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Some operations of graphs that preserve the property of well-covered by monochromatic paths

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Abstract

A graph is called well-covered if every maximal independent set of vertices of G is a maximum independent set; recall that S is independent if no two of its vertices are adjacent. In this paper we define the concept of well-covered by monochromatic paths graphs which is a variation of well-covered graphs. We consider some classical constructions of graphs: G-join of graphs and duplication of a subset of vertices. We also give necessary and sufficient conditions for well-coveredness by monochromatic paths of these graphs.

1 Introduction

For concepts not defined here see [2]. Let G be a finite connected graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. By a *path* from a vertex x_1 to a vertex x_n, $n \geq 2$ we mean a sequence of vertices $x_1, \ldots, x_n$ and edges $\{x_i, x_{i+1}\} \in E(G)$, for $i = 1, \ldots, n-1$ and for simplicity we denote it by $x_1 \ldots x_n$. A graph G is said to be *edge m-coloured* if its edges are coloured with m colours. A path is called *monochromatic* if all its edges are coloured alike. A subset $S \subset V(G)$ is said to be *independent by monochromatic paths* of the edge-coloured graph G if for any two different vertices $x, y \in S$ there is no monochromatic path between them. In addition a subset containing only one vertex, and the empty set are called independent by monochromatic paths sets of G. Note that every subset of an independent by monochromatic paths set of G also is an independent by monochromatic paths set of G. For convenience throughout this paper we will write an *imp-set of* G instead of an independent by monochromatic paths set of G. For the proper edge colouring of the graph G an imp-set of G is an independent set of G in the classical sense.

The concept of independence in graphs has existed in literature for a long time. There are many generalizations of the independence in graphs. The concept of independence by monochromatic paths was introduced in [4], studied for instance in [5], [8], [12], [13] and generalizes independence in the classical sense. A graph G is called *well-covered by monochromatic paths* if every maximal imp-set of G is a maximum imp-set of G. The concept of well-covered by monochromatic paths graphs is a variation of well-covered graphs. The well-covered graphs were introduced by Plummer in [7] and generalized on well-k-covered graphs by Favaron and Hartnell in [3]. Some interest in these graphs is motivated by the fact that a maximum independent set can be found efficiently in a well-covered graph whereas the independent set problem is NP-complete for general graphs.

Let G be an edge-coloured simple graph. By $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$, $t \geq 1$ we denote the family of all connected, maximal (with respect to set inclusion) monochromatic subgraphs of G. In [12] an uncoloured simple graph $G(\mathcal{Q})$ was defined as follows: $V(G(\mathcal{Q})) = V(G)$ and $E(G(\mathcal{Q})) = \{\{x_p, x_q\}; x_p, x_q \in V(Q_i), i = 1, \ldots, t\}$ with replacing multiple edges by one edge. Relationships between imp-sets in G and independent sets in $G(\mathcal{Q})$ were studied in [12]. It is easy to observe that a subset S is a maximal imp-set of G if and only if S is a maximal independent set of $G(\mathcal{Q})$. Then the next result is obvious:

Proposition 1 *An edge-coloured graph G is well-covered by monochromatic paths if and only if $G(\mathcal{Q})$ is well-covered.*

Let G be an edge-coloured graph with $V(G) = \{x_1, \ldots, x_n\}$, $n \geq 2$, and $\alpha = (G_i)_{i \in \{1,\ldots,n\}}$ be a sequence of vertex disjoint edge-coloured graphs on $V(G_i) = V = \{y_1, \ldots, y_p\}$, $p \geq 1$, $i = 1, \ldots, n$. Then the *G-join of the graph G and the sequence α* is the graph $G[\alpha]$ such that $V(G[\alpha]) = V(G) \times V$ and $E(G[\alpha]) = \{((x_s, y_j), (x_q, y_t))$ coloured ψ; ($x_s = x_q$ and $(y_j, y_t) \in E(G_s)$ coloured ψ) or ($(x_s, x_q) \in E(G)$ coloured ψ)$\}$. By G_i^c we mean a copy of the graph G_i in $G[\alpha]$. If all graphs from sequence α are isomorphic to the same graph H, then from the G-join we obtain the composition $G[H]$ of graphs G and H. Figure 1 contains a small example of $G[\alpha]$, for $\alpha = (G_1, G_2)$, where G_1, G_2 are different.

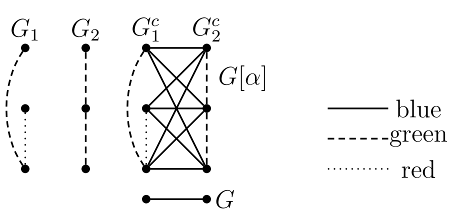

Fig. 1. The graph $G[\alpha]$

Maximal k-independent sets (i.e. maximal independent sets generalized in the distance sense) in G-join graphs were also studied in [11]. The well-coveredness of $G[\alpha]$ was considered in [9]. Well-covered products of graphs were studied in [10].

Let G be an edge-coloured graph and X be an arbitrary nonempty subset of $V(G)$. Let H be a graph isomorphic to a subgraph of G induced by X. The vertex from $V(H)$ that corresponds to $x \in X$ we will denote by x^c. The *duplication of X in G* denoted by G^X is the graph such that $V(G^X) = V(G) \cup V(H)$ and $E(G^X) = E(G) \cup E(H) \cup E$ where $E = \{\{x^c, y\}$ coloured ψ; $x^c \in V(H)$ and $\{x, y\} \in E(G)$ coloured $\psi\}$. A vertex $x^c \in V(H)$ (respectively a subset $S^c \subseteq V(H)$) we will call the *copy of the vertex $x \in X$* (resp. the copy of the subset $S \subseteq X$). The vertex $x \in X$ (resp. the subset $S \subseteq X$) will be named the *original of the vertex x^c* (resp. of the subset S^c) and if it is necessary the original of the vertex x^c (resp. of the subset S^c) we will denote by x^o (resp. S^o). The duplication of a vertex of a graph was introduced in [1] and in [6] the definition of the duplication of a subset of vertices of a graph was given as a generalization. We have applied this definition to edge-coloured graphs. Figure 2 contains a small example of G^X, where $V(G) \supset X = \{x, y, z\}$.

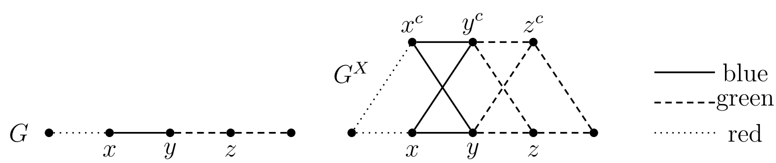

Fig. 2. The graph G^X

In this paper we study the well-coveredness by monochromatic paths in G-join of graphs and in the duplication G^X.

2 The well-coveredness by monochromatic paths of G-join of graphs

In this section we give necessary and sufficient conditions for the well-coveredness by monochromatic paths of the G-join of graphs. Our first lemma describes maximal imp-sets in G-join.

Lemma 1 *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, \ldots, n$. A subset $S^* \subset V(G[\alpha])$ is a maximal imp-set of $G[\alpha]$ if and only if S is a maximal imp-set of G such that $S^* = \bigcup_{i \in \mathcal{I}} S_i$, where $\mathcal{I} = \{i, x_i \in S\}$ and S_i is 1-element set containing an arbitrary vertex from $V(G_i^c)$, for every $i \in \mathcal{I}$.*

Proof: 1. Let S^* be a maximal imp-set of $G[\alpha]$. Denote $S = \{x_i \in V(G); S^* \cap V(G_i^c) \neq \emptyset\}$. At first we shall prove that S is an imp-set of G. We proceed by contradiction, suppose that S is not an imp-set of G. This means that there exist $x_i, x_j \in S$ such that there is a monochromatic path $x_i \ldots x_j$ in G. Hence by the definition of $G[\alpha]$ for each pair of vertices $(x_i, y_r) \in V(G_i^c)$ and $(x_j, y_q) \in V(G_j^c)$, where $1 \leq r, q \leq p$ there is a monochromatic path $(x_i, y_r) \ldots (x_j, y_q)$. By the definition of the set S we have that $S^* \cap V(G_i^c) \neq \emptyset$ and $S^* \cap V(G_j^c) \neq \emptyset$ so there exists a monochromatic path between vertices from S^*, contradiction with independence by monochromatic paths of S^*. Now we will prove that S is maximal. Suppose on contrary that S is not a maximal imp-set of G. Then there is $x_t \in (V(G) \setminus S)$ such that the set $S \cup \{x_t\}$ is an imp-set of G. Hence for every (x_t, y_m), $1 \leq m \leq p$ the set $S^* \cup \{(x_t, y_m)\}$ would be a greater imp-set of $G[\alpha]$, a contradiction that S^* is maximal. Evidently $S^* = \bigcup_{i \in \mathcal{I}} S_i$ where $\mathcal{I} = \{i; x_i \in S\}$. The definition of $G[\alpha]$ implies that for every two vertices from each copy G_i^c, $i = 1, \ldots, n$ there exists a monochromatic path between them in $G[\alpha]$. Hence at most one vertex from each copy G_i^c, $i \in \mathcal{I}$ can belong to the set S^*. So S_i is an 1-element set containing an arbitrary vertex from $V(G_i^c)$, for every $i \in \mathcal{I}$.

2. Let $S \subseteq V(G)$ be a maximal imp-set of G and let S_i, where $i \in \mathcal{I}$ and $\mathcal{I} = \{i; x_i \in S\}$ be an 1-element set containing an arbitrary vertex from $V(G_i^c)$. We will prove that $S^* = \bigcup_{i \in \mathcal{I}} S_i$ is a maximal imp-set of $G[\alpha]$. It is obvious from the definition of $G[\alpha]$ that S^* is an imp-set of $G[\alpha]$. Assume on the contrary that S^* is not a maximal imp-set of $G[\alpha]$. Then there is $(x_t, y_m) \in (V(G[\alpha]) \setminus S^*)$ such that the set $S^* \cup \{(x_t, y_m)\}$ is an imp-set of $G[\alpha]$. The definition of S^* implies that $x_t \notin S$ in otherwise contradiction with the assumption of $S_t, t \in \mathcal{I}$. Moreover the definition of $G[\alpha]$ implies that there does not exist a monochromatic path between x_t and x_i, for every $i \in \mathcal{I}$. So $S \cup \{x_t\}$ is an imp-set of G a contradiction with maximality of S.

Thus the lemma is proved. □

Theorem 1 *Let G be an edge-coloured graph on n vertices, $n \geq 2$ and α be a sequence of vertex disjoint edge-coloured graphs G_i, $i = 1, \ldots, n$. Then $G[\alpha]$ is well-covered by monochromatic paths if and only if G is well-covered by monochromatic paths.*

Proof: We begin by assuming that $G[\alpha]$ is a well-covered by monochromatic paths graph. Assume on the contrary that G is not well-covered by monochromatic paths. This means that there are maximal imp-sets of G say, S_1 and S_2 such that $|S_1| \neq |S_2|$. Let $\mathcal{I}_1 = \{i; x_i \in S_1\}$ and $\mathcal{I}_2 = \{j; x_j \in S_2\}$. By Lemma 1, it immediately follows that there exist maximal imp-sets $S_1^* = \bigcup_{i \in \mathcal{I}_1} S_i$ and $S_2^* = \bigcup_{j \in \mathcal{I}_2} S_j$ of $G[\alpha]$, where S_i, S_j are arbitrary 1-element sets of G_i^c, G_j^c, respectively. Consequently by assumptions of S_1,S_2 we have that $|S_1^*| \neq |S_2^*|$, contradiction with well-coveredness by monochromatic paths of $G[\alpha]$.

For the converse assume that G is a well-covered by monochromatic paths graph. We shall prove that $G[\alpha]$ is a well-covered by monochromatic paths graph. For this purpose assume that S_1^* and S_2^* are two arbitrary maximal imp-sets of $G[\alpha]$. Then Lemma 1 gives that $S_1^* = \bigcup_{i \in \mathcal{I}_1} S_i$ where $\mathcal{I}_1 = \{i; x_i \in S_1\}$ and $S_2^* = \bigcup_{j \in \mathcal{I}_2} S_j$ where $\mathcal{I}_2 = \{j; x_j \in S_2\}$ and S_1, S_2 are maximal imp-sets of G. Moreover $|S_1^*| = |S_1|$ and $|S_2^*| = |S_2|$. From well-coveredness by monochromatic paths of G every two maximal imp-sets of G has the same cardinality so it is obvious that $|S_1^*| = |S_2^*|$. Consequently $G[\alpha]$ is well covered by monochromatic paths, which completes the proof. □

3 The well-coveredness by monochromatic paths duplication G^X

In this section we give necessary and sufficient conditions for the well-coveredness by monochromatic paths of a duplication of a subset of vertices of a graph.

These results follows directly from the definition of G^X.

(1) Let G be an edge-coloured graph and $X \subseteq V(G)$. Let $x, y \in X$ and $x^c, y^c \in X^c$. Then the following conditions are equivalent:

(1.1) there is a monochromatic path $x \dots y$ in G

(1.2) there is a monochromatic path $x \dots y$ in G^X

(1.3) there is a monochromatic path $x^c \dots y^c$ in G^X

(1.4) there is a monochromatic path $x \dots y^c$ in G^X.

(2) Let G be an edge-coloured graph and $X \subseteq G$. Let $x \in X$, $x^c \in X^c$ and $u \in V(G) \setminus X$. Then the following conditions are equivalent:

(2.1) there is a monochromatic path $u \dots x$ in G

(2.2) there is a monochromatic path $u \dots x$ in G^X

(2.3) there is a monochromatic path $u \dots x^c$ in G^X.

(3) Let G be an edge-coloured graph and $X \subseteq G$. Let $u, v \in V(G) \setminus X$. There is a monochromatic path $u \dots v$ in G if and only if there is a monochromatic path $u \dots v$ in G^X.

The next corollary follows from the above facts:

Corollary 1 *Let G be an edge-coloured graph and $X \subseteq G$. Let $u, v \in V(G)$. There is a monochromatic path $u \dots v$ in G if and only if there is a monochromatic path $u \dots v$ in G^X.*

Lemma 2 *Let G be an edge-coloured graph, $X \subseteq V(G)$ and $S \subset V(G^X)$ be an arbitrary imp-set of G^X. For an arbitrary $x \in X$ and $x^c \in V(G^X)$ exactly one condition is fulfilled:*

(1) $x \notin S$ and $x^c \notin S$ or

(2) either $x \in S$ or $x^c \in S$, but not both.

Proof: Let $S \subset V(G^X)$ be an imp-set of G^X and assume on the contrary that there exists $x \in X$ such that $x \in S$ and $x^c \in S$. Because $x \in X \subseteq V(G)$ then there exists $y \in V(G)$ such that $\{x, y\} \in E(G)$ coloured ψ. From the definition of the duplication also $\{x^c, y\} \in E(G^X)$ coloured ψ. Hence there exists a monochromatic path xyx^c coloured ψ, contradiction with independence by monochromatic paths of S.

Thus the lemma is proved. □

Lemma 3 *Let G be an edge-coloured graph and $X \subseteq V(G)$. If S is a maximal imp-set of G then S is a maximal imp-set of G^X.*

Proof: Let S be a maximal imp-set of G. We shall show that S is a maximal imp-set of G^X. It is obvious that S is an imp-set of G^X. Assume on contrary that S is not maximal in G^X. This means that there is a vertex $x \in V(G^X)$ such that $S \cup \{x\}$ is an imp-set of G^X. We distinguish two possible cases:

(1) $x \in V(G)$.

From the maximality of the set S in G we deduce that there is a vertex $y \in S$ and a monochromatic path $x \dots y$ in G. Hence using Corollary 1 we obtain that there exists a monochromatic path $x \dots y$ in G^X, contradiction with the assumption.

(2) $x \in X^c$.

By Lemma 2 we obtain that the original x^o of x does not belong to S. Because S is a maximal imp-set of G there is a vertex $y \in S$ and a monochromatic path $x^o \dots y$ in G. Consequently by (2) there is a monochromatic path $x \dots y$ in G^X contradiction with the assumption.

Thus the lemma is proved. □

Lemma 4 *Let G be an edge-coloured graph and $X \subseteq V(G)$. If S^* is a maximal imp-set of G^X then there exists a maximal imp-set S of G such that $|S| = |S^*|$.*

Proof: Assume that $S^* \subset V(G^X)$ is a maximal imp-set of G^X. We will prove that $S = (S^* \cap V(G)) \cup (S^* \cap X^c)^o$ is a maximal imp-set of G. Clearly $|S| = |S^*|$. Let $S_1 = S^* \cap V(G)$ and $S_2 = (S^* \cap X^c)^o$. Hence $S_2^c = S^* \cap X^c$. Of course S_1 and S_2^c are imp-sets of G^X, so by the definition of the duplication S_1 and S_2 are imp-sets of G. Firstly we will prove that $S_1 \cup S_2$ is an imp-set of G. It is enough to prove that there does not exist a monochromatic path between x and y in G, for every $x \in S_1$ and $y \in S_2$. Assume on contrary that there exist $x \in S_1$ and $y \in S_2$ and a monochromatic path between them in G. Clearly $x, y^c \in S^*$. Consequently by (2) there exists a monochromatic path $x \dots y^c$ in G^X, a contradiction with the assumption of S^*. Now we shall show that $S_1 \cup S_2$ is a maximal imp-set of G. We proceed by contradiction, suppose that $S_1 \cup S_2$ is not maximal in G. This means that there exists $y \in V(G)$ such that $S_1 \cup S_2 \cup \{y\}$ is an imp-set of G. Of course $y \notin S^*$, hence from the maximality of the set S^* in G^X we obtain that there exists a vertex $x \in S^*$ such that a path $x \dots y$ is monochromatic in G^X. If $x \in S_1$ then by Corollary 1 we have that a path $x \dots y$ is monochromatic in G. Let $x \in S^* \cap X^c$. Evidently $x^o \in S_2$. Moreover if $y \in X$ then by (1) a path $x^o \dots y$ is monochromatic in G. If $y \in (V(G) \setminus X)$ then by (2) a path $x^o \dots y$ is monochromatic in G. All this together contradict that S is not maximal.

Thus the lemma is proved. □

Theorem 2 *Let G be an edge-coloured graph and $X \subseteq V(G)$. Then G^X is well-covered by monochromatic paths if and only if G is well-covered by monochromatic paths.*

PROOF: Let $\mathcal{S}$ be a family of maximal imp-sets of G and $\mathcal{S}^*$ be a family of maximal imp-sets of G^X. Assume that G is a well-covered by monochromatic paths graph and $X \subseteq V(G)$. We shall prove that the duplication G^X is well-covered by monochromatic paths. Let $S_1^*, S_2^* \in \mathcal{S}^*$. Then by Lemma 4 there are maximal imp-sets, say $S_1, S_2 \in \mathcal{S}$ such that $|S_1^*| = |S_1|$ and $|S_2^*| = |S_2|$. Because G is well-covered by monochromatic paths, $|S_1| = |S_2|$ so it immediately follows that $|S_1^*| = |S_2^*|$.

Let now G^X be a well-covered by monochromatic paths graph. We will prove that G is well-covered by monochromatic paths. Let $S_1, S_2 \in \mathcal{S}$. Then by Lemma 3 we have that $S_1, S_2 \in \mathcal{S}^*$ and by well-coveredness of G^X we obtain that $|S_1| = |S_2|$.

Thus the Theorem is proved. □

Let $X_1 \subseteq V(G)$ and G^{X_1} be the duplication of X_1 in G. For $n \geq 2$ by $G^{X_1,\ldots,X_n}$ we mean a duplication of X_n in $G^{X_1,\ldots,X_{n-1}}$.

Using Theorem 2 the next result is obvious:

Theorem 3 *Let G be an edge-coloured graph and $X_i \subseteq V(G^{X_1,\ldots,X_{i-1}})$, for $i = 1,\ldots,n$. Then $G^{X_1,\ldots,X_n}$ is well-covered by monochromatic paths if and only if G is well covered by monochromatic paths.*

4 Concluding remarks

Note that while many graphs are not well-covered, any graph can be trivially edge-coloured to make it well-covered by monochromatic paths (colour all the edges the same colour and any maximal imp-set is of size one, if the graph is connected). Also one could colour the edges of a well-covered graph in such a way that it would not be well-covered by monochromatic paths. There are a number of interesting open problems related to this area. It is natural to ask about a characterization of well-covered by monochromatic paths graphs when two or more colours are used (in particular if the number of colours is established).

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