shown by Corollary 2.6 and Theorem 4.1, and by the results for girth at least 6 by Finbow and Hartnell [9] and Caro and Hartnell [5].

It would be useful to have more evidence as to the structure of $\mathbf{Z}_m$-well-covered graphs with degree restrictions. Using the techniques of this paper, it would probably be possible to extend Ramey's characterisation of well-covered graphs of maximum degree at most 3 to $\mathbf{Z}_2$-well-covered graphs of maximum degree at most 3. However, this would be a long and tedious exercise. One alternative approach is to use a computer to determine the possible structures close to a vertex, From [4] we know that A-well-coveredness is a local property, determined by the structure of the graph out to distance 4 around each vertex. Once the possible local structures are found, a computer could also be used to determine which pairs of structures could belong to adjacent vertices. In this way, we might obtain enough information to give a structural characterisation without a tedious manual case analysis. Such an approach might also allow us to extend what is known about 4-regular graphs, or even graphs of maximum degree 4, both for well-covered graphs and for $\mathbf{Z}_m$-well-covered graphs.

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