AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **59(1)** (2014), Pages 157–160

Note on a class of $(q+1)$-sets of $\mathrm{PG}(3,q)$

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Abstract

The plane degree $g_{\mathcal{K}}(2)$ of a subset $\mathcal{K}$ of $\mathrm{PG}(3,q)$ is the greatest integer such that at least one plane intersecting $\mathcal{K}$ in exactly $g_{\mathcal{K}}(2)$ points exists. In this note, $(q+1)$-arcs of $\mathrm{PG}(3,q)$ (that is, twisted cubics when q is odd) are characterized as $(q+1)$-sets of type $(0,1,s)_1$ of $\mathrm{PG}(3,q)$ of minimal plane degree.

1 Introduction

A *k-arc* $\mathcal{K}$ in a d-dimensional finite projective space $\mathrm{PG}(d,q)$ over a finite field of order q is a set of k points, no $d+1$ of which belong to a hyperplane. A k-arc $\mathcal{K}$ is *complete* if there is no $(k+1)$-arc containing $\mathcal{K}$. A *rational normal curve* $\mathcal{C}$ in $\mathrm{PG}(d,q)$ is a complete $(q+1)$-arc.

For q odd, any $(q+1)$-arc of $\mathrm{PG}(3,q)$ is a twisted cubic (i.e. a rational normal curve for $d=3$) [9]. For $q=2^h$, $h \geq 3$, every $(q+1)$-arc of $\mathrm{PG}(3,q)$ is projectively equivalent to the set $K(r) = \{(1,t,t^r,t^{r+1}) \mid t \in GF(2^h)\} \cup \{(0,0,0,1)\}$ for some $r=2^n$, $(n,h)=1$ [3].

The *degree* [12], with respect to the dimension r, of a subset $\mathcal{K}$ of $\mathrm{PG}(d,q)$, is the greatest integer $g(r)=g_{\mathcal{K}}(r)$ such that subspaces of dimension r intersecting $\mathcal{K}$ in $g(r)$ points exist. For $r=2$ we speak of plane degree.

Let $0 \leq m_1 < \cdots < m_s$ be a finite increasing series of s non negative integers. A set $\mathcal{K}$ of points of $\mathrm{PG}(d,q)$ is of *class* $[m_1,\ldots,m_s]_1$ if $|\ell \cap \mathcal{K}| \in \{m_1,\ldots,m_s\}$ for any line ℓ. Moreover a set $\mathcal{K}$ of class $[m_1,\ldots,m_s]_1$ is of *type* $(m_1,\ldots,m_s)_1$ if for every m_j, $j=1,\ldots,s$, there exists a line ℓ intersecting $\mathcal{K}$ in m_j points. The integers $m_1,\ldots,m_s$ are the *intersection numbers* of $\mathcal{K}$ (with respect to the dimension $r=1$).

Most of the classical subsets of finite projective geometry, such as quadrics, algebraic varieties which are intersection of quadrics, subgeometries have few intersection

* This research was partially supported by G.N.S.A.G.A. of INdAM.

numbers with respect to some families of subspaces, such as for example those of lines or of hyperplanes, and so the question of characterizing such subsets in terms of their intersection numbers arises. There is a wide literature devoted to this question (e.g. [1, 5, 6, 8, 11, 12, 13, 14]), in particular for the case of two intersection numbers with respect to the hyperplane, because of its connections with coding theory (e.g. [2, 4, 15]).

Recently, Zannetti and Zuanni [16] have given the following characterization of twisted cubics of $\mathrm{PG}(3,q)$, q odd.

Theorem. (Zannetti-Zuanni, 2010) *A $(q+1)$-set of $\mathrm{PG}(3,q)$, q odd, of class $[a,b,c]_1$ such that $g(2)=g(1)+1$ is a twisted cubic.*

In this paper, we are going to prove the following result which generalizes the above one; our proof is also shorter than that of Zannetti and Zuanni.

Theorem I. *Let $\mathcal{K}$ be a $(q+1)$-set of $\mathrm{PG}(3,q)$. Then, $\mathcal{K}$ admits at least an external line, a tangent line and an s-line, and if it is of type $(0,1,s)_1$ then $g(2)\geq s^2-s+1 = g^2(1)-g(1)+1$. Moreover, $g(2)=s^2-s+1$ if and only if $\mathcal{K}$ is a $(q+1)$-arc of $\mathrm{PG}(3,q)$ (and so a twisted cubic if q is odd).*

2 Proof of Theorem I

Throughout this section $\mathcal{K}$ denotes a $(q+1)$-set of $\mathrm{PG}(3,q)$.

Let us start by proving that $\mathcal{K}$ admits both external and tangent lines. Indeed, if p is a point outside $\mathcal{K}$ on which there is no external line, counting k via the lines on p gives $q+1=k\geq q^2+q+1$, a contradiction. Similarly, if p is a point in $\mathcal{K}$ on which there is no tangent line, then $q+1=k\geq 1+q^2+q+1$, again a contradiction.

Since a set of class $[0,1]_1$ is a point, it follows that $\mathcal{K}$ admits at least an s-line.

From now on, assume that $\mathcal{K}$ is of type $(0,1,s)_1$. Then, $g(1)=s$.

The following lemmas give the proof of Theorem I.

Lemma 2.1 *Either $\mathcal{K}$ is a line or $g(2)\geq g^2(1)-g(1)+1=s^2-s+1$.*

PROOF. Let ℓ be an s-line. If $s=q+1$, namely $\ell=\mathcal{K}$. So, we may assume $s\leq q$. Thus, there is a point x_0 of $\mathcal{K}$ not in ℓ. Let π be the plane containing ℓ and x_0. Any line of π joining x_0 with a point of $\ell\cap\mathcal{K}$ is an s-line. So π contains at least s^2-s+1 points of $\mathcal{K}$. □

Lemma 2.2 $(s-1)|q$.

PROOF. Let x be a point of $\mathcal{K}$ and α be the number of s-lines on x. Counting k via the lines on x gives

$$q+1=k=1+\alpha(s-1)$$

from which the assertion follows. □

Thus, since $q = p^h$, with p a prime, it follows that $s - 1 = p^t$ with $t < h$.

Lemma 2.3 *If* $g(2) = s^2 - s + 1$ *then* $s = 2$ *and* $\mathcal{K}$ *is a* $(q+1)$*-arc of* $\mathrm{PG}(3, q)$.

PROOF. Since $g(2) = s^2 - s + 1$, every plane through an s-line with more than s points in common with $\mathcal{K}$ intersects $\mathcal{K}$ in exactly $s^2 - s + 1$ (call such a plane *big plane*).

Assume $s \geq 3$.

Let ℓ be an s-line and let α be the number of big planes on ℓ. Counting k via the planes through ℓ gives

$$q + 1 = k = s + \alpha(s^2 - 2s + 1)$$

so $(s-1)^2|q-(s-1)$. On the other hand, $(s-1)|q$. If $(s-1)^2|q$ then $(s-1)^2|s-1$, a contradiction. Hence $(s-1)^2$ does not divide q. That is $p^{2t} \geq p^h$. Therefore, $(s-1)^2 \geq q$. Hence,

$$s^2 - 2s + 2 \geq q + 1 = s + \alpha(s-1)^2,$$

a contradiction. Therefore $s = 2$. Thus, every plane intersects $\mathcal{K}$ in at most 3-points and so $\mathcal{K}$ is a $(q+1)$-arc. □

Corollary 2.4 (Zannetti-Zuanni, 2010) *A* $(q+1)$*-set of* $\mathrm{PG}(3, q)$, *q odd, of class* $[a, b, c]_1$ *such that* $g(2) = g(1) + 1$, *is a twisted cubic.*

PROOF. From the remarks at the beginning of the section, $\mathcal{K}$ is of type $(0, 1, c)_1$. Now, from $c + 1 = g(1) + 1 = g(2) \geq c^2 - c + 1$ it follows that $c = 2$, and by the Segre characterization of twisted cubics for q odd (see [9]), the assertion follows. □

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(Received 9 Aug 2013)