Orthogonal thickness of graphs

Maryam Tahmasbi* S. Mehdi Hashemi†

Abstract

In this paper, we introduce the concept of orthogonal thickness of graphs. Orthogonal thickness of a graph G is the minimum number of planar graphs with maximum degree less than or equal to 4 whose union is G. We compute the orthogonal thickness of complete graphs and planar graphs. Using this we compute lower and upper bounds for orthogonal thickness and show that they are tight.

Next, we introduce the concept of layered orthogonal drawing and develop an algorithm for layered orthogonal drawing of K_n in $\lceil n/4 \rceil$ layers. Then, we extend the results to develop an algorithm for three-dimensional orthogonal (line-) drawing of K_n in a $2n \times 2n \times \lceil n/4 \rceil$ box. This drawing has at most 2 bends per edge and a total number of $n(n-2)$ bends. In this drawing all vertices are drawn as parallel segments.

1 Introduction

Orthogonal drawing is a graph drawing technique with increasing applications in technology. A two-dimensional *orthogonal drawing* is a drawing of a graph in which every vertex is represented by a point and every edge is represented by a chain of horizontal and vertical segments. In this paper we consider *planar* orthogonal drawing, i.e. orthogonal drawings where no two edges cross except at a common end vertex. A planar graph with maximum degree less than or equal to k is called *k-planar*. Orthogonal drawings in three dimensions are defined similarly, in these drawings each vertex is represented by a point and each edge is represented by a chain of segments that are parallel to one of the x-, y-, or $z-$axis. In this type of orthogonal drawing the degree of each vertex must be less than or equal to 6. However, there are other ways to draw a general graph in an orthogonal way, one of them is *orthogonal (line-) drawing* that is a three-dimensional orthogonal drawing where each vertex is represented by an axis parallel line segment. Orthogonal drawings have direct applications in VLSI design and layout. In VLSI layout, a planar orthogonal drawing of a given graph is embedded on a plane. Orthogonal drawing has three disadvantages when applying to VLSI layouts:

- The degree of each vertex is restricted to 4 (in 2D drawings) or 6 (in 3D drawings). In orthogonal drawing of graphs with higher vertex degrees, the vertices are represented by boxes, line segments or cycles [8, 3]. This increases the number of bends and the area (volume) of the drawing.
- Non planar graphs do not have planar orthogonal drawings in 2-D plane. But in VLSI layouts are planar.
- VLSI technology limits circuits to have few layers, it means that the size of one of the dimensions in the drawing must be restricted; but this condition does not hold in most three-dimensional drawing approaches [1].

In this paper, we introduce the concept of orthogonal thickness. This concept has been studied by Bose and Prabhu[5] under the name thickness of graph with degree constrained vertices. They computed it for the graph K_n where $n \neq 4k+1$. We compute a tight lower bound for it. We also introduce the concept of layered orthogonal drawing and find such drawings for complete graphs. Then we transform it to a three-dimensional orthogonal (line-) drawing for the graph. In this drawing all vertices are represented by parallel segments. This type of orthogonal drawing is a way to overcome the disadvantages mentioned above: we can draw graphs with high vertex degree and we can also restrict the size of one of the dimensions of the final drawing. The problem is related to thickness, orthogonal drawing and geometric thickness of graphs [7, 9].

This paper is organized as follows: Basic definitions and results are in Section 2. The definition of orthogonal thickness and its upper and lower bounds are presented in Section 3. In Section 4 orthogonal layered drawing is introduced and an algorithm for layered orthogonal drawing of K_n is developed. Section 4.2 is devoted to constructing a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing. Results and conclusions are summarized in Section 5.

2 Preliminaries

We need some definitions and results from graph theory. We assume the definitions presented in [4]. A

*Department of Computer Science, Mathematical science faculty,Shahid Beheshti University, Tehran, Iran, M-Tahmasbi@sbu.ac.ir

†Department of Mathematics and Computer Science, Amirkabir University of Technology, hashemi@aut.ac.ir

planar graph with maximum degree Δ is called a Δ-*planar* graph. Let $G = (V(G), E(G))$ be a graph and $U \subseteq E(G)$. The subgraph of G whose vertex set is the set of vertices incident to the edges in U and whose edge set is U, is called the subgraph of G *induced* by U and is denoted by $G[U]$.

The *thickness* of a graph G, denoted by $\theta(G)$, is the minimum number of planar subgraphs of G whose union is G. There are two good surveys on the topic by Liebers [10] and Mutzel et.al [11]. An *edge coloring* of a graph G is an assignment of colors to edges of G such that no two adjacent edges have the same color. A graph is k-*edge-colorable* if it has an edge coloring with k colors.

In an orthogonal drawing, the place where an edge changes direction is a *bend*. Two important criteria in orthogonal drawings are small number of bends and small area (or volume). Research on the topic is extensive [8, 12].

3 Orthogonal thickness

Let G be a graph with the edge set E. An *orthogonal layering* of G is a decomposition of E into subsets $E_1, E_2, ..., E_k$ such that for each $1 \leq i \leq k$, $G[E_i]$ is 4-planar. For $1 \leq i \leq k$, $G[E_i]$ is called an *orthogonal layer* of G. The minimum number of orthogonal layers of G is called the *orthogonal thickness* of G and is represented by $\hat{\theta}(G)$.

3.1 Upper and lower bounds on orthogonal thickness of graphs

By definition of orthogonal thickness, $\hat{\theta}(G) = 1$ if and only if G is 4-planar. In this section we present lower and upper bounds on orthogonal thickness of graphs. Since each vertex has degree at most 4 in each layer, we can obtain a lower bound on orthogonal thickness of graphs.

Observation 1 *Let G be a graph of maximum vertex degree Δ. Then $\hat{\theta}(G) \geq \lceil \frac{\Delta}{4} \rceil$.*

We next prove that this bound is tight.

Theorem 1 *Let G be a planar graph with maximum degree Δ. Then $\hat{\theta}(G) = \lceil \frac{\Delta}{4} \rceil$.*

Proof. For $\Delta \geq 8$, a planar graph is Δ-edge-colorable [6]. Now let G_i be the subgraph induced by all edges of color $4i + 1, ..., 4i + 4$. Then this splits G into $\lceil \Delta/4 \rceil$ planar subgraphs of maximum degree 4.

For $4 < \Delta < 8$, the graph has a $\Delta + 1$ edge coloring. With the same decomposition as above, the graph splits to two planar subgraphs of maximum degree less than or equal to 4. □

On the other hand, Bose and Prabhu [5] studied the problem for complete graphs. They decomposed the graph K_n to $k = \lceil \frac{n}{4} \rceil$ planar subgraphs $G_1, ..., G_k$ as follows:

Let $n = 4p, p \in \mathbb{N}$ and $\{1, 2, ..., n\}$ be the vertex set of $G = K_n$. Suppose that the vertices of G are placed on a circle in the clockwise order. For $1 \leq k \leq n/2$, define the edge set $E^{(k)} = \{ij : i+j \equiv_n 2k+1 \text{ or } i+j \equiv_n 2k+2\}$ and $G^{(k)} = G[E^{(k)}]$. One can construct $G^{(i)}$ recursively by 1 unit counterclockwise shifting of vertices of $G^{(i-1)}$ around the circle. For $1 \leq k \leq n/2$, $G^{(k)}$ is a spanning path in G with end vertices $k + 1$ and $(n/2) + k + 1$. Now, for $1 \leq k \leq n/4$, let $G_k = G[E^{(2k)} \bigcup E^{(2k-1)}]$. G_k is planar. Figure 1 shows a planar drawing of G_1.

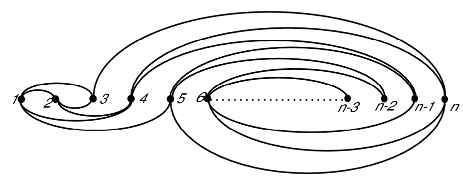

Figure 1: The graph G_1 in decomposing K_n to 4-planar graphs.

Theorem 2 *For $n \neq 4p + 1$, $\hat{\theta}(K_n) = \lceil n/4 \rceil$.*

Since every graph with n vertices is a subgraph of K_n, using theorems 1 and 2 we can introduce tight upper and lower bounds on orthogonal thickness.

Theorem 3 *Let G be a graph with n vertices and maximum vertex degree Δ. Then $\lceil \frac{\Delta}{4} \rceil \leq \hat{\theta}(G) \leq \lceil \frac{n}{4} \rceil$. The lower and upper bounds are obtained for planar graphs and complete graphs respectively, with $n \neq 4p+1$ where p is an integer.*

Taking a closer look to theorem 3 we state the following conjecture:

Conjecture 1 *For every graph G with maximum vertex degree Δ, $\lceil \Delta/4 \rceil \leq \hat{\theta}(G) \leq \lceil \Delta/4 \rceil + 1$*

4 Layered orthogonal drawing

Let G be a graph and $G_1, G_2, \ldots, G_k$ be the orthogonal layers of G. A drawing of G composed of 2D orthogonal drawings of the graphs $G_1, G_2, \ldots, G_k$ in k distinct planes where each vertex $v \in V(G)$ has the same x- and y-coordinate in the drawing of $G_1, G_2, \ldots, G_k$, is called a *layered orthogonal drawing* of G with k layers. Figure 2-(a) shows a layered orthogonal drawing.

In this section, we show that there exists an orthogonal layered drawing for every graph and then compute a layered orthogonal drawing of K_n with $\lceil n/4 \rceil$ layers.

In order to show that there is a layered orthogonal drawing for each orthogonal layer, we use the following lemma.

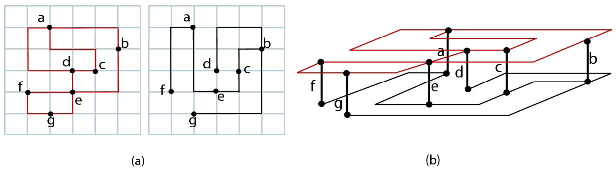

Figure 2: (a)Layered orthogonal drawing, (b) corresponding orthogonal (line-) drawing

Lemma 4 *[13] Each planar graph with n vertices admits a planar drawing that maps each vertex to a prespecified distinct location in the plane and each edge to a polygonal curve with $O(n)$ bends. More over, such a drawing can be constructed in $O(n^2)$ time.*

Theorem 5 *Let G be a graph with orthogonal layers $G_1, G_2, \ldots, G_k$. Then G has a layered orthogonal drawing with these layers.*

Proof. Each subgraph $G_1, G_2, \ldots, G_k$ is 4-planar, so it has an orthogonal drawing in the plane. We construct a layered orthogonal drawing Γ in an inductive manner. G_1 has an orthogonal drawing Γ_1. Let $P_1, P_2, \ldots, P_n$ be the location of vertices of G_1. By lemma 4, G_2 has a polygonal drawing where its vertices are at fixed points $P_1, P_2, \ldots, P_n$. We can transform the polygonal drawing to an orthogonal drawing, and after resizing the grid and Γ_1, we would get a layered orthogonal drawing of G_1 and G_2. We can continue and construct a drawing for G in $k-1$ steps. □

The number of bends in this drawing might be exponential. In the following section we develop an algorithm for drawings with small number of bends.

4.1 Layered orthogonal drawing of K_n

Using the same argument as in section 3.1, we can assume that $n = 4p$. Let $V = \{1, 2, \ldots, n\}$ be the vertex set of K_n. For $i = 1, 2, \ldots, n$, let the vertex i have coordinate $(2i-1, 2i-1)$ in the xy-plane. The straight line segment passing through all of these points decomposes the $2n \times 2n$ square to two triangles: one triangle on the top-left and the other on the bottom-right. For each $1 \leq k \leq \lceil n/4 \rceil$, we draw the edges of $E^{(2k-1)}$ in the top-left triangle and the edges of $E^{(2k)}$ in the other triangle.

We start drawing of $E^{(2k)}$ from the vertex $2k+1$ and the edge connecting it to $2k$. It can be drawn in the right down triangle with only one bend. The next edge that connects $2k$ and $2k+2$, can be drawn with two bends. All other edges of $E^{(2k)}$, except the last edge, can be drawn with two bends and the last edge can be drawn with one bend. We draw the edges in $E^{(2k-1)}$ in the bottom-right triangle. Figure 3 shows the orthogonal drawing of G_1. So we have the result:

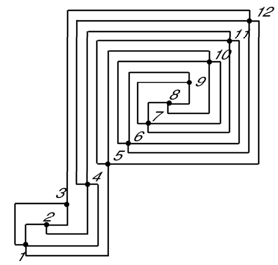

Figure 3: Orthogonal drawing of G_1 for K_{12}.

Theorem 6 *The graph K_n has a layered orthogonal drawing with $\lceil n/4 \rceil$ layers such that each layer is drawn in a $2n \times 2n$ square with $4n-8$ bends.*

4.2 Construction of a three-dimensional orthogonal (line-) drawing

In this section we construct a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing. In this drawing each vertex is represented by a z-axis parallel segment.

Let G be a graph, $G_1, G_2, \ldots, G_t$ be orthogonal layers of G and Γ be an orthogonal layered drawing of G with these layers. For $1 \leq k \leq t$, suppose that Γ_k is the restriction of Γ to the edges of G_k.

Construct a three-dimensional orthogonal (line-) drawing $\hat{\Gamma}$ of G in this way:

For each point (x, y) in Γ_k, let (x, y, k) be a point in $\hat{\Gamma}$ and for each vertex v of G with coordinate (a, b) in Γ_1, represent v by a z-axis parallel segment with end points $(a, b, 1)$ and (a, b, t). Figure 2 shows a layered orthogonal drawing and the related three-dimensional orthogonal drawing. Now we can translate Theorem 6 for three-dimensional orthogonal (line-) drawings:

Observation 2 *The graph K_n has a three-dimensional orthogonal (line-) drawing in which each vertex is drawn as a z-axis parallel segment. This drawing is enclosed in a $2n \times 2n \times (\lceil n/4 \rceil - 1)$ box and has at most $n(n-2)$ bends.*

There are other bounds on the volume of orthogonal drawing of K_n. Biedl et al.[2] proved that there is an orthogonal drawing of K_n with volume $O(n^{5/2})$ and at most three bends in each edge. In the same paper, Biedl et al. introduced a three-dimensional orthogonal (line-) drawing of K_n with at most two bends in each edge and with volume $O(n^3)$. In fact their drawing is bounded in the $\frac{n}{2} \times \frac{n}{2} \times \frac{n}{2}$ box.

As an application of layered orthogonal drawing in VLSI layout, suppose that the size of the final orthogonal drawing of a three-dimensional orthogonal drawing is bounded in two dimensions. The question is if it is possible to find a three-dimensional orthogonal (line-

) drawing with this restriction, and if so, find such a drawing with the smallest size of the third dimension.

Our result in this paper implies that if the drawing of K_n is restricted to have the size of two dimensions equal to $2n$, the size of the other dimension must be at least $\lceil \frac{n}{4} \rceil$ and at most two bends are needed in each edge.

5 Conclusion

In this paper we introduced concepts of orthogonal thickness and layered orthogonal drawing of graphs. We computed tight lower and upper bound for orthogonal thickness of a graph by computing it for planar graphs and complete graphs. Then, constructed a layered orthogonal drawing of complete graphs. We also constructed a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing. In this drawing, vertices are the only z-axis parallel parts of the drawing. We believe that the decomposition of edges and their special layout make this type of orthogonal (line-) drawing more applicable than the existing three-dimensional orthogonal drawing styles, since in constructing the physical model of the drawing, it is enough to construct each layer separately. Then locate the place where each vertex passes one of the layers, and insert a z-axis parallel segment (or connector) that passes through all layers.

On the other hand, this method can be applied when we want to keep the size of one or more of the dimensions small. Existing algorithms for three-dimensional orthogonal drawings try to keep the volume of the drawing small, so the drawing that they produce is much like a cube [2]. In our method the size of one of the dimensions is as small as possible. In VLSI technology, there is a limitation on the number of layers.

We computed a layered orthogonal drawing for complete graphs. The bounds in this type of drawing are upper bounds on the number of bends and volume in three-dimensional orthogonal drawings. But the problem of computing the orthogonal thickness for other non-planar graphs is still interesting. We are approaching the proof of conjecture 1 which is very interesting as it is very similar to the bounds for edge coloring. And there are some open problems:

1- Given two 4-planar graphs on the same vertex set, how can one construct a layered orthogonal drawing with these two layers?

2- Is there any class of (non trivial) forbidden subgraphs such that if the graph contains one of them as a minor, then the orthogonal thickness of the graph would be more than two?

6 Acknowledgement

The authors would like to thank Prof. Dr. Saeed Akbari for fruitful discussions on bounds for orthogonal thickness.

References

[1] T. Biedl, J.R. Johansen, T. Shermer, D.R. Wood, *Orthogonal Drawings with Few Layers*, proceeding of the Graph Drawing contest(GD 2001), LNCS 2265, pp. 297-311, 2002.

[2] T. Biedl, T. Shermer, S. Whitesides, S. Wismath, *Bounds for Orthogonal 3-D Graph Drawing*, J. Graph algorithms and applications, vol. 3, No. 4, pp. 63-79, 1999.

[3] T. Biedl, T. Thiele, D.R. Wood,*Three-Dimensional Orthogonal Graph Drawing with Optimal Volume*, Algorithmica vol. 44, pp. 233-255, 2006.

[4] J.A. Bondy, U.S.R Murthy, *Graph theory with applications*, Elsevier North-Holland, 1976.

[5] N.K. Bose, K.A. Prabhu,*Thickness of graphs with degree constrained vertices*, IEEE Trans. on circuit and systems, vol. cas. 24, No. 4, April 1977.

[6] M. Chrobak, T. Nishizeki *Improved edge-coloring algorithm for planar graphs* J. Algorithms 11, 102–116, 1990.

[7] M. B. Dillencourt, D. Eppstein, D.S. Hirschberg, *Geometric thickness of complete graphs*, Journal of Graph Algorithms and Applications, vol. 4, pp. 5-17, 2000.

[8] G. Di Battista, P. Eades, R. Tamassia, I.G. Tollis, *graph drawing, algorithms for visualization of graphs*, Prentice Hall, 1999.

[9] C. Erten, S.G. Kobourov, *Simultaneous Embedding of Planar Graphs with Few Bends*, J.Graph algorithms and applications,(9) no.3, pp.347364, 2005.

[10] Annegret Liebers, *Planarizing Graphs- A Survey and Annotated Bibliography*, J. Graph, algorithms and applications, vol. 5, no. 1, pp. 1-74, 2001.

[11] P. Mutzel, T. Odenthal, M. Scharbrodt, *The thickness of graphs: a survey*, Graphs and Combinatorics, 14 pp. 59-73,1998.

[12] T. Nishizeki, M.D. Rahman, *Planar graph drawing*, Lecture notes series on computing, vol.12, world scientific, 2004.

[13] J. Pach, R. Wenger, *Embedding planar graphs at fixed vertex locations*, Graphs and Combinatorics, 17 no. 4 , pp. 717-728, 2001.