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Forbidden families of configurations

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Abstract

A *simple* matrix is a $(0,1)$-matrix with no repeated columns. For a $(0,1)$-matrix F, we say that a $(0,1)$-matrix A has F as a *configuration* if there is a submatrix of A which is a row and column permutation of F (*trace* is the set system version of a *configuration*). Let $\|A\|$ denote the number of columns of A. Let $\mathcal{F}$ be a family of matrices. We define the extremal function $\text{forb}(m,\mathcal{F}) = \max\{\|A\| : A \text{ is } m\text{-rowed simple matrix and has no configuration } F \in \mathcal{F}\}$. We consider some families $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ such that individually each $\text{forb}(m, F_i)$ has greater asymptotic growth than $\text{forb}(m,\mathcal{F})$.

1 Introduction

We are initiating an exploration of families of forbidden configurations in this paper, as recommended in [13]. Some notation is needed. Define a matrix to be *simple* if it is a $(0,1)$-matrix with no repeated columns. Such a matrix can be viewed as an element-set incidence matrix. Given two $(0,1)$-matrices F, A, if there is a submatrix of A which is a row and column permutation of F then we say A has F as a *configuration* and write $F \prec A$. In set terminology the notation *trace* would be appropriate. For a subset of rows S, define $A|_S$ as the submatrix of A consisting of rows S of A. Define $[n] = \{1, 2, \ldots, n\}$. If F has k rows and A has m rows and $F \prec A$, then there is a k-subset $S \subseteq [m]$ such that $F \prec A|_S$. For two m-rowed matrices A, B, use $[A \,|\, B]$ to denote the concatenation of A, B yielding a larger m-rowed matrix. Define $t \cdot A$ to be the matrix obtained from concatenating t copies of A. These two operations need not yield simple matrices. Let A^c denote the $(0,1)$-complement of A.

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Define $\|A\|$ to be the number of columns of A. For a set of matrices $\mathcal{F}$, define our extremal problem as follows:

$$\text{Avoid}(m, \mathcal{F}) = \{A : A \text{ is } m\text{-rowed, simple}, F \nprec A \text{ for all } F \in \mathcal{F}\},$$

$$\text{forb}(m, \mathcal{F}) = \max_A\{\|A\| : A \in \text{Avoid}(m, \mathcal{F})\}.$$

When $|\mathcal{F}| = 1$ and $\mathcal{F} = \{F\}$, we write Avoid(m, F) and forb(m, F). A conjecture of Anstee and Sali [3] for a single configuration sometimes makes the correct predictions for the asymptotic growth of forb$(m, \mathcal{F})$. Let I_k denote the $k \times k$ identity matrix and let T_k denote the $k \times k$ triangular simple matrix with a 1 in position (i, j) if and only if $i \leq j$. For an $m_1 \times n_1$ simple matrix A and a $m_2 \times n_2$ simple matrix B, we define the 2-fold product $A \times B$ to be the $(m_1 + m_2) \times n_1 n_2$ simple matrix whose columns are obtained from placing a column of A on top of a column of B in all possible ways. This generalizes to p-fold products. For a configuration F, define $X(F)$ as the smallest value of p such that $F \prec A_1 \times A_2 \times \cdots \times A_p$ for every p-fold product where $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$.

Conjecture 1.1 *[3] The extremal function forb(m, F) is $\Theta(m^{X(F)-1})$.*

This conjecture will help in guessing asymptotic bounds for forb$(m, \mathcal{F})$. Define $X(\mathcal{F})$ as the smallest value of p such that for every every p-fold product $A_1 \times A_2 \times \cdots \times A_p$ where $A_i \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$ there is some $F \in \mathcal{F}$ with $F \prec A_1 \times A_2 \times \cdots \times A_p$. We might expect that forb$(m, \mathcal{F})$ is $\Theta(m^{X(\mathcal{F})-1})$ but Theorem 1.7 and Theorem 1.8 are quick counterexamples.

Two easy remarks are the following. The definition of A^c is above.

Remark 1.2 *We have forb$(m, \{F_1, F_2, \ldots, F_t\})$ = forb$(m, \{F_1^c, F_2^c, \ldots, F_t^c\})$.*

Remark 1.3 *Let $\mathcal{F} \subseteq \mathcal{G}$. Then forb$(m, \mathcal{G}) \leq$ forb$(m, \mathcal{F})$.*

Remark 1.4 *Let $\mathcal{F}$ be given with $F \in \mathcal{F}$. Let F' be given with $F \prec F'$, Then forb$(m, \mathcal{F} \cup \{F'\})$ = forb$(m, \mathcal{F})$.*

In view of Remark 1.4, define $\mathcal{F}$ to be *minimal* if there are no pairs $F, F' \in \mathcal{F}$ with $F \prec F'$.

Some examples are in order. Balanced and totally balanced matrices are classes of matrices which can each be defined using an infinite family of forbidden configurations. Let C_k denote the vertex-edge incidence matrix of the cycle of length k. Thus

$$\text{e.g.} \quad C_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, C_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

A matrix A is *balanced* if has no configuration C_k for k odd and a matrix is *totally balanced* if it has no configuration C_k for all $k \geq 3$. These are important classes of matrices. While the definitions do not require the matrices to be simple, it is still of interest how many different columns can there be in a balanced (resp. totally balanced) matrix on m rows. We obtain an upper bound using Remark 1.3 and the lower bound follows from the result that any $m \times \text{forb}(m, C_3)$ matrix $A \in \text{Avoid}(m, C_3)$ is necessarily totally balanced.

Theorem 1.5 *[1] We have:*

$$forb(m, C_3) = forb(m, \{C_3, C_4, C_5, C_6, \ldots\}) = forb(m, \{C_3, C_5, C_7, C_9, \ldots\}).$$

The result $\text{forb}(m, C_3) = \binom{m}{2} + \binom{m}{1} + \binom{m}{0}$ is due to Ryser [14]. Note that $X(\{C_3, C_4, C_5, C_6, \ldots\}) = X(\{C_3, C_5, C_7, C_9, \ldots\}) = 3$ where the construction $T_{m/2} \times T_{m/2}$ avoids C_k for all $k \geq 3$. From another point of view, the result suggests that the bound for a forbidden family might arise from the most restrictive configuration in the family (i.e. $\text{forb}(m, \mathcal{F}) = \min_{F \in \mathcal{F}} \text{forb}(m, F)$ or its asymptotic equivalent) but this is generally not true. The following examples suggest that forbidden families can behave quite differently. Consider the fundamental extremal function $\text{ex}(m, H)$ which denotes the maximum number of edges in a (simple) graph on m vertices that has no subgraph H. Let $\mathbf{1}_k$ denote the $k \times 1$ column of 1's. There is a connection to forbidden families as follows. Note that each $A \in \text{Avoid}(m, \mathbf{1}_3)$ consists of columns of column sum 0,1,2. There are at most $m+1$ columns of column sum 0 or 1 on m rows. The columns of sum 2 in A can be interpreted as a vertex-edge incidence matrix of a graph. For a graph H, let $\text{Inc}(H)$ denote its vertex-edge incidence matrix.

Lemma 1.6 *We have $forb(m, \{\mathbf{1}_3, Inc(H)\}) = ex(m, H) + m + 1$.*

Two sample results concerning $\text{ex}(m, H)$ yield the following where the vertex-edge incidence matrix of the cycle of length k is C_k.

Theorem 1.7 *[11] We have $forb(m, \{\mathbf{1}_3, C_4\})$ is $\Theta(m^{3/2})$.*

Theorem 1.8 *[8] We have $forb(m, \{\mathbf{1}_3, C_6\})$ is $\Theta(m^{4/3})$.*

Simonovits refers to an unpublished upper bound of Erdős as the 'Even Circuit Theorem' so the origins of the results are partly folklore. The analogue of Conjecture 1.1 for forbidden families is failing spectacularly on these examples ($X(\{\mathbf{1}_3, C_4\}) = X(\{\mathbf{1}_3, C_6\}) = 2$) and also on the following example. You might note that $I_2 \times I_2$ is the same as C_4 after a row and column permutation.

Theorem 1.9 *[4] We have $forb(m, \{I_2 \times I_2, T_2 \times T_2\})$ is $\Theta(m^{3/2})$.*

Balogh and Bollobás proved the following useful bound which is consistent with Conjecture 1.1. For fixed k, $X(\{I_k, I_k^c, T_k\}) = 1$ since all 1-fold products contain some element of $\{I_k, I_k^c, T_k\}$.

Theorem 1.10 *[7] Let k be given. Then there is a constant c_k so that forb$(m, \{I_k, I_k^c, T_k\}) = c_k$.*

The following lemma is straightforward and quite useful.

Lemma 1.11 *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_\ell\}$. Assume that for every G_i, there is some F_j with $F_j \prec G_i$. Then forb$(m, \mathcal{F}) \leq$ forb$(m, \mathcal{G})$.*

Proof: Assume $\|A\| >$ forb$(m, \mathcal{G})$. Then for some $i \in [t]$, $G_i \prec A$. But by hypothesis there is some $F_j \in \mathcal{F}$ with $F_j \prec G_i$. But then $F_i \prec A$, verifying that forb$(m, \mathcal{F}) \leq$ forb$(m, \mathcal{G})$. ∎

Now combining with Theorem 1.10, we obtain a surprising classification.

Theorem 1.12 *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ be given. There are two possibilities. Either forb$(m, \mathcal{F})$ is $\Omega(m)$ or there exist ℓ, i, j, k with $F_i \prec I_\ell$, with $F_j \prec I_\ell^c$ and with $F_k \prec T_\ell$ in which case there is a constant c with forb$(m, \mathcal{F}) = c$.*

Proof: Let F_i be $a_i \times b_i$ and let $\ell = \max_{i \in [t]}(a_i + b_i)$. Let $\mathcal{G} = \{I_\ell, I_\ell^c, T_\ell\}$. Then $F_j \not\prec I_\ell$ implies $F_j \not\prec I_m$ for any $m \geq \ell$. Thus if $F_j \not\prec I_\ell$ for $j = 1, 2, \ldots t$, then forb$(m, \mathcal{F})$ is $\Omega(m)$ using the construction I_m. The same holds for I^c and T. ∎

This paper considers all pairs of forbidden configurations drawn from Table 1. The listed nine configurations are *minimal quadratic* configurations, namely those Q for which forb(m, Q) is $\Theta(m^2)$ yet for any submatrix Q' of Q, where $Q' \neq Q$, has forb(m, Q') being $O(m)$. The results which yield this list are in [5]. Configurations Q_1, Q_2, Q_4, Q_5 reduce the list dramatically. All other minimal quadratic configurations on 3 or more rows must be simple and cannot have 5 rows. The minimal quadratic configurations of Table 1 have the virtue of having few possible 2-fold constructions avoiding them and so avoiding the configurations in pairs (or larger families) results in interesting interactions. Table 1 lists *all* the 2-fold (quadratic) product constructions (of I, I^c, T) that yield the quadratic lower bounds. There are no 3-fold products to consider since for all i, $X(Q_i) = 3$. This allows you to compute $X(\{Q_i, Q_j\})$ for pairs Q_i, Q_j in the table. The asymptotic growth rates of forb$(m, \{Q_i, Q_j\})$ are collected together in Table 2 and the complete analysis for any non-empty $\mathcal{F} \subset \{Q_1, Q_2, \ldots, Q_9\}$ is in Theorem 5.7. Section 2 handles those pairs with $X(\{Q_i, Q_j\}) = 3$ for which it is immediate that forb$(m, \{Q_i, Q_j\})$ is $\Theta(m^2)$. The section also consider those cases where Lemma 1.11 when applied with Theorem 1.10 yield that forb$(m, \{Q_i, Q_j\})$ is $O(1)$. Section 3 considers how to apply Lemma 1.6 more generally to help with forb$(m, \{Q_5, Q_j\})$. Section 4 provides a new standard induction introduced in [6] that is useful in this context and helps with forb$(m, \{Q_8, Q_j\})$ and forb$(m, \{Q_3, Q_j\})$. Section 5 considers the structures that arise from forbidding Q_9 and then uses this to obtain results on forb$(m, \{Q_9, Q_j\})$.

	Configuration Q_i	forb(m, Q_i)	Construction(s)	Reference
Q_1	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I^c \times I^c$	[9]
Q_2	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I \times I$	[9]
Q_3	$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$	$\lfloor \frac{m^2}{4} \rfloor + m + 1$	$I \times I^c$	[2]
Q_4	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I^c \times I^c$	[15, 16, 17]
Q_5	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I \times I$	[15, 16, 17]
Q_6	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I^c \times I^c$ $I^c \times T$ $T \times T$	[14]
Q_7	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$\binom{m}{2}+\binom{m}{1}+\binom{m}{0}$	$I \times I$ $I \times T$ $T \times T$	[14]
Q_8	$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	$\lfloor \frac{m^2}{4} \rfloor + m + 1$	$T \times T$	[5]
Q_9	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\binom{m}{2} + 2m - 1$	$I \times T$ $I^c \times T$	[12]

Table 1: Minimal Quadratic Configurations

	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9
Q_1	$\Theta(1)$ Th 2.6	$\Theta(m)$ Cor 4.8	$\Theta(m^2)$ Th 2.1	$\Theta(1)$ Th 2.6	$\Theta(m^2)$ Th 2.1	$\Theta(1)$ Th 2.6	$\Theta(m)$ Cor 4.2	$\Theta(m)$ Cor 5.3
Q_2		$\Theta(m)$ Cor 4.8	$\Theta(1)$ Th 2.6	$\Theta(m^2)$ Th 2.1	$\Theta(1)$ Th 2.6	$\Theta(m^2)$ Th 2.1	$\Theta(m)$ Cor 4.2	$\Theta(m)$ Cor 5.3
Q_3			$\Theta(m)$ Th 3.6	$\Theta(m)$ Th 3.6	$\Theta(m)$ Cor 4.8	$\Theta(m)$ Cor 4.8	$\Theta(m)$ Cor 4.4	$\Theta(m)$ Cor 5.3
Q_4				$\Theta(1)$ Th 2.6	$\Theta(m^2)$ Th 2.1	$\Theta(1)$ Th 2.6	$\Theta(m)$ Th 3.6	$\Theta(m)$ Th 3.6
Q_5					$\Theta(1)$ Th 2.6	$\Theta(m^2)$ Th 2.1	$\Theta(m)$ Th 3.6	$\Theta(m)$ Th 3.6
Q_6						$\Theta(m^2)$ Th 2.2	$\Theta(m^2)$ Th 2.2	$\Theta(m^2)$ Th 2.3
Q_7							$\Theta(m^2)$ Th 2.2	$\Theta(m^2)$ Th 2.3
Q_8								$\Theta(m)$ Th 5.5

Table 2: Asymptotic growth rates of $\text{forb}(m, \{Q_i, Q_j\})$.

2 Quadratic and Constant Bounds

First we are interested in pairs with $X(\{Q_i, Q_j\}) = 3$ for which it follows that $\text{forb}(m, \{Q_i, Q_j\})$ is $\Theta(m^2)$ (the upper bound follows from Remark 1.3 using the fact that $\text{forb}(m, \{Q_i\})$ is $O(m^2)$ for all $i \in [9]$).

Theorem 2.1 *We have forb*$(m, \{Q_1, Q_4, Q_6\}) =$ *forb*$(m, \{Q_2, Q_5, Q_7\})$ *is* $\Theta(m^2)$.

Proof: Use the construction $I^c_{m/2} \times I^c_{m/2} \in \text{Avoid}(m, \{Q_1, Q_4, Q_6\})$ to deduce that $X(\{Q_1, Q_4, Q_6\}) = 3$ and $I_{m/2} \times I_{m/2} \in \text{Avoid}(m, \{Q_2, Q_5, Q_7\})$ to show that $X(\{Q_2, Q_5, Q_7\}) = 3$. ∎

Theorem 2.2 *We have forb*$(m, \{Q_6, Q_7, Q_8\})$ *is* $\Theta(m^2)$.

Proof: The construction $T_{m/2} \times T_{m/2} \in \text{Avoid}(m, \{Q_6, Q_7, Q_8\})$ shows that $X(\{Q_6, Q_7, Q_8\}) = 3$. ∎

Theorem 2.3 *We have forb*$(m, \{Q_6, Q_9\})$ *and forb*$(m, \{Q_7, Q_9\})$ *are* $\Theta(m^2)$.

Proof: Use the construction $I^c_{m/2} \times T_{m/2} \in \text{Avoid}(m, \{Q_6, Q_9\})$ to deduce that $X(\{Q_6, Q_9\}) = 3$ and $I_{m/2} \times T_{m/2} \in \text{Avoid}(m, \{Q_7, Q_9\})$ yields $X(\{Q_7, Q_9\}) = 3$. ∎

Families $\mathcal{F}$ for which $\text{forb}(m, \mathcal{F})$ is $O(1)$ must arise from applying Lemma 1.11 and Theorem 1.10 in view of Theorem 1.12. There are no 2-fold or 1-fold product constructions in common for Q_1, Q_2 so that $X(\{Q_1, Q_2\}) = 1$. Using Theorem 1.10 and Lemma 1.11 yields a constant bound but perhaps recording a general result is in order. Let $0_{a,b}$ denote the $a \times b$ matrix of 0's and let $J_{a,b}$ denote the $a \times b$ matrix of 1's.

Theorem 2.4 *Let k, ℓ, p, q be given. Then there exists some constant $c_{k\ell pq}$ such that for $m \geq c_{k\ell pq}$, $forb(m, \{0_{k,\ell}, J_{p,q}\}) = \ell + q - 2$.*

Proof: Let $d = \max\{k, \ell, p, q\}$. Then $0_{k,\ell} \prec T_{2d}, 0_{k,\ell} \prec I_{2d}$ and $J_{p,q} \prec I^c_{2d}$. Thus by Theorem 1.12, $\text{forb}(m, \{0_{k,\ell}, J_{p,q}\})$ is $O(1)$. The claim is that $\text{forb}(m, \{0_{k,\ell}, J_{p,q}\}) = \ell+q-2$. Let $B \in \text{Avoid}(m, \{0_{k,\ell}, J_{p,q}\})$ with $n = \|B\| > \ell+q-2$. Delete columns from B if necessary to obtain a matrix $A \in \text{Avoid}(m, \{0_{k,\ell}, J_{p,q}\})$ with $n = \|A\| = \ell+q-1$. From Lemma 2.5, the right side of (1) is constant based on n, k, ℓ, p and q. The right hand side of the inequality in (1) is at least m since the summands of the left side will be at least 1 unless $a_r < \ell$ and $b_r < q$ which is impossible because $a_r + b_r = \ell + q - 1$. So for sufficiently large m, this is a contradiction. Hence there exists a constant $c_{k\ell pq}$ so that for $m \geq c_{k\ell pq}$, $\text{forb}(m, \{0_{k,\ell}, J_{p,q}\}) \leq \ell + q - 2$.

It remains to give a construction $A \in \text{Avoid}(m, \{0_{k,\ell}, J_{p,q}\})$ with $\|A\| = \ell + q - 2$. Assume $m = \binom{\ell+q-2}{q-1} + t$ for some $t \geq 0$. Let the first $\binom{\ell+q-2}{q-1}$ rows of A consist of all possible rows of $\ell + q - 2$ entries with exactly $q - 1$ 1's. For the remaining rows of A simply repeat the row of $q - 1$ 1's followed by $\ell - 1$ 0's $m - \binom{\ell+q-2}{q-1}$ times. The matrix is seen to be simple and cannot have $0_{k,\ell}$ since each row has $\ell - 1$ 0's and cannot have $J_{p,q}$ since each row has $q - 1$ 1's. Thus $\text{forb}(m, \{0_{k,\ell}, J_{p,q}\}) \geq q + \ell - 2$. This yields the result. ∎

Lemma 2.5 *Let k, ℓ, p, q be given. Let $A \in$ Avoid$(m, \{0_{k,\ell}, J_{p,q}\})$, with $\|A\| = n$. Also let a_r denote the number of 0's in row r of A, and b_r the number of 1's in row r so that $a_r + b_r = n$. Then:*

$$\sum_{r=1}^{m} \left(\binom{a_r}{\ell} + \binom{b_r}{q} \right) \leq (k-1)\binom{n}{\ell} + (p-1)\binom{n}{q}. \tag{1}$$

Proof: Consider the columns of A. Take all ℓ-subsets of the columns and call them 0-buckets. Similarly, we take all q-subsets of the columns as 1-buckets. There are $\binom{n}{\ell}$ 0-buckets and $\binom{n}{q}$ 1-buckets. Process the rows of A one by one, considering all possible ℓ-subsets and q-subsets of columns on that row. If one of these subsets contains all 0's or all 1's, it makes a contribution to the appropriate 0-bucket or 1-bucket. Thus if there are a 0's in a row, and b 1's (where $a + b = n$), then the row will make contributions to $\binom{a}{\ell}$ 0-buckets and $\binom{b}{q}$ 1-buckets. The left side of (1) is thus the total number of contributions over the rows of A. Each of our $\binom{n}{\ell}$ 0-buckets can have a maximum of $k-1$ contributions, and similarly, our $\binom{n}{q}$ 1-buckets can have a maximum of $p - 1$ contributions, which produces the right side of the inequality. ∎

Theorem 2.6 *We have forb*$(m,\{Q_1,Q_2\})$, *forb*$(m,\{Q_1,Q_5\})$, *forb*$(m,\{Q_1,Q_7\})$, *forb*$(m,\{Q_2,Q_4\})$, *forb*$(m,\{Q_2,Q_6\})$, *forb*$(m,\{Q_4,Q_5\})$, *forb*$(m,\{Q_4,Q_7\})$ *and forb*$(m,\{Q_5,Q_6\})$ *are all bounded by* $O(1)$.

Proof: Apply Lemma 1.11 with $\mathcal{G}=\{I_4,I_4^c,T_4\}$ and also Theorem 1.10. Two examples are the following. For the family $\{Q_1,Q_5\}$, note that $Q_1\prec I_4$, $Q_5\prec I_4^c$ and $Q_1\prec T_4$. For the family $\{Q_5,Q_6\}$, note that $Q_6\prec I_4$, $Q_5\prec I_4^c$ and $Q_5\prec T_4$. ■

The exact values for forb$(m,\{Q_1,Q_2\})$ are recorded below. The function forb$(m,\{Q_1,Q_2\})$ has a surprising non-monotonicity in m.

Theorem 2.7 *[10] We have*

$$\textit{forb}\left(m,\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}\right)=\begin{cases}2 & \textit{if } m=1 \textit{ or } m\geq 7\\ 4 & \textit{if } m=2,5,6\\ 6 & \textit{if } m=3,4\end{cases}.$$

3 Graph Theory

Consider a family $\mathcal{F}=\{\mathbf{1}_3,F\}$ for some F. Note that $Q_5=\mathbf{1}_3$. Now forb$(m,\{\mathbf{1}_3,F\})$ is $O(m^2)$ since forb$(m,\mathbf{1}_3)$ is $O(m^2)$. In this section we consider those F which are $(0,1)$-matrices with column sums 0,1 or 2. If F has a repeated column of sum 2 then $2\cdot\mathbf{1}_2\prec F$ and then forb$(m,\{\mathbf{1}_3,F\})$ is $\Theta(m^2)$ (the construction $I_{m/2}\times I_{m/2}$ yields the lower bound). So assume F has no repeated columns of sum 2 and so these columns can be viewed as the incidence matrix of some graph. Lemma 1.6 is readily extended to those F with columns having sum 0,1 or 2. The following remark describes a useful construction.

Remark 3.1 *Let F be a $k\times\ell$ $(0,1)$-matrix with column sums $\in\{0,1,2\}$. Assume $2\cdot\mathbf{1}_2\not\prec F$. Let a_i be the number of columns of F of sum 1 with a 1 in row i, and let b be the number of columns of F of all 0's. Form a graph G with $V(G)=[k+\sum_{i\in[k]}a_i+b+1]$ as follows. For $i,j\in[k]$, $i,j\in E(G)$ if and only if there is a column of F with 1's in rows i,j. Also, for each $i\in[k]$, add a_i edges to G joining $i\in[k]$ to a_i vertices chosen from $[k+\sum_{i\in[k]}a_i+b+1]\backslash[k]$ (each of which has degree 1). Finally on the remaining $b+1$ vertices add b edges in the form of a tree. Then $F\prec$ Inc(G).*

Proof: By construction, $F\prec\text{Inc}(G)|_{[k]}$. ■

The remark demonstrates some of the differences between a 'subgraph' and a 'configuration'. The following lemma is certainly folklore.

Lemma 3.2 *Let T be a graph on k vertices and assume T has no cycles (i.e., T is a forest). Then ex(m,T) is $O(m)$.*

Proof: Folklore says if a graph G on m vertices has at least km edges then T is a subgraph of G. Assume G has at least km edges. First obtain a subgraph G' of G with minimum degree k which is obtained by removing vertices whose degree is at most $k-1$. Each vertex deleted removes at most $k-1$ edges. Thus the process must stop with a non-empty subgraph G' of G with minimum degree k. Since T has no cycles, there is an ordering of the vertices $v_1, v_2, \ldots, v_k$ of T so that for each v_i there is at most one v_j with $j < i$ such that (v_j, v_i) is an edge of T. Assume we have found in G' a subgraph T, namely vertices $x_1, x_2, \ldots, x_p \in V(G')$ such that $(x_i, x_j) \in E(G')$ if $(v_i, v_j) \in E(T)$ where $1 \leq i < j \leq p$. If $p = k$, we are done. If $p < k$, then consider v_{p+1}. If v_{p+1} is not joined in T to anything in $v_1, v_2, \ldots, v_p$ then select x_{p+1} as any vertex in G' (say adjacent to x_p) which has not already been selected. If v_{p+1} is joined to v_i with $i \leq p$, then choose x_{p+1} as any vertex adjacent x_i which has not already been selected. Use that minimum degree in G' is at least $k > p$. Continue until $p = k$. This yields $\mathrm{ex}(m, T) < km$. ■

The above lemma applies to certain configurations.

Theorem 3.3 *Let F be a given $k \times \ell$ $(0,1)$-matrix such that every column has at most 2 1's. Assume that $2 \cdot \mathbf{1}_2 \not\prec F$ and assume $C_t \not\prec F$ for any $t \geq 3$. Then forb$(m, \{\mathbf{1}_3, F\})$ is $O(m)$.*

Proof: Use Remark 3.1 to obtain a graph G from F. The graph G has no cycles and hence by Lemma 3.2, $\mathrm{ex}(m, G)$ is $O(m)$. We note that $F \prec \mathrm{Inc}(G)$. Now applying Lemma 1.6 yields $\mathrm{forb}(m, \{\mathbf{1}_3, \mathrm{Inc}(G)\})$ is $O(m)$ and so, by Lemma 1.11, $\mathrm{forb}(m, \{\mathbf{1}_3, F\})$ is $O(m)$. ■

The following is a weak version of the extremal graph results of Erdős, Stone and Simonovits since it only considers asymptotic growth rates.

Theorem 3.4 *Let F be a given $k \times \ell$ $(0,1)$-matrix such that every column has at most two 1's. Let t be given. Assume $2 \cdot \mathbf{1}_2 \prec F$ or there is some $t \geq 1$ with $C_{2t+1} \prec F$. Then forb$(m, \{\mathbf{1}_3, F\})$ is $\Theta(m^2)$.*

Proof: The upper bound $O(m^2)$ is easy. The construction $I_{m/2} \times I_{m/2}$ yields the matching lower bound. ■

Let H be a bipartite graph. Then $\mathrm{ex}(m, H)$ is $o(m^2)$. The following result extends this to configurations.

Theorem 3.5 *Let F be a given $k \times \ell$ $(0,1)$-matrix such that every column has at most 2 1's. Let F be given with and also with the property that $2 \cdot \mathbf{1}_2 \not\prec F$ and for all $t \geq 1$, $C_{2t+1} \not\prec F$. Then forb$(m, \{\mathbf{1}_3, F\})$ is $o(m^2)$.*

Proof: Form a graph G as described in Remark 3.1. Since for all $t \geq 1$, $C_{2t+1} \not\prec F$, the resulting graph G will be a bipartite graph. Then for some s, t, G is a subgraph of the complete bipartite graph $K_{s,t}$. Now $\text{ex}(m, K_{s,t})$ is $o(m^2)$. Thus $\text{ex}(m, G)$ is $o(m^2)$. Now $F \prec \text{Inc}(G)$ and so by Lemma 1.6, $\text{forb}(m, \{\mathbf{1}_3, F\})$ is $o(m^2)$. ■

One could imagine trying to obtain similar results for $\text{forb}(m, \{\mathbf{1}_k, F\})$ where F has columns sums at most $k-1$. It is still very much an open problem to determine the exact asymptotic growth $\text{ex}(m, C_{2t})$ for various $t \geq 2$ with two results noted Theorem 1.7, Theorem 1.8. Theorem 3.3 combined with Remark 1.2, yields the following.

Theorem 3.6 *We have forb*$(m, \{Q_5, Q_3\})$*, forb*$(m, \{Q_5, Q_8\})$*, forb*$(m, \{Q_5, Q_9\})$*, forb*$(m, \{Q_4, Q_3\})$*, forb*$(m, \{Q_4, Q_8\})$*, forb*$(m, \{Q_4, Q_9\})$ *are all* $O(m)$*.*

Theorem 3.4 yields that $\text{forb}(m, \{Q_5, Q_7\})$ is $\Omega(m^2)$, a fact which has already been noted.

4 New Standard Induction

Our standard induction argument proceeds as follows. Let $A \in \text{Avoid}(m, \mathcal{F})$ with $\|A\| = \text{forb}(m, \mathcal{F})$. Choose $r \in [m]$ and delete row r from A. The result may have repeated columns in pairs and matrix C_r contains one copy of each pair. After permuting rows and columns, the following decomposition is obtained:

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \begin{bmatrix} 0\,0\cdots 0 & 1\,1\cdots 1 \\ B_r\ C_r & C_r\ D_r \end{bmatrix}. \tag{2}$$

Both $[B_r\, C_r\, D_r]$ and C_r are simple. Thus $[B_r\, C_r\, D_r] \in \text{Avoid}(m-1, \mathcal{F})$ suggesting an induction. Now $[0\,1] \times C_r$ is in A. Define $\mathcal{G}$ as a *minimal* set of configurations F' such that $F \prec [0\,1] \times F'$ for some $F \in \mathcal{F}$ (definition of minimal appears after Remark 1.4). Thus $C_r \in \text{Avoid}(m-1, \mathcal{G})$. This yields the following induction formula

$$\text{forb}(m, \mathcal{F}) = \|A\| = \|[B_r C_r D_r]\| + \|C_r\| \leq \text{forb}(m-1, \mathcal{F}) + \text{forb}(m-1, \mathcal{G}). \tag{3}$$

This means any upper bound on $\|C_r\|$ (as a function of m) automatically yields an upper bound on A by induction. Thus to show $\text{forb}(m, \mathcal{F})$ is $O(m)$ it suffices to show $\|C_r\|$ is bounded by a constant. A new standard induction has been discovered by Anstee and Lu [6] that, by extending the argument to matrices with multiple columns, yields a more powerful induction formula (4). Let A be an m-rowed $(0,1)$-matrix (not necessarily simple) and α be an $m \times 1$ column. Let $\mu(\alpha, A)$ denote the multiplicity of column α in A. Define A to be *s-simple* if every column α of A has $\mu(\alpha, A) \leq s$. Let $\text{Avoid}(m, \mathcal{F}, s)$ denote the m-rowed s-simple matrices with no $F \in \mathcal{F}$. Define

$$\text{forb}(m, \mathcal{F}, s) = \max_A \{\|A\| \, : \, A \in \text{Avoid}(m, \mathcal{F}, s)\}.$$

Note that $\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{F}, s) \leq s \cdot \text{forb}(m, \mathcal{F})$ and so the asymptotic growth rate of $\text{forb}(m, \mathcal{F})$ and $\text{forb}(m, \mathcal{F}, s)$ are the same (for fixed s). Associate with A the simple matrix supp(A) where $\mu(\alpha, \text{supp}(A)) = 1$ if and only if $\mu(\alpha, A) \geq 1$. Given $\mathcal{F}$, let t be the maximum multiplicity of a column in F over all $F \in \mathcal{F}$, i.e. each $F \in \mathcal{F}$ is t-simple but some $F \in \mathcal{F}$ is not $(t-1)$-simple. Assume for (4) that some $F \in \mathcal{F}$ is not simple and so $t \geq 2$. Define $s = t - 1$. Assume $A \in \text{Avoid}(m, \mathcal{F}, s)$. First decompose A using row r as follows:

$$A = \begin{array}{c} \text{row } r \\ \\ \end{array} \begin{bmatrix} 0\,0\cdots 0 & 1\,1\cdots 1 \\ G & H \end{bmatrix}.$$

Then $\mu(\alpha, G) \leq s$ and $\mu(\alpha, H) \leq s$. The following decomposition of A belongs to $\text{Avoid}(m, \mathcal{F}, s)$ based on deleting row r and rearranging by selecting certain columns for C_r so that if $\mu(\alpha, G) + \mu(\alpha, H) \geq s + 1$, then $\mu(\alpha, C_r) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}$. We again obtain (2) with the property that $[B_r\, C_r\, D_r]$ and C_r are both s-simple. Thus $\|[B_r\, C_r\, D_r]\| \leq \text{forb}(m, \mathcal{F}, s)$. Since each column in C_r appears at least $s + 1$ times in $[B_r C_r C_r D_r]$, then C_r has no configuration in $\mathcal{F}' = \{\text{supp}(F) : F \in \mathcal{F}\}$. In the case that each $F \in \mathcal{F}$ is simple then $\mathcal{F}' = \mathcal{F}$. This yields the following useful inductive formula:

$$\text{forb}(m, F, s) = \|A\| = \|[B_r C_r D_r]\| + \|C_r\| \leq s \cdot (\text{forb}(m-1, \mathcal{F}) + \text{forb}(m-1, \mathcal{F}' \cup \mathcal{G}))\,. \tag{4}$$

The extra value here, as compared with (3), is in forbidding in C_r the configurations supp(F) for each $F \in \mathcal{F}$.

Theorem 4.1 *Let k, ℓ be given. Then forb$(m, \{Q_8, [0\,1] \times 0_{k,\ell}\})$ is $O(m)$.*

Proof: Let $A \in \text{Avoid}(m, \{Q_8, [0\,1] \times 0_{k,\ell}\})$. Applying the decomposition of (2), deduce that $C_r \in \text{Avoid}(m-1, \{I_2, 0_{k,\ell}\})$. We note that $Q_8 = [0\,1] \times I_2$ and deduce that $\mathcal{G} = \{I_2, 0_{k,\ell}\}$. With $I_2 \not\prec C_r$, this yields $C_r \prec [\mathbf{0}_{m-1}|T_{m-1}]$ (i.e. C_r is a selection of columns from the triangular matrix). Then, if $\|C_r\| \geq k + \ell$, $0_{k,\ell} \prec C_r$. Hence $\|C_r\| \leq k + \ell - 1$ and then induction on m (using (3)) yields forb$(m, \{Q_8, [0\,1] \times 0_{k,\ell}\})$ is $O(m)$. ■

Corollary 4.2 *We have forb$(m, \{Q_1, Q_8\})$, forb$(m, \{Q_2, Q_8\})$, forb$(m, \{Q_4, Q_8\})$ and forb$(m, \{Q_5, Q_8\})$ are $O(m)$.*

Proof: Note that $Q_1 \prec [0\,1] \times 0_{1,2}$ and $Q_4 \prec [0\,1] \times 0_{2,1}$. Also Q_8^c is the same configuration as Q_8 and $Q_1^c = Q_2$, $Q_4^c = Q_5$, so apply Remark 1.2. ■

Theorem 4.3 *Let $t \geq 2$ be given. Then forb$(m, \{Q_8, t \cdot ([0\,1] \times [0\,1])\})$ is $O(m)$.*

Proof: Let $A \in \text{Avoid}(m, \{Q_8, t \cdot ([0\,1] \times [0\,1])\})$. Apply the decomposition obtained as (2) and deduce that $C_r \in \text{Avoid}(m-1, \{I_2, t \cdot [0\,1]\}$. Note that $Q_8 = [0\,1] \times I_2$ and deduce that $\mathcal{G} = \{I_2, t \cdot [0\,1]\}$. With $I_2 \not\prec C_r$, it follows that $C_r \prec [\mathbf{0}_{m-1}|T_{m-1}]$. With $\|C_r\| \geq 2t$, this forces $t \cdot [0\,1] \prec C_r$. This is a contradiction and so $\|C_r\| \leq 2t - 1$. Induction on m (using (3)) yields forb$(m, \{Q_8, t \cdot ([0\,1] \times [0\,1])\})$ is $O(m)$. ■

Note that $Q_3 \prec 2 \cdot ([0\,1] \times [0\,1])$ to obtain the following.

Corollary 4.4 *We have forb*$(m, \{Q_3, Q_8\})$ *is* $O(m)$.

Note that $Q_3 \not\prec I \times I^c$ and Q_3 is a configuration in the other five 2-fold products. Also $Q_6 \not\prec I^c \times I^c$, $Q_6 \not\prec I^c \times T$ and $Q_6 \not\prec T \times T$ and Q_6 is a configuration in the other three 2-fold products. Note that both T and I^c are 1-fold products avoiding Q_3 and Q_6. Let

$$F_2(1,t,t,1) = \left[\begin{array}{c} 0\,\overbrace{1\,1\cdots 1}^{t}\,\overbrace{0\,0\cdots 0}^{t}\,1 \\ 0\,0\,0\cdots 0\,1\,1\cdots 1\,1 \end{array}\right].$$

This notation is from [5]. Thus $Q_3 = F_2(1,2,2,1)$. We have $F_2(1,t,t,1) \not\prec I \times I^c$ and $F_2(1,t,t,1)$ is a configuration in the other five 2-fold products. Similarly to $Q_6 = I_3$, the configuration $t \cdot I_k$ is not in the $(k-1)$-fold products consisting solely of the terms I^c and T but is in every 2-fold product using I. Thus we might guess (using the forbidden family analog of Conjecture 1.1) that forbidding $F_2(1,t,t,1)$ and $t \cdot I_k$ results in a linear bound. This is true and is proven using two lemmas. The following idea has been used before.

Lemma 4.5 *Let A be a simple matrix with ℓ rows $a_1, a_2, \ldots, a_\ell$ such that there are at most t columns containing $\begin{smallmatrix} a_i \\ a_{i+1} \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for each $i = 1, 2, \ldots, \ell-1$ and also there are at most t columns containing $\begin{smallmatrix} a_\ell \\ a_1 \end{smallmatrix}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Then there are at most ℓt columns from A (as described) and $\ell - 1$ rows $a_1, a_2, \ldots, a_{\ell-1}$ such that deleting these columns and rows from A yields a simple matrix.*

Proof: Consider the matrix A' obtained from A by deleting the special columns described of which there are at most ℓt. Then we deduce that $A'|_{\{a_1,a_2,\ldots,a_\ell\}}$ consists of columns of all 0's and columns of all 1's. Now deleting from A the special columns and the $\ell - 1$ rows $a_1, a_2, \ldots, a_{\ell-1}$ will result in a simple matrix. ■

Lemma 4.6 *Let C be a matrix having row sums at most $t-1$. Assume each column sum is at least 1. Assume $\|C\| \geq (t-1)k$, Then $I_k \prec C$.*

Proof: We could phrase this with sets corresponding to the rows. For row r we form a subset $S_r \subseteq \{1, 2, \ldots, \|C\|\}$ with $q \in S_r$ if and only if there is a 1 in row r and column q. Our induction is on k with the result being trivial for $k = 1$. We can greedily select sets $S_1, S_2, \ldots S_p$ so that $S_j \backslash \left(\cup_{i=1}^{j-1} S_i\right) \neq \emptyset$ for $j \in [p]$ and so that $\cup_{i=1}^{p} S_i \geq (t-1)k$. Begin by choosing an element $a_k \in S_p \backslash \left(\cup_{i=1}^{p-1} S_i\right) \neq \emptyset$. Delete, from our p sets, the elements of S_p (there are at most $t-1$ such elements) and then delete any sets which are now $\emptyset$. We now have sets $S'_1, S'_2, \ldots S'_q$ so that $S'_i \backslash \cup_{i=1}^{j-1} S'_i \neq \emptyset$ and so that $\cup_{i=1}^{q} S_i \geq (t-1)(k-1)$ and $|S_i| \leq t-1$. If we form a set-element incidence matrix C' from these q sets, we find that each row sum of C' is at most $t-1$ ($|S'_i| \leq |S_i| \leq t-1$). Moreover each column sum is at least 1 (we deleted columns corresponding to elements of S_p) and $\|C\| \geq (t-1)(k-1)$ (we only deleted the elements of S_p and $|S_p| \leq t-1$). By induction on k, $I_{k-1} \prec C'$. Now the pth row of C is 0's on columns not in S_p and in column a_k has 0's on all rows except row p for which it is 1. Now we find $I_k \prec C$. ■

The following result applies the lemmas.

Theorem 4.7 *Let $k, t \geq 2$ be given. Then forb$(m, \{t \cdot I_k, F_2(1,t,t,1)\})$ is $\Theta(m)$.*

Proof: Use induction on m. Let $A \in \text{Avoid}(m, \{t \cdot I_k, F_2(1,t,t,1)\}, t-1)$. Use $s = t-1$ and then for any row $r \in [m]$, obtain the decomposition (2). Use (4). With $\mathcal{F} = \{t \cdot I_k, F_2(1,t,t,1)\}$ and $s = t-1$, then $\mathcal{F}' = \{I_k, F_2(1,1,1,1)\}$ (since $F_2(1,t,t,1) \prec t \cdot F_2(1,1,1,1)$) and $\mathcal{G} = \{t \cdot [1\,0], t \cdot [\mathbf{0}_{k-1} \,|\, I_{k-1}]\}$ (since $F_2(1,t,t,1) \prec [0\,1] \times (t \cdot [0\,1])$). Thus $C_r \in \text{Avoid}(m, \{I_k, F_2(1,1,1,1), t \cdot [1\,0], t \cdot [\mathbf{0}_{k-1} \,|\, I_{k-1}]\}, t-1)$. The second configuration ($F_2(1,1,1,1)$) and the fourth configuration ($t \cdot [\mathbf{0}_{k-1} \,|\, I_{k-1}]$) do not get used in the proof. Form a digraph on $[m]$ by setting $r \to q$ if there are at most $t-1$ columns of A with $\begin{array}{c} r \\ q \end{array}\!\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If there is a row q of C_r with one 0 and at least t 1's then, by considering the forbidden configuration $F_2(1,t,t,1)$, it can be deduced that $r \to q$ (else $F_2(1,t,t,1) \prec A|_{\{r,s\}}$). Given a row r, assume no such row q exists. Then all rows of C_r have either at most $t-1$ 1's or are all 1's.

Assume $\|C_r\| \geq tk$. Now remove from C_r any rows of all 1's to obtain a simple matrix C' and obtain a simple matrix C from C' by deleting a column of 0's if it exists. Deduce that each row of C has at most $t-1$ 1's and each column of C has at least one 1. Also $\|C\| \geq tk - 1 \geq (t-1)k$. Then by Lemma 4.6, C_r has I_k, a contradiction. So a row s exists. Since for each row $r \in [m]$ there is some row $q \in [m]$ with $r \to q$, deduce that there is a directed cycle. Now apply Lemma 4.5 to show that $\|A\|$ is $O(m)$. ■

Corollary 4.8 *We have forb$(m, \{Q_1, Q_3\})$, forb$(m, \{Q_2, Q_3\})$, forb$(m, \{Q_3, Q_6\})$ and forb$(m, \{Q_3, Q_7\})$ are $O(m)$.*

Proof: We use Lemma 1.11 with $\mathcal{G} = \{F(1,t,t,1), t \cdot I_k\}$. For example $Q_1 \prec t \cdot I_k$ and $Q_3 \prec F(1,t,t,1)$ and also $Q_6 \prec t \cdot I_k$. We also use Remark 1.2 noting that $\{Q_1^c, Q_3^c\}$ and $\{Q_2, Q_3\}$ are the same as sets of configurations and $\{Q_3^c, Q_6^c\}$ and $\{Q_3, Q_7\}$ are the same as sets of configurations. ■

5 Structure that arises from forbidding Q_9

The following result gives some of the structure of matrices $A \in \text{Avoid}(m, Q_9)$. Let A_k denote the columns of A of column sum k. Then A_k is of one of the following two types. Define A_k to be of *type* 1 if there is a partition of the rows $[m] = X_k \cup Y_k \cup Z_k$ such that all columns in A_k are 1's on rows X_k, 0's on rows Z_k and each column of A_k has exactly one 1 in rows Y_k Thus $A_k|_{Y_k}$ is $I_{|Y(k)|}$. In that case, by examining column sums, $|X_k| + 1 = k$. Define A_k to be of *type* 2 if there is a partition of the rows $[m] = X_k \cup Y_k \cup Z_k$ such that all columns in A_k are 1's on rows X_k, 0's on rows Z_k and each column of A has exactly one 0 in rows Y_k Thus $A_k|_{Y_k}$ is $I^c_{|Y(k)|}$. In that case, by examining column sums, $|X_k| + |Y_k| - 1 = k$. In either type, when $\|A_k\| > 2$ we have $\|A_k\| = |Y_k|$.

Lemma 5.1 *[12] Let $A \in \text{Avoid}(m, Q_9)$. Let A_k denote the columns of column sum k. Then A_k is of type 1 or type 2.*

For $\|A_k\| \le 2$, then A_k is of type 1 and of type 2.

Consider the following $(t+1) \times (2t+2)$ matrix $F(t)$ whose first two rows coincide with $F_2(1, t, t, 1)$:

$$F(t) = \begin{bmatrix} 0\,\overbrace{1\,1 \cdots 1}^{t}\,\overbrace{0\,0 \cdots 0}^{t}\,1 \\ 0\,0\,0 \cdots 0\,1\,1 \cdots 1\,1 \\ \vdots \quad \vdots \qquad \vdots \quad \vdots \\ 0\,0\,0 \cdots 0\,1\,1 \cdots 1\,1 \end{bmatrix}.$$

Lemma 5.2 *Let $t \ge 1$ be given. Then forb$(m, \{Q_9, F(t)\})$ is $O(m)$.*

Proof: Let $A \in \text{Avoid}(m, \{Q_9, F(t)\})$. We will show that $\|A\| \le (7t+1)m$. let A_k denote the columns of column sum k. For $j = 1, 2$, let $W(j) = \{k : A_k \text{ is of type } j, \|A_k\| \ge t+2\}$ and let $V(j)$ be the concatenation of A_k for $k \in W(j)$ so that $\|V(j)\| = \sum_{k \in W(j)} \|A_k\|$.

First note that for $a < b$ that $|X_a \backslash X_b| \le 1$. This is because if $|X_a \backslash X_b| \ge 2$ and $r, s \in X_a \backslash X_b$ then any column α from A_a has 1's on rows r, s. We can choose a column β from A_b with 0's on rows r, s using $r, s \in Y_b \cup Z_b$ and the fact that $\|A_b\| \ge t+2$. But β has more 1's than α and so $Q_9 \prec [\alpha \,|\, \beta]$.

Assume $\|V(1)\| \ge 3tm+1$. Then there are $3t$ indices $\{s(1), s(2), \ldots, s(3t)\} \subseteq W(1)$ where $s(1) < s(2) < \cdots < s(3t)$ so that there is a row r with $r \in \cap_{i=1}^{3t} Y_{s(i)}$. Find a set of rows S with $|S| = t$ such that $S \subseteq X_{s(3t)} \cap \big(\cup_{i=1}^{t} (Y_{s(i)} \cup Z_{s(i)})\big)$. We have $|X_{s(3t)} \backslash X_{s(t)}| \ge 2t$. Using $|X_{s(i)} \backslash X_{s(t)}| \le 1$, we have $|X_{s(3t)} \backslash \big(\cup_{i=1}^{t} X_{s(i)}\big)| \ge t$ and so we can find S as claimed. Now we obtain $F(t)$ as follows. For each i with $1 \le i \le t$, we have $r \in Y_{s(i)}$ and $S \subseteq Y_{s(i)} \cup Z_{s(i)}$. We choose one column from $A_{s(1)}$ with a 0 on row r where we choose the column so it also has 0's on rows S (which is possible since $|Y_{s(i)}| = \|A_{s(i)}\| \ge t+2$ yields $r \cup S \subsetneq Y_{s(i)}$). We choose one column from each $A_{s(i)}$ for $i \in [t]$, with a 1 on row r and necessarily 0's on rows S. All columns from $A_{s(3t)}$ are 1's on rows $S \subseteq X_{s(3t)}$. With $\|A_{s(i)}\| \ge t+2$, we can find $t+1$ columns in $A_{s(3t)}$ of which t are 0 on row r and one is 1 on row r. This yields all of $F(t)$, a contradiction. Thus $\|V(1)\| \le 3tm$.

Noting that $Q_9^c, F(t)^c$ are the same as $Q_9, F(t)$ when considered as configurations, we deduce that $\|V(2)\| \le 3tm$. Now A consists of $V(1)$ and $V(2)$ plus at most $(t+1)m$ columns (to account for $\|A_k\|$ where $\|A_k\| \le t+1$) and so $\|A\| \le (7t+1)m$. ■

Corollary 5.3 *We have forb$(m, \{Q_1, Q_9\})$, forb$(m, \{Q_2, Q_9\})$, forb$(m, \{Q_3, Q_9\})$, forb$(m, \{Q_4, Q_9\})$ and forb$(m, \{Q_5, Q_9\})$ are $O(m)$.*

Proof: We note that Q_1, Q_2, Q_3, Q_4, Q_5 are all configurations in $F(2)$. ■

The following lemma helps with forb$(m, \{Q_8, Q_9\})$.

Lemma 5.4 *Let I be an ordered set. Let $\mathcal{Y} = \{Y_i : i \in I\}$ be a system of distinct sets $Y_i \subseteq [m]$ and $|Y_i| \geq 2$ for $i \in I$. Assume $|Y_i \cap Y_j| \leq 1$ for $i, j \in I$. Assume for all triples $a, b, c \in I$ with $a < b < c$ with the property that $Y_c \cap Y_b = r$ and $Y_c \cap Y_a = s$, must have $r = s$. Then $\sum_{i \in I} |Y_i| \leq 2m$.*

Proof: Use induction on m with the result being easy for $m = 1, 2$. Let d be the maximum index in I.

Our first case is that $Y_d \cap Y_i = \emptyset$ for all $i \in I \backslash d$. Form a new set family $\mathcal{Y}' = \mathcal{Y} \backslash Y_d$, whose sets are indexed by $I' = I \backslash d$, and whose sets are contained in $[m] \backslash Y_d$. Thus $\sum_{i \in I \backslash d} |Y_i| \leq 2(m - |Y_d|)$ and so $\sum_{i \in I} |Y_i| \leq 2(m - |Y_d|) + |Y_d| \leq 2m$.

Our second case assumes $Y_d \cap Y_j = \{q\}$ for some $j \in I \backslash d$. Our properties yield $Y_d \cap Y_i = \emptyset$ or $Y_d \cap Y_i = \{q\}$ for all $i \in I \backslash d$. Form a new set family $\mathcal{Y}' = \mathcal{Y} \backslash Y_d$, whose sets are indexed by $I' = I \backslash d$. We have $(Y_d \backslash q) \cap Y_i = \emptyset$ for $i \in I \backslash d$. Thus the sets of $\mathcal{Y}'$ are contained in $[m] \backslash (Y_d \backslash q)$. It is easy to verify that $\mathcal{Y}'$ satisfies the hypotheses of the lemma with I replaced by I' and $[m]$ replaced by $[m] \backslash (Y_d \backslash q)$. The hypothesis $|Y_d| \geq 2$ yields $|[m] \backslash (Y_d \backslash q)| < m$. By induction $\sum_{i \in I \backslash d} |Y_i| \leq 2(m - |Y_d| + 1)$ and so $\sum_{i \in I} |Y_i| \leq 2(m - |Y_d| + 1) + |Y_d| \leq 2m$. ■

Theorem 5.5 *We have forb$(m, \{Q_8, Q_9\})$ is $O(m)$.*

Proof: Let $A \in \text{Avoid}(m, \{Q_8, Q_9\})$ and let A_k denote the columns of column sum k. For $j = 1, 2$, let $W(j, \text{even}) = \{k : A_k \text{ is of type } j, \|A_k\| \geq 3, j \text{ is even}\}$ and let $V(j, \text{even})$ be the concatenation of A_k for $k \in W(j, \text{even})$. Define $W(j, \text{odd})$ and $V(j, \text{odd})$ similarly. This more complicated definition ensures that for $a, b \in W(j, \text{even})$ (or $a, b \in W(j, \text{odd})$) with $a < b$ that $a < a + 1 < b$ (column sums differ by at least 2).

We first claim $\|V(1, \text{even})\| \leq 2m$. A number of properties are established before using an interesting induction. Assume that for $i < j$ and $i, j \in W(1, \text{even})$, that $|X_i \backslash X_j| \leq 1$ else there is a copy of Q_9 in $[A_i \,|\, A_j]$ as described in proof of Lemma 5.2.

Assume $|Y_i \cap Y_j| \leq 1$ for all pairs $i, j \in W(1, \text{even})$. Otherwise assume $|Y_i \cap Y_j| \geq 2$ for some pair $i < j$ with $i, j \in W(1, \text{even})$. Let $r, s \in Y_i \cap Y_j$. Now $|X_i| < |X_j|$ and so we can choose a third row $p \in X_j \backslash X_i$. There is a copy of Q_8 in $[A_i \,|\, A_j]$ in rows p, r, s, a contradiction

Now assume $|Y_i \cap Y_j| = 1$ for some pair $i < j$. We claim $X_i \subset X_j$. Otherwise, choose $r \in X_i \backslash X_j$ and $p = Y_i \cap Y_j$. There is $\begin{bmatrix}1\\1\end{bmatrix}$ in rows p, r of some column of A_i and $\begin{bmatrix}0\\0\end{bmatrix}$ in rows p, r of some column of A_j. Give $i < j$, this is Q_9 in $[A_i \,|\, A_j]$, a contradiction.

Finally assume indices $a, b, c \in W(1, \text{even})$ with $a < b < c$ have $Y_a \cap Y_c = \{r\}$ and $Y_b \cap Y_c = \{s\}$. Then $r = s$. If not, recall that $X_a \subset X_c$ and $X_b \subset X_c$ and $a < a + 1 < b < b + 1 < c$. Now $|X_c \backslash X_b| \geq 2$ and $|X_a \backslash X_b| \leq 1$ so there is a $p \in X_c \backslash (X_b \cup X_a)$. Then Q_8 is in rows p, r, s of $[A_a \,|\, A_b \,|\, A_c]$ by taking two columns of A_c with I_2 on rows r, s and 1's on row p and then one column of A_b with 1 on row r and so 0's on rows s, p and one column of A_a with a 1 on row s and so 0's on rows r, p.

Now our claim $\|V(1,\text{even})\| \le 2m$ is the same as asserting $\sum_{i\in W(1,\text{even})} |Y_i| \le 2m$. Consider the set system $\mathcal{Y}$ with sets Y_i for $i \in W(1,\text{even})$. Set $I = W(1,\text{even})$ and appeal to Lemma 5.4 to obtain $\sum_{i\in W(1,\text{even})} |Y_i| \le 2m$, establishing our claim $|V(1,\text{even})| \le 2m$.

Similarly $\|V(1,\text{odd})\| \le 2m$ since the argument never used the parity other than to ensure for $a,b \in W(j,\text{odd})$ that $|a-b| \ge 2$. Also the same holds for $V(2,\text{even})$, $V(2,\text{odd})$ by taking $(0,1)$-complements. Thus $\|V(1,\text{odd})\| \le 2m$, $\|V(2,\text{even})\| \le 2m$ and $\|V(2,\text{odd})\| \le 2m$. Now this has included all columns of A with the exception of A_k for which $\|A_k\| \le 2$ and hence for at most 2m columns. We now conclude that A has at most $10m$ columns. ∎

The following result is needed to complete our knowledge of $\text{forb}(m,\mathcal{F})$ for $\mathcal{F} \subset \{Q_1, Q_2, \ldots, Q_9\}$.

Theorem 5.6 *We have forb*$(m, \{Q_6, Q_7, Q_9\})$ *is* $O(m)$.

Proof: Let $A \in \text{Avoid}(m, \{Q_6, Q_7, Q_9\})$. Proceed as above letting A_k be the columns of sum k and apply Lemma 5.1. If A_k is of type 1 then $\|A_k\| \le 2$ else $Q_6 \prec A_k$. Similarly if A_k is of type 2 then $\|A_k\| \le 2$ else $Q_7 \prec A_k$. Thus $\|A\| \le 2m-2$. ∎

Theorem 5.7 *Let* $\mathcal{F} \subset \{Q_1, Q_2, \ldots, Q_9\}$ *with* $\mathcal{F} \ne \emptyset$. *If* $\mathcal{F} \subseteq \{Q_1, Q_4, Q_6\}$ *or if* $\mathcal{F} \subseteq \{Q_2, Q_5, Q_7\}$ *or if* $\mathcal{F} \subseteq \{Q_6, Q_7, Q_8\}$ *or if* $\mathcal{F} \subseteq \{Q_6, Q_9\}$ *or if* $\mathcal{F} \subseteq \{Q_7, Q_9\}$ *or if* $\mathcal{F} = Q_3$ *then forb*$(m,\mathcal{F})$ *is* $\Theta(m^2)$. *In all other cases, forb*$(m,\mathcal{F})$ *is* $O(m)$. *In those cases forb*$(m,\mathcal{F})$ *is* $\Theta(m)$ *or* $\Theta(1)$ *and Theorem 1.12 will determine the asymptotic growth rate of forb*$(m,\mathcal{F})$ *as either* $\Theta(m)$ *or* $\Theta(1)$ *in those cases where forb*$(m,\mathcal{F})$ *is* $O(m)$.

Proof: Given that $\text{forb}(m, Q_i)$ is $\Theta(m^2)$ for $i \in [9]$, it suffices to show $\text{forb}(m,\mathcal{F})$ is $O(m)$ in the other cases. The results in Table 2 identify all pairs Q_i, Q_j with $\text{forb}(m, \{Q_i, Q_j\})$ being $O(m)$. Consider this as yielding a graph on a vertex set [9]. Any subset $S \subset [9]$ which contains one of these pairs has $\text{forb}(m, \bigcup_{i\in S} Q_i)$ being $O(m)$ by Remark 1.3. For example, any superset of $\{Q_1, Q_4, Q_6\}$ contains a pair Q_i, Q_j with $\text{forb}(m, \{Q_i, Q_j\})$ being $O(m)$. In particular $\text{forb}(m, \{Q_1, Q_2\})$, $\text{forb}(m, \{Q_1, Q_3\})$, $\text{forb}(m, \{Q_1, Q_5\})$, $\text{forb}(m, \{Q_1, Q_7\})$, $\text{forb}(m, \{Q_1, Q_8\})$, and $\text{forb}(m, \{Q_1, Q_9\})$ are all $O(m)$. For example, any superset of $\{Q_6, Q_9\}$ either contains a pair Q_i, Q_j with $\text{forb}(m, \{Q_i, Q_j\})$ being $O(m)$ or is a triple Q_i, Q_j, Q_k with $\text{forb}(m, \{Q_i, Q_j, Q_k\})$ being $O(m)$. Thus $\text{forb}(m, \{Q_1, Q_9\})$, $\text{forb}(m, \{Q_2, Q_6\})$, $\text{forb}(m, \{Q_3, Q_6\})$, $\text{forb}(m, \{Q_4, Q_9\})$, $\text{forb}(m, \{Q_5, Q_6\})$, and $\text{forb}(m, \{Q_8, Q_9\})$ are all $O(m)$. There are two exceptional pairs $\{Q_6, Q_7\}$ and $\{Q_7, Q_9\}$ but the triple $\{Q_6, Q_7, Q_9\}$ for which $\text{forb}(m, \{Q_6, Q_7, Q_9\})$ is $O(m)$ (by Theorem 5.6) handles these two cases. ∎

We may summarize our investigations by saying the Conjecture 1.1, when applied to a forbidden family, predicts the correct asymptotic growth for a number of elementary cases. Perhaps the cases where Conjecture 1.1 does not correctly predict the asymptotic growth, such as Theorem 1.9, are rare. It is premature to conjecture an analog of Conjecture 1.1 for forbidden families.

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