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Absorbent sets and kernels by monochromatic directed paths in m-colored tournaments

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Abstract

In this paper, we consider the following problem due to Erdős: for each $m \in \mathbb{N}$, is there a (least) positive integer $f(m)$ so that every finite m-colored tournament contains an absorbent set S by monochromatic directed paths of $f(m)$ vertices? In particular, is $f(3) = 3$? We prove several bounds for absorbent sets of m-colored tournaments under certain conditions on the number of colors of the arcs incident to every vertex from its in-neighborhood (respectively, ex-neighborhood). In particular, we establish the validity of Erdős' problem for 3-colored tournaments with this condition. It is also proven that a 3-colored tournament containing no heterochromatic directed triangles with at most bichromatic ex-neighborhoods (respectively, in-neighborhoods) has a kernel by monochromatic directed paths. Previous results are generalized.

1 Introduction

Let $D = (V, A)$ be a finite digraph, where V and A denote the sets of vertices and arcs of D respectively. The in- and ex-neighborhood of a vertex v of D are denoted by $N^-(v, D)$ and $N^+(v, D)$, and define $d^-(v) = |N^-(v, D)|$ and $d^+(v) = |N^+(v, D)|$. For $\varnothing \neq S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the subset S.

A digraph D is said to be m-colored if the arcs of D are colored with m colors. Given $u, v \in V(D)$, a directed path from u to v of D is *monochromatic* if all its arcs have the same color and it is denoted by $u \rightsquigarrow_m v$. A nonempty set $S \subseteq V(D)$ is an *absorbent set by monochromatic directed paths* (m.d.p.) if for every vertex $u \in V(D) - S$ there exists $v \in S$ such that $u \rightsquigarrow_m v$. A *kernel* K of D is an independent set of vertices so that for every $u \in V(D) - K$ there exists $(u, v) \in A(D)$, where $v \in K$. A digraph D is called *kernel-perfect* if every induced subdigraph of D has a kernel.

Let D be an m-colored digraph. A set $K \subseteq V(D)$ is called a *kernel by m.d.p.* if

(i) for every $u, v \in K$ there is no m.d.p between u and v, and

(ii) for every $x \in V(D) - K$ there exists $y \in K$ such that $x \rightsquigarrow_m y$.

A tournament T with n vertices is an orientation of the complete graph K_n.

The study of absorbent sets by m.d.p. in m-colored tournaments goes back to the statement of a classical problem due to Erdős.

Problem 1. *For each $m \in \mathbb{N}$, is there a (least) positive integer $f(m)$ so that every finite m-colored tournament contains an absorbent set S by m.d.p of $f(m)$ vertices? In particular, is $f(3) = 3$?*

In [8], Sands et al. proved that $f(2) = 1$, that is,

Theorem 1 ([8], Corollary 1)**.** *Let T be a finite tournament whose arcs are colored with two colors. Then there is a vertex v of T such that for every other vertex x of T there exists $x \rightsquigarrow_m v$.*

More generally, they showed that a 2-colored digraph has a kernel by m.d.p. and posed the following

Problem 2. *Let T be a 3-colored (in general, m-colored) tournament not containing 3-colored directed triangles. Must T contain a vertex v such that for every other vertex x of T there exists $x \rightsquigarrow_m v$? (Or equivalently, must T have a kernel by m.d.p.?)*

In 1988, Shen [9] proved that

Theorem 2 ([9])**.** *If an m-colored tournament T does not contain 3-colored directed triangles ($\overrightarrow{C}_3$) or transitive tournaments of order 3 (TT_3) then $f(m) = 1$, that is, T has a kernel by m.d.p.*

Moreover, for $m \geq 4$ the condition on T not containing 3-colored $\overrightarrow{C}_3$ or TT_3 cannot be improved. The case of 4-colored tournaments is solved in [4], where it is proved that for every $n \geq 6$, there exists a 4-colored tournament T of n vertices satisfying that T does not contain 3-colored directed triangles and T does not have a kernel by m.d.p. Counterexamples for the case of m-colored tournaments with $m \geq 5$ were constructed in [9]. For $m = 3$ the problem is still open. Similar results for tournaments and digraphs in general were obtained in [2] and [3].

Recently, Galeana-Sánchez and Rojas-Monroy [5] showed that if a 3-colored tournament T does not contain 3-colored directed triangles and the number of colors assigned to the arcs incident to every vertex of T is at most 2, then T has a kernel by m.d.p. Further, it is proven that if an m-colored ($m \geq 4$) tournament T does not contain 3-colored directed triangles and the number of colors assigned to all the arcs incident to every vertex of T is at most 2, then T has a kernel by m.d.p.

Extensions of known results and new approaches to this kind of questions (multi-tournaments and underlying digraphs) are studied in [6].

In Section 2 of this paper, we prove several bounds for absorbent sets of m-colored tournaments under weaker conditions on the number of colors of the arcs incident to every vertex from its in-neighborhood (resp. ex-neighborhood). In particular, we establish the validity of Erdős' problem for 3-colored tournaments with this condition. In Section 3, we generalize the results of [5].

Throughout this paper, we will use the following definitions and notations.

We say that a digraph D is *monochromatic kernel perfect (m.k.p.)* if every induced subdigraph of D has a kernel by m.d.p. An arc $(u, v) \in A(D)$ is *asymmetrical* (resp. *symmetrical*) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). A directed cycle γ of D is said to be *asymmetrical* (resp. *symmetrical)* if every arc of γ is asymmetrical (resp. symmetrical). A *semicomplete digraph* is a digraph with no non-adjacent vertices and thus, tournaments are asymmetrical semicomplete digraphs.

Theorem 3 ([1], Théorème 4.2)**.** *If every directed cycle of a digraph D has a symmetrical arc, then D is kernel-perfect.*

If $D = (V, A)$ is an m-colored digraph, then the *closure of* D, denoted by $\mathfrak{C}(D)$, is the m-colored digraph defined by

$$V(\mathfrak{C}(D)) = V(D) \text{ and}$$

$$A(\mathfrak{C}(D)) = A(D) \cup \{(u, v) \text{ of color } i : \exists\, u \rightsquigarrow_m v \text{ of color } i \text{ in } D\}.$$

Remark 1. *(i) For every digraph D, $\mathfrak{C}(D)$ is isomorphic to $\mathfrak{C}(\mathfrak{C}(D))$.*

(ii) D has a kernel by m.d.p. if and only if $\mathfrak{C}(D)$ has a kernel.

Let T be an m-colored tournament, $\{1, 2, ..., m\}$ the set of colors and $z \in V(T)$. We define

$$A^+(z) = \{(z, v) \in A(D) : v \in V(T)\} \text{ (resp. } A^-(z) = \{(v, z) \in A(D) : v \in V(T)\}\text{)}, \tag{$\triangle$}$$

$C^+(z)$ (resp, $C^-(z)$) the set of colors appearing in $A^+(z)$ (resp. in $A^-(z)$) and $\xi^+(z) = |C^+(z)|$ (resp. $\xi^-(z) = |C^-(z)|$). When $\xi^+(z) \leq 2$ (resp. $\xi^-(z) \leq 2$), we will refer to the *at most bichromatic ex-neighborhood* (resp. *in-neighborhood*) of vertex z.

2 Absorbent sets by m.d.p. in m-colored tournaments

Theorem 4. *Let T be an m-colored tournament and suppose that $\xi^+(z) \leq 2$ for every vertex $z \in V(T)$. Then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq \binom{m}{2}$.*

Proof. For all pair of colors $i, j \in \{1, 2, ..., m\}$ $(i \neq j)$, we define T_{ij} to be the subtournament of T induced by the set

$$\left\{z \in V(T) : C^+(z) \subseteq \{i, j\}\right\}.$$

Clearly, T_{ij} is a 2-colored subtournament of T (every arc of T_{ij} is colored i or j). By Theorem 1, there exists a vertex $x_{ij} \in V(T_{ij})$ which absorbs every other vertex of T_{ij} by m.d.p. Since

$$V(T) = \bigcup_{\{i,j\} \subseteq \{1,2,...,m\}} V(T_{ij}),$$

the set

$$S = \{x_{ij} : i, j \in \{1, 2, ..., m\}, i \neq j\}$$

is an absorbent set by m.d.p. such that $|S| \leq \binom{m}{2}$. □

Corollary 1. *Let T be a 3-colored tournament and suppose that $\xi^+(z) \leq 2$ for every vertex $z \in V(T)$. Then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq 3$.*

Observe that the analogous results of Theorem 4 and Corollary 1 wtih $\xi^-(z) \leq 2$ are also true.

So, we have proved that Erdős' conjecture (Problem 1) is true for all 3-colored tournaments with at most bichromatic ex-neighborhoods (resp. in-neighborhoods).

Theorem 5. *Let T be an m-colored tournament so that for every vertex $z \in V(T)$, at least one of the following properties holds: $\xi^+(z) \leq 2$ or $\xi^-(z) \leq 2$. Then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq 2\binom{m}{2}$.*

Proof. Let T^+_{ij} (resp. T^-_{ij}) be the subtournament of T induced by the set

$$\left\{z \in V(T) : C^+(z) \subseteq \{i, j\}\right\} \text{ (resp. } \left\{z \in V(T) : C^-(z) \subseteq \{i, j\}\right\}\text{)}.$$

Clearly, T^+_{ij} (resp. T^-_{ij}) is an at most 2-colored tournament. By Theorem 1, there exists a vertex x^+_{ij} (resp. x^-_{ij}) which absorbs every other vertex of T^+_{ij} (resp. T^-_{ij}) by m.d.p. Since

$$V(T) = \bigcup_{\{i,j\} \subseteq \{1,2,...,m\}} \left(V(T^+_{ij}) \cup V(T^-_{ij})\right),$$

the set

$$S = \{x_{ij}^{+} : i, j \in \{1, 2, ..., m\}, i \neq j\} \cup \{x_{ij}^{-} : i, j \in \{1, 2, ..., m\}, i \neq j\}$$

is an absorbent set by m.d.p. such that $|S| \leq 2\binom{m}{2}$. □

Theorem 6. *Let T be an m-colored tournament and suppose that $\xi^{+}(z) \leq 2$ for every vertex $z \in V(T)$. If T contains no 3-colored directed triangles, then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq m-1$ if m is even and $|S| \leq m$ if m is odd.*

Proof. For every pair of colors $i, j \in \{1, 2, ..., m\}$ $(i \neq j)$, we define T_{ij} to be the subtournament of T induced by the set $\{z \in V(T) : C^{+}(z) \subseteq \{i, j\}\}$. Clearly, T_{ij} is a 2-colored subtournament of T (every arc of T_{ij} is colored i or j). By Theorem 1, there exists a vertex $x_{ij} \in V(T_{ij})$ which absorbs every other vertex of T_{ij} by m.d.p.

Claim 1. *Let i, j, k and l be distinct integers such that $\{i, j\} \subseteq \{1, 2, ..., m\}$, $\{k, l\} \in \{1, 2, ..., m\}$. If $(x_{kl}, x_{ij}) \in A(T)$, then x_{ij} absorbs every vertex of T_{kl} by m.d.p. (and every other vertex of T_{ij} as observed before).*

We know that x_{kl} absorbs every other vertex of T_{kl} by m.d.p. and T_{kl} is at most 2-colored with color k and l. Since $x_{kl} \in V(T_{kl})$ and $C^{+}(x_{kl}) \subseteq \{k, l\}$, the arc (x_{kl}, x_{ij}) is colored k or l. Without loss of generality, we can suppose that (x_{kl}, x_{ij}) is colored k. It follows that x_{ij} absorbs by m.d.p of color k all those vertices of T_{kl} that x_{kl} absorbs by m.d.p of color k. Consider a vertex $z \in V(T_{kl})$ such that there exists $z \rightsquigarrow_m x_{kl}$ of color l. Let $(z_0, z_1, ..., z_r)$ be such a path, where $z_0 = z$ and $z_r = x_{kl}$.

(1) If $(z, x_{ij}) \in A(T)$, then x_{ij} absorbs z by a m.d.p.

(2) If $(z, x_{ij}) \notin A(T)$, then $(x_{ij}, z) \in A(T)$ (note that T is a tournament). Since $(z_r, x_{ij}) \in A(T)$ and $(x_{ij}, z_0) \in A(T)$, there exists $t \in \{0, 1, ..., r\}$ such that $(x_{ij}, z_t) \in A(T)$ and $(z_{t+1}, x_{ij}) \in A(T)$.

(2.1) If (z_{t+1}, x_{ij}) is colored l, then $(z_0, z_1, ..., z_{t+1}) \cup (z_{t+1}, x_{ij})$ is a $z \rightsquigarrow_m x_{ij}$.

(2.2) If (z_{t+1}, x_{ij}) is colored k, then $\overrightarrow{C}_3 = (z_{t+1}, x_{ij}, z_t, z_{t+1})$ is 3-colored contradicting the hypothesis of the theorem (observe that (z_{t+1}, x_{ij}) and (z_t, z_{t+1}) are colored k and l respectively, and (x_{ij}, z_t) is colored i or j, where $\{i, j\} \cap \{k, l\} = \varnothing$).

This concludes the proof of the claim.

Let $\mathcal{P} = \{M_1, M_2, ..., M_{n(m)}\}$ be a partition of $E(K_m)$ (the set of edges of the complete graph with m vertices) of minimum cardinality into maximal matchings. When m is even, these matchings are perfect and $n(m) = m - 1$. When m is odd, the matchings are of cardinality $\frac{m-1}{2}$ and $n(m) = m$. Define $M_i = \{e_i^1, e_i^2, ..., e_i^{g(m)}\}$, where $g(m) = \frac{m}{2}$ if m is even and $g(m) = \frac{m-1}{2}$ if m is odd. We denote x_{st} by $x(e_i^j)$, where $e_i^j = st$

(that is, s and t are the ends of edge e_i^j and recall that x_{st} is a selected vertex of T_{st} absorbing every other vertex of T_{st}). Let T_i be the subtournament generated by the set $\{x(e_i^j) : 1 \leq j \leq g(m)\}$.

Claim 2. *T_i has a kernel.*

Since T_i is a tournament, it is enough to prove that T_i does not contain directed triangles. By contradiction, suppose that T_i contains a directed triangle, say $\overrightarrow{C}_3 = (u, v, w, u)$. Therefore,

$$u = x(e_i^r),\ v = x(e_i^p) \text{ and } w = x(e_i^q),$$

where r, p and q are all different and $\{p, q, r\} \subseteq \{1, 2, ..., g(m)\}$. Since e_i^r, e_i^p and e_i^q are three distinct edges of a same matching of K_m, it follows that

$$u = x(e_i^r) = x_{ab},\ v = x(e_i^p) = x_{cd} \text{ and } w = x(e_i^q) = x_{ef},$$

where $\{a, b, c, d, e, f\} \subseteq \{1, 2, ..., m\}$,

$$\{a, b\} \cap \{c, d\} = \varnothing,\ \{a, b\} \cap \{e, f\} = \varnothing \text{ and } \{c, d\} \cap \{e, f\} = \varnothing.$$

Moreover, $\overrightarrow{C}_3 = (u, v, w, u) = (x_{ab}, x_{cd}, x_{ef}, x_{ab})$. So, (x_{ab}, x_{cd}) is an arc of T colored a or b, (x_{cd}, x_{ef}) is an arc of T colored c or d and (x_{ef}, x_{ab}) is an arc of T colored e or f (recall that arcs of type (x_{ij}, z) are colored i or j). We conclude that $\overrightarrow{C}_3 = (x_{ab}, x_{cd}, x_{ef}, x_{ab})$ is a 3-colored directed triangle contained in T, a contradiction to the theorem hypothesis. Thus T_i does not contain directed triangles which implies that T_i is a transitive tournament and therefore T_i has a kernel.

Let $z_i \in V(T_i)$ such that z_i is a kernel of T_i and therefore $(w, z_i) \in A(T)$ for every $w \in V(T_i)$. Let $u(e_i^j)$ and $v(e_i^j)$ be the ends of the edge e_i^j. So, $z_i = x(e_i^j) = x_{u(e_i^j)v(e_i^j)}$ for some $j \in \{1, 2, ..., g(m)\}$ and we have that $(x(e_i^k), x(e_i^j)) \in A(T)$ for every $k \in \{1, 2, ..., m\} - \{j\}$. By Claim 1, z_i absorbs every vertex of $\bigcup T_{x(e_i^k)}$ for every $k \in \{1, 2, ..., g(m)\}$ and $i \in \{1, 2, ..., n(m)\}$. Since $\mathcal{P} = \{M_1, M_2, ..., M_{n(m)}\}$ is a partition of $E(K_m)$ into maximal matchings, we conclude that $S = \{z_i : i \in \{1, 2, ..., n(m)\}\} \subseteq V(T)$ is an absorbent set by m.d.p. □

Theorem 7. *Let T be an m-colored tournament and suppose that $\xi^-(z) \leq 2$ for every vertex $z \in V(T)$. If T contains no 3-colored directed triangles, then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq m-1$ if m is even and $|S| \leq m$ if m is odd.*

Proof. Analogous to the proof of Theorem 6. □

Theorem 8. *Let T be an m-colored tournament so that for every vertex $z \in V(T)$, at least one of the following properties holds: $\xi^+(z) \leq 2$ or $\xi^-(z) \leq 2$. If T contains no 3-colored directed triangles, then there exists an absorbent set by m.d.p. $S \subseteq V(T)$ such that $|S| \leq 2m - 2$ if m is even and $|S| \leq 2m$ if m is odd.*

Proof. Let T^+ and T^- be the subtournaments of T generated by the sets $\{z \in V(T) : \xi^+(z) \leq 2\}$ and $\{z \in V(T) : \xi^-(z) \leq 2\}$ respectively. By Theorem 6, there exists an absorbent set by m.d.p. $S^+ \subseteq V(T^+)$ such that $|S^+| \leq m-1$ if m is even and $|S^+| \leq m$ if m is odd. By Theorem 7, there exists an absorbent set by m.d.p. $S^- \subseteq V(T^-)$ such that $|S^-| \leq m-1$ if m is even and $|S^-| \leq m$ if m is odd. Clearly, the set $S = S^+ \cup S^-$ satisfies the required properties. □

3 Kernels by m.d.p. in m-colored tournaments

Denote by $[z, w]$ the arc (z, w) or (w, z) in a tournament T. For the subsequent results, we need the following

Lemma 1. *Let T be an m-colored tournament, $M = \{1, 2, ..., m\}$ the set of colors,*

$$M' \subseteq M^2 = \{S \subseteq M : |S| = 2\}$$

and $\mathcal{Z} = \{V_{ij} : \{i, j\} \in M'\}$ a family of (not necessarily distinct) nonempty subsets of $V(T)$ such that for every $V_{ij} \in \mathcal{Z}$:

(1) $V_{ij} \subsetneq V(T)$,

(2) $T[V_{ij}]$ is m.k.p. and

(3) for every $z \in V_{ij}$ and $w \in V(T) - V_{ij}$, the arc $[z, w]$ is colored i or j.

Then for every $V_{ij}, V_{kl} \in \mathcal{Z}$ at least one of the following properties holds:

(i) $V_{ij} = V_{kl}$.

(ii) There exists $x_{ij} \in V_{ij}$ which absorbs every other vertex of V_{ij} by m.d.p. and such that $(y, x_{ij}) \in A(T)$ for every $y \in V_{kl}$.

(iii) There exists $x_{kl} \in V_{kl}$ which absorbs every other vertex of V_{kl} by m.d.p. and such that $(x, x_{kl}) \in A(T)$ for every $x \in V_{ij}$.

(iv) $\{i\} = \{i, j\} \cap \{k, l\}$ and there exist $x \in V_{ij}$ and $y \in V_{kl}$ such that x (resp. y) absorbs every vertex of $V_{ij} \cup V_{kl}$ by m.d.p. of color i and alternating between vertices of V_{ij} and V_{kl}. Moreover $\{z \in V_{kl} : (z, x) \in A(T)\} \neq \varnothing$ and $\{z \in V_{ij} : (z, y) \in A(T)\} \neq \varnothing$.

Proof. It is based on the following two claims.

Claim 1. *If $\{i, j\} \cap \{k, l\} = \varnothing$, then $V_{ij} = V_{kl}$.*

First we prove that $V_{ij} \subseteq V_{kl}$. By contradiction, suppose that there exists $x \in V_{ij} - V_{kl}$. In this case, we will show that $V_{kl} \subseteq V_{ij}$. If there exists $y \in V_{kl} - V_{ij}$, then by property (3), the arc $[x, y]$ is colored k or l in V_{kl} and colored i or j in V_{ij}, a contradiction since

$\{i,j\} \cap \{k,l\} = \varnothing$. So, $V_{kl} \subseteq V_{ij}$. We prove now that $V_{ij} = V(T)$. Let $z \in V_{kl} \subseteq V_{ij}$. If there exists $w \in V(T) - V_{ij}$, then by property (3), the arc $[z,w]$ is colored k or l in V_{kl} and colored i or j in V_{ij}, a contradiction again since $\{i,j\} \cap \{k,l\} = \varnothing$. So, $V_{ij} = V(T)$, contradicting property (1). Therefore $V_{ij} \subseteq V_{kl}$. Analogously, $V_{kl} \subseteq V_{ij}$.

Claim 1 is proved.

Suppose now that $\{i,j\} \cap \{k,l\} \neq \varnothing$ and, without loss of generality, that $\{i\} = \{i,j\} \cap \{k,l\}$ (observe that in this case, every arc between V_{ij} and V_{kl} is colored i by means of property (3)). If either (ii) or (iii) holds, we are done. So, we assume that (ii) and (iii) do not hold and prove that (iv) is satisfied. For this purpose, we first show the following claim.

Claim 2. *There exists $z \in V_{ij} \cup V_{kl}$ such that*

(i) *z is a kernel by m.d.p. in V_{ij} if $z \in V_{ij}$, or in V_{kl} if $z \in V_{kl}$ and*

(ii) *z absorbs every vertex of $V_{ij} \cup V_{kl}$ by m.d.p. of color i and alternating between vertices of V_{ij} and V_{kl}.*

Let

$$\begin{aligned} N &= \{z \in V_{ij} : z \text{ is a kernel by m.d.p. in } T[V_{ij}]\} \\ &\quad \cup \{z \in T_{kl} : z \text{ is a kernel by m.d.p. in } T[V_{kl}]\}. \end{aligned}$$

For some $w \in N$, we define

$$\mathcal{M}(w) = \{x \in V_{ij} \cup V_{kl} : \exists\, x \rightsquigarrow_m w \text{ of color } i \text{ alternating between } V_{ij} \text{ and } V_{kl}\}$$

and $\mathcal{A}(w) = |\mathcal{M}(w)|$. Let $n \in N$ be such that $\mathcal{A}(n) = \max\{\mathcal{A}(z) : z \in N\}$. Without loss of generality, we can assume that $n \in V_{ij}$. We will prove that n absorbs every vertex of $V_{ij} \cup V_{kl}$ by m.d.p. of color i and alternating between vertices of V_{ij} and V_{kl}. By contradiction, suppose that there exists $w \in V_{ij} \cup V_{kl}$ so that there is no $w \rightsquigarrow_m n$ of color i alternating between V_{ij} and V_{kl}. We have two cases:

Case 1. $w \in V_{kl}$.

Observe that by property (3) of the hypothesis and since $\{i\} = \{i,j\} \cap \{k,l\}$, every arc between V_{ij} and V_{kl} is colored i. Since n does not absorb w by m.d.p. of color i alternating between V_{ij} and V_{kl}, we have that $(n,w) \in A(T)$ when w is a kernel by m.d.p. in V_{kl} and we obtain that $\mathcal{A}(w) > \mathcal{A}(n)$, a contradiction to the maximality of $\mathcal{A}(n)$. So, there exists $w_1 \in V_{kl}$ such that there is no $w_1 \rightsquigarrow_m w$ contained in T. Therefore $(n, w_1) \in A(T)$ (otherwise, the path from w_1 through n to w is a $w_1 \rightsquigarrow_m w$ in T which is a contradiction).

If w_1 is a kernel by m.d.p in $T[V_{kl}]$, then $w_1 \in N$ and since $\mathcal{A}(w_1) > \mathcal{A}(n)$, we have a contradiction. So, w_1 is not a kernel by m.d.p. in $T[V_{kl}]$ and there exists $w_2 \in V_{kl}$ such that there is no $w_2 \rightsquigarrow_m w_1$ in T. Moreover, $(n, w_2) \in A(T)$ (otherwise, the path from w_2 through n to w_1 is a $w_2 \rightsquigarrow_m w_1$ in T which is a contradiction).

If w_2 is a kernel by m.d.p in $T[V_{kl}]$, then $w_2 \in N$ and since $\mathcal{A}(w_2) > \mathcal{A}(n)$, we have a contradiction. So, w_2 is not a kernel by m.d.p. in $T[V_{kl}]$ and there exists $w_3 \in V_{kl}$ such that there is no $w_3 \rightsquigarrow_m w_2$ in T. Moreover, $(n, w_3) \in A(T)$ (otherwise, the path from w_3 through n to w_2 is a $w_3 \rightsquigarrow_m w_2$ in T which is a contradiction).

Continuing this procedure, we obtain a sequence of vertices $w = w_0$, w_1, w_2, w_3, $\ldots$ such that there is no $w_{i+1} \rightsquigarrow_m w_i$ in T. Since T is finite, there exist indices r and s ($r < s$) such that $w_r = w_s$ and $(w_r, w_{r+1}, \ldots, w_s)$ is an asymmetrical directed cycle in $\mathfrak{C}(T[V_{kl}])$, contradicting that $T[V_{kl}]$ is m.k.p. (see Theorem 3 and Remark 1(ii)).

Case 2. $w \in V_{ij}$.

First, observe that there exists $z \in V_{kl}$ such that $(n, z) \in A(T)$, otherwise, we have that $(z, n) \in A(T)$ for every $z \in V_{kl}$, and option (ii) of the Lemma would be satisfied (recall that previous to this claim, it was supposed that (ii) and (iii) do not hold).

Subcase 2.1. $(w, z) \in A(T)$.

Since $\mathcal{A}(z) > \mathcal{A}(n)$ (note that $w \in \mathcal{M}(z) \setminus \mathcal{M}(n)$), z is not a kernel by m.d.p. in $T[V_{kl}]$. Hence there exists $z_1 \in V_{kl}$ such that there is no $z_1 \rightsquigarrow_m z$ in T. Observe that $(n, z_1) \in A(T)$ (resp. $(w, z_1) \in A(T)$), because otherwise, the path from z_1 through n to z (resp. the path from z_1 through w to z) is a $z_1 \rightsquigarrow_m z$ in T, a contradiction.

If z_1 is a kernel by m.d.p. in $T[V_{kl}]$, we have a contradiction since $\mathcal{A}(z_1) > \mathcal{A}(n)$. So, z_1 is not a kernel by m.d.p. in $T[V_{kl}]$ and there exists $z_2 \in V_{kl}$ such that there is no $z_2 \rightsquigarrow_m z_1$ in T. Recall again that $(n, z_2) \in A(T)$ (resp. $(w, z_2) \in A(T)$), because otherwise, the path from z_2 through n to z_1 (resp. the path from z_2 through w to z_1) is a $z_2 \rightsquigarrow_m z_1$ in T, a contradiction.

If z_2 is a kernel by m.d.p. in $T[V_{kl}]$, we have a contradiction since $\mathcal{A}(z_2) > \mathcal{A}(n)$. So, z_2 is not a kernel by m.d.p. in $T[V_{kl}]$ and there exists $z_3 \in V_{kl}$ such that there is no $z_3 \rightsquigarrow_m z_2$ in T. Recall again that $(n, z_3) \in A(T)$ (resp. $(w, z_3) \in A(T)$), because otherwise, the path from z_3 through n to z_2 (resp. the path from z_3 through w to z_2) is a $z_3 \rightsquigarrow_m z_2$ in T, a contradiction.

Continuing this procedure, we obtain a sequence of vertices $z = z_0$, z_1, z_2, z_3, $\ldots$ such that there is no $z_{i+1} \rightsquigarrow_m z_i$ in T. Since T is finite, there exist indices r and s ($r < s$) such that $z_r = z_s$ and $(z_r, z_{r+1}, \ldots, z_s)$ is an asymmetrical directed cycle in $\mathfrak{C}(T[V_{kl}])$, contradicting that $T[V_{kl}]$ is m.k.p. (see Theorem 3 and Remark 1(ii)).

Therefore this subcase is impossible.

Subcase 2.2. $(z, w) \in A(T)$.

This subcase is impossible too, the proof is analogous to the subcase before (take w and V_{ij} instead of z and V_{kl}).

Claim 2 is proved.

By Claim 1, we can suppose without loss of generality that $z \in V_{ij}$ is a kernel by m.d.p. in $T[V_{ij}]$ and z absorbs every vertex of $V_{ij} \cup V_{kl}$ by m.d.p. of color i alternating between vertices of V_{ij} and V_{kl}. Recall that part (ii) of the lemma does not hold, so there exists $y \in V_{kl}$ such that $(y, z) \notin A(T)$ and therefore $(z, y) \in A(T)$. Clearly, z and y are the vertices satisfying (iv). This concludes the proof of the lemma. □

Theorem 9. *Let T be an m-colored tournament, $M = \{1, 2, ..., m\}$ the set of colors,*

$$M' \subseteq M^2 = \{S \subseteq M : |S| = 2\}$$

and $\mathcal{Z} = \{V_{ij} : \{i, j\} \in M'\}$ a family of (not necessarily distinct) nonempty subsets of $V(T)$ such that:

(1) $V(T) = \bigcup_{V_{ij} \in \mathcal{Z}} V_{ij}$,

(2) $T[V_{ij}]$ is m.k.p. and

(3) for every $z \in V_{ij}$ and $w \in V(T) - V_{ij}$, the arc $[z, w]$ is colored i or j.

If T contains no 3-colored directed triangles, then T has a kernel by m.d.p.

Proof. Clearly, we can suppose that $V_{ij} \neq V(T)$ for every $\{i, j\} \in M'$, otherwise the theorem is true by property (2). So, the conditions (1), (2) and (3) of Lemma 1 are satisfied. We define a digraph $D_{\mathcal{Z}}$ as follows: $V(D_{\mathcal{Z}}) = \mathcal{Z}$ (take a vertex for each element of $\mathcal{Z}$ without repetition) and

$$\begin{aligned}(V_{kl}, V_{ij}) &\in A(D_{\mathcal{Z}}) \text{ of color } i \text{ if (ii) of Lemma 1 holds and } \{k, l\} \cap \{i, j\} = \{i\},\\ (V_{ij}, V_{kl}) &\in A(D_{\mathcal{Z}}) \text{ of color } i \text{ if (iii) of Lemma 1 holds and } \{k, l\} \cap \{i, j\} = \{i\}\end{aligned}$$

and

$$\{(V_{ij}, V_{kl}); (V_{kl}, V_{ij})\} \subseteq A(D_{\mathcal{Z}}) \text{ of color } i \quad \begin{array}{l}\text{if (iv) of Lemma 1 holds}\\ \text{and } \{k, l\} \cap \{i, j\} = \{i\}.\end{array}$$

Claim 1. *Every directed triangle in $D_{\mathcal{Z}}$ is either monochromatic or symmetrical.*

Let $\overrightarrow{C}_3 = (V_{kl}, V_{ij}, V_{mn}, V_{kl})$ be a non symmetrical directed triangle in $D_{\mathcal{Z}}$. We will prove that $\overrightarrow{C}_3$ is monochromatic. Then $\overrightarrow{C}_3$ has at least one arc, say (V_{kl}, V_{ij}), satisfying (ii) of Lemma 1, that is, there exists $x_{ij} \in V_{ij}$ which absorbs every vertex of $T[V_{ij}]$ by m.d.p. and such that $(y, x_{ij}) \in A(T)$ for every $y \in V_{kl}$. Since $(V_{ij}, V_{mn}) \in A(D_{\mathcal{Z}})$ and $(V_{mn}, V_{kl}) \in A(D_{\mathcal{Z}})$, there exist $x_{mn} \in V_{mn}$ and $x_{kl} \in V_{kl}$ such that $(x_{ij}, x_{mn}) \in A(T)$ and $(x_{mn}, x_{kl}) \in A(T)$. So, we have a directed triangle $\gamma = (x_{kl}, x_{ij}, x_{mn}, x_{kl})$ contained in T. By hypothesis, γ is not 3-colored, hence γ has at least two arcs of the same color. Without loss of generality, suppose that (x_{kl}, x_{ij}) and (x_{ij}, x_{mn}) are colored i. By condition (3) of the Theorem, $\{k, l\} \cap \{i, j\} = \{i\}$ and $\{i, j\} \cap \{m, n\} = \{i\}$, therefore $\{k, l\} \cap \{m, n\} = \{i\}$. So, every arc between V_{kl} and V_{mn} is colored i, in particular, (x_{mn}, x_{kl}) is colored i. We conclude that $\overrightarrow{C}_3 = (V_{kl}, V_{ij}, V_{mn}, V_{kl})$ is monochromatic.

Claim 2. *If D is a semicomplete m-colored digraph such that every directed triangle is monochromatic or symmetrical, then D has a kernel by m.d.p.*

Since D is semicomplete, then $\mathfrak{C}(D)$ is semicomplete. Moreover, since every directed triangle in D is monochromatic or symmetrical, it follows that every directed triangle in $\mathfrak{C}(D)$ is symmetrical. Thus, $\mathfrak{C}(D)$ has a kernel and therefore D has a kernel by m.d.p.

Let $V_{ij} \in \mathcal{Z}$. For every $V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}})$, we denote by $x_{ij,kl}$ a chosen vertex in V_{ij} such that:

(i) If (ii) of Lemma 1 holds, then $(y, x_{ij,kl}) \in A(T)$ for every $y \in V_{kl}$ and $x_{ij,kl}$ is a kernel by m.d.p. of $T[V_{ij}]$.

(ii) If (iv) of Lemma 1 holds, then $x_{ij,kl}$ absorbs every vertex of $V_{ij} \cup V_{kl}$ by m.d.p. of color i.

Claim 3. *There exists $z_{ij} \in V_{ij}$ such that there is a $y \rightsquigarrow_m z_{ij}$ of color i in T, for every $V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}})$ and all $y \in V_{kl}$ for which $(y, x_{ij,kl}) \in A(T)$ of color i.*

For every $w \in V_{ij}$, define

$$\begin{aligned}\mathcal{B}(w) = \mid \{V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}}) : \forall\, y \in V_{kl} \text{ with } (y, x_{ij,kl}) \in A(T) \text{ of color } i \\ \exists\, y \rightsquigarrow_m w \text{ of color } i \text{ in } T\} \mid .\end{aligned}$$

Let $z_{ij} \in V_{ij}$ such that $\mathcal{B}(z_{ij}) = \max\{\mathcal{B}(w) : w \in V_{ij}\}$. We will prove that z_{ij} satisfies the conditions of the claim. If $\mathcal{B}(w) = d^-(V_{ij}, D_{\mathcal{Z}})$, the claim holds. If it is not the case, there exists $V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}})$ such that for some $y \in V_{kl}$ with $(y, x_{ij,kl}) \in A(T)$ of color i there is no $y \rightsquigarrow_m z_{ij}$ in T. So, $(z_{ij}, y) \in A(T)$. Moreover, there exists $x \in V_{mn}$ ($V_{mn} \in N^-(V_{ij}, D_{\mathcal{Z}})$) such that $(x, z_{ij}) \in A(T)$ and there is no $x \rightsquigarrow_m x_{ij,kl}$ in T. If this m.d.p. exists, it is colored the same color of arc (x, z_{ij}), since every arc between V_{mn} and V_{ij} are of the same color, and we have that $\mathcal{B}(x_{ij,kl}) > \mathcal{B}(z_{ij})$, a contradiction. Therefore (x, z_{ij}, y) and $(y, x_{ij,kl}, x)$ are two directed paths of length 2 in T. We have two possibilities:

(1) If $(x, y) \in A(T)$, then $\overrightarrow{C}_3 = (y, x_{ij,kl}, x, y)$ is a directed triangle. Since T does not contain 3-colored directed triangles, $\overrightarrow{C}_3$ has at least two arcs of the same color. By property (3) of the theorem, $\overrightarrow{C}_3$ is monochromatic and we have that $(x, y, x_{ij,kl})$ is a $x \rightsquigarrow_m x_{ij,kl}$ in T, a contradiction.

(2) If $(y, x) \in A(T)$, then $\overrightarrow{C}_3 = (x, z_{ij}, y, x)$ is a directed triangle in T. Following the argument of (1), $\overrightarrow{C}_3$ is monochromatic and we have that (y, x, z_{ij}) is a $y \rightsquigarrow_m z_{ij}$ in T, a contradiction.

Claim 3 is proved.

Claim 4. *There exists $z_{ij} \in V_{ij}$ such that z_{ij} absorbs every vertex of V_{ij} and there is a $y \rightsquigarrow_m z_{ij}$ of color i in T, for every $V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}})$ and all $y \in V_{kl}$ for which $(y, x_{ij,kl}) \in A(T)$ of color i.*

By Claim 3, there exists $z_{ij} \in V_{ij}$ such that there is a $y \rightsquigarrow_m z_{ij}$ of color i in T, for every $V_{kl} \in N^-(V_{ij}, D_{\mathcal{Z}})$ and all $y \in V_{kl}$ for which $(y, x_{ij,kl}) \in A(T)$ of color i. Let S be the subset of vertices of V_{ij} which satisfy the property above and n a kernel by m.d.p. of $T[S]$. It remains to prove that n absorbs every vertex of $V_{ij} - S$ by m.d.p. Let $w \in V_{ij} - S$ and by contradiction, suppose that there is no $w \rightsquigarrow_m n$ in T. Thus $(n, w) \in A(T)$. Observe that if $(x, n) \in A(T)$ and (x, n) is colored r, for example, then $(x, w) \in A(T)$ and (x, w) is colored r. Otherwise, we would have that $(w, x) \in A(T)$ and (w, x) is colored r (recall that every arc between x and the vertices of V_{ij} is of the same color by property (3)) and hence (w, x, n) is a $w \rightsquigarrow_m n$ in T, a contradiction. So, $(x, w) \in A(T)$ for every $x \in V_{ij}$ for which $(x, n) \in A(T)$. Clearly, it follows that $w \in S$, a contradiction. This concludes the proof of the claim.

Finally, by Claims 1 and 2, we have that $D_{\mathcal{Z}}$ has a kernel by m.d.p. and by Lemma 1, the definition of $D_{\mathcal{Z}}$ and Claim 4, T has a kernel by m.d.p. □

The following corollary positively answers Problem 2 for all m-colored tournaments with bichromatic ex-neighborhoods (resp. in-neighborhoods) and generalizes Theorem 6 and 9 of [5] and Corollary 1 of [9]

Corollary 2. *Let T be an 3-colored tournament and suppose that $\xi^+(z) \leq 2$ (resp. $\xi^-(z) \leq 2$) for every vertex $z \in V(T)$. If T contains no 3-colored directed triangles, then T has a kernel by m.d.p.*

We define $A(z) = A^+(z) \cup A^-(z)$ (see $(\triangle)$), $C(z)$ the set of colors appearing in $A(z)$ and $\xi(z) = |C(z)|$. As a consequence of Theorem 9, we have Corollary 1 of [9] and Theorem 6 and 9 of [5]:

Corollary 3 ([9], Corollary 1)**.** *Let T be a 2-colored tournament. Then T has a kernel by m.d.p.*

Corollary 4 ([5], Theorem 6)**.** *Let T be an 3-colored tournament and suppose that $\xi(z) \leq 2$ for every vertex $z \in V(T)$. If T contains no 3-colored directed triangles, then T has a kernel by m.d.p.*

Corollary 5 ([5], Theorem 9)**.** *If T is an m-colored tournament with $m \geq 4$ and suppose that $\xi(z) \leq 2$ for every vertex $z \in V(T)$. Then T has a kernel by m.d.p.*

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