On the star arboricity of hypercubes[*]

Negin Karisani   E.S. Mahmoodian   Narges K. Sobhani

Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11155–9415, Tehran
I.R. IRAN
karisani@alum.sharif.edu   n.sobhanikhakestar@gmail.com
emahmood@sharif.edu

Abstract

A hypercube Q_n is a graph in which the vertices are all binary vectors of length n, and two vertices are adjacent if and only if their components differ in exactly one place. A galaxy or a star forest is a union of vertex disjoint stars. The star arboricity of a graph G, $\mathrm{sa}(G)$, is the minimum number of galaxies which partition the edge set of G. In this paper among other results, we determine the exact values of $\mathrm{sa}(Q_n)$ for $n \in \{2^k - 3, 2^k + 1, 2^k + 2, 2^i + 2^j - 4\}$, $i \geq j \geq 2$. We also improve the last known upper bound of $\mathrm{sa}(Q_n)$ and show the relation between $\mathrm{sa}(G)$ and square coloring.

1 Introduction and preliminaries

Hypercubes have numerous applications in computer science such as studying networks. Their architecture has played an important role in the development of parallel processing and is still quite popular and influential [10]. An *n-cube* or *n-dimensional hypercube*, Q_n, is a graph in which the vertices are all binary vectors of length n, and two vertices are adjacent if and only if the Hamming distance between them is 1, i.e. their components differ in 1 place. Q_n is also defined recursively in terms of the cartesian product of two graphs as follows:

$$Q_1 = K_2$$
$$Q_n = Q_{n-1} \square K_2,$$

where $\square$ stands for the cartesian product.

A *galaxy* or a *star forest* is a union of vertex disjoint stars. The *star arboricity* of a graph G, denoted by $\mathrm{sa}(G)$, is the minimum number of galaxies which partition the

---

edge set of G. The study of decomposing graphs into galaxies is naturally suggested by the analysis of certain communication networks such as radio networks. As an example, suppose that we need to transmit once along every edge, in order to check that there is indeed a connection between each adjacent pair. It is explained in [2] that the minimum number of steps in which we can finish all the required transmissions is precisely $\mathrm{sa}(G)$. Star arboricity was introduced by Akiyama and Kano in 1982 [1]. They called it star decomposition index. In other literature some authors have used concepts and notations such as galactic number, $\mathrm{gal}(G)$, and star number, $\mathrm{st}(G)$ or $\mathrm{s}(G)$.

Star arboricity is closely related to *arboricity*, the minimum number of forests which partition the edges of a graph G and is denoted by $\mathrm{arb}(G)$. But unlike arboricity which is easy, even determining whether the star arboricity of an arbitrary graph is at most 2, is shown to be an NP-complete problem [4], also see [6]. Clearly, by definition $\mathrm{arb}(G) \leq \mathrm{sa}(G)$. Furthermore, it is easy to see that any tree can be covered by two star forests, thus $\mathrm{sa}(G) \leq 2\mathrm{arb}(G)$. Alon et al. [3] showed that for each k, there exists a graph G_k with $\mathrm{arb}(G_k) = k$ and $\mathrm{sa}(G_k) = 2k$. They also showed that for any graph G, $\mathrm{sa}(G) \leq \mathrm{arb}(G) + O(\log_2 \Delta(G))$, where $\Delta(G)$ is the maximum degree of G.

In [4] and [11] the star arboricity of Q_n is studied and it is shown that $\mathrm{sa}(Q_{2^n-2}) = 2^{n-1}$ and $\mathrm{sa}(Q_{2^n-1}) = 2^{n-1} + 1$. Here by extending earlier results, we find exact values of $\mathrm{sa}(Q_n)$ for $n \in \{2^k - 4, 2^k - 3, 2^k, 2^k + 1, 2^k + 2, 2^k + 4, 2^k + 2^j - 4\}$. Also we introduce a new upper bound and show a relation between $\mathrm{sa}(G)$ and square coloring.

2 Some earlier results

In this section we mention some earlier results about the star arboricity of general graphs which are used in the next section.

In the following theorem Akiyama and Kano found an exact value for the star arboricity of complete graphs K_n.

Theorem A. ([1]) *Let $n \geq 4$. Then the star arboricity of the complete graph of order n is $\lceil \frac{n}{2} \rceil + 1$, i.e. $\mathrm{sa}(K_n) = \lceil \frac{n}{2} \rceil + 1$.*

In the next lemma an upper bound for the star arboricity of product of two graphs is given.

Lemma 1. ([4]) *The star arboricity of the cartesian product of two graphs satisfies $\mathrm{sa}(G \square H) \leq \mathrm{sa}(G) + \mathrm{sa}(H)$.*

Next we state some of Truszczyński's results [11] which will be used in this paper.

Theorem B. ([11]) *Let G be an n-regular graph, $n \geq 2$. Then $\mathrm{sa}(G) \geq \lceil \frac{n}{2} \rceil + 1$.*

Lemma 2. ([11]) *Let G be an n-regular graph, where n is an even number. If $\chi(G) > \frac{n}{2} + 1$ or if $\frac{n}{2} + 1$ does not divide $|V(G)|$, then $\mathrm{sa}(G) \geq \frac{n}{2} + 2$.*

The following question is also raised about the upper bound for $\mathrm{sa}(G)$.

Problem 1. ([11]) *Is it true that for every n-regular graph G,*

$$\left\lceil \frac{n}{2} \right\rceil + 1 \leq \mathrm{sa}(G) \leq \left\lceil \frac{n}{2} \right\rceil + 2 \ ?$$

Lemma 3. ([11]) *If $k \geq 2$, then there is a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_{2^{k-1}}\}$ of $V(Q_{2^k-2})$ such that*

   (i) *for every i, $1 \leq i \leq 2^{k-1}$, A_i is independent,*

   (ii) *for every i, j, $1 \leq i < j \leq 2^{k-1}$, the subgraph of Q_{2^k-2} induced by $A_i \cup A_j$, is 2-regular.*

Proof of Lemma 3 in [11] is constructive and as an example a decomposition of Q_6 into 4 sets is presented in Table 1. This will be used in the next section.

A_1	110011	110000	000000	101001	100110	001111	010110	100101
	111111	011001	111100	001100	000011	010101	101010	011010
A_2	000110	101111	101100	010011	100011	110101	110110	000101
	001001	100000	010000	011100	111001	011111	111010	001010
A_3	000111	111000	100010	101101	010010	110100	110111	001000
	101110	100001	010001	011101	011110	000100	111011	001011
A_4	110010	001110	110001	000001	100100	100111	010111	010100
	011000	111110	111101	001101	101011	101000	011011	000010

Table 1: A vertex decomposition of Q_6 by Lemma 3.

In the earlier results the star arboricity of Q_n was determined in just two cases.

Theorem C. ([11]) $\mathrm{sa}(Q_{2^k-2}) = 2^{k-1}$ *for $k \geq 2$.*

By Theorem B, Lemma 2 and Theorem C we have,

Corollary 1. $\mathrm{sa}(Q_n) \geq \lfloor \frac{n}{2} \rfloor + 2,$ *except for $n = 2^a - 2$, $a \geq 2$.*
For $n = 2^a - 2$, we have $\mathrm{sa}(Q_{2^a-2}) = \lfloor \frac{n}{2} \rfloor + 1 = 2^{a-1}$.

Corollary 2. (Also in [4].) $\mathrm{sa}(Q_{2^k-1}) = 2^{k-1} + 1$, $k \geq 2$.

Proof. $\mathrm{sa}(Q_{2^k-1}) \geq 2^{k-1} + 1$ by Corollary 1, and $\mathrm{sa}(Q_{2^k-1}) \leq \mathrm{sa}(Q_{2^k-2}) + 1 = 2^{k-1} + 1$. ☐

The following bounds also are given in [11].

Theorem D. ([11]) $\lceil \frac{n+1}{2} \rceil + 1 \leq \mathrm{sa}(Q_n) \leq \lceil \frac{n}{2} \rceil + \log_2 n,$ *for every $n \geq 3$, (except for $n = 2^a - 2$ and $a \geq 2$).*

In the next section we introduce more exact values of star arboricity of some Q_n.

3 Hypercubes

In this section we focus on the star arboricity of hypercubes and extend earlier results. Based on the results, we conjecture that:

Conjecture 1. *For $n \neq 2^a - 2$, $\mathrm{sa}(Q_n) = \lfloor \frac{n}{2} \rfloor + 2$, and for $n = 2^a - 2$, $\mathrm{sa}(Q_{2^a-2}) = \lfloor \frac{n}{2} \rfloor + 1 = 2^{a-1}$, $a \geq 2$.*

Note that the second part of Conjecture 1 is known to be true (Theorem C).

Theorem 1. *If Conjecture 1 holds for an odd integer n, then it holds for $n-1$ and $n+1$.*

Proof. If $n - 1 = 2^a - 2$ or $n + 1 = 2^a - 2$ for some a, then the statement follows. Otherwise let $n = 2k + 1$. For $n - 1 = 2k$, we have $\mathrm{sa}(Q_{2k}) \leq \mathrm{sa}(Q_{2k+1}) = k + 2$; the statement follows by Corollary 1. For $n + 1 = 2k + 2$, we have $\mathrm{sa}(Q_{2k+2}) \leq \mathrm{sa}(Q_{2k+1}) + 1 = k + 3$ and again by Corollary 1 the statement follows. $\square$

By Theorem 1, one only needs to show Conjecture 1 for odd numbers.

3.1 Exact values

Proposition 1. $\mathrm{sa}(Q_{2^k+2^j-4}) = 2^{k-1} + 2^{j-1}$, *for* $k \geq j \geq 2$.

Proof. By Lemma 1 and Theorem C, we have $\mathrm{sa}(Q_{2^k+2^j-4}) \leq \mathrm{sa}(Q_{2^k-2}) + \mathrm{sa}(Q_{2^j-2}) = 2^{k-1} + 2^{j-1}$. Also by Corollary 1, $\mathrm{sa}(Q_{2^k+2^j-4}) \geq 2^{k-1} + 2^{j-1}$. So $\mathrm{sa}(Q_{2^k+2^j-4}) = 2^{k-1} + 2^{j-1}$. $\square$

The following lemma is a useful tool for the next theorem.

Lemma 4. *If a graph G satisfies the following conditions then $\mathrm{sa}(G) = 2$.*

1. *G is tripartite with $V(G) = V_1 \cup V_2 \cup V_3$.*

2. *Each vertex in V_1 or V_2 has degree 4 and each vertex in V_3 has degree 2.*

3. *Each vertex in V_1 or V_2 has exactly 2 neighbours in V_3 and each vertex in V_3 is adjacent to both V_1 and V_2.*

Proof. We decompose the edges of G into two galaxies in such a way that all of the stars are $K_{1,3}$. The induced subgraph on $H = \langle V_1 \cup V_2 \rangle$ is a bipartite graph. This bipartite graph must be a disjoint union of some even cycles. So we can partition edges of H into the sets M_1 and M_2, such that each of them is a perfect matching in H. Now we partition the edges of G into two galaxies G_1 and G_2: the first one, G_1, is the union of M_1 with an induced subgraph of $\langle V_1 \cup V_3 \rangle$. In a similar way, G_2 is the union of M_2 with an induced subgraph of $\langle V_2 \cup V_3 \rangle$. $\square$

Theorem 2. $\mathrm{sa}(Q_{2^k+1}) = 2^{k-1} + 2$, *for* $k \geq 2$.

Proof. By Corollary 1, $\mathrm{sa}(Q_{2^k+1}) \geq 2^{k-1} + 2$. So it suffices to partition the edges of Q_{2^k+1} into $2^{k-1} + 2$ galaxies. We know that $Q_{2^k+1} = Q_{2^k-2} \square Q_3$. Also by Lemma 3 the vertices of Q_{2^k-2} can be partitioned into 2^{k-1} sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_{2^{k-1}}\}$, such that an induced subgraph between each two sets is a 2-regular subgraph. Now we need some conventions and notation. For a fixed 3-bit codeword c from Q_3, we extend each set A_i, $1 \leq i \leq 2^{k-1}$, to a set $A_i(c)$, which has codewords of length $2^k + 1$, such that for each codeword $c' \in A_i$, we append c to the end of c' in $A_i(c)$. Therefore for each pair i and j, the induced subgraph on $A_i(c) \cup A_j(c)$ in Q_{2^k+1} is a 2-regular graph which can be decomposed into two perfect matchings. We denote them by $A_i(c) \rightarrow A_j(c)$ and $A_j(c) \rightarrow A_i(c)$. Also for any two 3-bit codewords c_1 and c_2 which are different in only one bit, the induced subgraph of two sets $A_i(c_1)$ and $A_i(c_2)$ is a perfect matching between those sets and we denote it by $A_i(c_1) \parallel A_i(c_2)$. Also for any 3-bit codeword c, we denote by c^i the 3-bit codeword which is different from c in exactly the i-th bit, and by $\overline{c}$, the complement of c that differs from c in all bits. Also the set of all 3-bit codewords with even weights is denoted by E_c, i.e. $E_c = \{000, 011, 110, 101\}$. Now we are ready to introduce our 2^{k-1} galaxies. For each i, $1 \leq i \leq 2^{k-1}$, define G_i as follows:

$$G_i = \bigcup_{c \in E_c} \{[\underset{j \neq i}{\cup} \left(A_i(c) \rightarrow A_j(c) \right)] \cup [A_i(c) \parallel A_i(c^1)] \cup [\underset{k \neq i, i+1}{\cup} \left(A_{i+1}(c^1) \rightarrow A_k(c^1) \right)]\},$$

$$1 \leq j, k \leq 2^{k-1}.$$

Note that in the above formula the indices i, j, k are considered modulo 2^{k-1}.

Next we prove that the following statements hold for G_i:

Statement 1. Every G_i is a galaxy.

Statement 2. The remaining edges satisfy the conditions of Lemma 4.

By using these two statements, we derive $\mathrm{sa}(Q_{2^k+1}) \leq 2^{k-1} + 2$; therefore the statement of the theorem will hold.

Before proving these statements, as an example, we illustrate our construction in the case of Q_9. We have $Q_9 = Q_6 \square Q_3$. Previously in Table 1 a decomposition of Q_6 into 4 sets as in Lemma 3 is presented. In Figure 1:(a), we have shown a Q_3 and a figure in which each of these partitioned sets is a vertex of $2K_4$, where each edge stands for a perfect matching between two corresponding sets. In Figure 1:(b) the galaxy G_1 is represented, where again each edge represents a perfect matching. To illustrate more, we have also shown G_3 in Figure 1:(c). Figure 1:(d) is for the last two galaxies obtained from the remaining edges. Each of these presented galaxies can be mapped to a galaxy of Q_9 by a blow up.

Proof of Statement 1. By the definitions, it is obvious that for each G_i and for each $c \in E_c$, the independent sets $A_i(c)$, $A_j(c)$, $A_i(c^1)$, $A_{i+1}(c^1)$ and $A_k(c^1)$,
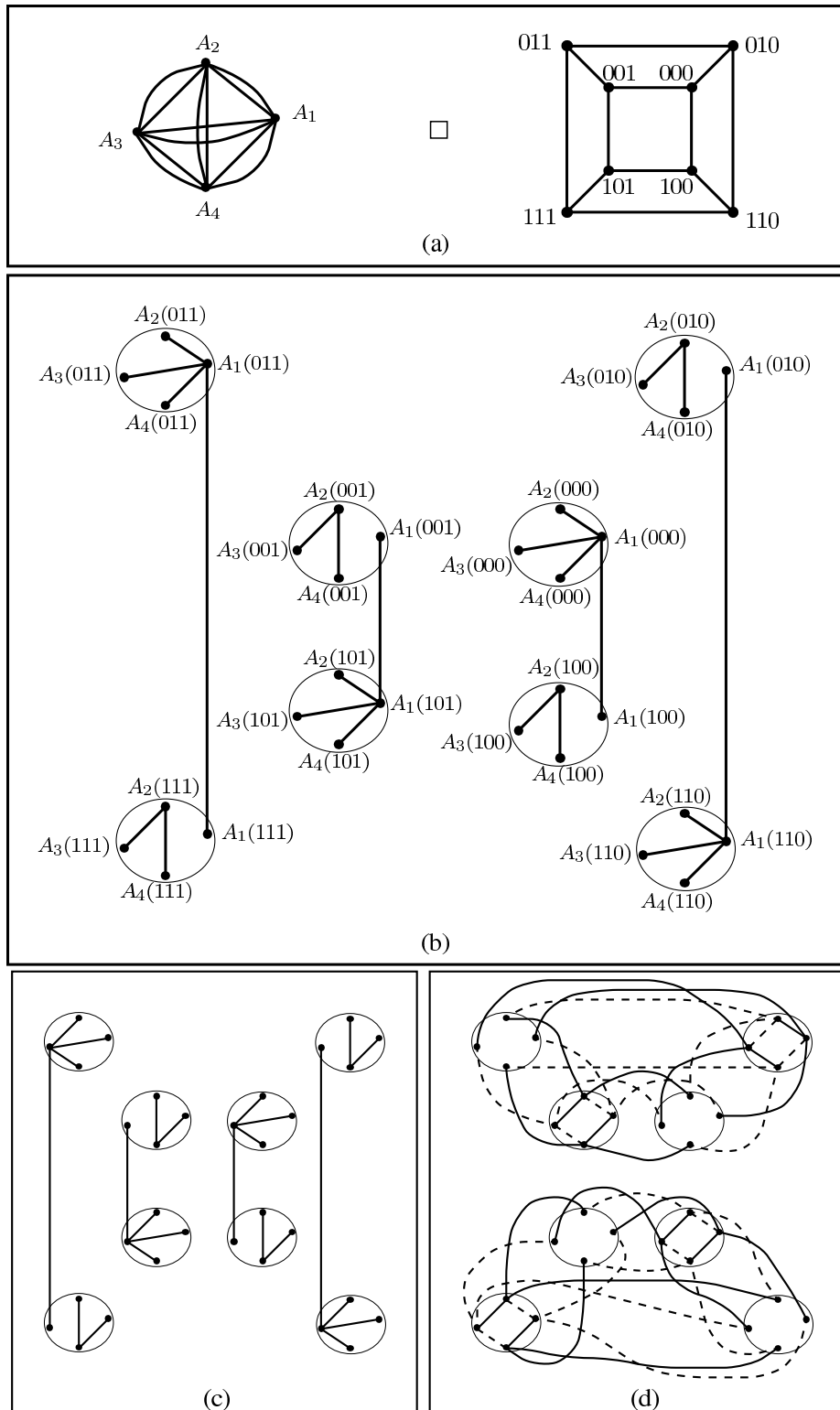
Figure 1: (a) $Q_6 \Box Q_3$, (b) G_1, (c) G_3, (d) Galaxies obtained in Statement 2.

$1 \leq j,\ k \leq 2^{k-1},\ j \neq i,\ k \neq i, i+1$, are mutually disjoint. By construction, every component is a star, and $[\cup_{j \neq i} (A_i(c) \to A_j(c))] \cup [A_i(c) \parallel A_i(c^1)]$ is a union of stars with centers at the vertices in $A_i(c)$, and similarly $\cup_{k \neq i, i+1} (A_{i+1}(c^1) \to A_k(c^1))$ is a union of stars with centers at the vertices in $A_{i+1}(c^1)$. Since here every c^1 corresponds to exactly one c, it follows that these stars have no overlaps.

Proof of Statement 2. We must prove that the remaining edges make a new graph which satisfies conditions of Lemma 4. Let

$$V_1 = \bigcup_{1 \leq s \leq 2^{k-2}} \left(A_{2s}(001) \cup A_{2s}(100) \cup A_{2s-1}(010) \cup A_{2s-1}(111) \right),$$
$$V_2 = \bigcup_{1 \leq s \leq 2^{k-2}} \left(A_{2s-1}(001) \cup A_{2s-1}(100) \cup A_{2s}(010) \cup A_{2s}(111) \right),$$
$$V_3 = \bigcup_{c \in E_c} \left(\cup_{1 \leq i \leq 2^{k-1}} A_i(c) \right).$$

First, we show that in the graph of remaining edges, V_3 is a set of vertices with degree 2. By construction, each vertex in V_3, i.e. $A_i(c)$, $c \in E_c$, is a center with degree $2^{k-1} - 1 + 1$ in a star of G_i, and is a leaf in any other G_j, $j \neq i$, $1 \leq i, j \leq 2^{k-1}$. So each vertex in $A_i(c)$ is covered in the given galaxies by totally $(2^{k-1} - 1 + 1) + (2^{k-1} - 1) \times 1 = 2^k - 1$ adjacent edges. Since Q_{2^k+1}, is $(2^k + 1)$-regular graph, so in the remaining graph the degree of each vertex in $A_i(c)$ is 2. Next we show that each of the remaining vertices which are in $V_1 \cup V_2 = \bigcup_{c \in E_c} \left(\cup_i A_i(\overline{c}) \right)$, $1 \leq i \leq 2^{k-1}$, has degree 4. For each vertex v in $A_i(\overline{c})$, it is clear that the galaxy G_{i-1} covers $2^{k-1} - 2$ edges of v and each of the other galaxies, G_j, $j \neq i - 1$, covers 1 edge of v. So $(2^{k-1} - 2) + (2^{k-1} - 1) \times 1 = 2^k - 3$ edges of each vertex in $A_i(\overline{c})$ is covered by all G_j, $1 \leq j \leq 2^{k-1}$. Thus each vertex in $A_i(\overline{c})$ has 4 uncovered edges. Therefore each of the vertices in $V_1 \cup V_2$ has degree 4 and the vertices in V_3 have degree 2, hence Condition (2) is satisfied.

For Condition (3), by the given construction we note that in the first 2^{k-1} galaxies each vertex of V_3, i.e. $A_i(c)$ (i fixed and $c \in E_c$) is adjacent just to one of the vertices of $V_1 \cup V_2$, i.e. $A_i(c^1)$. So in the remaining graph each vertex in V_3, which has degree 2, is adjacent to both a vertex with degree 4 in $A_i(c^2)$, and another vertex with degree 4 in $A_i(c^3)$. Since $A_i(c^2) \cup A_i(c^3) \subseteq V_1 \cup V_2$, so vertices of V_3 are independent. Since c^2 and c^3 are different in the last two bits, so $A_i(c^2)$ and $A_i(c^3)$ can not be in the same V_j, $j \in \{1, 2\}$. Also each vertex in $A_i(\overline{c})$, which is in V_1 or V_2, is adjacent to a vertex with degree 2 in $A_i(\overline{c}^2)$ and another vertex with degree 2 in $A_i(\overline{c}^3)$. Thus each vertex in V_1 and V_2 has exactly 2 neighbors in V_3. Hence Condition (3) holds.

To prove Condition (1), it remains to show that each of V_1 and V_2 is an independent set. As we have seen each vertex in $A_i(\overline{c})$ has degree 4 and two of its neighbors are in V_3. The other two neighbors are in the sets $A_{i+1}(\overline{c})$ and $A_{i-1}(\overline{c})$. By the definition of V_1 and V_2, it is obvious that $A_i(\overline{c})$ is not in the same V_j, $j \in \{1, 2\}$, as $A_{i+1}(\overline{c})$ and $A_{i-1}(\overline{c})$ are. $\square$

Lemma 5. $\mathrm{sa}(Q_n) = \lfloor \frac{n}{2} \rfloor + 2$ *for* $n = 2^k + 4$ *and* $2^k - 4 \leq n \leq 2^k + 2$ *except for* $2^k - 2$.

Proof. We can check that the statement holds for $n \leq 10$, see Table 2.

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{sa}(Q_n)$	1	2	3	4	4	4	5	6	6	7

Table 2: $\mathrm{sa}(Q_n)$ for $n \leq 10$.

So let $k \geq 3$, other than previous mentioned cases in Theorem C, Theorem 2 and Corollary 2, for the remaining cases the statement holds as follows:

- $n = 2^k + 4$, (by Proposition 1 for $j = 3$),

- $n = 2^k + 2$, (by Theorem 2 and Theorem 1 for $n = 2^k + 1$),

- $n = 2^k$, (by Proposition 1 for $j = 2$) and (by Theorem 2 and Theorem 1 for $n = 2^k + 1$),

- $n = 2^k - 3$, ($\mathrm{sa}(Q_{2^k-3}) \leq \mathrm{sa}(Q_{2^k-2}) = 2^{k-1}$ and by Corollary 1 for $n = 2^k - 3$),

- $n = 2^m - 4$, (by Proposition 1 for $k = j = m - 1$) and (by Theorem 1 for $n = 2^m - 3$).
  $\square$

The value of $\mathrm{sa}(Q_{2^k+3})$ is left open, but we know that $2^{k-1}+3 \leq \mathrm{sa}(Q_{2^k+3}) \leq 2^{k-1}+4$. So the smallest unknown case is $\mathrm{sa}(Q_{11})$.

3.2   An upper bound

In the following theorem, we improve the known upper bound on $\mathrm{sa}(Q_n)$ by a method similar to the proof of Theorem D.

Theorem 3. *Let n be an even integer. We can write n as $n = \sum_{j=1}^{l}(2^{i_j} - 2) + r$, where r is in $\mathcal{R} = \{2^k + 2, 2^s + 2^t - 4\}$, $s \geq t \geq 2$, $i_1 > i_2 > \cdots > i_l$ and l is the smallest number with this property. Also we have*

$$\mathrm{sa}(Q_n) \leq \frac{n}{2} + l + 2.$$

Proof. If $n \in \mathcal{R}$ then $l = 0$ and the statement holds by Lemma 5 or by Proposition 1. Otherwise it is easy to see that n can be written as $2^{i_1} - 2 + r_1$, where i_1 is the largest possible integer and r_1 is the remainder. If $r_1 = 0$ then the statement holds by Theorem C, else $4 < r_1 < 2^{i_1} - 2$. If $r_1 \in \mathcal{R}$ then $l = 1$ and by Lemma 5 and Proposition 1, $\mathrm{sa}(Q_{r_1}) = \frac{r_1}{2} + 2$ and $\mathrm{sa}(Q_n) \leq \mathrm{sa}(Q_{2^{i_1}-2}) + \mathrm{sa}(Q_{r_1}) = 2^{i_1-1} + \frac{r_1}{2} + 2 =$

$\frac{n}{2} + 3$, and we are done. Else, again r_1 can be written as $2^{i_2} - 2 + r_2$, where i_2 is the largest possible integer and so on. Thus assume $n = \sum_{j=1}^{l}(2^{i_j} - 2) + r$, $r \in \mathcal{R}$, then

$$
\begin{aligned}
\text{sa}(Q_n) \ &\leq \ \text{sa}(Q_{2^{i_1}-2}) + \text{sa}(Q_{2^{i_2}-2}) + \cdots + \text{sa}(Q_{2^{i_l}-2}) + \text{sa}(Q_r) \\
&= \ 2^{i_1-1} + 2^{i_2-1} + \cdots + 2^{i_l-1} + \frac{r}{2} + 2 \\
&= \ \frac{2^{i_1}-2}{2} + \frac{2^{i_2}-2}{2} + \cdots + \frac{2^{i_l}-2}{2} + l + \frac{r}{2} + 2 \\
&= \ \frac{n-r}{2} + l + \frac{r}{2} + 2 \\
&= \ \frac{n}{2} + l + 2.
\end{aligned}
$$

$\square$

Corollary 3. *Let $n \geq 1$, then $\text{sa}(Q_n) \leq \lceil \frac{n}{2} \rceil + l + 2$, where l is obtained for n or $n-1$ as in Theorem 3, whether n is even or odd, respectively.*

Corollary 4. $\text{sa}(Q_n) \leq \lceil \frac{n}{2} \rceil + \lfloor \log_2 n \rfloor - 1$ *for* $n \geq 5$.

Proof. For $n = 5$ or 6 it follows from Lemma 5. If the statement holds for an even number n, then it holds for $n + 1$ as follows

$$
\begin{aligned}
\text{sa}(Q_{n+1}) \ &= \ \text{sa}(Q_n \square K_2) \\
&\leq \ \text{sa}(Q_n) + 1 \\
&\leq \ \left\lceil \frac{n}{2} \right\rceil + \lfloor \log_2 n \rfloor - 1 + 1 \\
&= \ \left\lceil \frac{n+1}{2} \right\rceil + \lfloor \log_2(n+1) \rfloor - 1 \\
&\qquad \text{(since for even } n,\ \lfloor \log_2(n+1) \rfloor = \lfloor \log_2 n \rfloor).
\end{aligned}
$$

So it suffices to prove the corollary for an even number n.

It is easy to see that n can be represented as a sum of $\lfloor \log_2 n \rfloor$ numbers of the form $2^k - 2$, i.e. $n = \sum_{j=1}^{m}(2^{i_j} - 2)$, $m \leq \lfloor \log_2 n \rfloor$. In Theorem 3, we represented $n = \sum_{j=1}^{l}(2^{i_j} - 2) + r$. So $l \leq \lfloor \log_2 n \rfloor - \lfloor \log_2 r \rfloor$. As in Theorem 3, $\text{sa}(Q_n) \leq \frac{n-r}{2} + \lfloor \log_2 n \rfloor - \lfloor \log_2 r \rfloor + \text{sa}(Q_r)$, $r \in \mathcal{R}$.
Now if $r = 2^k + 2$, then we have

$$
\begin{aligned}
\text{sa}(Q_n) \ &\leq \ \frac{n - 2^k - 2}{2} + \lfloor \log_2 n \rfloor - k + 2^{k-1} + 3 \\
&= \ \frac{n}{2} - 2^{k-1} - 1 + \lfloor \log_2 n \rfloor - k + 2^{k-1} + 3 \\
&= \ \frac{n}{2} + \lfloor \log_2 n \rfloor - k + 2.
\end{aligned}
$$

If $r = 2^k + 2^j - 4$ and $k \geq j \geq 2$,

$$
\begin{aligned}
\mathrm{sa}(Q_n) \;&\leq\; \frac{n - 2^k - 2^j + 4}{2} + \lfloor \log_2 n \rfloor - k + 2^{k-1} + 2^{j-1} \\
&=\; \frac{n}{2} - 2^{k-1} - 2^{j-1} + 2 + \lfloor \log_2 n \rfloor - k + 2^{k-1} + 2^{j-1} \\
&=\; \frac{n}{2} + \lfloor \log_2 n \rfloor - k + 2.
\end{aligned}
$$

As we have seen in both cases $\mathrm{sa}(Q_n) \leq \frac{n}{2} + \lfloor \log_2 n \rfloor - k + 2$. In both cases k can be considered greater than or equal to 3. As an example for $r = 2^k + 2^j - 4$, assume $k = 2$, then $j = 2$ and $r = 4$. Hence the last two numbers in $\sum_{j=1}^{l}(2^{i_j} - 2) + r$, are $2^{i_l} - 2$ and r where $i_l \geq 3$. So we have $2^{i_l} - 2 + r = 2^{i_l} - 2 + 4 = 2^{i_l} + 2$ which is a contradiction the choice of r. Therefore $\mathrm{sa}(Q_n) \leq \frac{n}{2} + \lfloor \log_2 n \rfloor - k + 2 \leq \frac{n}{2} + \lfloor \log_2 n \rfloor - 1$. $\qquad\square$

4 Coloring and star arboricity

The connection of star arboricity with other colorings such as incidence coloring and acyclic coloring are studied (see [7] and [8]). In this section we consider the connection between square coloring and star arboricity of graphs.

Square of a graph G is a graph denoted by G^2 with $\mathrm{V}(G) = \mathrm{V}(G^2)$ and two vertices are adjacent if their distance in G is at most 2. A square-coloring of G is a proper coloring of G^2. Let $\chi(G^2)$ be the minimum number of colors used in any square-coloring of G.

Theorem 4. *If* $\chi(G^2) \leq k$ *then* $\mathrm{sa}(G) \leq \lceil \frac{k}{2} \rceil + 1$, $k \geq 4$.

Proof. Let c be a proper k-coloring of G^2 with color classes $C_1, C_2, \ldots, C_k$. We show that the degree of vertices in any induced subgraph on each pair of color classes is at most 1. Assume to the contrary that there are two classes C_i and C_j, $1 \leq i, j \leq k$, such that an induced subgraph on them has a vertex of degree at least 2. Without loss of generality let v be a vertex in C_i with $\deg_{\langle C_i \cup C_j \rangle} v = 2$. So v has at least two neighbors u and w in C_j. But in G^2, u and w are adjacent, that is contradiction with c being a proper coloring. Thus the vertices of G are partitioned into k independent sets such that an induced subgraph on each pair of them is a matching.

Now using G we construct a graph H as follows. Each vertex of H corresponds to a color class of G and two vertices are adjacent if there is an edge between their corresponding color classes. Clearly H is a subgraph of K_k. Thus from Theorem A, $\mathrm{sa}(H) \leq \lceil \frac{k}{2} \rceil + 1$. By a blow up each galaxy of H can be mapped to a galaxy of G. $\qquad\square$

Note that the result in the theorem can be sharp. As an example for $Q_{2^t - 1}$, we have $\chi(Q_{2^t - 1}^2) = 2^t$ (see [5] and [9]) which implies that $\mathrm{sa}(Q_{2^t - 1}) \leq 2^{t-1} + 1$.

Acknowledgements

References

[1] J. Akiyama and M. Kano, Path factors of a graph, In *Graphs and Applications (Boulder, Colo., 1982)*, Wiley-Intersci. Publ., pp. 1–21, Wiley, New York, 1985.

[2] N. Alon, Probabilistic methods in coloring and decomposition problems, *Discrete Math.* 127 (1-3) (1994), 31–46.

[3] N. Alon, C. McDiarmid and B. Reed, Star arboricity, *Combinatorica* 12 (4) (1992), 375–380.

[4] M. Fellows, M. Hoover and F. Harary, On the galactic number of a hypercube, *Math. Comput. Modelling* 11 (1988), 212–215, in Mathematical modelling in science and technology (St. Louis, MO, 1987).

[5] M. Ghebleh, L. A. Goddyn, E,. S. Mahmoodian and M. Verdian-Rizi, Silver cubes, *Graphs Combin.* 24 (5) (2008), 429–442.

[6] D. Gonçalves and P. Ochem, On star and caterpillar arboricity, *Discrete Math.* 309 (11) (2009), 3694–3702.

[7] B. Guiduli, On incidence coloring and star arboricity of graphs, *Discrete Math.* 163 (1-3) (1997), 275–278.

[8] S. L. Hakimi, J. Mitchem and E. Schmeichel, Star arboricity of graphs, *Discrete Math.* 149 (1-3) (1996), 93–98.

[9] P. R. J. Östergård, On a hypercube coloring problem, *J. Combin. Theory Ser. A* 108 (2) (2004), 199–204.

[10] B. Parhami, *Introduction to Parallel Processing: Algorithms and Architectures*, Kluwer Academic Publishers, 2002.

[11] M. Truszczyński, Decomposing graphs into forests of stars, in *Proc. 17th Southeastern Int. Conf. Combin., Graph Theory, Comput. (Boca Raton, Fla., 1986)*, Vol. 54 (1986), 73–86.