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Decompositions of complete multipartite graphs via generalized graceful labelings

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Abstract

We prove the existence of infinite classes of cyclic Γ-decompositions of the complete multipartite graph, Γ being a caterpillar, a hairy cycle or a cycle. All results are obtained by the construction of d-divisible α-labelings of Γ, introduced in [A. Pasotti, On d-graceful labelings, *Ars Combin.* 111 (2013), 207–223] as a generalization of classical α-labelings. It is known that such labelings imply the existence of cyclic Γ-decompositions of certain complete multipartite graphs.

1 Introduction

As usual, we denote by K_v and $K_{m(n)}$ the *complete graph on v vertices* and the *complete m-partite graph with parts of size n*, respectively. For any graph Γ we write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. If $|E(\Gamma)| = e$, we say that Γ has *size e*. If Γ is a path or a cycle its size is more usually called its *length*.

Given a subgraph Γ of a graph K, a Γ-*decomposition of* K is a set of graphs isomorphic to Γ, called *blocks*, whose edges partition the edge-set of K. Such a decomposition $\mathcal{D}$ is said to be *cyclic* if, up to isomorphisms, the vertices of K are the elements of $\mathbb{Z}_v$ and if $B+1 \in \mathcal{D}$ for any block $B \in \mathcal{D}$ (where by $B+1$ one means the graph obtained from B by replacing each vertex $x \in V(B)$ by $x+1$). A Γ-decomposition of K_v is also called a Γ-*system of order v*. For a survey on graph decompositions see [9].

The concept of a *graceful labeling* of a graph Γ, introduced by A. Rosa in [26], is a useful tool for determining the existence of infinite classes of Γ-systems. Rosa proved that if a graph Γ of size e admits a graceful labeling then there exists a cyclic Γ-system of order $2e+1$ and if Γ admits an α-labeling then there exists a cyclic Γ-system of order $2en+1$ for any positive integer n. Labeled graphs are models for a broad range of applications; see for instance [7, 8]. For a very rich survey on graceful labelings we refer to [17]. Here we recall the basic definition. A *graceful labeling* of a graph Γ of size e is an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, e\}$ such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, \ldots, e\}.$$

In the case where Γ is bipartite and f has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one, f is said to be an *α-labeling.*

Many variations of graceful labelings have been considered over the years. In particular Gnana Jothi [18] defines an *odd graceful labeling* of a graph Γ of size e as an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, 2e - 1\}$ such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 3, 5, \ldots, 2e - 1\}.$$

In a recent paper, see [25], the second author introduced the following new definition of a *d-divisible graceful labeling* which is, at the same time, a generalization of the concepts of a graceful labeling (when $d = 1$) and of an odd graceful labeling (when $d = e$).

Definition 1.1. *Let Γ be a graph of size $e = d \cdot m$. A d-*divisible graceful labeling *of Γ is an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that*

$$\begin{aligned}\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{&1, 2, 3, \ldots, d(m + 1) - 1\} \\ &\setminus \{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}.\end{aligned}$$

By saying that d is *admissible* we will mean that it is a divisor of e and so it makes sense to investigate the existence of a d-divisible graceful labeling of Γ. We note that α-labelings can be generalized in a similar way, as in the following definition.

Definition 1.2. *A d-*divisible α-labeling *of a bipartite graph Γ is a d-divisible graceful labeling of Γ having the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one.*

Results on the existence of d-divisible α-labelings can be found in [24, 25]. In particular, using the notion of a $(v, d, \Gamma, 1)$-difference family introduced in [13], in [25] it is proved that the existence of a d-divisible (α-)labeling of a graph Γ implies the existence of cyclic Γ-decompositions. In fact:

Theorem 1.3. *If there exists a d-divisible graceful labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{\frac{e+d}{d}(2d)}$.*

Theorem 1.4. *If there exists a d-divisible α-labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{\frac{e+d}{d}(2dn)}$ for any positive integer n.*

In this paper we deal with the existence of d-divisible α-labelings of caterpillars, hairy cycles and cycles. In Section 2, we construct a d-divisible α-labeling of a caterpillar for any admissible value of d. We also introduce a generalization of d-divisible α-labelings, called α_S-labelings, and show that such labelings exist whenever Γ is a caterpillar. Then, in Section 3, we use these labelings to obtain d-divisible α-labelings of certain classes of hairy cycles and cycles. In particular, we show that bipartite hairy cycles admit an odd α-labeling and, when all the vertices of the cycle have the same degree, they admit a d-divisible α-labeling for any admissible value of d. In the case of cycles, we show that, for any positive integer k, C_{4k} admits a d-divisible α-labeling for any admissible value of d. We note that when Γ is a caterpillar or cycle and $d = 1$, the d-divisible α-labelings of Γ presented here are precisely the α-labelings given by Rosa in [26].

The existence of these d-divisible α-labelings allows us to obtain new infinite classes of cyclic decompositions of the complete multipartite graph. In particular, we obtain a cyclic Γ-decomposition of $K_{m(n)}$ for infinite pairs (m, n), whenever Γ is a caterpillar, a bipartite hairy cycle or a cycle of length divisible by 4. We recall that the existence problem for k-cycle decompositions of the complete graph has been completely solved in [2] for the case k odd and [27] for the case k even. An alternative easier proof for the odd case has been given in [10]. On the other hand we have only very partial results on the existence question for *cyclic* k-cycle decompositions of K_v. The most important result answers in the affirmative when $v \equiv 1 \pmod{2k}$ or $v \equiv k \pmod{2k}$ with k odd. Many authors contributed to this result (see [32] and the references therein). Also, the existence problem for a decomposition of the complete multipartite graph into cycles of an assigned length has not been solved yet, but a great progress has been recently done; see for instance [5, 6, 14, 15, 23, 28, 29, 30, 31]. Very little is known about the same problem with the additional constraint that the decomposition be cyclic. We have a complete solution in the following very special cases: the length of the cycles is equal to the size of the parts [11]; the cycles are Hamiltonian and the parts have size two [12, 20]. Finally, the literature is quite poor about results on decompositions (not necessarily cyclic) of the complete graph or the multipartite complete graph into caterpillar or hairy cycles. For some partial results see [1, 19, 21, 22, 26].

The results contained in this paper were already briefly presented in [4].

2 Caterpillars

Definition 2.1. *A* caterpillar Γ *is a tree, namely a graph without cycles, with the property that the removal of its vertices of degree one leaves a path.*

By the definition, any path is a caterpillar too. In [26], Rosa proved that any caterpillar admits an α-labeling. In this section we show, more generally, that any caterpillar admits a d-divisible α-labeling for any admissible value of d.
First of all we introduce the following notations to emphasize that a caterpillar is a bipartite graph and that any caterpillar Γ can be represented as a path $P = [v_1, \ldots, v_n]$ (called the *spine* of Γ, see [33]) with $n_i \geq 0$ pendant edges incident with the vertex v_i, $i = 1, \ldots, n$. For instance, the caterpillar Γ of Figure 1 can be seen as:

- a path of length 4 with $[n_1, n_2, n_3, n_4, n_5] = [0, 2, 0, 3, 0]$, see Figure 2(a);
- a path of length 3 with $[n_1, n_2, n_3, n_4] = [0, 2, 0, 4]$ or $[n_1, n_2, n_3, n_4] = [3, 0, 3, 0]$, see Figure 2(b) or 2(c), respectively;
- a path of length 2 with $[n_1, n_2, n_3] = [3, 0, 4]$, see Figure 2(d).

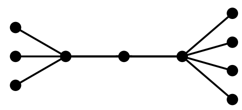


Figure 1: A caterpillar of size 9

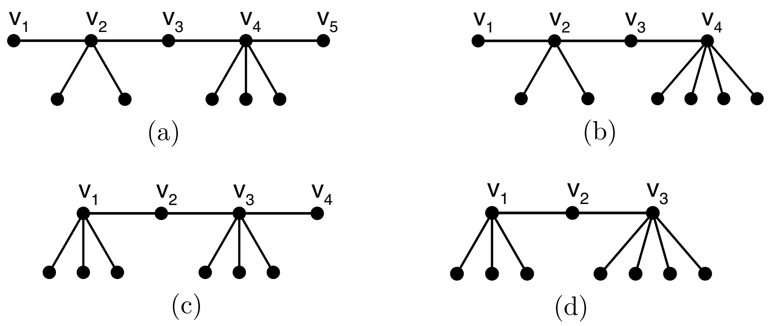


Figure 2: Some representations of the caterpillar of Figure 1

We note however that the results presented below are independent of the particular representation chosen for Γ. Let Γ be a caterpillar and let P be the path associated with its chosen representation. If the size of P is even, set $P = [x_1, y_1, x_2, y_2, \ldots, x_t, y_t]$. Let n_i and m_i be the number of pendant edges incident the vertex x_i and y_i, respectively, for $i = 1, \ldots, t$. Let $x_i^1, x_i^2, \ldots, x_i^{n_i}$ be the pendant vertices adjacent to the vertex x_i and $y_i^1, y_i^2, \ldots, y_i^{m_i}$ be the pendant vertices adjacent to the vertex y_i, for $i = 1, \ldots, t$; see Figure 3. Hence the two bipartite sets are

$$\begin{aligned} A &= \{x_1, y_1^1, y_1^2, \ldots, y_1^{m_1}, x_2, y_2^1, y_2^2, \ldots, y_2^{m_2}, x_3, \ldots, y_{t-1}^1, \\ &\quad y_{t-1}^2, \ldots, y_{t-1}^{m_{t-1}}, x_t, y_t^1, y_t^2, \ldots, y_t^{m_t}\} \end{aligned} \tag{1}$$

and

$$\begin{aligned} B &= \{x_1^1, x_1^2, \ldots, x_1^{n_1}, y_1, x_2^1, x_2^2, \ldots, x_2^{n_2}, y_2, x_3^1, x_3^2, \ldots, x_3^{n_3}, \\ &\quad y_3, \ldots, x_t^1, x_t^2, \ldots, x_t^{n_t}, y_t\}. \end{aligned} \tag{2}$$

We denote such a caterpillar by $C[n_1, m_1, n_2, m_2, \ldots, n_t, m_t]$.

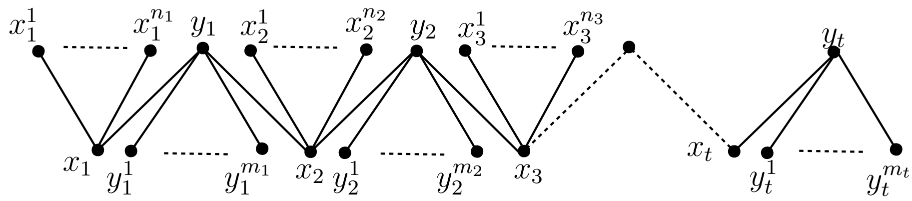


Figure 3: The caterpillar $C[n_1, m_1, n_2, m_2, \dots, n_t, m_t]$

Notice that the elements of $E(\Gamma)$ can be taken in a natural order from $[x_1, x_1^1]$ up to $[y_t, y_t^{m_t}]$, that is we can write

$$\begin{aligned} E(\Gamma) \;=\; & \{[x_1, x_1^1], [x_1, x_1^2], \dots, [x_1, x_1^{n_1}], [x_1, y_1], [y_1, y_1^1], \dots, [y_1, y_1^{m_1}], \\ & [y_1, x_2], [x_2, x_2^1], \dots, [x_2, x_2^{n_2}], [x_2, y_2], \dots, [x_t, x_t^{n_t}], [x_t, y_t], \\ & [y_t, y_t^1], [y_t, y_t^2], \dots, [y_t, y_t^{m_t}]\}. \end{aligned} \tag{3}$$

In what follows, by *consecutive* we will mean two vertices of A or of B, or two edges of $E(\Gamma)$, which are consecutive with respect to the order assumed in (1), (2) or (3), respectively.

Note that if the length of P is odd, we can use similar notation.

We now introduce a useful new concept which generalizes that of a d-divisible α-labeling.

Definition 2.2. *Let Γ be a bipartite graph of size e with parts A and B, let S be a set of e positive integers and let $\bar{s}$ be the largest element of S. An* α_S-labeling of Γ *is an injective function $f : V(\Gamma) \to \{0, 1, 2, \dots, \bar{s}\}$ such that*

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = S \quad \textit{and} \quad \max_A f < \min_B f.$$

Theorem 2.3. *Any caterpillar Γ of size e admits an α_S-labeling for any set S of e positive integers.*

Proof. The case $e = 1$ is trivial. Suppose then Γ is a caterpillar of size $e \geq 2$. We can choose without loss of generality a representation of Γ with a spine of even length, namely $\Gamma = C[n_1, m_1, n_2, m_2, \dots, n_t, m_t]$, where its parts A and B are defined as in (1) and (2), respectively. We label the edges of Γ, taken in the same order of (3), with the elements of S in ascending order. Let $f : V(\Gamma) \to \{0, 1, 2, \dots, \bar{s}\}$ (where, as above, $\bar{s}$ denotes the largest element of S) be the function defined by setting $f(y_t^{m_t}) = 0$ (with $f(x_t) = 0$ if $m_t = 0$) and so that, given $x \in A$ and $y \in B$, the label of the edge $[x, y]$ is $f(y) - f(x)$. Note that if we consider the elements of A in the same order of (1) their labels are in descending order and if we consider the elements of B in the same order of (2) their labels are in ascending order. So, it is easy to see that f is an injective function and that $\max_A f < \min_B f$. Hence f is an α_S-labeling of Γ. □

Definition 2.4. *We will call the function f constructed in the proof of Theorem 2.3* a standard α_S-labeling of Γ.

We have to point out that the function constructed in the proof of Theorem 2.3 depends on the chosen representation of the caterpillar, as shown in Example 2.5.

Example 2.5. *Consider the two different representation of the same caterpillar given in Figure 2 (b) and (c). Note that in both cases the path associated to the representation has even size and the difference depends only on the choice of A and B. Given a set S and following the proof of Theorem 2.3 we obtain two different standard α_S-labelings of Γ. For example, if we take $S = \{1, 3, 4, 5, 10, 12, 13, 15, 18\}$ we obtain the standard α_S-labelings shown in Figure 4.*

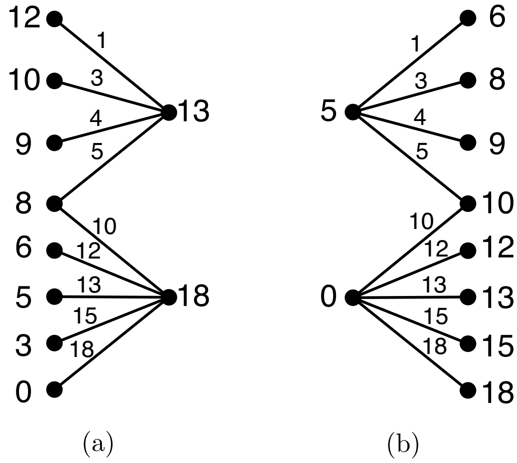


Figure 4: Standard α_S-labelings

Thanks to Theorem 2.3, we can completely solve the problem of the existence of d-divisible α-labelings of caterpillars.

Corollary 2.6. *Any caterpillar admits a d-divisible α-labeling for any admissible value of d.*

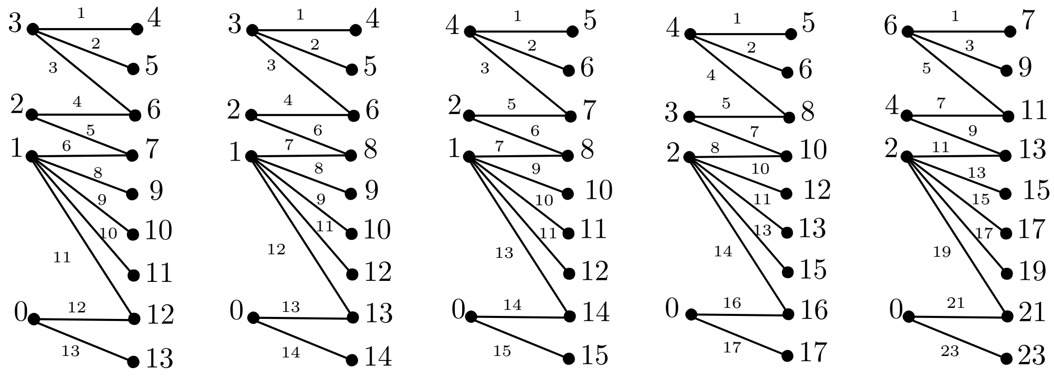
Proof. Let Γ be a caterpillar with $e = d \cdot m$ edges. Let f be a standard α_S-labeling of Γ where $S = \{1, 2, \ldots, d(m+1) - 1\} \setminus \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$. It is easy to see that f is a d-divisible α-labeling of Γ. □

As an immediate consequence of Theorem 1.4 and Corollary 2.6 we have:

Theorem 2.7. *Let Γ be a caterpillar with e edges. There exists a cyclic Γ-decomposition of $K_{\frac{e+d}{d}(2dn)}$ for any divisor d of e and any positive integer n.*

Remark 2.8. *We point out that if $d = 1$ the d-divisible α-labeling of the proof of Corollary 2.6 is nothing but the α-labeling obtained by Rosa in [26]. Also, if the caterpillar is indeed a path, we find again the d-divisible α-labeling constructed in [25].*

Example 2.9. *We consider a caterpillar* Γ *with* 12 *edges. In Figure 5 we show all possible d-graceful* α*-labelings of* Γ*, other than the classical one, obtained following the proof of Corollary 2.6.*



(a) (b) (c) (d) (e)

Figure 5: d-divisible α-labelings of a caterpillar, with $d = 2, 3, 4, 6, 12$

Definition 2.10. *Let* f *be an* α_S*-labeling of a graph* Γ *and let* $X \subseteq V(\Gamma)$*. We call each element of the set* $\{n \in \mathbb{N} \mid \min_X f \leq n \leq \max_X f\} \setminus f(X)$ *a* missing vertex label *in* $f(X)$*, or* mv-label *for short.*

We point out that, following the construction of the α_S-labeling provided in Theorem 2.3, each missing integer in S causes a corresponding mv-label in $f(A) \cup f(B)$. For instance, look at Figure 5(d). Now $S = \{1, 2, \ldots, 17\} \setminus \{3, 6, 9, 12, 15\}$ and the missing integers $\{3, 6, 9, 12, 15\}$ cause the mv-labels $\{7, 9, 11, 14, 1\}$, respectively.

3 Hairy cycles and cycles from caterpillars

Definition 3.1. *A graph* H *with exactly one cycle* $C \subset H$ *is called a* hairy cycle *if the deletion of any edge* $[x, y]$ *in* C *gives a caterpillar* Γ*.*

We will call $C \setminus \{[x, y]\}$ *the path associated to* Γ. So, both cycles and hairy cycles can always be seen as a suitable caterpillar Γ with one extra edge: the one connecting the ending vertices of the associated path.

Following this line and keeping in mind the construction of Rosa (see [26]), in [3] Barrientos proves that all hairy cycles are graceful and that when the graph is bipartite, namely when the cycle has even length, the labeling is an α-labeling.

Following the same line and using the results of Section 2, here we give a method for constructing d-divisible α-labelings of bipartite hairy cycles and cycles. We will draw hairy cycles and cycles as in Figure 6(b), instead of the classical way of Figure 6(a), and we denote the hairy cycle obtained by adding the edge $[x_1, y_t]$ to the caterpillar $C[n_1, m_1, n_2, m_2, \ldots, n_t, m_t]$ by $HC(n_1, m_1, n_2, m_2, \ldots, n_t, m_t)$.

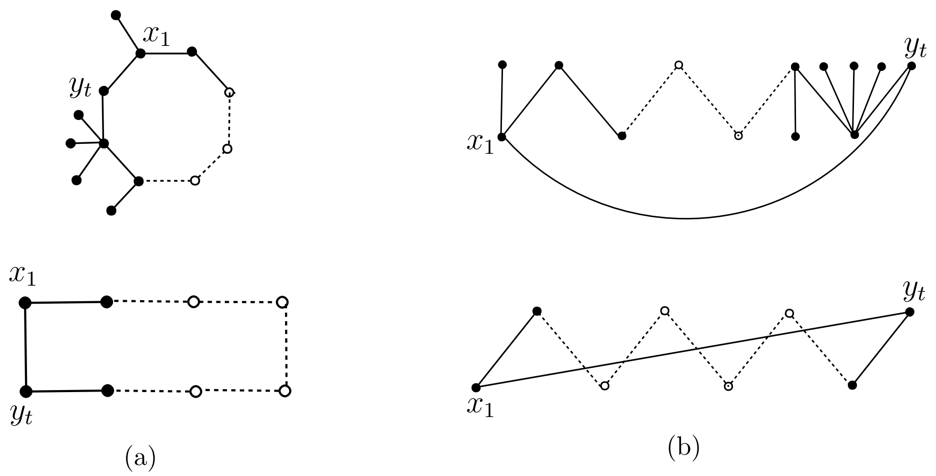


Figure 6: A hairy cycle and a cycle

Remark 3.2. *Consider the graphs of Figure 6(b), both of size* e. *Removing the edge* $[x_1, y_t]$, *in both cases we obtain caterpillars of size* $e-1$, *say* Γ. *Let* $S = \{a_1, a_2, \ldots, a_e\}$ *be a set of* e *integers. Set* $S' = S \setminus \{a_c\}$, $1 \le c \le e$, *and consider an* $\alpha_{S'}$*-labeling of* Γ, *say* f. *If* $f(y_t) - f(x_1) = a_c$ *we can extend* f *to an* α_S*-labeling of the original graph in a natural way simply adding the edge* $[x_1, y_t]$ *to* Γ.

In the sequel Γ is a caterpillar (see Figure 3), S is an arbitrary set of positive integers of size $e = |E(\Gamma)|$, f is the standard α_S-labeling of Γ. If $X, Y \subseteq V(\Gamma)$, we will write $f(X) < f(Y)$ instead of $\max_X f < \min_Y f$, for short, and $f(X)+h$ as the set obtained adding the integer h to all the elements of $f(X)$. If x and y are consecutive vertices of A or of B, $f(x) + h = f(y)$ means that there are $h-1$ mv-labels between $f(x)$ and $f(y)$. Obviously, the positions of the mv-labels affect the value of $f(y_t) - f(x_1)$. Our aim is to modify f by changing the position of some mv-labels in such a way as to obtain a new α_S-labeling, g say, with $g(y_t) - g(x_1) \neq f(y_t) - f(x_1)$. Here we describe some methods for achieving this.

- $[\mathbf{O_1}]$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$.

Set

$$\begin{aligned}
H_{11} &= \{x_{s+1}, y^1_{s+1}, y^2_{s+1}, \ldots, y^{m_{s+1}}_{s+1}, x_{s+2}, \ldots, x_t, y^1_t, y^2_t, \ldots, y^{m_t}_t\} \subseteq A\\
H_{12} &= \{x^2_{s+1}, \ldots, x^{n_{s+1}}_{s+1}, y_{s+1}, x^1_{s+2}, \ldots, y_{t-1}, x^1_t, x^2_t, \ldots, x^{n_t}_t, y_t\} \subseteq B\\
H_{21} &= \{x_1, y^1_1, y^2_1, \ldots, y^{m_1}_1, x_2, \ldots, x_s, y^1_s, y^2_s, \ldots, y^{m_s}_s\} \subseteq A\\
H_{22} &= \{x^1_1, x^2_1, \ldots, x^{n_1}_1, y_1, x^1_2, \ldots, y_{s-1}, x^1_s, x^2_s, \ldots, x^{n_s}_s, y_s\} \subseteq B.
\end{aligned}$$

Set also $H_1 = H_{11} \cup H_{12}$ and $H_2 = H_{21} \cup H_{22}$.
If there exists $s \in \{1, 2, \ldots, t-1\}$ *such that the following conditions are true:*
(1) $f(y_s) + 1 = f(x^1_{s+1})$, *(2)* $f(y_s) = f(x^{n_s}_s) + h$, *with* $h \ge 2$,
then, the function g *defined as follows:*

- $g(x^1_{s+1}) = f(x^1_{s+1}) - 1$

- $g(z) = f(z) \quad \forall z \in H_1$
- $g(z) = f(z) + 1 \quad \forall z \in H_2$

results in again an α_S*-labeling of* Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_{s+1}, y_s]$ and $[x_{s+1}, x^1_{s+1}]$, that commute. So, the set of the edge labels remains the same.

2. If $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$ for $z_1, z_2 \in H_1$ or $z_1, z_2 \in H_2$, since f is injective. Moreover, by the definition of f we know that $f(H_{11}) < f(H_{21}) < f(H_{22}) < f(H_{12})$ and $\max f(H_{22}) + 1 < \min f(H_{12})$. This implies $f(H_{11}) < f(H_{21}) + 1 < f(H_{22}) + 1 < f(H_{12})$, that is $g(H_{11}) < g(H_{21}) < g(H_{22}) < g(H_{12})$. So $g(z_1) \neq g(z_2)$ also when $z_1 \in H_1$ and $z_2 \in H_2$. What about $g(x^1_{s+1})$? Clearly, from the definition of g, it could be $g(x^1_{s+1}) = g(y_s)$, excluded by (1), or $g(x^1_{s+1}) = g(x^{n_s}_s)$, excluded by (1) and (2), so g results in injective.

3. $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$, so g results in an α_S-labeling of Γ.

• $[\mathbf{O_2}]$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2$.
Define H_1 and H_2 as in $[\mathbf{O_1}]$.
If there exists $s \in \{1, 2, \ldots, t-1\}$ *such that the following conditions are true:*
(1) $f(y_s) + 2 = f(x^1_{s+1})$, *(2)* $f(y_s) = f(x^{n_s}_s) + h$, *with* $h \geq 3$,
then, the function g *defined as follows:*

- $g(x^1_{s+1}) = f(x^1_{s+1}) - 2$
- $g(z) = f(z) \quad \forall z \in H_1$
- $g(z) = f(z) + 2 \quad \forall z \in H_2$

results in again an α_S*-labeling of* Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_{s+1}, y_s]$ and $[x_{s+1}, x^1_{s+1}]$, that commute. So, the set of the edge labels remains the same.

2. Exactly as in the previous case, one can show that g results in injective.

3. $\max_A g = \max_A f + 2 < \min_B f + 2 = \min_B g$, so g results in an α_S-labeling of Γ.

• $[\mathbf{O_3}]$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2$.
Set H_1 and H_2 as in $[\mathbf{O_1}]$.
If there exists $s \in \{1, 2, \ldots, t-1\}$ *such that the following conditions are true:*
(1) $f(y_s) + 2 = f(x^1_{s+1}) + 1 = f(x^2_{s+1})$, *(2)* $f(y_s) = f(x^{n_s}_s) + h$, *with* $h \geq 3$,
then, the function g *defined as follows:*

- $g(x^2_{s+1}) = f(x^2_{s+1}) - 2$, $f(x^1_{s+1}) = g(x^1_{s+1})$
- $g(z) = f(z) \quad \forall z \in H_1 \setminus \{x^2_{s+1}\}$
- $g(z) = f(z) + 2 \quad \forall z \in H_2$

results in again an α_S-labeling of Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_{s+1}, y_s]$ and $[x_{s+1}, x^2_{s+1}]$, that commute. So, the set of the edge labels remains the same.

2. As in $[O_1]$, it can be easily seen that if $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$ for $z_1, z_2 \in H_1 \cup H_2$. Now, what about $g(x^1_{s+1})$ and $g(x^2_{s+1})$? Clearly, from the definition of g, each of them may be equal to $g(y_s)$ or $g(x^{n_s})$. But $g(y_s) = \begin{cases} g(x^1_{s+1}) \\ g(x^2_{s+1}) \end{cases}$ implies $f(y_s) + 2 = \begin{cases} f(x^1_{s+1}) \\ f(x^2_{s+1}) - 2 \end{cases}$, excluded by (1), as well as $g(x^{n_s}_s) = \begin{cases} g(x^1_{s+1}) \\ g(x^2_{s+1}) \end{cases}$ and (1) imply $f(x^{n_s}_s) + 2 = \begin{cases} f(y_s) + 1 \\ f(y_s) \end{cases}$, excluded by (2). Again, g results in injective.

3. $\max_A g = \max_A f + 2 < \min_B f + 2 = \min_B g$, so g results in an α-labeling of Γ.

• $[\mathbf{O_4}]$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - (f(y_t) - f(x^{n_t}_t))$.
If the following conditions are true: (1) $n_t \neq 0$, (2) $m_t = 0$,
then, the function g defined as follows:

- $g(z) = f(z) \quad \forall z \in V(\Gamma) \setminus \{y_t, x^{n_t}_t\}$
- $g(y_t) = f(x^{n_t}_t)$ *and* $g(x^{n_t}_t) = f(y_t)$

results in again an α_S-labeling of Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_t, y_t]$ and $[x_t, x^{n_t}_t]$, that commute. So, the set of the edge labels remains the same.

2. g is injective because $g(z) = f(z)$ for each $z \in V(\Gamma)$ with the exception of y_t and $x^{n_t}_t$, whose images are swapped.

3. $\max_A g = \max_A f < \min_B f = \min_B g$, so g results in an α_S-labeling of Γ;

• $[\mathbf{O_5}]$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$.

Set

$$\begin{aligned} J_1 &= \{y_s^1, y_s^2, \ldots, y_s^{m_s}, x_{s+1}, \ldots, x_t, y_t^1, y_t^2, \ldots, y_t^{m_t}\} \cup \\ & \quad \{y_s, x_{s+1}^1, x_{s+1}^2, \ldots, x_{s+1}^{n_{s+1}}, y_{s+1}, \ldots, y_t\}, \\ J_2 &= \{x_1, y_1^1, y_1^2, \ldots, y_1^{m_1}, x_2, \ldots, y_{s-1}^1, \ldots, y_{s-1}^{m_{s-1}}, x_s\} \cup \\ & \quad \{x_1^1, x_1^2, \ldots, x_1^{n_1}, y_1, x_2^1, \ldots, y_{s-1}, x_s^1, x_s^2, \ldots, x_s^{n_s-1}\}. \end{aligned}$$

If there exists $s \in \{1, 2, \ldots, t-1\}$ such that the following conditions are true:
(1) $f(y_s) = f(x_{s+1}^1) - 2$,   (2) $f(y_s) = f(x_s^{n_s}) + 1$,
then, the function g defined as follows:

- $g(x_s^{n_s}) = f(x_s^{n_s}) + 2$,
- $g(z) = f(z) \quad \forall z \in J_1$
- $g(z) = f(z) + 1 \quad \forall z \in J_2$

results in again an α_S-labeling of Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_s, x_s^{n_s}]$ and $[x_s, y_s]$, that commute. So, the set of the edge labels remains the same.

2. As in $[O_1]$, it is easy to see that if $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$ for $z_1, z_2 \in J_1 \cup J_2$. Now, what about $g(x_s^{n_s})$? Clearly, from the definition of g, it may occur that $g(x_s^{n_s}) = \begin{cases} g(y_s) \\ g(x_{s+1}^1) \end{cases}$ that is
$f(x_s^{n_s}) + 2 = \begin{cases} f(y_s) \text{ excluded by (2)} \\ f(x_{s+1}^1) = f(y_s) + 2 \text{ excluded as } f \text{ is injective} \end{cases}$.
Thus, g results in injective.

3. $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$, so g results in an α_S-labeling of Γ.

• $[\mathbf{O_5}]_\mathbf{4}$ gives g such that $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1$.
Set J_1 and J_2 as in $[\mathbf{O_5}]$.
If there exists $s \in \{1, 2, \ldots, t-1\}$ such that the following conditions are true:
(1) $\exists j \in \{1, 2, \ldots, m_s\}$ such that $f(y_s^j) = f(x_s) - 4$,   (2) $f(y_s) = f(x_s^{n_s}) + 1$,
(3) $f(x_{s+1}^i) = f(y_s) + i$, $i = 1, 2, 3, 4$, and $\begin{cases} f(x_{s+1}^5) = f(y_s) + 6 \text{ if } n_{s+1} \geq 5 \\ f(y_{s+1}) = f(y_s) + 6 \text{ if } n_{s+1} = 4 \end{cases}$ then,
the function g defined as follows:

- $g(x_s^{n_s}) = f(x_s^{n_s}) + 6$, $g(y_s^j) = f(x_s)$
- $g(z) = f(z) \quad \forall z \in J_1 \setminus \{y_s^j\}$
- $g(z) = f(z) + 1 \quad \forall z \in J_2$

results in again an α_S-labeling of Γ.

In fact:

1. The labels of the edges of Γ induced by g are the same of those induced by f with the exception of those of $[x_s, x_s^{n_s}]$, $[x_s, y_s]$ and $[y_s, y_s^j]$, that cyclically permute. So, the set of the edge labels remains the same.

2. As in $[O_1]$, one can show that if $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$ for $z_1, z_2 \in (J_1 \setminus \{y_s^j\}) \cup J_2$. It remains to observe that, by definition, $g(y_s^j)$ and $g(x_s^{n_s})$ are different and do not belong to $g((J_1 \setminus \{y_s^j\}) \cup J_2)$, to conclude that g results in injective.

3. $\max_A g = \max_A f + 1 < \min_B f + 1 = \min_B g$, so g results in an α_S-labeling of Γ.

Remark 3.3. *Obviously, we can apply more than one of the previous operations to the same standard α_S-labeling, as long as they operate on disjoint set of edges.*

3.1 Hairy cycles

In this subsection we focus our attention on hairy cycles other than cycles, namely with at least one pendant edge.

In [3] Barrientos gives a labeling for any hairy cycle and when the graph is bipartite such a labeling is an α-labeling.

Here we show that any bipartite hairy cycle H admits an odd α-labeling (namely an e-divisible α-labeling, e being the size of H). Then, for any admissible value of d, we will prove the existence of d-divisible α-labelings of an infinite class of hairy cycles, the *coronas* $C_{2t} \odot \lambda K_1$, see [16].

Remark 3.4. *Let $H = HC(n_1, m_1, n_2, m_2, \ldots, n_t, m_t)$, with $t \geq 2$, be a bipartite hairy cycle. We view H as the hairy cycle obtained from the caterpillar Γ of Figure 3 by adding the edge $[x_1, y_t]$. Clearly, the number of edges of H is $e = 2t + \sum_{i=1}^{t} n_i + \sum_{i=1}^{t} m_i$. Let $k = t + \sum_{i=1}^{t} n_i$, that is $k = |B|$. Suppose $e = d \cdot m$ and let $\Delta = \{1, 2, \ldots, e+d-1\}$ and $\Delta' = \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$. Set $S = (\Delta \setminus \Delta') \setminus \{c\}$, where $1 < c < e+d-1$ and $c \notin \Delta'$, and let f be the standard α_S-labeling of Γ. The missing integers of Δ' and the removal of c cause d mv-labels in $f(A) \bigcup f(B)$, let d_A and d_B be the number of mv-labels in $f(A)$ and $f(B)$, respectively. Now $f(y_t) - f(x_1) = f(y_t) - (f(x_1)+1) + 1 = |B| + d_B = k + d_B$. Thus if $k + d_B = c$, in a natural way we can extend f to a d-divisible α-labeling of H.*

3.1.1 Odd (e-divisible) α-labelings of hairy cycles

Theorem 3.5. *A hairy cycle with at least a pendant edge admits an odd α-labeling if and only if it is bipartite.*

Proof. By definition a hairy cycle with an odd α-labeling is necessarily bipartite.

Suppose then $H = HC(n_1, m_1, n_2, m_2, \ldots, n_t, m_t)$ is a bipartite hairy cycle with at least one n_i or one m_i greater than zero. We start from the above Remark 3.4 and consider the case when $d = e$, $m = 1$ and thus $\Delta \setminus \Delta' = \{1, 3, 5, \ldots, 2e-1\}$. Let Γ be the caterpillar obtained from H deleting the edge $[x_1, y_t]$ and let f be the standard α_S-labeling of Γ, where $S = (\Delta \setminus \Delta') \setminus \{c\}$. It is not hard to see that $d_B = k - 1$ or $d_B = k + 1$, where $k = |B|$, hence $f(y_t) - f(x_1) = 2k - 1$ or $2k + 1$, respectively. Therefore, it will be convenient to choose $c \in \{2k - 1, 2k + 1\}$ (note that this choice is always possible since $\{2k - 1, 2k + 1\} \subseteq \Delta \setminus \Delta'$). There are four possible cases.

• Case (1) If the k-th edge is a pendant edge from a vertex in B, for both choices of c we have $d_B = k - 1$, so $f(y_t) - f(x_1) = 2k - 1$. Hence, choosing $c = 2k - 1$ the α_S-labeling of Γ can be extended to an odd α-labeling of H.

• Case (2) If the k-th edge is a pendant edge from a vertex in A, for both choices of c we have $d_B = k + 1$, thus $f(y_t) - f(x_1) = 2k + 1$. Hence, choosing $c = 2k + 1$ the α_S-labeling of Γ can be extended to an odd α-labeling of H.

• Case (3) If the k-th edge is an edge of the cycle of the form $[x_s, y_{s-1}]$, $c = 2k - 1$ implies $d_B = k - 1$, so $f(y_t) - f(x_1) = 2k - 1$, while $c = 2k + 1$ implies $d_B = k + 1$, so $f(y_t) - f(x_1) = 2k + 1$. In both cases, the choice of the value of c results in an α_S-labeling of Γ which can be extended to an odd α-labeling of H.

• Case (4) If the k-th edge is an edge of the cycle of the form $[x_s, y_s]$, $c = 2k - 1$ implies $d_B = k+1$, so $f(y_t) - f(x_1) = 2k+1$, while $c = 2k+1$ implies $d_B = k-1$, so $f(y_t) - f(x_1) = 2k - 1$. Thus no choice of c is appropriate because $f(y_t) - f(x_1) \neq c$. To solve the problem, we have to distinguish two subcases: Case (4_1) $n_{s+1} \neq 0$, and Case (4_2) $n_t \neq 0$ and $m_t = 0$. If we are able to define an odd α-labeling in both previous subcases, we can do so in anycase. In fact, if the sequence $(n_1, m_1, \ldots, n_t, m_t)$ does not contain zeros, we refer to the Case (4_1). If the sequence $(n_1, m_1, \ldots, n_t, m_t)$ contains at least one zero we refer to the Case (4_2) because we can always choose a suitable representation of H so that it results in $n_t \neq 0$ and $m_t = 0$, H being not a cycle.

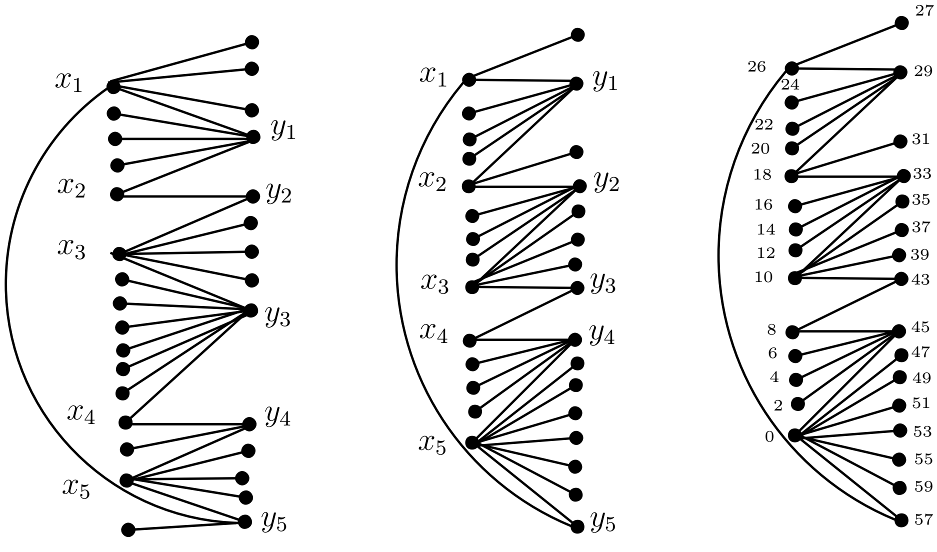
Case (4_1) We choose $c = 2k - 1$ and applying $[O_2]$ we obtain another α_S-labeling g of Γ with $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = 2k + 1 - 2 = 2k - 1$. $[O_2]$ can be used because both (1) and (2) of its definition hold, in fact we know that $f(y_s) - f(x_s) = 2k + 1$ and $c = 2k - 1$ has been removed, so $f(x_s^{n_s}) - f(x_s) = 2k - 3$. Hence $f(x_s^{n_s}) + 4 = f(y_s)$ and (2) holds. Moreover, from $n_{s+1} \neq 0$ we have $f(x_{s+1}^1) - f(x_{s+1}) = f(y_s) - f(x_{s+1}) + 2$. Hence $f(x_{s+1}^1) = f(y_s) + 2$ and (1) holds.

Case (4_2) We choose $c = 2k - 1$ and applying $[O_4]$ we obtain another α_S-labeling g of Γ with $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 2 = 2k - 1$. $[O_4]$ can be used because both (1) and (2) of its definition hold by the hypothesis.

So, in both cases, we obtain $g(y_t) - g(x_1) = c$, hence g can be extended to an odd α-labeling of H. □

For an explicit definition of the odd α-labeling described in the above theorem see the Appendix.

Example 3.6. *In Figure* 7(*a*) *we have the graph* $H = HC(3, 3, 0, 0, 3, 6, 0, 1, 3, 1)$. *Clearly,* $e = 30$, $k = 14$ *and the* k*-th edge is* $[x_3, y_3]$. *So, following the notation of the proof of Theorem 3.5 Case* (4), *we have* $s = 3$. *Since* $n_{s+1} = n_4 = 0$ *and* $m_5 \neq 0$, *we have to rearrange the representation of* H *as* $HC(1, 3, 1, 3, 3, 0, 0, 3, 6, 0)$. *It just*



(a) (b) (c)

Figure 7: An odd α-labeling of $HC(3,3,0,0,3,6,0,1,3,1)$

so happens that the k-th edge (now $k = 16$) is still $[x_3, y_3]$, but now we are in the hypotheses of the Case (4_2), see Figure 7(b). The odd α-labeling of H constructed following the proof of the previous theorem is shown in Figure 7(c). Obviously, if after the rearrangement the k-th edge is not of the form $[x_s, y_s]$ we are in one of the other cases and we apply the corresponding construction.

As a consequence of Theorems 1.4 and 3.5 we have:

Theorem 3.7. *Let Γ be a bipartite hairy cycle of size e. There exists a cyclic Γ-decomposition of $K_{2(2en)}$ for any positive integer n.*

3.1.2 d-divisible α-labelings of $C_{2t} \odot \lambda K_1$

For convenience, by $H(2t, \lambda)$ we will denote the hairy cycle $HC(\underbrace{\lambda, \lambda, \ldots, \lambda}_{2t})$, with $t \geq 2$ and $\lambda \geq 1$ (namely, the *corona* of C_{2t} with λK_1, denoted by $C_{2t} \odot \lambda K_1$ in [16]). Obviously $H(2t, \lambda)$ has $2t(\lambda + 1)$ edges.

Theorem 3.8. *The hairy cycle $H(2t, \lambda)$ admits a d-divisible α-labeling for any admissible value of d.*

Proof. If $d = e$ the result follows from Theorem 3.5. So, from now on, we can assume $d \neq e$. Let Γ be the caterpillar obtained deleting the edge $[x_1, y_t]$ from $H(2t, \lambda)$ and let P be the path associated to Γ. We start again from the Remark 3.4 and we notice that now $e = d \cdot m = 2t(\lambda + 1)$, $k = e/2$ and the k-th edge is always an edge of

the path P: of the form $[x_r, y_{r-1}]$ if t is even, of the form $[x_r, y_r]$ if t is odd, with $r = \lfloor \frac{t+2}{2} \rfloor$.
We label all the edges of Γ except the k-th one, starting from $[x_1, x_1^1]$ (or from $[x_1, y_1]$ if $n_1 = 0$), by the elements of $(\Delta \setminus \Delta') \setminus \{c_1 = \lfloor \frac{e+d-1}{2} \rfloor, c_2 = \lfloor \frac{e+d+2}{2} \rfloor\}$ taken in ascending order. It is easy to check that $c_1, c_2 \in \Delta \setminus \Delta'$. Now, in order to obtain a d-divisible α-labeling of $H(2t, \lambda)$ we label the k-th edge by c_1 or c_2 and then we have to ensure that $f(y_t) - f(x_1) = c_2$ or $f(y_t) - f(x_1) = c_1$, respectively. Because of the form of the graph, the mv-labels due to the $d-1$ missing elements of Δ', if d is odd, as well as those due to the $d-2$ missing elements of $\Delta' \setminus \{\frac{e+d}{2}\}$, if d is even, are equally distributed in $f(A)$ and $f(B)$. The removal of c_1 or c_2 causes an extra mv-labels which will be in $f(A)$ or $f(B)$ and which, in the case of d even, will drag with itself the mv-label due to the element $\{\frac{e+d}{2}\} \in \Delta'$.
Now we continue by distinguishing several cases.

• Case (1) Let t be even, hence the k-th edge is of the form $[x_r, y_{r-1}]$ with $r = \frac{t+2}{2}$.
Let $S = (\Delta \setminus \Delta') \setminus \{c_2\}$. Let f be the standard α_S-labeling of Γ. The choice of c_2 implies that the extra mv-label is always in $f(B)$, so $d_B = \lfloor \frac{d+2}{2} \rfloor$ hence $f(y_t) - f(x_1) = k + d_B = c_2$ and, in a natural way, f can be extended to a d-divisible α-labeling of $H(2t, \lambda)$.
• Case (2) Let t be odd, hence the k-th edge is of the form $[x_r, y_r]$ with $r = \frac{t+1}{2}$.
Let $S = (\Delta \setminus \Delta') \setminus \{c\}$ where c can be chosen in $\{c_1, c_2\}$. Let f be the standard α_S-labeling of Γ. The choice $c = c_1$ implies that the extra mv-label is always in $f(B)$, so, as above, $f(y_t) - f(x_1) = c_2$. The choice $c = c_2$ implies that the extra mv-label is always in $f(A)$, so $d_B = \lfloor \frac{d-1}{2} \rfloor$ and hence $f(y_t) - f(x_1) = c_1$. In both cases f cannot be extended to a d-divisible α-labeling of $H(2t, \lambda)$ since $f(y_t) - f(x_1) \neq c$. We can proceed by distinguishing two subcases.

Case (2_1) Let d be odd. Now $\{c_1, c_2\} = \{\frac{e+d-1}{2}, \frac{e+d+1}{2}\}$. Choose $c = c_1$, consider the corresponding standard α_S-labeling of Γ and apply $[O_1]$ to f with $s = \frac{t+1}{2}$. This is possible because conditions (1) and (2) of $[O_1]$ are satisfied. In fact, now we have $f(y_s) - f(x_s) = c_2$ and $f(x_s^{n_s}) - f(x_s) = c_1 - 1$ if $c_1 - 1 \notin \Delta'$ or $f(x_s^{n_s}) - f(x_s) = c_1 - 2$ if $c_1 - 1 \in \Delta'$, so $f(y_s) - f(x_s^{n_s}) = 2$ or 3 and (2) is true. Also, if (1) was not true, there would be a mv-label between $f(y_s)$ and $f(x_{s+1}^1)$ and, symmetrically, there would be a mv-label between $f(x_s)$ and $f(y_{s-1}^{m_s-1})$. Thus m should be a divisor of $2\lambda + 3$, an odd number, while m is obviously even as now d is odd. Applying $[O_1]$ we obtain an α_S-labeling g of Γ with $g(y_t) - g(x_1) = f(y_t) - f(x_1) - 1 = c_2 - 1 = c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t, \lambda)$.

Case (2_2) Let d be even. Now $\{c_1, c_2\} = \{\frac{e+d-2}{2}, \frac{e+d+2}{2}\}$. We have to split the proof in several subcases.

Case (2_{2A}) Let $\lambda \geq 2$ and $\lambda \not\equiv m-2, m-3 \pmod{m}$.
Choose $c = c_1$ and let f be the corresponding standard α_S-labeling of Γ. Apply $[O_3]$ to f with $s = \frac{t+1}{2}$. This is possible because conditions (1) and (2) of $[O_3]$ are satisfied. In fact, now we have $f(y_s) - f(x_s) = c_2$ and $f(x_s^{n_s}) - f(x_s) = c_1 - 1$ if $c_1 - 1 \notin \Delta'$ or $f(x_s^{n_s}) - f(x_s) = c_1 - 2$ if $c_1 - 1 \in \Delta'$, so $f(y_s) - f(x_s^{n_s}) = 3$ or 4 and (2) is true. Also, if (1) was not true, there would be a mv-label either between $f(y_s)$ and $f(x_{s+1}^1)$ (excluded as $\lambda \not\equiv m-2 \pmod{m}$) or between $f(x_{s+1}^1)$ and $f(x_{s+1}^2)$

(excluded as $\lambda \not\equiv m-3 \pmod{m}$). Applying $[O_3]$ we obtain an α_S-labeling g of Γ with $g(y_t)-g(x_1)=f(y_t)-f(x_1)-2=c_2-2=c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t,\lambda)$.

Case (2_{2B}) Let $\lambda \geq 2$ and $\lambda \equiv m-2 \pmod{m}$.
Choose $c=c_1$ and let f be the corresponding standard α_S-labeling of Γ. Apply $[O_2]$ to f with $s=\frac{t+1}{2}$. This is possible because conditions (1) and (2) of $[O_2]$ are satisfied. In fact, now we have $f(y_s)-f(x_s)=c_2$ and $f(x_s^{n_s})-f(x_s)=c_1-1$ if $c_1-1\notin\Delta'$ or $f(x_s^{n_s})-f(x_s)=c_1-2$ if $c_1-1\in\Delta'$, so $f(y_s)-f(x_s^{n_s})=3$ or 4 and hence (2) is true. Also, there is a mv-label between $f(y_s)$ and $f(x_{s+1}^1)$ as $\lambda\equiv m-2 \pmod{m}$, so (1) is true. Applying $[O_2]$ we obtain an α_S-labeling g of Γ with $g(y_t)-g(x_1)=f(y_t)-f(x_1)-2=c_2-2=c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t,\lambda)$.

Case (2_{2C}) Let $\lambda\equiv m-3 \pmod{m}$ and

- $(\lambda \geq 4$ and $m\geq 4$ and $t\geq 5)$ or $(\lambda\geq 7$ and $m\geq 4$ and $t=3)$.

Choose $c=c_1$, consider the corresponding standard α_S-labeling of Γ, say f, and apply $[O_1]$ with $s=\frac{t+1}{2}$ and $[O_5]_4$ with $s=1$ to obtain an α_S-labeling g of Γ. We can apply $[O_1]$ because $\lambda\equiv m-3 \pmod{m}$, and we can apply $[O_5]_4$ because we have also $\lambda\geq 4$ and $m\geq 4$. We can apply both $[O_1]$ and $[O_5]_4$, in any order, as $t\geq 5$ or $t=3$ and $7\leq\lambda$, and this ensures that there is no edge affected by both the operations. Thus we have $g(y_t)-g(x_1)=(f(y_t)-f(x_1)-1)-1=c_2-2=c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t,\lambda)$.

- $4\leq\lambda\leq 7$ and $m\geq 4$ and $t=3$.

From our hypotheses we have

$4\leq\lambda<7$	$\lambda+3$	$e=6(\lambda+1)$	$m\geq 4$ divides $\lambda+3$ and e
4	7	30	there is no value
5	8	36	4 unacceptable, as d is even
6	9	42	there is no value

Hence this case can not occur.

- $m=2$ and $t\geq 3$.

Choose $c=c_1$, consider the corresponding standard α_S-labeling of Γ, say f, and apply $[O_1]$ with $s=\frac{t+1}{2}$ and $[O_5]$ with $s=1$ to obtain an α_S-labeling g of Γ. We can apply $[O_1]$ because $\lambda\equiv m-3 \pmod{m}$ and we can apply $[O_5]$ as m divides $2(\lambda+1)$. We can apply both $[O_1]$ and $[O_5]$, in any order, as $\lambda\geq 2$ and this ensures that there is no edge affected by both the operations.
Thus we have $g(y_t)-g(x_1)=(f(y_t)-f(x_1)-1)-1=c_2-2=c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t,\lambda)$.

- $m=3$ and $t\geq 3$.

Choose $c=c_1$ and consider the corresponding standard α_S-labeling of Γ, say f. We can apply $[O_1]$ with $s=\frac{t+1}{2}$ because $\lambda\equiv m-3 \pmod{m}$ and we can also apply $[O_1]$ with $s=1$, as m divides λ. We can apply twice $[O_1]$ with $s=1$ and $s=\frac{t+1}{2}$, in any order, as $\lambda\geq 2$ and this ensures that there is no edge affected by both the operations.
Thus we have $g(y_t)-g(x_1)=(f(y_t)-f(x_1)-1)-1=c_2-2=c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t,\lambda)$.

• $m \geq 4$ and $\lambda = 2$.
It happens only when $m = 5$ and $\lambda = 2$. Then $e = 6t$ implies that t have to be an odd multiple of 5. We choose $c = c_1$ and consider the corresponding standard α_S-labeling of Γ, say f. It easy to see that we can apply both $[O_1]$ with $s = \frac{t+1}{2}$ and $[O_5]$ with $s = \frac{t+3}{2}$, in any order, being sure that there is no edge affected by both the operations.
Thus we have $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t, \lambda)$.

• $m \geq 4$ and $\lambda = 3$.
It happens only when $m = 6$ and $\lambda = 3$. Then $e = 8t$ implies that t have to be an odd multiple of 3. If $t = 3$, it is not hard to construct directly a d-divisible α-labeling of $H(2t, \lambda)$. If $t \geq 5$, we choose $c = c_1$ and consider the corresponding standard α_S-labeling of Γ, say f. It is easy to see that we can apply both $[O_1]$ with $s = \frac{t+1}{2}$ and $[O_5]$ with $s = 3$, in any order, being sure that there is no edge affected by both the operations.
Thus we have $g(y_t) - g(x_1) = (f(y_t) - f(x_1) - 1) - 1 = c_2 - 2 = c_1$. So, we can extend g to a d-divisible α-labeling of $H(2t, \lambda)$.

Case (2_{2D}) Let $\lambda = 1$. Also in this case the foregoing construction can be applied, by distinguishing four cases according to the congruence class of m modulo 4 and applying in a suitable way the $[O_i]$'s. In Appendix the reader can found an explicit 2-divisible α-labeling of $H(2t, 1)$ for any t odd. □

Example 3.9. *Here we show a* 6*-divisible* α*-labeling of the hairy cycle* $HC(10, 2)$*. Since* $t = 5$*,* $d = 6$*,* $m = 5$ *and* $\lambda = 2$*, we are in the Case* (2_{2C}) *of Theorem 3.8. In Figure* 8(a) *we have the standard* α_S*-labeling* f *of the caterpillar* Γ *obtained from* $HC(10, 2)$ *by deleting the edge* $[x_1, y_5]$ *with* $S = (\Delta \backslash \Delta') \backslash \{c\}$ *where* $\Delta = \{1, 2, \ldots, 35\}$*,* $\Delta' = \{6, 12, 18, 24, 30\}$ *and* $c = 17$*. Now* $f(y_5) - f(x_1) = 19 \neq c$*, so we cannot extend* f *to a* 6*-divisible* α*-labeling of* $HC(10, 2)$*. It is easy to see that it is possible to apply* $[O_1]$ *with* $s = 3$*, the* α_S*-labeling* g' *of* Γ *so obtained is shown in Figure* 8(b)*. Now* $g'(y_5) - g'(x_1) = 18 \neq c$*. Then we can apply* $[O_5]$ *with* $s = 4$*. After this we obtain the* α_S*-labeling* g *of* Γ *shown in Figure* 8(c)*. Finally* $g(y_5) - g(x_1) = 17 = c$*, so we can extend* g *to a* 6*-divisible* α*-labeling of* $HC(10, 2)$*.*

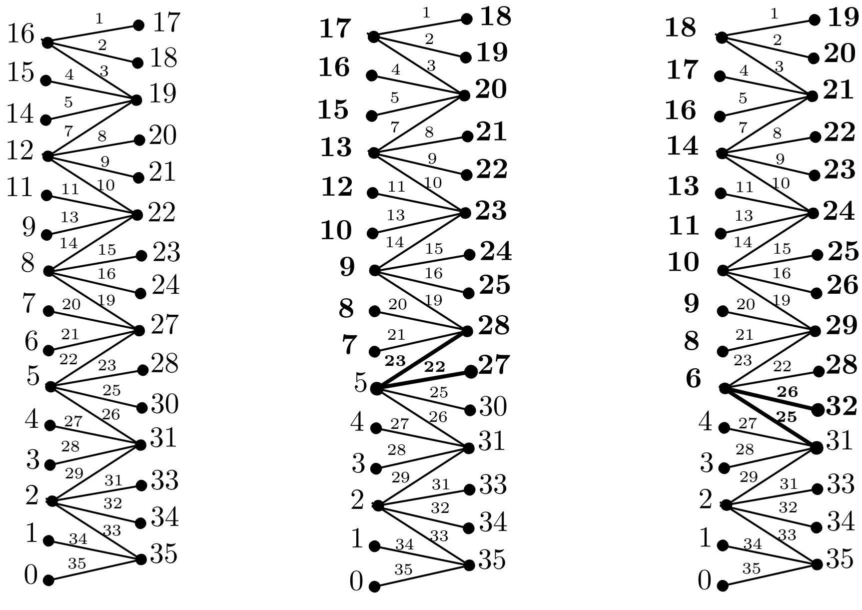


Figure 8: A 6-divisible α-labeling of $HC(10, 2)$

The following result is an immediate consequence of Theorems 1.4 and 3.8.

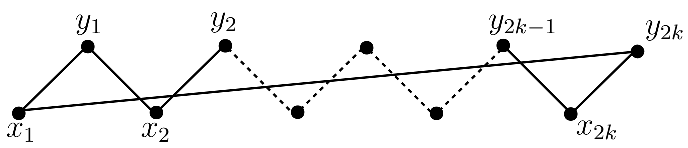
Theorem 3.10. *Given $t \geq 2$ and $\lambda \geq 1$, there exists a cyclic $H(2t, \lambda)$-decomposition of $K_{\frac{2t(\lambda+1)+d}{d}(2dn)}$ for any admissible d and for any positive integer n.*

3.2 Cycles

As usual, we will denote the cycle on k vertices by C_k, $k \geq 3$. It is obvious that C_k is a graph of size k and that it is bipartite if and only if k is even.
In [26] Rosa proved that C_k has an α-labeling if and only if $k \equiv 0 \pmod 4$. In [25], the second author proved that C_{4k} admits a 2-divisible and a 4-divisible α-labeling for any positive integer k. Here, generalizing these results, we prove that C_{4k} admits a d-divisible α-labeling for any divisor d of $4k$.

Theorem 3.11. *For any positive integer k, the cycle C_{4k} admits a d-divisible α-labeling for any admissible value of d.*

Proof. Consider the cycle C_{4k} as a bipartite graph as follows:



and set $A = \{x_1, x_2, \ldots, x_{2k}\}$ and $B = \{y_1, y_2, \ldots, y_{2k}\}$ the two bipartite sets. Let Γ be the caterpillar obtained form C_{4k} deleting the edge $[x_1, y_{2k}]$. Let $4k = d \cdot m$, $\Delta = \{1, 2, \ldots, 4k+d-1\}$ and $\Delta' = \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$. Choose an element in $\Delta \setminus \Delta'$, say c. Let f be the standard α_S-labeling of Γ where $S = (\Delta \setminus \Delta') \setminus \{c\}$. In order to show that f can be naturally extended to a d-divisible α-labeling of C_{4k} it remains to prove that we can choose the element c so that $c = f(y_{2k}) - f(x_1)$; see Remark 3.2.

Let d_x and d_y denote the number of mv-labels in $f(A)$ and $f(B)$, respectively. As in Remark 3.4, the condition $c = f(y_{2k}) - f(x_1)$ becomes $c = |B| + d_y = 2k + d_y$. In what follows, we are able to determine d_y and, consequently, c.

If $m = 4k/d$ is even, the elements $m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)$ of Δ' are alternately odd and even, so c lies between two elements of different parity. The deletion of any element of Δ' less than c produces a mv-label in $f(B)$, while deleting an element of Δ' greater than c we have a mv-label in $f(A)$. Thus we have necessarily $d_y(m + 1) < c < (d_y + 1)(m + 1)$, where $c = d_y + 2k$ and $2k = md/2$. With a simple calculation we obtain $d_y m < md/2 \leq (d_y + 1)m$, from which $2d_y < d \leq 2d_y + 2$. So, d even implies $d_y = (d - 2)/2$ and d odd implies $d_y = (d - 1)/2$.
If $m = 4k/d$ is odd, all the elements of Δ' are even. If c was even too, we would have $d_x = d_y = d/2$, so $c = 2k + d/2 = d/2(m + 1) \in \Delta'$, but c must not belong to Δ'. Thus c must be odd and we have $d_x - d_y = \pm 2$. In addition, we know that $d_x + d_y = d$, so $d_y = (d \mp 2)/2$. □

Remark 3.12. *It is known that if f is a (d-divisible) α-labeling of a bipartite graph Γ of size e, the function $g : V(\Gamma) \to \{0, 1, \ldots, e\}$, defined by $g(x) = e - f(x)$, $\forall x \in V(\Gamma)$ is again a (d-divisible) α-labeling of Γ. We point out that if $d = 1$, f is the α-labeling constructed in the above theorem and g is the classical α-labeling of C_{4k} given by A. Rosa in [26], then $g(x) = e - f(x)$, $\forall x \in V(C_{4k})$. Also, when $d \in \{2, 4\}$ the same relation holds between the d-divisible α-labeling constructed in the above theorem and that given by the second author in [25].*

Example 3.13. *In Figure 9 we show the d-divisible α-labelings of C_{24} described in Theorem 3.11 for $d = 3$ and $d = 8$.*
If $d = 3$, we have $m + 1 = 9$ and $c = 13$. If $d = 8$ then $m + 1 = 4$ and we can choose $c = 15$ or $c = 17$ (as in Figure 9).

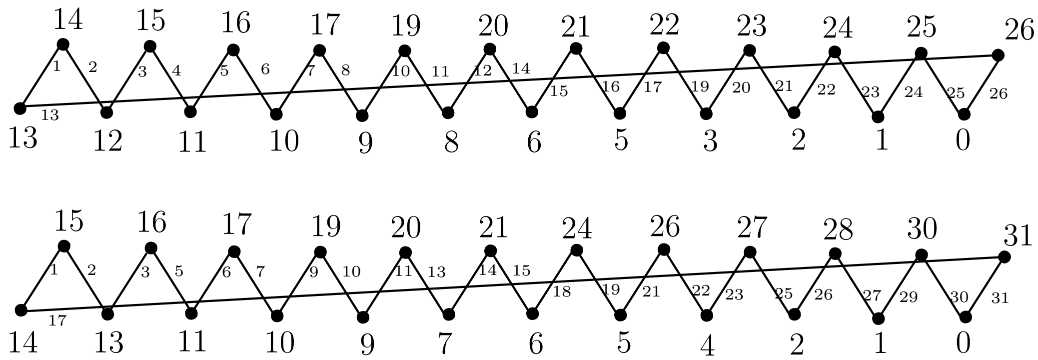


Figure 9: A d-divisible α-labeling of C_{24} for $d = 3$ and $d = 8$

The following result immediately follows from Theorems 1.4 and 3.11.

Theorem 3.14. *There exists a cyclic C_{4k}-decomposition of $K_{\frac{4k+d}{d}(2dn)}$ for any positive integers k, n and any divisor d of $4k$.*

Appendix

Here we give an explicit representation of the odd α-labeling f of a bipartite hairy cycle $H = HC(n_1, m_1, n_2, m_2, \ldots, n_t, m_t)$ of size e constructed in Theorem 3.5. We give the definition of the function $f : V(H) \to \{0, 1, \ldots, 2e-1\}$ in each of the possible cases.

Case (1): the k-th edge is a pendant edge from a vertex in B, say $[y_s, y_s^j]$.

$$f(x_r) = \begin{cases} 2\sum_{\ell=r}^{t} m_\ell + 2(t-r) + 2, & r = 1, \ldots, s \\ 2\sum_{\ell=r}^{t} m_\ell + 2(t-r), & r = s+1, \ldots, t \end{cases}$$

$$f(y_r) = f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r - 1, \qquad r = 1, \ldots, t$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1, & r = 1 \text{ and } i = 1, \ldots, n_1 \\ f(y_{r-1}) + 2i, & r = 2, \ldots, t \text{ and } i = 1, \ldots, n_r \end{cases}$$

$$f(y_r^i) = \begin{cases} f(x_r) - 2i, & r \neq s \text{ and } i = 1, \ldots, m_r \\ & r = s \text{ and } i = 1, \ldots, j-1 \\ f(x_r) - 2(i+1), & r = s \text{ and } i = j, \ldots, m_s. \end{cases}$$

Case (2): the k-th edge is a pendant edge from a vertex in A, say $[x_s, x_s^j]$.

$$f(x_r) = 2\sum_{\ell=r}^{t} m_\ell + 2(t-r), \qquad r = 1, \ldots, t$$

$$f(y_r^i) = f(x_r) - 2i, \qquad r = 1, \ldots, t \text{ and } i = 1, \ldots, m_r$$

$$f(y_r) = \begin{cases} f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r - 1, & r = 1, \ldots, s-1 \\ f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r + 1, & r = s, \ldots, t \end{cases}$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1, & r = 1 \text{ and } i = 1, \ldots, n_1 \\ f(y_{r-1}) + 2i, & r \neq 1, s \text{ and } i = 1, \ldots, n_r \\ & r = s \text{ and } i = 1, \ldots, j \\ f(y_{r-1}) + 2i + 2, & r = s \text{ and } i = j+1, \ldots, n_s, \text{ if } j \neq n_s. \end{cases}$$

Case (3): the k-th edge is an edge of the cycle of the form $[x_s, y_{s-1}]$.

$$f(x_r) = \begin{cases} 2\sum_{\ell=r}^{t} m_\ell + 2(t-r) + 2, & r = 1, \ldots, s-1 \\ 2\sum_{\ell=r}^{t} m_\ell + 2(t-r), & r = s, \ldots, t \end{cases}$$

$$f(y_r) = f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r - 1, \qquad r = 1, \ldots, t$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1, & r = 1 \text{ and } i = 1, \ldots, n_1 \\ f(y_{r-1}) + 2i, & r = 2, \ldots, t \text{ and } i = 1, \ldots, n_r \end{cases}$$

$$f(y_r^i) = f(x_r) - 2i, \qquad r = 1, \ldots, t \text{ and } i = 1, \ldots, m_r.$$

Case (4): the k-th edge is an edge of the cycle of the form $[x_s, y_s]$.
Case (4_1) : $n_{s+1} \neq 0$.

$$f(x_r) = \begin{cases} 2\sum_{\ell=r}^{t} m_\ell + 2(t-r) + 2, & r = 1, \ldots, s \\ 2\sum_{\ell=r}^{t} m_\ell + 2(t-r), & r = s+1, \ldots, t \end{cases}$$

$$f(y_r^i) = f(x_r) - 2i, \qquad r = 1, \ldots, t \text{ and } i = 1, \ldots, m_r$$

$$f(y_r) = \begin{cases} f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r - 1, & r \neq s \\ f(x_1) + 2\sum_{\ell=1}^{s} n_\ell + 2s + 1, & r = s \end{cases}$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1, & r = 1 \text{ and } i = 1, \ldots, n_1 \\ f(y_{r-1}) + 2i, & r = 2, \ldots, s, s+2, \ldots, t \text{ and } i = 1, \ldots, n_r \\ f(y_s) - 2, & r = s+1 \text{ and } i = 1 \\ f(y_s) + 2(i-1), & r = s+1 \text{ and } i = 2, \ldots, n_{s+1}. \end{cases}$$

Case (4_2) : $n_t \neq 0$, $m_t = 0$.

$$f(x_r) = 2\sum_{\ell=r}^{t} m_\ell + 2(t-r), \qquad r = 1, \ldots, t$$

$$f(y_r^i) = f(x_r) - 2i, \qquad r = 1, \ldots, t-1 \text{ and } i = 1, \ldots, m_r$$

$$f(y_r) = \begin{cases} f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r - 1, & r = 1, \ldots, s-1 \\ f(x_1) + 2\sum_{\ell=1}^{r} n_\ell + 2r + 1, & r = s, \ldots, t-1 \\ 2e - 3 & r = t \end{cases}$$

$$f(x_r^i) = \begin{cases} f(x_1) + 2i - 1, & r = 1 \text{ and } i = 1, \ldots, n_1 \\ f(y_{r-1}) + 2i, & r = 2, \ldots, t-1 \text{ and } i = 1, \ldots, n_r \\ & r = t \text{ and } i = 1, \ldots, n_t - 1 \\ 2e - 1, & r = t \text{ and } i = n_t. \end{cases}$$

In the following we give an explicit construction of a 2-divisible α-labeling of $H(2t, 1)$ for $1 < t$ odd, whose existence has been considered in Theorem 3.8. We have to distinguish two cases according to the congruence class of t modulo 4.
Case (1): $t \equiv 1 (mod\ 4)$

$$f(x_r) = \begin{cases} 2t + 3 - 2r, & r = 1, \ldots, \frac{t+3}{4} \\ 2t + 2 - 2r, & r = \frac{t+7}{4}, \ldots, \frac{t+1}{2} \\ 2t + 1 - 2r, & r = \frac{t+3}{2}, \ldots, t \end{cases} \qquad f(y_r^1) = \begin{cases} 2t + 2 - 2r, & r = 1, \ldots, \frac{t-1}{4} \\ 2t + 1 - 2r, & r = \frac{t+3}{4}, \ldots, \frac{t-1}{2} \\ 2t - 2r, & r = \frac{t+1}{2}, \ldots, t \end{cases}$$

$$f(x_r^1) = \begin{cases} 3t + 2, & r = 1 \\ 2t - 1 + 2r, & r = 2, \ldots, \frac{t+1}{2} \\ 3t + 1, & r = \frac{t+3}{2} \\ 2t + 2r, & r = \frac{t+5}{2}, \ldots, t \end{cases} \qquad f(y_r) = \begin{cases} 2t + 2r, & r = 1, \ldots, \frac{t-1}{2} \\ 3t + 3, & r = \frac{t+1}{2} \\ 2t + 1 + 2r, & r = \frac{t+3}{2}, \ldots, t. \end{cases}$$

Case (2): $t \equiv 3 (mod\ 4)$

$$f(x_r) = \begin{cases} 2t + 3 - 2r, & r = 1, \ldots, \frac{t+1}{4} \\ 2t + 2 - 2r, & r = \frac{t+5}{4}, \ldots, \frac{t+1}{2} \\ 2t + 1 - 2r, & r = \frac{t+3}{2}, \ldots, t \end{cases} \qquad f(y_r^1) = \begin{cases} 2t + 2 - 2r, & r = 1, \ldots, \frac{t+1}{4} \\ 2t + 1 - 2r, & r = \frac{t+5}{4}, \ldots, \frac{t+1}{2} \\ 2t - 2r, & r = \frac{t+3}{2}, \ldots, t \end{cases}$$

$$f(x_r^1) = \begin{cases} 3t + 1, & r = 1 \\ 2t - 1 + 2r, & r = 2, \ldots, \frac{t+3}{2} \\ 2t + 2r, & r = \frac{t+5}{2}, \ldots, t, \text{ if } t > 3 \end{cases} \qquad f(y_r) = \begin{cases} 2t + 2r, & r = 1, \ldots, \frac{t-1}{2} \\ 3t + 3, & r = \frac{t+1}{2} \\ 2t + 1 + 2r, & r = \frac{t+3}{2}, \ldots, t. \end{cases}$$

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