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k-Automatic Reals

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Introduction

1.1 Preface

In mathematics and theoretical computer sciences sequences, both finite and infinite, are ubiquitous. Hence there are numerous sequences of mathematical interest. For example, the website

`http://www.research.att.com/∼njas/sequences/index.html`

gives access to more than $100,000$ such sequences. A lot of mathematical work is devoted to the studies of sequence-related topics.

There are all kind of sequences. Some, such as periodic sequences, are highly ordered and very easy to describe, while others, such as random sequences, are unordered and have no simple descriptions.

In this thesis we are especially interested in the connections of some sequences and number theory. We will look at the properties of numbers using the fact that their base-k expansions or their continued fraction expansions are sequences with certain characteristics.

It is well known that for any integer $k \geq 2$ the base-k expansion of a rational number should be ultimately periodic, but a long-standing problem, apparently asked for the first time by Borel [18], is the following: how regular or random (depending on the viewpoint) is the base-k expansion of an algebraic irrational number?

A general conjecture claims that it should be totally random, requiring an algebraic irrational number to be normal in any base $k \geq 2$ (i.e., each block of length ℓ occurs with frequency $\frac{1}{k^\ell}$). Though this conjecture has not yet been proven and is considered as out of reach, some results have already been known in this direction for more than 70 years [5, 17, 26, 28, 47, 48]. These results, proved with different methods, express the following idea: if the base-k expansion of an irrational number could be obtained by a too regular process, then this number is transcendental. The term 'too regular' has various interpretations and the goal of this thesis is to study some of those processes.

The sequences we want to study lie somewhat between periodicity (order) and chaos (disorder). As said above, the numbers with base-k expansions that are periodic sequences are the rational numbers. Concerning continued fractions, the rational numbers are exactly those with a finite continued fraction expansion. The infinite continued fractions expansions that are (ultimately) periodic represent numbers that are quadratic irrational. So we want to study sequences with a low complexity that are not (ultimately) periodic.

The sequences with the lowest complexity that are still not periodic are the *Sturmian sequences*. The first one to investigate what are now generally called Sturmian words was Johann Bernoulli III (1744–1807). In 1772, he studied one particular word and found a connection with continued fractions. However, he did not provide any proofs. At the end of the 19th century Christoffel [19] and Smith [67] independently found similar results. Markoff [50] proved Bernoulli's assertions. The term "Sturmian" was introduced by Morse and Hedlund [37] in their work on symbolic dynamics. (The term is rather unfortunate in that Sturm apparently never worked on these sequences.) Also see [36]. Sturmian words have also received some attention in computer graphics and image processing literature. They appear in many areas of mathematics. For an excellent survey of Sturmian words see [16].

Considering sequences one step higher in the complexity scale we get to *k-automatic sequences*. Finite automata — a simple model for computation similar to Turing machines — produce these sequences. The first one to systematically study k-automatic sequences was Cobham [22]. He called them "uniform tag sequences". The first occurrence of the term "automatic sequence" (in French) appears to be in a paper of Deshouillers [29]. The review journals *Mathematical Reviews* and *Zentralblatt für Mathematik* have assigned the classifications 11B85 to "automata sequences".

Cobham proved that the k-automatic sequences are exactly those which are fixed points of k-uniform morphisms. Hence they are a special case of *morphic sequences*, i.e., infinite sequences that are fixed points of a morphism.

In this thesis we are going to examine the properties of numbers whose base-k expansion or continued fraction expansion is given by one of those sequences mentioned above.

1.2 The Structure of This Thesis

This thesis is structured as follows. The first two sections of Chapter 2 provide us with some basic definitions and results from the fields of stringology and number theory that we need later on. In the third section of that chapter we give the definition of numerations system and prove that every real number has a base-k representation. We also present a short proof of the well-known fact that the sequence representing the fractional part of a rational number are exactly those that are ultimately periodic. The fourth section provides us with a deep result commonly know as Schmidt's Subspace Theorem and its p-adic generalization.

In chapter 3 we give a short introduction to a simple model of computation that everyone interested in computer sciences should be familiar with, the finite automata, and how to use these to build elementary functions called *finite-state functions*.

The next chapter — number 4 — establishes the concept of the *automatic sequence* and proves some basic properties of those sequences.

After the first introductory chapters have provided us with the necessary concepts and tools we can finally start to investigate the numbers we are interested in: chapter 5 deals with numbers whose base-k expansion is a Sturmian sequence. We show that those numbers are all transcendental as one might expect. This result is due to Ferenczi and Mauduit [34].

After having looked at the numbers with the 'simplest' base-k expansions that are not ultimately periodic in chapter 5, we then turn to the next more complex numbers, the k-automatic reals. A good introduction to this subject can be found in the book of Allouche and Shallit [9]. Chapter 6 first gives a proper definition and some basic properties like the fact that k-automatic reals form a vector space over $\mathbb{Q}$ (a result that is due to Lehr [44]). Thereupon we investigate the transcendence of these numbers. We start by proving the transcendence of three examples: The first, the number $F = \sum_{n\geq 0} B^{-2^n}$, is sometimes called "Fredholm number", although Fredholm apparently never studied it. Our proof of the transcendence of F is due to Knight [40]. For other proofs, see for example Kempner [39], Mahler [48] or Loxton and van der Poorten [46]. The second example is the so-called Komornik-Loreti constant. It was introduced by Komornik and Loreti in [42] using some results by Erdős et al. [30, 31] and its transcendence proved by Allouche and Cosnard in [7] resting upon a result by Mahler [48, p. 363]. The third example is the Thue-Morse number. This is the number whose binary expansion is given by the famous Thue-Morse sequence with values 0 and 1. This sequence was introduced in Thue [69], although it was hinted at sixty years earlier by Prouhet [55]. The sequence itselfs appears to be somewhat ubiquitous — for a description of many of its apparently unrelated occurrences, see [10]. Our proof of the transcendence of the Thue-Morse number is essentially that of Dekking [28].

After looking at those three explicit examples we start with a general approach. Following the work of Allouche and Zamboni [11] we prove that a positive real number whose binary expansion is a fixed point of a morphism on the alphabet $\{0, 1\}$ that is either of constant length ≥ 2 or primitive is either rational or transcendental. This nice result shows that some of the irrational morphic numbers, namely those produced by this kind of morphisms, are indeed transcendental. It is a widely believed conjecture that all irrational morphic numbers are transcendental. This conjecture has not been proved yet. But still, we do confirm this conjecture for a wide class of morphisms. Namely we prove that irrational k-automatic numbers are transcendental and that binary algebraic irrational numbers cannot be generated by a morphism. Adamczewski and Bugeaud [2] proved this with a new transcendence criterion derived from the Schmidt Subspace Theorem [64] (see also [63]) as formulated by Evertse [32].

In chapter 7 we look at our numbers out of the continued fraction expansion view. Again we are interested in the differences between algebraic and transcendent irrationals (those being the numbers with infinite continued fraction expansions). Once more using a theorem of Schmidt [65], we prove that if the sequence of partial quotients of the continued fraction expansion of a positive real number takes only two values, and begins with arbitrarily long blocks which are "almost squares", then this number is either quadratic or transcendental. This result applies in particular to real numbers whose partial quotients form a Sturmian (or quasi-Sturmian) sequence, or are given by the sequence $(1 + (\lfloor n\alpha \rfloor \bmod 2))_{n\geq 0}$, or are a "repetitive" fixed point of a binary morphism satisfying some technical conditions. So again it is the "not too random" sequences that lead us to transcendent numbers. This first part of chapter 7 is due to Allouche, Davison, Queffélec and Zamboni [8]. We further establish two more transcendence criteria for continued fractions that also rely upon repetitions in sequences of the continued fraction expansions. The proofs we give again rest on the Schmidt Subspace Theorem [63, 64]. It is essentially that of Adamczewski and Bugeaud [3].

1.3 Acknowledgements

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Chapter 2

Basics

In this chapter we introduce the basic objects and concepts we will use later on, as well as some needed fundamental results.

2.1 Stringology

2.1.1 Words

The fundamental mathematical object we will study here is here is the *word*. We will start with a definition.

Definition 2.1 *Let Σ be a nonempty set of symbols, or an alphabet. A* word *or* string *is a finite or infinite list of symbols chosen from Σ.*

One alphabet of special importance is the following, therefore it has its own symbol: if k is an integer ≥ 2 , then $\Sigma_k := \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \ldots, k-1\}$.

To be more precise, a word is a map from $\{0, 1, \ldots, n-1\}$ or $\{1, 2, \ldots, n\}$ to Σ. If $n = 0$, the result is the *empty word*, which is denoted by ε. The set of all words made up by letters chosen from Σ is denoted by Σ^*. For example, if $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, then $\Sigma^* = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}, \mathtt{aaa} \ldots\}$.

Let Σ^+ denote the set of all nonempty words over Σ.
If w is a finite word, then its *length* is defined as the number of symbols it contains and is denoted by $|w|$. For example, if $w = \mathtt{two}$, then $|w| = 3$. And always $|\varepsilon| = 0$.

It is also possible to count the occurrences of a particular letter in a word. If $a \in \Sigma$ and $w \in \Sigma^*$, then $|w|_a$ denotes the number of occurrences of a in w. Thus, for example, if $w = \mathtt{babaa}$, then $|w|_{\mathtt{a}} = 3$ and $|w|_{\mathtt{b}} = 2$.

The basic operation on words is *concatenation*. To concatenate two words w and x, simply juxtapose their symbols, denoted by wx. Concatenation of words is, in general, not commutative. However, it is associative. Notationally, concatenation is treated like multiplication, so that w^n denotes the word $www \cdots w$ (n times). The set Σ^* together with concatenation becomes a *monoid*, with the empty word ε as the identity element.

A word y is called a *subword* of a word w if there are words x, z such that $w = xyz$. x is called a *prefix* of w if there exists y such that $w = xy$, and x is a *proper prefix* of w if $y \neq \varepsilon$. A *suffix* is defined in an analogue way.

A *language* over Σ is a (finite or infinite) set of words, hence a subset of Σ^*.

And now to the introduction of infinite words (also called infinite sequences). Let $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}^+$ denote the positive integers and $\mathbb{Z}^-$ the negative integers, and $\mathbb{N}$ the non-negative integers. Then a *one-sided* right-infinite word $\mathbf{a} = a_0a_1a_2\ldots$ is a map from $\mathbb{N}$ to Σ. An infinite word can be formed by concatenating infinitely many finite words, e.g.

$$\prod_{i\geq 1} w_i$$

denotes a word $w_1w_2w_3\ldots$, which is infinite if and only if $w_i \neq \varepsilon$ infinitely often.

Sometimes it is useful to start the indices with 1 instead of 0 as the following example shows:

Example 2.2 *Define*

$$\boldsymbol{p} = (p_n)_{n\geq 1} = \mathsf{0110101000101}\cdots,$$

the characteristic sequence of the prime numbers.

The set of all one-sided right-infinite words over Σ is denoted by Σ^ω. For both finite and infinite words, define $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$.

A *left-infinite word* $\cdots a_{-3}a_{-2}a_{-1}$ is a map from $\mathbb{Z}^-$ to Σ. The set of all left-infinite words is denoted by ${}^\omega\Sigma$.

A *two-sided infinite word* is a map from $\mathbb{Z}$ to Σ. Such words are denoted

$$\cdots c_{-3}c_{-2}c_{-1}c_0.c_1c_2c_3\cdots$$

where the decimal point is a notational convention and not a part of the word itself. The set of all two-sided infinite words over Σ is denoted by $\Sigma^{\mathbb{Z}}$.

The notions of subword, prefix, and suffix for finite words have evident analogues for infinite words. Let $\mathbf{w} = a_0a_1a_2\cdots$ be an infinite word. For

$i \geq 0$ we define $\mathbf{w}[i] = a_i$. For $i \geq 0$ and $j \geq i-1$, we define $\mathbf{w}[i..j] = a_i a_{i+1} \cdots a_j$. We also define $\mathbf{w}[i..\infty] = a_i a_{i+1} \cdots$. If

$$\lim_{n\to\infty} \frac{|\mathbf{w}[0..n-1]|_b}{n}$$

exists and equals r, then the *frequency* of the symbol b in $\mathbf{w}$ is defined to be r.

If x is a nonempty finite word, then x^ω is the right-infinite word $xxx\cdots$. Such a word is called *purely periodic*. An infinite word $\mathbf{w}$ of the form xy^ω for $y \neq \varepsilon$ is called *ultimately periodic*. If $\mathbf{w}$ is ultimately periodic, then it can be written in the form xy^ω for finite words x, y with $y \neq \varepsilon$. Then x is called a *preperiod* of $\mathbf{w}$, and y is called a *period*. If $|x|$, $|y|$ are chosen as small as possible, then x is called the *least preperiod*, and y is called the *least period*.

The infinite word $\mathbf{w}$ is called *recurrent* when every subword of $\mathbf{w}$ occurs infinitely often in $\mathbf{w}$. The infinite word $\mathbf{w}$ is *minimal* when every subword of $\mathbf{w}$ occurs infinitely often in $\mathbf{w}$ and with bounded gaps.

In conjunction with an infinite word $\mathbf{w}$ the *language* $L_n(\mathbf{w})$ is the set of all subwords of the length n of $\mathbf{w}$. The *language of the infinite word* is the reunion of all the $L_n(\mathbf{w})$.

The *complexity function* of the infinite word $\mathbf{w}$ is the function associating to n the cardinality of $L_n(\mathbf{w})$, denoted by $p(n)$.

An *overlap* is a word of the form $cxcxc$, where x is a word, possibly empty, and c is a single letter. E.g. the English word `alfalfa` is an overlap with $c = \texttt{a}$ and $x = \texttt{lf}$. If a word w contains no overlap, it is called *overlap-free*.

2.1.2 Morphisms

Now we will introduce a fundamental tool when working with words, the *homomorphism*, or just *morphism*.

Definition 2.3 *Let Σ and Δ be alphabets. A* morphism *is a map φ from Σ^* to Δ^* that obeys the identity $\varphi(xy) = \varphi(x)\varphi(y)$.*

It's clear that if φ is a morphism, then $\varphi(\varepsilon) = \varepsilon$. Furthermore, if φ is defined for all elements of Σ, then it can be uniquely extended to a map from Σ^* to Δ^*. Thus for defining a morphism, it will suffice to specify its actions on Σ.

Example 2.4 *Let $\Sigma = \{n, t, u\}$ and $\Delta = \{a, c, e, k, r\}$ and define*

$$\begin{aligned} \varphi(n) &= cra \\ \varphi(t) &= cker \\ \varphi(u) &= \varepsilon \end{aligned}$$

Then $\varphi(\texttt{nut}) = \texttt{cracker}$.

If $\Sigma = \Delta$, it's possible to iterate the application of φ. Define $\varphi^0(a) = a$ and $\varphi^i(a) = \varphi(\varphi^{i-1}(a))$ for all $a \in \Sigma$, $i \geq 1$.

Example 2.5 *Let* $\Sigma = \Delta = \{\texttt{0,1}\}$. *Define the* Thue-Morse-morphism $\mu(\texttt{0}) = \texttt{01}$ *and* $\mu(\texttt{1}) = \texttt{10}$. *Then* $\mu^2(\texttt{0}) = \texttt{0110}$ *and* $\mu^3(\texttt{0}) = \texttt{01101001}$.

Morphisms can be classified into groups, as follows: If there is a constant k such that $|\varphi(a)| = k$ for all $a \in \Sigma$, then φ is called *k-uniform* (or just *uniform*, if k is clear from the context). A 1-uniform morphism is called a *coding*. A morphism is said to be *expanding* if $|\varphi(a)| \geq 2$ for all $a \in \Sigma$.

A morphism $\varphi : \Sigma^* \to \Sigma^*$ is called *primitive* if there exists an integer $n \geq 1$ such that for all $a, b \in \Sigma$, a occurs in $\varphi^n(b)$.

A finite or infinite word w is called a *fixed point* of the morphism φ if $\varphi(w) = w$. If for some letter $a \in \Sigma$ the word $\varphi(a)$ begins with a and has at least length 2, then the sequence of words $a, \varphi(a), \varphi^2(a), \ldots$ converges, in the limit, to the infinite word $\varphi^\infty(a) \in \Sigma^\omega$ which is also a fixed point of φ, i.e. $\varphi(\varphi^\infty(a)) = \varphi^\infty(a)$. Moreover, it is easy to see that $\varphi^\infty(a)$ us the unique fixed point of φ which starts with a.

2.2 Number Theory & Algebra

This section is a conglomeration of some basic results from algebra and number theory we will need later on.

Let α be a real number. If $\alpha = \frac{a}{b}$ for some integers a, b, then α is rational; otherwise it is irrational.

A complex number is said to be *algebraic* if it is the root of an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ and $a_n \neq 0$. The set of algebraic numbers includes all rational numbers, and other numbers as for example the complex unit i or numbers like $\sqrt{2}$, $\sqrt[3]{2}$, etc. If a complex number is not algebraic, then it is called *transcendental*.

The set of all algebraic numbers is countable. Thus almost all real numbers are transcendental, since the real algebraic numbers, being countable, form a set of measure 0.

If $\theta \in \mathbb{R}$, then θ is said to be *approximable by rationals to order* n if there exists a constant $c(\theta)$ such that the inequality

$$\left| \frac{p}{q} - \theta \right| < \frac{c(\theta)}{q^n}$$

has infinitely many distinct rational solutions $\frac{p}{q}$.

Theorem 2.6 *A real algebraic number of degree n is not approximable to any order greater than n.*

Proof. Suppose that θ is a real number satisfying

$$f(\theta) = a_n\theta^n + \cdots + a_1\theta + a_0 = 0$$

with $a_0, a_1, \ldots, a_n \in \mathbb{Z}, a_n \neq 0$. Then there exists a number $M(\theta)$ such that

$$|f'(x)| < M(\theta) \quad \text{for}\, \theta - 1 < x < \theta + 1.$$

Suppose that $\frac{p}{q}$ is an approximation to θ. Without loss of generality we may assume that $\theta - 1 < \frac{p}{q} < \theta + 1$, and that $\frac{p}{q}$ is closer to θ than any other root of f, so $f\left(\frac{p}{q}\right) \neq 0$. Then

$$\left|f\left(\frac{p}{q}\right)\right| = \frac{|a_np^n + \cdots + a_1p + a_0|}{q^n} \geq \frac{1}{q^n}$$

and, by the mean value theorem, we have

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\theta) = \left(\frac{p}{q} - \theta\right) f'(x)$$

for some x lying between $\frac{p}{q}$ and θ. Thus

$$\left|\frac{p}{q} - \theta\right| = \frac{\left|f\left(\frac{p}{q}\right)\right|}{|f'(x)|} > \frac{1}{M(\theta)q^n}.$$

So θ is not approximable to any order higher than n. ∎

Corollary 2.7 (Liouville, 1844) *The number $\theta = \sum_{k\geq 1} 10^{-k!} = 0.110001000\ldots$ is transcendental.*

Proof. Define $\theta_n = \sum_{1\leq k\leq n} 10^{-k!} = \frac{p}{10^{n!}} = \frac{p}{q}$. Now

$$0 < \theta - \frac{p}{q} = \theta - \theta_n = \sum_{k\geq n+1} 10^{-k!} < 2 \cdot 10^{-(n+1)!} \leq 2q^{-n}.$$

Thus θ is approximable to order n for any n. Hence by Theorem 2.6, θ cannot be algebraic. ∎

Another useful lemma we will need later on.

Lemma 2.8 *Let β be an algebraic number of degree g. For each $N \geq 1$, there exists a constant $C > 0$ that depends only on β and N, such that, for every polynomial Q of degree N with integer coefficients, we have*

$$Q(\beta) = 0 \quad or \quad |Q(\beta)| \geq \tfrac{C}{||Q||^{g-1}},$$

where $|| \sum_{0 \leq j \leq N} a_j X^j || = \max_{0 \leq j \leq N} |a_j|$.

Proof. Let P be the minimal polynomial of β having integer coefficients and leading coefficient $a_g \geq 1$. Let $\beta_1 = \beta, \beta_2, \ldots, \beta_g$ be the conjugates of β, i.e., the roots of P, taken with multiplicity. Let Q be a polynomial of degree N, with leading coefficient c_N, such that $Q(\beta) \neq 0$, and let $\gamma_1, \gamma_2, \ldots, \gamma_N$ be its roots. We note that the β_i's and the γ_j's are all different. We have

$$\begin{aligned} 0 < \left| \frac{1}{a_g^N} \prod_{1 \leq j \leq N} P(\gamma_j) \right| &= \left| \prod_{1 \leq j \leq N} \prod_{1 \leq i \leq g} (\gamma_j - \beta_i) \right| \\ &= \left| \prod_{1 \leq i \leq g} \prod_{1 \leq j \leq N} (\gamma_j - \beta_i) \right| = \left| \frac{1}{c_N^g} \prod_{1 \leq i \leq g} Q(\beta_i) \right|. \end{aligned}$$

Hence

$$0 < |Q(\beta)| = \frac{\left| c_N^g \prod_{1 \leq j \leq N} P(\gamma_j) \right|}{a_g^N \left| \prod_{2 \leq i \leq g} Q(\beta_i) \right|}. \tag{2.1}$$

Now the expression $\prod_{1 \leq j \leq N} P(\gamma_j)$ is a symmetric polynomial in the γ_j's with integer coefficients, so the quantity $|c_N^g \prod_{1 \leq j \leq N} P(\gamma_j)|$ is a positive integer. Hence we deduce from Equation (2.1) that

$$Q(\beta) \geq \frac{1}{a_g^N (\sup_{2 \leq i \leq g} |Q(\beta_i)|)^{g-1}}.$$

Note that

$$\sup_{2 \leq i \leq g} |Q(\beta_i)|)^{g-1} \leq \|Q\|^{g-1} \sup_{1 \leq i \leq g} (1 + |\beta_i| + \cdots + |\beta_i|^N).$$

Hence

$$Q(\beta) \geq \frac{C}{\|Q\|^{g-1}},$$

where C depends only on β and N. ∎

Two integers k, $l \geq 1$ are called *multiplicatively independent* if $\log k$ and $\log l$ are linearly independent over $\mathbb{Q}$. Otherwise k and l are *multiplicatively dependent*.

2.3 Enumeration Systems

Now we will take a look at how numbers can be represented by strings of finite words. First, how to represent integers.

A *numeration system* is a way of representing an integer n as a finite linear combination $n = \sum_{0 \le i \le r} a_i u_i$ of base elements u_i. The a_i are called *digits* and the finite string of digits $a_r a_{r-1} \cdots a_1 a_0$ is said to be a *representation* of the number n.

For example, in the well-know decimal notation the base elements are the powers of 10 and every non-negative integer can be expressed as a non-negative linear combination $\sum_{0 \le i \le r} a_i 10^i$ with $0 \le a_i < 10$.

A minor problem with numeration systems is the *leading-zeros problem.* For example, each of the strings `101`,`0101`,`00101`,... represents the number 5 in base 2. Unless otherwise stated we will assume that the leading digit of a representation, if it exists, is nonzero. Hence the empty string ε is the representation for 0 in every numeration system.

And now to a more formal Definition of a numeration system.

Definition 2.9 *A* numeration system *for a semiring S is a triple $\mathcal{N} = (U, D, R)$, where $U = \{u_0, u_1, u_2, \dots\}$ is an infinite sequence of elements of S called the* base sequence*, D is a finite subset of S, called the* digit set *and $R \subseteq D^*$ is the set of* valid representations*. The mapping $[w]_U$ from D^* to S is defined as follows: if $w = a_r a_{r-1} \cdots a_1 a_0$, then $[w]_U = \sum_{0 \le i \le r} a_i u_i$.*

There are two properties a numeration system $\mathcal{N}$ should have:

1. there is at least one valid representation for every element of the underlying semiring - in this case $\mathcal{N}$ is *complete*

2. every element has no more than one valid representation - in this case $\mathcal{N}$ is *unambiguous*

If $\mathcal{N}$ is both complete and unambiguous, then it is *perfect.* In a perfect enumeration system the mapping $[w]_U$ is invertible.

Let $k \ge 2$ be an integer. Then every non-negative integer has a unique representation of the form $N = \sum_{0 \le i \le r} a_i k^i$ where $a_r \ne 0$ and $0 \le a_i < k$ for $0 \le i \le r$. Thus for every k we get a perfect enumeration system with the base sequence $U = \{k^0, k^1, k^2, \dots\}$, the digit set $D = \Sigma_k = \{\mathtt{0}, \mathtt{1}, \dots, k-1\}$ and the set of valid representations R being the set which contains the empty word ε and all elements of Σ^* which do not start with `0`.

For the unique representation of the integer N being $N = \sum_{0 \le i \le r} a_i k^i$ the *canonical base-k representation* is defined as $(N)_k = a_r a_{r-1} \cdots a_1 a_0$. For example, $(23)_2 =$ `10111`.

Here it is possible to define the inverse operation: For $w = a_r a_{r-1} \cdots a_1 a_0$, define $[w]_k = \sum_{0 \le i \le r} a_i k^i$. Clearly $[(N)_k]_k = N$.

Until now we only discussed how to represent integers in base k. The following theorem deals with representations of real numbers.

Theorem 2.10 (representation of real numbers) *Let k be an integer ≥ 2. Every $x \in \mathbb{R}$ can be represented in the form*

$$\lfloor x \rfloor + \sum_{i \ge 1} a_i k^{-i}$$

where $0 \le a_i < k$ [1]*. If x is not of the form $\frac{b}{k^r}$ for some integers b, r with $r \ge 0$, then the representation is unique. If x is of the form $\frac{b}{k^r}$ with $r \ge 0$, then there are two different representations, one where $a_i = 0$ for $i > r$, and another where $a_i = k - 1$ for $i > r$.*

Proof. The following algorithm provides one base-k representation for x_0:

```
RealRep(k,x0)
a0 := ⌊x0⌋
i := 0
while a0 ≠ xi do
        x(i+1) := k(xi − ai)
        i := i + 1
        ai := ⌊xi⌋
        output(ai)
```

If the algortithm terminates on input (k, x), then it is clear that $x = a_0 + \sum_{1 \le i \le r} a_i k^{-i}$ for some $r \ge 0$. On the other hand, if the algorithm does not terminate, then it is easy to see that the sequence $(a_0 + \sum_{1 \le i \le r} a_i k^{-i})_{r \ge 1}$ tends to x from below. Hence every number has at least one representation. Suppose there are integers b, r such that $x = \frac{b}{k^r}$. Then i can write $x = \lfloor x \rfloor + \{x\}$, where $\{x\} = \frac{c}{k^r}$ for some integer $c \ge 0$. Let the base-k representation of the integer c be $w = (c)_k$, and let $w' = 0^{r-|w|} w$. Then if $w' = d_1 d_2 \cdots d_r$, we have

$$x = a_0 + \sum_{i \ge 1} a_i k^{-i} = a'_0 + \sum_{i \ge 1} a'_i k^{-i}$$

where $a_0 = \lfloor x \rfloor$, and $a_i = d_i$ for $1 \le i \le r$, and $a_i = 0$ for $i > r$, and $a'_i = d_i$ for $1 \le i < r$; $a'_r = d_r - 1$, and $a'_i = k - 1$ for $i > r$. Finally, $a'_0 = a_0$ unless

[1] $\lfloor x \rfloor = \max\{c \in \mathbb{Z}, c \le x\}$, cf. $\{x\}$, the fractional part of the real number x, $\{x\} = x - \lfloor x \rfloor$

$r = 0$, in which case $a'_0 = a_0 - 1$. Now it's simple to verify that these two representations are the only ones possible.
Now suppose that there exist no integers b, r sucht that $x = \frac{b}{k^r}$, and assume that x has at least two different representations, say

$$x = a_0 + \sum_{i \geq 1} a_i k^{-i}$$

and

$$x' = a'_0 + \sum_{i \geq 1} a'_i k^{-i}$$

where $x = x'$. Since these representations differ, there must exist a smallest index $j \geq 0$ such that $a_j \neq a'_j$. Without loss of generality assume $a_j < a'_j$. Then there exists an index $l > j$ sucht that $a_l < k - 1$; for if not, we would have $x = \frac{b}{k^r}$ for some integers b, r. Then $x' - x > k^{-l}$, contradicting the assumption that $x = x'$. ■

The following theorem shows that the rational numbers are exactly those which base-k expansion, for every k, is ultimately periodic.

Theorem 2.11 *Let k be an integer ≥ 2, and let $\{x\} = 0.a_1a_2a_3\ldots$ be the base-k representation of the fractional part of x. Then x is a rational number if and only if the infinite word*

$$\boldsymbol{a} = a_1a_2a_3\ldots$$

is ultimately periodic.

Proof. Suppose **a** is ultimately periodic. Then we can write

$$\{x\} = .a_1a_2\ldots a_r(a_{r+1}\ldots a_{r+s})^\omega$$

for some integers r, s with $r \geq 0$ and $s > 0$. Then it is easy to verify that

$$\{x\} = k^{-r}\left([a_1a_2\ldots a_r]_k + \frac{[a_{r+1}a_{r+2}\ldots a_{r+s}]_k}{k^s - 1}\right),$$

so x is rational.

On the other hand, if x is rational, then $\{x\} = \frac{b}{c}$ for some integers b, c with $b \geq 0$, $c > 0$. Each step of the algorithm RealRep produces a new digit a_i and an x_i of the form $\frac{b_i}{c}$, with $0 \leq b_i < c$. If $b_i = 0$, the algorithm terminates, which corresponds to an ultimately periodic representation with period equal to the single digit 0. If the algorithm does not terminate, there are at most c different possibilities for b_i; when one occurs for the second time, the output of the algorithm becomes ultimately periodic. ■

2.4 Schmidt's Subspace Theorem

In this section we state two versions of the Subspace Theorem of Schmidt. We will use these deep results in the transcendence proofs in Chapter 6 and 7. We start with the version of Schmidt.

Theorem 2.12 (W.M. Schmidt) *Let $m \geq 2$ be an integer. Let $L_1, \ldots, L_m$ be linearly independent linear forms in $\mathbf{x} = (x_1, \ldots, x_m)$ with algebraic coefficients. Let ε be a positive real number. Then, the set of solutions $\mathbf{x} = (x_1, \ldots, x_m)$ in $\mathbb{Z}^m$ to the inequality*

$$|L_1(\mathbf{x}) \ldots L_m(\mathbf{x})| \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbb{Q}^m$.

Proof. See e.g. [63] or [64]. The case $m = 3$ has been established earlier in [65]. ■

The second version of the Schmidt Subspace Theorem we will need is the p-adic generalization due to Schlickewei [61, 62] and Evertse [32]. Before we state it we have to establish the terms *absolute values* and *height* in this context.

We normalize absolute values an height as follows. Let $\mathbf{K}$ be an algebraic number field of degree d. Let $M(\mathbf{K})$ denote the set of places on $\mathbf{K}$. For $x \in \mathbf{K}$ and a place $v \in M(\mathbf{K})$, define the absolute value $|x|_v$ by

1. $|x|_v = |\sigma(x)|^{\frac{1}{d}}$ if v corresponds to the embedding $\sigma : \mathbf{K} \to \mathbb{R}$;
2. $|x|_v = |\sigma(x)|^{\frac{2}{d}} = |\overline{\sigma}(x)|^{\frac{2}{d}}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \overline{\sigma} : \mathbf{K} \to \mathbb{C}$;
3. $(N\mathfrak{p})^{\frac{-ord_{\mathfrak{p}}(x)}{d}}$ if v corresponds to the prime ideal $\mathfrak{p}$ of $O_{\mathbf{K}}$.

These absolute values satisfy the product formula

$$\prod_{v \in M(\mathbf{K})} |x|_v = 1 \text{ for } x \in \mathbf{K}^*.$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be in $\mathbf{K}^n$ with $x \neq 0$. For a place $v \in M(\mathbf{K})$, put

$$\begin{aligned} |\mathbf{x}|_v &= \left(\sum_{i=1}^{n} |x_i|_v^{2d}\right)^{\frac{1}{2d}} \quad \text{if } v \text{ is real infinite;} \\ |\mathbf{x}|_v &= \left(\sum_{i=1}^{n} |x_i|_v^{d}\right)^{\frac{1}{d}} \quad \text{if } v \text{ is complex infinite;} \\ |\mathbf{x}|_v &= \max\{|x_1|_v, \ldots, |x_n|_v\} \quad \text{if } v \text{ is finite.} \end{aligned}$$

Now define the *height* of $\mathbf{x}$ by

$$H(\mathbf{x}) = H(x_1, \ldots, x_n) = \prod_{v \in M(\mathbf{K})} |\mathbf{x}|_v$$

We stress that $H(\mathbf{x})$ depends only on $\mathbf{x}$ and not on the choice of the number field $\mathbf{K}$ containing the coordinates of $\mathbf{x}$, see e.g. [32].

The following theorem is the p-adic generalization of the Subspace Theorem. We assume that the algebraic closure of $\mathbf{K}$ is $\overline{\mathbb{Q}}$. We choose for every place v in $M(\mathbf{K})$ a continuation of $|\cdot|_v$ to $\overline{\mathbb{Q}}$, that we denote also by $|\cdot|_v$.

Theorem 2.13 *Let* $\mathbf{K}$ *be an algebraic number field. Let* $m \geq 2$ *be an integer. Let* S *be a finite set of places on* $\mathbf{K}$ *containing all infinite places. For each* v *in* S*, let* $L_{1,v}, \ldots, L_{m,v}$ *be linear forms with algebraic coefficients and with rank*$\{L_{1,v}, \ldots, L_{m,v}\} = m$*. Let* ε *be real with* $0 < \varepsilon < 1$*. Then, the set of solutions* $\mathbf{x}$ *in* $\mathbf{K}^m$ *to the inequality*

$$\prod_{v \in S} \prod_{i=1}^{m} \frac{|L_{i,v}(\mathbf{x})|_v}{|\mathbf{x}|_v} \leq \prod_{v \in S} (|\det(L_{1,v}, \ldots, L_{m,v})|_v)\, H(\mathbf{x})^{-m-\varepsilon}$$

lies in finitely many proper subspaces of $\mathbf{K}^m$.

A proof of Theorem 2.13 can be found in [32], where a quantitative version is established.

Chapter 3

Finite Automata

Now we will introduce some basic notions and facts about finite automata.

A *deterministic finite automaton*, or *DFA*, is a very simple model for computation. It is an *acceptor*, i.e. strings are given as inputs and are either accepted or rejected.

A DFA starts in an *initial state* and then after reading the input can be in one of a finite number of states. It takes a word w as input and then, reading the symbols of w from left to right, moves from state to state. If after reading all the symbols of w the DFA is in a distinguished state called an *accepting state*, then the string is accepted; otherwise it is rejected. The language accepted by the DFA is the set of all accepted strings.

A DFA can be displayed by a directed graph called a *transition diagram*. A directed edge labeled with a letter indicates the new state of the machine if the given letter is read. By convention, the initial state is drawn with an unlabeled arrow entering the state, and accepting states are drawn with double circles.

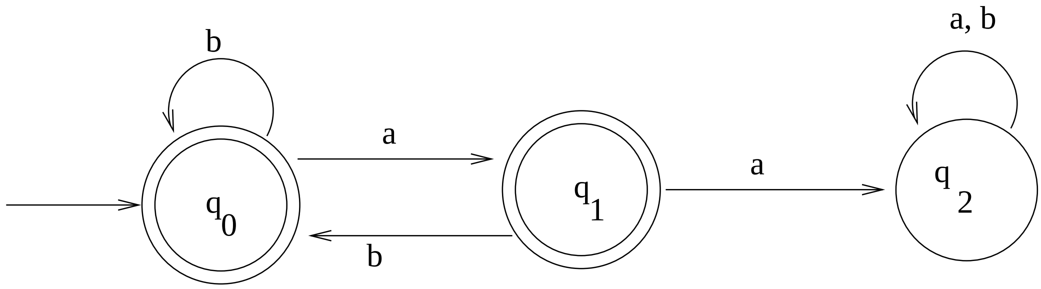

Figure 3.1: A DFA which accepts words with no two consecutive `a`'s.

Example 3.1 *Figure 3.1 shows a simple DFA: It takes words made out of* *the alphabet* $\{\mathtt{a}, \mathtt{b}\}$ *as input. A word is accepted if, and only if, it does not contain two consecutive* $\mathtt{a}$*'s.*

And now for a more formal definition:

Definition 3.2 (deterministic finite automaton (DFA)) *A* DFA M *is defined to be a 5-tupel*

$$M = (Q, \Sigma, \delta, q_0, F)$$

where
Q *is a finite set of states,*
Σ *is the finite input alphabet,*
$\delta : Q \times \Sigma \to Q$ *is the* transition function*,*
$q_0 \in Q$ *is the* initial state*, and*
$F \subseteq Q$ *is the set of accepting states.*

It is assumed that δ is defined for all pairs in its range, or in other words, that the DFA is *complete*. To be able to speak of acceptance of strings the domain of δ needs to be extended to $Q \times \Sigma^*$. This is done in the natural way: first, $\delta(q, \varepsilon) = q$ for all $q \in Q$, and then $\delta(q, xa) = \delta(\delta(q, x), a)$ for all $q \in Q$, $x \in \Sigma^*$ and $a \in \Sigma$.

The language accepted by M is then the set $L(M) = \{w \in \Sigma^* : \delta(q_0, w) \in F\}$.

A state q for which there exists some $x \in \Sigma^*$ such that $\delta(q_0, x) = q$ is called *reachable*, otherwise *unreachable*. Obviously, we can delete all unreachable states without changing the language accepted by the DFA.

Until now we said that a DFA *accepts* or *rejects* a word. Mathematically the DFA can be seen as a function $f : \Sigma^* \to \{\mathtt{0,1}\}$, where `1` represents acceptance and `0` rejection. Of course it is possible to use more general sets than $\{\mathtt{0,1}\}$ which leads to finite automata with output. They work like this: Given the input word w, the automaton moves from state to state according to its transition function δ, while reading the symbols of w. As soon as the end of w is reached, the automaton halts in a state q. Now the automaton uses its output mapping τ to give the symbol $\tau(q)$ as the output.

Again, a formal definition:

Definition 3.3 (deterministic finite automaton with output (DFAO)) *A DFAO* M *is defined to be a 6-tupel*

$$M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$$

where Q, Σ, δ, q_0 *are as in the definition 3.2,* Δ *is the output alphabet, and* $\tau : Q \to \Delta$ *is the output function.*

A function $f : \Sigma^* \to \Delta$ that uses M to be computed like this

$$f(w) = f_M(w) = \tau(\delta(q_0, w))$$

is called a *finite-state function*.

DFAOs can be represented with a transition diagram just like DFAs; the only difference is that states are labeled like q/a indicating that if the automaton halts in q the output is the symbol a.

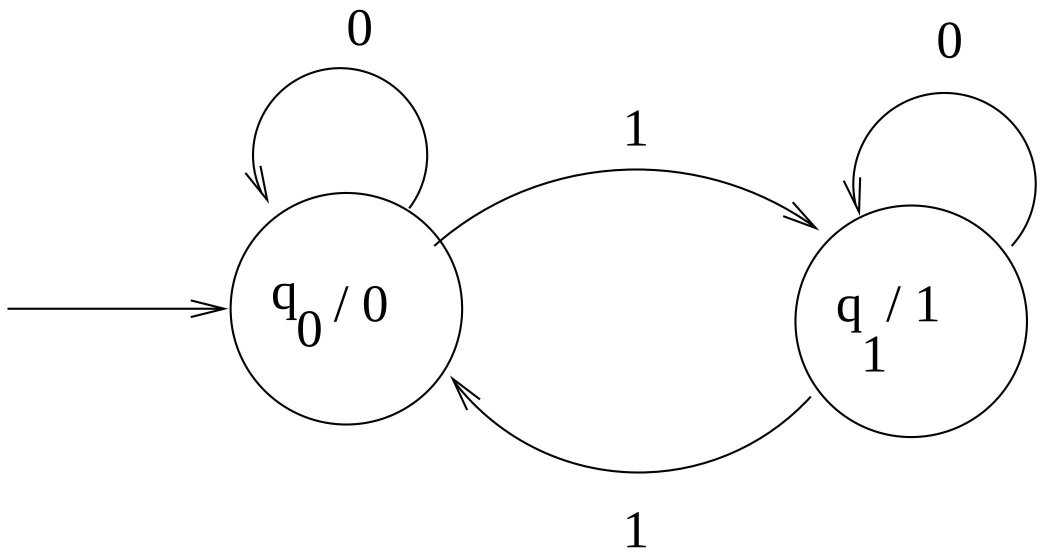

Figure 3.2: A DFAO which computes the mod-2 sum of its binary input bits.

Example 3.4 *The DFAO in figure 3.2 computes the sum* mod2 *of the bits of the input word* $w \in \{0, 1\}^*$.

Chapter 4

Automatic Sequences

Now we will introduce the concept of the *automatic sequence*. In the last chapter we introduced DFAOs and finite state functions. Now we are especially interested in the case where the input is a number in base k, which means the input alphabet $\Sigma = \Sigma_k := \{0, 1, 2, \ldots, k-1\}$, with $k \in \mathbb{N}$, $k \geq 2$. The DFAO is then called a k-DFAO.

And now to the fundamental concept of the k-automatic sequence: Informally, a sequence $(a_n)_{n \geq 0}$ is called k-automatic if a_n is a finite-state function of the base-k digits of n. In other words, a_n is computed by feeding a finite automaton with the base-k representation of n, starting with the most siginificant digit, and then applying an output mapping τ to the last state reached. Here comes the formal definition:

Definition 4.1 (k-automatic sequence) *A sequence $(a_n)_{n \geq 0}$ over a finite alphabet Δ is called k-*automatic *if there exists a k-DFAO $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ such that $a_n = \tau(\delta(q_0, w))$ for all $n \geq 0$ and all w with $[w]_k = n$.*

If M is such a k-DFAO, then it is said that M *generates* the sequence $(a_n)_{n \geq 0}$. Definition 4.1 requires that the automaton returns the correct answer even if the input possesses leading zeros; this condition is not a problem and can be relaxed, see theorem 4.3.

There is a multitude of sequences of mathematical interest which are k-automatic for some integer $k \geq 2$, including an example we already mentioned in section 2.1.2:

Example 4.2 (The Thue-Morse Sequence) *The sequence $\boldsymbol{t} = (t_n)_{n \geq 0}$ generated by the morphism that we introduced in example 2.5, $\boldsymbol{t} = \mu^{\infty}(0)$, counts the number of 1's $\pmod 2$ in the base-2 representation of n. Here are the first few terms:*

$$\begin{array}{ccccccccccccccccc} n & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \dots \\ \mathbf{t}_n & = & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & \dots \end{array}$$

The Thue-Morse sequence can be generated by the DFAO in figure 4.1, hence it is 2*-automatic.*

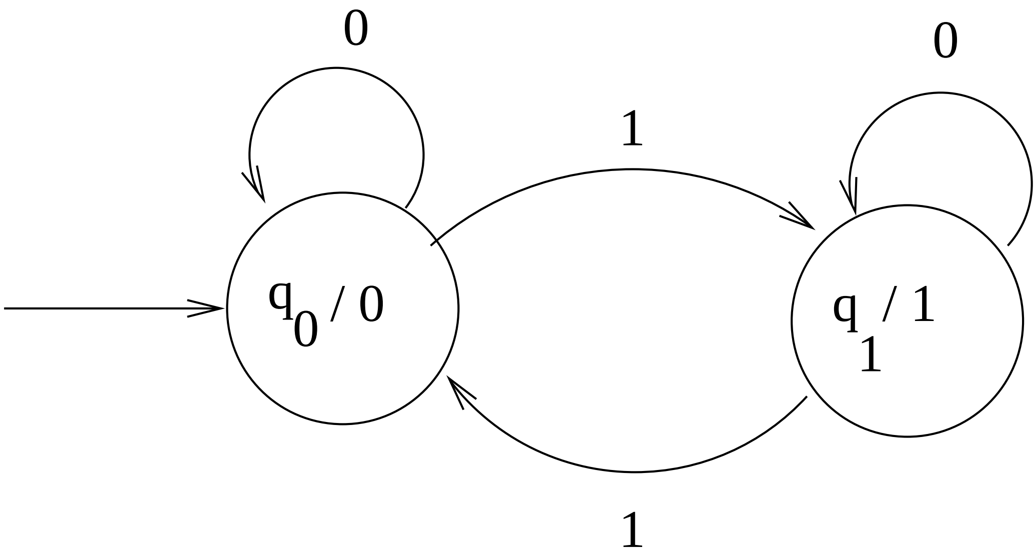

Figure 4.1: The DFAO Generating the Thue-Morse Sequence.

To understand that this DFAO actually generates $\mathbf{t}$*, consider that being in state* q_0 *means that the bits of the input seen so far sum to* $0 \pmod 2$*, while being in state* q_1 *means that the bits of the input seen so far sum to* $1 \pmod 2$*, as one can easily prove by induction.*

Definition 4.1 demanded that our machine M computes a_n correctly no matter which base-k representation of n is input. More precisely, M must give the same answer even if the input has leading zeros. This is a strong requirement, but as the next theorem shows, it is not necessary. In fact, it suffices that the DFAO compute the correct output just for the canonical representation of n in base k (those lacking leading zeros).

Theorem 4.3 *The sequence* $(a_n)_{n\geq 0}$ *is* k*-automatic if and only if there exists a* k*-DFAO* M *such that* $a_n = \tau(\delta(q_0, (n)_k))$ *for all* $n \geq 0$*. Moreover, we may choose* M *such that* $\delta(q_0, 0) = q_0$*.*

Proof. $\Longrightarrow$: Trivial.
$\Longleftarrow$: Let $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$. Define $M' = (Q', \Sigma_k, \delta', q_0', \Delta, \tau')$ as follows:

$$\begin{array}{rcll} Q' &=& Q \cup \{q_0'\}, & \\ \delta'(q,a) &=& \delta(q,a) & \text{for all } q \in Q,\ a \in \Sigma_k, \\ \delta'(q_0',a) &=& \begin{cases} \delta(q_0,a) & \text{if } a \neq 0, \\ q_0' & \text{if } a = 0, \end{cases} & \\ \tau'(q_0') &=& \tau(q_0), & \\ \tau'(q') &=& \tau(q) & \text{for all } q \in Q. \end{array}$$

Then we claim that $\tau'(\delta'(q_0', \mathtt{0}^i(n)_k)) = \tau(\delta(q_0,(n)_k))$ for all $i \geq 0$. This follows from the property $\delta'(q_0', \mathtt{0}^i(n)_k) = \delta'(q_0',(n)_k) = \delta(q_0,(n)_k)$ for $n \neq 0$. ∎

k-automatic sequences have a lot of interesting and useful properties. For example, one of them is stated in the following theorem.

Theorem 4.4 *If a sequence $(v_n)_{n\geq 0}$ differs only in finitely many terms from a k-automatic sequence $(a_n)_{n\geq 0}$, then it is k-automatic.*

Proof. The assertion follows easily from a lemma which shows that a sequence $(a_n)_{n\geq 0}$ over Δ is k-automatic if and only if each of the sets $\{(n)_k : a_n = d\}$ is a regular language for all $d \in \Delta$ (a proof of this can be found in [9], p. 160). ∎

This will help us in the proof of the following theorem.

Theorem 4.5 *If $(a_n)_{n\geq 0}$ is an ultimately periodic sequence, then it is k-automatic for all $k \geq 2$.*

Proof. From theorem 4.4 it suffices to show this is the case where $(a_n)_{n\geq 0}$ is purely periodic of period t, i.e., $a_{tn+i} = a_i$ for $0 \leq i < t$ and $n \geq 0$. Now we define the k-automaton $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$, where $\Sigma = \{\mathtt{0}, \mathtt{1}, \ldots, k-1\}$, as follows:

$$\begin{array}{rcl} Q &=& \{0,1,\ldots,t-1\}, \\ \delta(q,b) &=& (kq+b) \pmod t \ \ \textit{for all } q \in Q, b \in \Sigma \\ \tau(q) &=& a_q \ \ \textit{for } 0 \leq q < t. \end{array}$$

Then it is easy to see by induction that $\delta(q, b_0 b_1 \cdots b_j) = [b_0 b_1 \cdots b_j]_k \pmod t$ and the result follows. ∎

Theorem 4.6 *Let $\boldsymbol{u} = (u_n)_{n\geq 0}$ be a k-automatic sequence, and let ρ be a coding. Then the sequence $\rho(\boldsymbol{u})$ is also k-automatic.*

Proof. By the definition of k-automatic, there exists a k-DFAO $M = (Q, \Sigma, \delta, q_0, \Gamma, \tau)$ such that $u_n = \tau(\delta(q_0, (n)_k))$ for all $n \geq 0$. Now consider the k-DFAO $M' = (Q, \Sigma, \delta, q_0, \Gamma, \rho \circ \tau)$. Clearly this DFAO generates $\rho(\mathbf{u})$. ■

Theorem 4.7 *Let $\boldsymbol{a} = (a_n)_{n\geq 0}$ and $\boldsymbol{b} = (b_n)_{n\geq 0}$ be two k-automatic sequences with values in Δ and Δ', respectively. Then $\boldsymbol{a} \times \boldsymbol{b} = ([a_n, b_n])_{n\geq 0}$ is k-automatic.*

Proof. Let $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$ generate $(a_n)_{n\geq 0}$ and $M' = (Q', \Sigma, \delta', q_0', \Delta', \tau')$ generate $(b_n)_{n\geq 0}$. Then $M'' = (Q \times Q', \Sigma, \delta'', [q_0, q_0'], \Delta \times \Delta', \tau'')$ generates $\mathbf{a} \times \mathbf{b}$, where

$$\delta''([q, q'], c) = [\delta(q, c), \delta'(q', c)] \quad \text{for all } q \in Q, q' \in Q', c \in \Sigma$$

and

$$\tau''([q, q']) = [\tau(q), \tau'(q')].$$

■

Theorem 4.8 *Let $\boldsymbol{a} = (a_n)_{n\geq 0}$ and $\boldsymbol{b} = (b_n)_{n\geq 0}$ be two k-automatic sequences with values in finite sets Δ and Δ', respectively. Let $f : \Delta \times \Delta' \to \Delta''$ be any function into a finite set Δ''. Then the sequence $(f(a_n, b_n))_{n\geq 0}$ is k-automatic.*

Proof. Combine the previous two theorems. ■

Chapter 5

Sturmian Real Numbers

5.1 Introduction

It is a well known fact that, for any integer $k \geq 2$, the base-k expansion of a rational number is ultimately periodic. If we decide to measure the *complexity* of a base-k expansion by counting the number of blocks of digits of length n which appear in it, we may then say that the complexity of the expansion in base k of a rational number is very low. In this chapter we consider the following problem: can there exist irrational algebraic numbers whose expansion in some base k is of lowest complexity? This low complexity represents in a way the opposite situation to normal numbers in base k.

Our main result here shows that if the k-adic expansion of an irrational number has the lowest possible complexity—in that case we say it is a *Sturmian* real number—, then this number is transcendental.

For some time this result was known only in some particular cases ([5], [26], [43] correcting [51]). The method we are proposing in this chapter uses a combinatorial translation of a result of Ridout ([49], chap. 9, pp. 147-148), Theorem 5.2, stating that, if the expansion of a number contains infinitely many $(2+\varepsilon)$-powers of blocks (that is, a block followed by itself and then by its beginning of relative length at least ε), at distances from the origin which are not too much larger than the lengths of the considered blocks, then it is transcendental. The transcendence in the Sturmian case is then a consequence of this criterion and of the combinatorial properties of Sturmian sequences.

5.2 Sturmian Numbers and Transcendence

We recall a famous result of Hedlund and Morse([37]): considering the infinite sequence $\mathbf{w}$ and its complexity function $p(n)$, if there exists n such that $p(n+1) = p(n)$, or such that $p(n) \leq n$, then $\mathbf{w}$ is ultimately periodic.

We are looking at non-ultimately periodic sequences with the lowest possible complexity:

Definition 5.1 *The sequence* $\boldsymbol{w}$ *is called* Sturmian *(on two letters) if* $p(n) = n + 1$ *for every* $n \in \mathbb{N}\backslash\{0\}$*, or, more generally, for any integer* $l \geq 2$*, the sequence* $\boldsymbol{w}$ *is* Sturmian on l letters *if* $p(n) = n + l - 1$ *for every* $n \in \mathbb{N}\backslash\{0\}$*.*

The general definition of a Sturmian sequence is equivalent to the following: $p(1) = l$ (and we may then suppose the alphabet Σ has cardinality l) and $p(n+1) - p(n) = 1$ for any $n \geq 1$. A Sturmian sequence cannot be ultimately periodic, and the equation $p(n+1) - p(n) = 1$ is equivalent to the following property: there exists one word in $L_n(\mathbf{w})$ which is a prefix of two different words in $L_{n+1}(\mathbf{w})$, and each of the other words in $L_n(\mathbf{w})$ is a prefix of one and only one word in $L_{n+1}(\mathbf{w})$.

Now we start by stating the combinatorial criterion for transcendence used in this chapter.

Theorem 5.2 *If* θ *is an irrational number and, for every* $n \in \mathbb{N}$*, the expansion of* θ *in base* k *begins by* $0.U_nV_nV_nV_n'$*, where* U_n *is a possibly empty and* V_n *is a nonempty word on an alphabet* $\Sigma \subset \{0, \ldots, k-1\}$*,* V_n' *is a prefix of* V_n*,* $|V_n| \to +\infty$*,* $\limsup \left(\frac{|U_n|}{|V_n|}\right) < +\infty$ *and* $\liminf \left(\frac{|V_n'|}{|V_n|}\right) > 0$*, then* θ *is a transcendental number.*

Proof. Let $r_n = |U_n|$, $s_n = |V_n|$, and choose $0 < \varepsilon < \liminf(\frac{|V_n'|}{|V_n|})$. Let t_n be the number whose expansion in base k is $0.U_nV_n \cdots V_n \cdots$; then

$$t_n = \frac{p_n}{k^{r_n}(k^{s_n} - 1)}$$

for some integer p_n, while for n large enough

$$|\theta - t_n| \leq \frac{1}{k^{r_n + (\varepsilon+2)s_n}}$$

Now, suppose θ were algebraic irrational. Then, from a theorem of Ridout ([49], chap. 9, pp. 147-148), if there exist infinitely many rational numbers $\frac{P_n}{Q_n}$, with $Q_n = k^{m_n}Q_n'$ (the numbers k, m_n and Q_n' being integers), such that

$$\left|\frac{P_n}{Q_n} - \theta\right| < c_1(Q_n)^{-\rho}$$

and

$$Q'_n < c_2(Q_n)^\mu,$$

where c_1 and c_2 are positive constants, then $\rho \leq 1+\mu$. Since $\liminf(\frac{s_n}{r_n+s_n}) = \frac{1}{\limsup(\frac{r_n}{s_n}+1)} > 0$, and up to restricting n to a strictly increasing sequence of integers, one can suppose that $\frac{s_n}{r_n+s_n} \to \eta > 0$. In particular, there exist two numbers ρ and μ such that, for all n in some infinite set,

$$1 + \frac{s_n}{r_n + s_n} < 1 + \mu < \rho < 1 + (1+\varepsilon)\frac{s_n}{r_n + s_n}.$$

This choice of μ and ρ together with the choice $P_n = p_n$, $Q_n = k^{r_n}(k^{s_n} - 1)$, $m_n = r_n$, and $Q'_n = k^{s_n} - 1$ gives us the desired contradiction. Hence θ is transcendental. ■

After having proven the transcendence criterion we now look at the main focus of this chapter, the Sturmian real numbers. We start with two lemmata that will bring us to the main result of this section, Theorem 5.5.

Lemma 5.3 *If $\boldsymbol{u} = u_0u_1u_2\ldots$ is Sturmian on l letters and not recurrent, $\boldsymbol{u}$ is ultimately equal to a Sturmian recurrent sequence on $l' < l$ letters.*

Proof. Suppose a word w, of length m, does not occur an infinite number of times in $\mathbf{u}$; then there exists an N such that the complexity of the sequence $(v_n)_{n\geq 0}$ equal to $(u_n)_{n\geq N}$ satisfies $p(m) \leq m+l-2$; but for each n, $L_n(\mathbf{v}) \subset L_n(\mathbf{u})$, hence every word in $L_n(\mathbf{v})$ is a prefix of almost one word in $L_{n+1}(\mathbf{v})$, except maybe one which is a prefix of two words, and hence $p(n+1)-p(n) \leq 1$; but also $p(n+1) - p(n) > 0$ as this sequence is not ultimately periodic; hence it must have complexity $n + l_1 - 1$ for all $n \geq 1$, for some $0 \leq l_1 < l$; if it is recurrent, the lemma is proved; if it is not recurrent, we iterate the process, and, after at most l steps, it shows that $\mathbf{u}$ is ultimately equal to some recurrent Sturmian sequence. ■

The same proof shows that every binary Sturmian sequence is recurrent; in fact, every binary Sturmian sequence, and, more generally every recurrent Sturmian sequence, can be shown to be minimal.

The same method can be used to prove that a non-recurrent Sturmian sequence $\mathbf{u}$ is of the form $u_0 \cdots u_p\mathbf{v}$, where $\mathbf{v}$ is a recurrent Sturmian sequence on an alphabet Σ', and $u_0, \ldots, u_p$ are distinct elements of $\Sigma\backslash\Sigma'$; this characterizes completetly the non-recurrent Sturmian sequences.

The following lemma is essentially the same as in [12], pp.205–208, though we need a different labelling of the edges in the graph and a different definition of what they call a segment; this yields correspondingly different, though of course equivalent, results.

Lemma 5.4 *If $\boldsymbol{u}$ is a recurrent Sturmian sequence, there exist two words w_0 and w_1, and a sequence of integers $a_n \geq 1$, $n \geq 1$, such that if the words w_n, $n \in \mathbb{N}$ are given by the recursion formulas*

$$w_{n+1} = w_n^{a_n} w_{n-1}$$

for $n \geq 1$. Then, for any $N \geq 1$ and $n \geq 1$, the word $u_0 u_1 \cdots u_{N-1}$ is of the form $X_0 X_1 \cdots X_k$, where $X_1, X_2, \ldots, X_{k-1}$ are equal either to w_n or to w_{n+1}, X_0 is a (possibly empty) suffix of either w_n or w_{n+1}, X_k is a (possibly empty) prefix of either w_n or w_{n+1}. This decomposition, which is not unique, is independent of N for fixed n.

Proof. Let Σ be the alphabet of the sequence. Let $L_n(\mathbf{u})$ be the set of all subwords of $\mathbf{u}$ of length n; let Γ_n be the graph whose vertices are the elements of $L_n(\mathbf{u})$, and where there is an arrow from E to F, with label b, whenever $E = bH$, $F = Ha$, with $b \in \Sigma, a \in \Sigma$, and $bHa \in L_{n+1}(\mathbf{u})$. The property $p(n+1) - p(n) = 1$ implies there is one vertex D_n with two outgoing arrows, and from every other vertex leaves only one arrow; the property $p(n+1) - p(n) = 1$ and the recurrence imply that there is one vertex G_n with two incoming arrows, and to every other vertex arrives only one arrow; hence Γ_n has one of the following forms:

$$\begin{array}{ccccccc}
 & \leftarrow & & & & \leftarrow & \\
\downarrow & & \uparrow & & \downarrow & & \uparrow \\
G_n & \rightarrow & D_n & \text{or} & G_n & = & D_n \\
\downarrow & & \uparrow & & \uparrow & & \downarrow \\
 & \rightarrow & & & & \rightarrow &
\end{array}$$

We call an *n-segment* any finite sequence $(E_0, \ldots, E_k)$ of vertices of Γ_n such that $E_0 = G_n, E_i \rightarrow E_{i+1}, E_k = G_n$, and to each E_i, $1 \leq i \leq k-1$ arrives only one arrow. The *name* of an n-segment is the word made with the labels of the arrows $E_0 \rightarrow E_1, \ldots, E_{k-1} \rightarrow E_k$. There are exactly two n-segments for each n, and their names generate the language $L_n(\mathbf{u})$. Let K_n and J_n be the names of the two n-segments.

There are two cases for going from Γ_n to Γ_{n+1}: in the first one, $G_n \neq D_n$. Then for any word $X \neq D_n$ in $L_n(\mathbf{u})$, there exists a unique word Xa in $L_{n+1}(\mathbf{u})$, and if $Yb \rightarrow Xa$, then $Y \rightarrow X$. We must have $G_{n+1} = G_n\gamma$ and $D_{n+1} = \delta D_n$, where γ and δ are uniquely determined by the graph Γ_n. The graph Γ_{n+1} is then known entirely, and we check that $K_{n+1} = K_n$ and $J_{n+1} = J_n$.

In the second case, $G_n = D_n$; let the two n-segments be $(G_n, G'_n a, \ldots, bG''_n, G_n)$ with name K_n, and $(G_n, G'_n c, \ldots, dG''_n, G_n)$, with name J_n. Then $G_{n+1} =$

$G_n\gamma$ and $D_{n+1} = \delta G_n$ but now Γ_n does not determine Γ_{n+1}. To lift the indetermination, we suppose for example that $\gamma = a$; then, because of the recurrence of $\mathbf{u}$, we must have $D_{n+1} = bD_n$, the other possible choice giving a non-transitive graph. This implies that $K_{n+1} = K_n$, $J_{n+1} = K_nJ_n$; the other choice for γ leads to $K_{n+1} = J_nK_n$, $J_{n+1} = J_n$.

Hence the names of the two n-segments are w_n and w_{n+1}, with the recursion formulas we claimed; and now, for any fixed n and N, the word $u_0 \cdots u_{N-1}$ is the path in Γ_n starting at $u_0 \cdots u_{n-1}$, continuing through $u_1 \cdots u_n, u_2 \cdots u_{n+1}, \ldots$ and ending at $u_{N+1} \cdots u_{N+n}$. So it has the required decomposition. ■

With these two lemmata and the transcendence criterion, i.e. Theorem 5.2, we are now ready to prove the main result of this chapter:

Theorem 5.5 *If there exists k such that the expansion of θ in base k is a Sturmian sequence, then θ is a transcendental number.*

Proof. As the transcendence does not depend on the initial values of $\mathbf{u}$, it is enough, because of Lemma 5.3, to prove our claim if $\theta = S_k(\mathbf{u})$ for a recurrent Sturmian sequence $\mathbf{u}$. Let then a_n and w_n be as in lemma 5.4.

Then, for each n, a suitable initial segment of $\mathbf{u}$ is $X_0X_1 \ldots X_{k-1}$ as in lemma 5.4; X_0 is either a suffix of w_n, denoted by T_n, or a suffix of w_{n+1}, which may be a suffix of w_{n-1}, denoted again by T_n, or is of the form $T_nw_n^{c_n}w_{n-1}$ with T_n a suffix of w_n and $0 \le c_n \le a_n$ an integer (every considered suffix may be empty). Then the first b_n words among $X_1, \ldots, X_{k-1}$ are w_n for some integer $b_n \ge 0$, and then comes one w_{n+1} (if not, $\mathbf{u}$ would be ultimately periodic).

Hence, for every n, $\mathbf{u}$ begins by either

1. the word $T_nw_n^{b_n+a_n}w_{n-1}$ or
2. the word $T_nw_n^{c_n}w_{n-1}w_n^{b_n+a_n}w_{n-1}$,

where T_n is a suffix of w_n or of w_{n-1} and b_n and c_n are non-negative integers. Let q_n be the length of w_n, satisfying $q_{n+1} = q_na_n + q_{n-1}$. Then

- if, for infinitely many n, the case 2 occurs with $c_n \ge 3$, Theorem 5.2 applied for this sequence with $U_n = T_n$ and $V_n = V_n' = W_n$ yields the transcendence of $S_k(\mathbf{u})$;
- if not, we take U_n to be the T_n in case 1 and $T_nw_n^{c_n}w_{n-1}$ in case 2, so we have ultimately (i.e., for each n large enough) $|U_n| \le 5q_n$. And
    - if $a_n + b_n \ge 3$ for infinitely many n, we take $V_n = w_n$ and apply Theorem 5.2, with $V_n' = V_n$, which yields the result;

- if $a_n + b_n \leq 2$ ultimately but $a_n + b_n = 2$ infinitely often, then $q_{n-1} \geq \frac{q_n}{3}$ ultimately, and Theorem 5.2 with $V_n = w_n$ and $V_n' = w_{n-1}$ yields the result;
- finally, in the remaining case we must have $b_n = 0$ and $a_n = 1$ ultimately. In this case, where one will recognize the Fibonacci recursion, we have also $w_n w_{n-1} = w_{n-1} w_{n-1} w_{n-4} w_{n-3}$, and we apply Theorem 5.2 with $V_n = w_{n-1}$ and $V_n' = w_{n-4}$, as $|V_n| = q_{n-1}$ is then larger than $\frac{|U_n|}{10}$ and smaller than $8|V_n'|$.

■

So now we have proven that the Sturmian real numbers, i.e. the real numbers whose expansion in some base k is a Sturmian sequence, are transcendental.

Note that the method used to prove this uses only the combinatorial properties of the Sturmian sequences, and that Lemma 5.4, which gives only an explicit characterization of the *language* of $\mathbf{u}$ but not of the sequence itself, is sufficient to yield the main result. There exist, however, more precise results than Lemma 5.4, and they have been used to give the transcendence result in some particular cases.

There is a characterization due to Morse and Hedlund [37] of Sturmian sequences on $\Sigma_2 = \{0, 1\}$: a sequence $(t_n)_{n\geq 0}$ is a Sturmian sequence if and only if it satisfies, for some irrational $\alpha \in (0, 1)$ and some real number β, either $t_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$ for all $n \geq 0$ or $t_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$ for all $n \geq 0$. We will use this characterization in chapter 7.

More on algebraic expressions of Sturmian sequences can be found in [34]. There they also give a generalization of the methods used in this chapter to some sequences that share part of the combinatorial properties of the Sturmian sequences like the *Arnoux-Rauzy* sequences. See also [12], [47] and [68].

Chapter 6

Automatic Real Numbers

In this chapter we will now introduce the concept of the *automatic real numbers*. In Turing [70] the computability of real numbers is discussed, saying that a real number α is computable if there is a Turing machine that, given the input i, will compute a rational approximation to α that lies within $[\alpha - 2^{-i}, \alpha + 2^{-i}]$. The automatic real numbers are an analogous concept. A single real number x is associated with a DFAO as follows: Given the input n represented in base k, the DFAO outputs the nth digit of the base-b expansion of the fractional part of x. Here we will explore the properties of such numbers.

6.1 Definition

Here comes a proper definition:

Definition 6.1 (automatic real numbers) *Let k, $b \in \mathbb{N}$ and ≥ 2. Let r be a real number, and suppose*

$$r = a_0 + \sum_{i \geq 1} a_i b^{-i}$$

with $a_i \in \mathbb{Z}$ for $i \geq 0$ and $0 \leq a_i < b$ for $i \geq 1$. Then r is (k, b)-automatic *if the sequence of digits $(a_i)_{i \geq 0}$ is a k-automatic sequence. $L(k, b)$ denotes the set of all (k, b)-automatic reals.*

6.2 Basic Properties

First we will show that $\mathbb{Q} \subseteq L(k, b)$ for all $k, b \geq 2$.

Theorem 6.2 *If r is rational, then $r \in L(k,b)$ for all $k, b \geq 2$.*

Proof. Let r be a rational number. Then by Theorem 2.11 the expansion of r in base b is ultimately periodic. Hence by Theorem 4.5 , the sequence $(a_i)_{i\geq 0}$ is k-automatic. ■

Now let's take a look at the other direction:

Theorem 6.3 *Let j, $k \geq 2$ be multiplicatively independent integers. Then $L(j,b) \cap L(k,b) = \mathbb{Q}$.*

Proof. $\supseteq$: use Theorem 6.2.
$\subseteq$: follows immediately from Cobham's theorem which states the following:

Let k, l be multiplicatively independet integers, and suppose the sequence $\mathbf{s} = (s_n)_{n\geq 0}$ is both k- and l-automatic. Then $\mathbf{s}$ is ultimately periodic.

A proof of this can be found in [9], 346-350. ■

And now some theorems that will show that $L(k,b)$ forms a vector space over $\mathbb{Q}$.

Theorem 6.4 *Let k, $b \in \mathbb{N}$ and ≥ 2. If $x \in L(k,b)$, then $-x \in L(k,b)$.*

Proof. Clearly, this is true for $x \in \mathbb{Z}$. For $x \notin \mathbb{Z}$, we write $x = a_0 + \sum_{i\geq 1} a_i b^{-i}$ with $0 \leq a_i < b$ for $i \geq 1$. Then by hypothesis $(a_i)_{i\geq 0}$ is a k-automatic sequence. Now we consider the coding $h : \{0,1,2,\ldots,b-1\} \to \{0,1,2,\ldots,b-1\}$ defined as follows: $h(i) = b - 1 - i$. Let $c_i = h(a_{i+1})$ for $i \geq 0$, and define $y = \sum_{i\geq 0} c_i b^{-(i+1)}$. A simple calculation now gives $y = a_0 + 1 - x$.

The shifted sequence $(a_{i+1})_{i\geq 0}$ is k-automatic because a shifted k-automatic sequence is still k-automatic. By Theorem 4.6, $(c_i)_{i\geq 0}$ is k-automatic. Now we define

$$d_i = \begin{cases} -(a_0+1) & \text{if } i = 0, \\ c_{i-1} & \text{if } i \geq 1. \end{cases}$$

Then $(d_i)_{i\geq 0}$ is also k-automatic, since it is nothing more than a shift of $(c_i)_{i\geq 0}$ with an arbitrary element as d_0.

And $\sum_{i\geq 0} d_i b^{-i} = -x$ which concludes the proof. ■

Theorem 6.5 *If r, $s \in L(k,b)$, then so is $r + s$.*

Proof. The base-b expansions of r and s can be added digit by digit, using Theorem 4.8 and the function $f(a,c) = a + c$. this gives an "unnormalized" base-b expansion $\sum_{n\geq 0} u_n b^{-n}$ with $u_i \in \{0,1,\ldots,2b-2\}$ for $i \geq 0$. The results then follows from the fact that for a positive integer C and any k-automatic sequence $(a_n)_{n\geq 1}$ of integers with $0 \leq a_i \leq C$ for all $i \leq 1$, the number $y := \sum_{i\geq 0} a_i b^{-i}$ is a (k,b)-automatic real number. ■

Theorem 6.6 *Let $x \in L(k, b)$, i.e. x is a number whose base-b expansion is k-automatic. If c is a nonzero integer, then $\frac{x}{c} \in L(k, b)$.*

Proof. We construct a 1-uniform transducer that transforms the sequence $x_1, x_2, \ldots, x_i, \ldots$ into the sequence $y_1, y_2, \ldots, y_i, \ldots$, where $x = .x_1 x_2 \ldots$, $y = \frac{x}{c} = y_1 y_2 \cdots$ in base b. Since automatic sequences are closed under 1-uniform transducers, it then follows that $y = \frac{x}{c} \in L(k, b)$.

Define the transducer $T = (Q, \Sigma, \delta, q_0, \Delta, p)$ by $Q = \{0, 1, 2, \ldots, c-1\}$, $\Sigma = \{0, 1, 2, \ldots, b-1\}$, $\delta(d, a) = (bd + a) \bmod c$ for $d \in Q$, $a \in \Sigma$, $q_0 = 0$, $\Delta = \{0, 1, \ldots, b-1\}$, $p(d, a) = \lfloor \frac{bd+a}{c} \rfloor$. Then this transducer essentially divides its input by c using the ordinary pencil-and-paper method of long division. ■

Corollary 6.7 *The set $L(k, b)$ forms a vector space over $\mathbb{Q}$.*

6.3 Transcendence

In this section we will now deal with questions concerning the transcendence of automatic real numbers.

6.3.1 Examples

A simple example

So, let's start with a simple example. We will show that the automatic real number $F = \sum_{n\geq 0} B^{-2^n}$ is transcendent.

Theorem 6.8 *The real number $F = \sum_{n\geq 0} B^{-2^n}$ is transcendental for all integers $B \geq 2$.*

Proof. Assume that F is algebraic and satisfies the polynomial equation

$$c_e F^e + \cdots + c_1 F + c_0 = 0 \tag{6.1}$$

where $\forall\ 0 \leq i \leq e : c_i \in \mathbb{Z}$ and $c_e > 0$. Let $H = \max_{0\leq i\leq e} |c_i|$.

Now rewrite (6.1) in the following way:

$$c_e F^e + \cdots = b_s F^s + \ldots \tag{6.2}$$

where the coefficients on both sides are ≥ 0 and $0 \leq s < e$.

Now we define $f(X) = \sum_{n\geq 0} X^{2^n}$. For $r,\ k \geq 0$ let $a(r,k)$ denote the coefficient of X^r in $f(X)^k$. Note that $a(r,k)$ is the number of ways that r can be written as a sum of k powers of 2, where different orderings are counted as distinct.

Lemma 6.9 *Let e, m be fixes integers, and let k be an integer with $1 \leq k \leq e$. Define $N = (2^e - 1) \cdot 2^m$. Then for $N - (2^{m-1} - 1) \leq r \leq N + 2^m - 1$ we have*

$$a(r,k) = \begin{cases} e! & if\ r = N\ and\ k = e \\ 0 & otherwise. \end{cases} \tag{6.3}$$

Proof. We have $(N)_2 = \mathtt{1}^e\mathtt{0}^m$. Then for $N - (2^{m-1} - 1) \leq r < N$ we have

$$(r)_2 = \mathtt{1}^{e-1}\mathtt{01}x,$$

where the string x contains at least one $\mathtt{1}$.

For $N < r \leq N + (2^m - 1)$ we have

$$(r)_2 = \mathtt{1}^e x'$$

where the string x' also has at least one `1`. Hence for all $r \neq N$ in the specified range, r has at least $e+1$ `1`'s in its binary expansion, and hence $a(r,k) = 0$.

If, on the other hand, $r = N$, then $a(r,k) = 0$ for $1 \leq k < e$. If $k = e$, then $a(r,k) = e!$, since then N can be written as the sum of e distinct powers of 2, and all $e!$ permutations of these will work. ∎

Now consider Equation (6.2) as a number in base B, with both sides thought of for the moment without carries. The left-hand side will have, in digit positions specified by the interval $I := [N - (2^{m-1} - 1), N + (2^m - 1)]$, all zeros, except at position N. The right-hand side will have all zeros in these positions.

It now remains to consider the effect of the carries.

Lemma 6.10 *For integers $k, r \geq 1$ we have*

$$a(r,k) \leq (1 + \log_2 r)^k$$

Proof. We can use powers from 2^0 up to $2^{\lfloor \log_2 r \rfloor}$ in the summands to represent r, which gives $1 + \lfloor \log_2 r \rfloor$ different choices; each choice can be used at most k times. ∎

We now show that, for m sufficiently large, the carries do not extend significantly into the positions in I.

The term at digit N, on the left-hand side of (6.2), is $c_e \cdot e!$, which is independent of m. Hence for all large m, its carries are bounded by $1 + \lfloor \log_B (c_e \cdot e!) \rfloor$, which occupies only a small portion of I.

On the other hand, the carries occurring in positions to the right of those in I will never come close to position N. For we have, considering a single term in (6.2),

$$\begin{aligned}
\sum_{r \geq N+2^m} \frac{a(r,k)}{B^r} &\leq \sum_{r \geq N+2^m} \frac{(1+\log_2 r)^k}{B^r} \\
&\leq \sum_{r \geq N+2^m} \frac{r}{B^r} \quad \text{(for } m \text{ sufficiently large)} \\
&\leq \frac{N+2^m}{B^{N+2^m-2}}
\end{aligned}$$

which gives carries to at most $\lfloor \log_B (N + 2^m) \rfloor + 3$ positions to the left of position $N + 2^m$. Now multiply by H and sum $e+1$ terms, to get carries at most to position $\lfloor \log_b (N + 2^m) \rfloor + 4 + \lfloor \log_B H(e+1) \rfloor$.

As $m \to \infty$, these cannot come close to position N.

It follows that the left-hand side of (6.2) looks, in base B, like

...	0's	Expression of $c_e \cdot e!$	0's	Spillover from positions to right of $N + 2^m$

while the right-hand side of (6.2) looks like

...	0's	0's	0's	Spillover from positions to right of $N + 2^m$

(the second block in both cases representing the positions in interval I)

so they cannot be equal. ■

The Kormornik-Loreti constant

We continue with another example, the so-called Kormornik-Loreti constant. In [42], Komornik and Loreti proved the existence and uniqueness of a constant q found in Theorem 6.11. It was then shown in [7] that this number is also transcendental.

So this section is dedicated to the Komornik-Loreti constant. To introduce it we need some basics.

Given a real number $1 < q \leq 2$, by a *q-development* we mean a series

$$\sum_{n=1}^{\infty} \varepsilon_n q^{-n} = 1$$

where $\varepsilon_n = 0$ or 1 for every n. One such development can be obtained easily by the so-called *greedy algorithm*: we choose $\varepsilon_n = 1$ whenever it is possible. More precisely, set $\varepsilon_1 = 1$ and then define $\varepsilon_2, \varepsilon_3 \ldots$ recursively by the formula

$$\varepsilon_n := \begin{cases} 1 & \text{if } \varepsilon_1 q^{-1} + \cdots + \varepsilon_{n-1} q^{1-n} + q^{-n} \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

If $q = 2$, then we obtain $\varepsilon_n = 1$ for all n and obviously this is the only 2-development. On the other hand, it is natural to expect that for each $1 < q < 2$ there are many different q-developments because we may drop infinitely many terms q^{-n} such that the sum of the rest is still greater than 1.

Indeed, for almost all $1 < q < 2$ there exist 2^ω different q-developments. However, rather surprisingly, there exist 2^ω exceptional $q \in (0, 1)$ for which there is only one q-development: see [30] or [31] for proofs and related results. We determine the smallest number q having this curious uniqueness property:

Theorem 6.11 (Komornik-Loreti) *There is a smallest number $1 < q < 2$ for which there is only one q-development. This q is the unique positive solution of the equation*

$$1 = \sum_{i=1}^{\infty} \delta_i q^{-i},$$

where the sequence $(\delta_i)_{i \geq 1}$ of zeroes and ones is defined recursively as follows: First set $\delta_1 = 1$. If $n \geq 0$ and if $\delta_1, \ldots, \delta_{2^n}$ are already defined, then set $\delta_{2^n+k} = 1 - \delta_k$ for $1 \leq k < 2^n$ and $\delta_{2^{n+1}} = 1$.

Comparing the definition of the δ_i with the sequence defined by $t_0 = 0, t_{i+2^m} = 1 - t_i$ if $0 \leq i < 2^m$ — a definition of the famous Thue-Morse sequence equivalent the one we introduced in 4.2 — we can see that actually $\delta_i = t_i$ for $i \geq 1$ so the δ_i are nothing else than the Thue-Morse sequence. We will also show that this definition of the δ_i is actually equivalent to the definition of $\mathbf{t}$ in 4.2 at the beginning of the proof of Lemma 6.15.

So it is easy to compute the sequence $(\delta_i)_{i \geq 1}$: It begins with

$$1101\ 0011\ 0010\ 1101\ 0010\ 1100\ 1101\ 0011\ 0010\ 1100.$$

It follows that

$$1.7872316501 < q < 1.7872316505.$$

Our proof is based on a characterization of the unique q-developments by using the lexicographic order. Given two sequences $(\eta_i)_{i \geq 1}$ and $(\varepsilon_i)_{i \geq 1}$ of zeroes and ones, we write $(\eta_i)_{i \geq 1} < (\varepsilon_i)_{i \geq 1}$ or $\eta_1 \eta_2 \cdots < \varepsilon_1 \varepsilon_2 \ldots$ if there exists an integer $n \geq 1$ such that $\eta_i = \varepsilon_i$ for all $1 \leq i < n$ but $\eta_n < \varepsilon_n$. This is a complete ordering.

For example, it follows from (6.4) that if $1 < q < \tilde{q} < 2$, then the corresponding greedy developments satisfy the inequality $(\varepsilon_n)_{n \geq 1} < (\tilde{\varepsilon}_n)_{n \geq 1}$.

In the sequel we write for brevity $\overline{\varepsilon_i}$ instead of $1 - \varepsilon_i$ and also $\overline{s}$ instead of $\overline{\varepsilon_1 \varepsilon_2} \ldots$ if $s = \varepsilon_1 \varepsilon_2 \ldots$ is a finite or infinite sequence of zeroes and ones.

Let us introduce the following

Definition 6.12 *A sequence $\varepsilon_1, \varepsilon_2, \ldots$ of numbers 0 and 1 is called* admissible *if the following two conditions are fulfilled:*

$$\varepsilon_{n+1}\varepsilon_{n+2} \cdots < \varepsilon_1 \varepsilon_2 \ldots \quad \textit{whenever } \varepsilon_n = 0 \tag{6.5}$$

$$\overline{\varepsilon_{n+1}\varepsilon_{n+2} \cdots} < \varepsilon_1 \varepsilon_2 \ldots \quad \textit{whenever } \varepsilon_n = 1 \tag{6.6}$$

It is easy to construct admissible sequences. For example, if $(\varepsilon_i)_{i\geq 1}$ begins with $N \geq 2$ consecutive 1 digits and if there are neither N consecutive 1 digits nor N consecutive 0 digits later, then the sequence $(\varepsilon_i)_{i\geq 1}$ is admissible.

We recall the following theorem from [31].

Theorem 6.13 *The formula* (6.4) *establishes a strictly increasing bijection between the numbers $q \in (1,2)$ for which there is only one q-development and the admissible sequences $(\varepsilon_n)_{n\geq 1}$.*

In view of this result Theorem 6.11 is equivalent to the

Theorem 6.14 *The sequence $(\delta_n)_{n\geq 1}$ given in Theorem 6.11 is the smallest admissible sequence.*

So now we can continue with the proof of Theorem 6.14 instead of proving Theorem 6.11.

Proof. Consider the sequence $(\delta_n)_{n\geq 1}$ defined in Theorem 6.11. First we prove the following lemma.

Lemma 6.15 *The sequence $(\delta_n)_{n\geq 1}$ is admissible.*

Proof.

1. Let us give an equivalent definition of the sequence $(\delta_n)_{n\geq 1}$. Considering the dyadic expansion of the positive integers i given by

$$i = \varepsilon_l 2^l + \cdots + \varepsilon_0$$

with $\varepsilon_l = 0$ or 1 for every l, we claim that

$$\delta_i := \begin{cases} 1 & \text{if } \varepsilon_l + \cdots + \varepsilon_0 \text{ is odd,} \\ 0 & \text{if } \varepsilon_l + \cdots + \varepsilon_0 \text{ is even.} \end{cases}$$

Denoting temporarily by $(\Delta_i)_{i\geq 1}$ the sequence defined by the right-hand side of this formula, we have $\Delta_{2^n} = 1$ for all $n \geq 0$ because $\varepsilon_l + \cdots + \varepsilon_0 = \varepsilon_n = 1$. Furthermore, changing k to $2^n + k$, where $1 \leq k < 2^n$, the sum $\varepsilon_l + \cdots + \varepsilon_0$ increases by $\varepsilon_n = 1$ so its parity changes. Hence $\Delta_{2^n+k} = \overline{\Delta_k}$. Using the definition of δ_i we conclude that $\Delta_i \equiv \delta_i$. So the δ_i are actually the Thue-Morse sequence.

2. Let $\delta_i = 0$ for some i. Then $\varepsilon_l + \cdots + \varepsilon_0 \geq 2$ and we may write $i = 2^n + 2^m + j$ with $n > m$ and $0 \leq j < 2^m$. Observe that $\delta_j (= \delta_i) = 0$ if $j \neq 0$. We claim that

$$\delta_{i+1}\delta_{i+2}\cdots < \delta_{j+1}\delta_{j+2}\ldots \tag{6.7}$$

We distinguish two cases. If $n \geq m+2$, then using 1 we obtain that

$$\delta_{i+k} = \delta_{j+k} \quad \text{for} \quad 1 \leq k < 2^m - j$$

and

$$\delta_{i+2^m-j} = \delta_{2^n+2^{m+1}} = 0 < 1 = \delta_{2^m} = \delta_{j+2^m-j}$$

so that (6.7) is satisfied. If $n = m+1$, then the same reasoning gives

$$\delta_{i+k} = \delta_{j+k} \quad \text{for} \quad 1 \leq k < 2^{m+1} - j$$

and

$$\delta_{i+2^{m+1}-j} = \delta_{2^{m+2}+2^m} = 0 < 1 = \delta_{2^{m+1}} = \delta_{j+2^{m+1}-j}$$

which imply (6.7) again. Iterating (6.7) we eventually obtain (6.5) (for $(\delta_i)_{i\geq 1}$).

3. Let $\delta_i = 1$ for some $i \geq 1$ and write $i = 2^m + j$, $0 \leq j < 2^m$. Observe that $\delta_j = 0$ if $j \neq 0$. Using 1 we have

$$\overline{\delta_{i+k}} = \delta_{j+k} \text{ for } 1 \leq k < 2^m - j$$

and

$$\overline{\delta_{i+2^m-j}} = \overline{\delta_{2^{m+1}}} = 0 < 1 = \delta_{2^m} = \delta_{j+2^m-j}.$$

Hence

$$\overline{\delta_{i+1}\delta_{i+2}\ldots} < \delta_{j+1}\delta_{j+2}\ldots \tag{6.8}$$

If $j = 0$, then this proves (6.6) (for $(\delta_i)_{i\geq 1}$). If not, then (6.6) follows from (6.8) combined with (6.5).

■

For the proof of the minimality of $(\delta_i)_{i\geq 1}$ we also use the lexicographic order $r < s$ between *finite* sequences r and s having the same length (defined in the same way as for infinite sequences) and we write $r \leq s$ if $r < s$ or $r = s$. We start by proving the following lemma.

Lemma 6.16 *And admissible sequence $(\varepsilon_i)_{i\geq 1}$ cannot begin with a block of the form $s\overline{s}$ ending with 0.*

Proof. Assume on the contrary that there exists such an admissible sequence and write $s = \varepsilon_1 \ldots \varepsilon_n$. Then $\varepsilon_n = 1$ and $\varepsilon_{2n} = 0$. We show that then $(\varepsilon_i)_{i\geq 1}$ is necessarily periodic with period $s\overline{s}$. However, this is impossible: and admissible sequence cannot be periodic with a period $\varepsilon_1 \ldots \varepsilon_m$ ending with $\varepsilon_m = 0$ because this sequence does not satisfy the condition (6.5) for $\varepsilon_m = 0$.

Assume that the $(\varepsilon_i)_{i\geq 1}$ begins with $k \geq 1$ consecutive blocks of $s\overline{s}$ followed by a block r of n digits. Applying the condition (6.6) for $\varepsilon_n = 1$ we find that

$$\begin{aligned} \overline{\overline{s}s\ldots\overline{s}s\overline{s}r\ldots} &< s\overline{s}\ldots s\overline{s}s\overline{s}\ldots \\ \Longrightarrow s\overline{s}\ldots s\overline{s}s\overline{r}\ldots &< s\overline{s}\ldots s\overline{s}s\overline{s}\ldots \\ \Longrightarrow \overline{r} &\leq \overline{s} \\ \Longrightarrow r &\geq s. \end{aligned}$$

Now applying the condition (6.5) for $\varepsilon_{2n} = 0$ we find that

$$\begin{aligned} s\overline{s}\ldots s\overline{s}r\ldots &< s\overline{s}\ldots s\overline{s}s\ldots \\ \Longrightarrow r &\leq s. \end{aligned}$$

Therefore $r = s$.

Next assume that $(\varepsilon_i)_{i\geq 1}$ begins with $k \geq 1$ consecutive blocks of $s\overline{s}$ followed by a block sr of $2n$ digits. Applying the condition (6.6) for $\varepsilon_n = 1$ we find that

$$\begin{aligned} \overline{\overline{s}s\ldots\overline{s}sr\ldots} &< s\overline{s}\ldots s\overline{s}s\ldots \\ \Longrightarrow s\overline{s}\ldots s\overline{sr}\ldots &< s\overline{s}\ldots s\overline{s}s\ldots \\ \Longrightarrow \overline{r} &\leq s \\ \Longrightarrow r &\geq \overline{s}. \end{aligned}$$

Now applying the condition (6.5) for $\varepsilon_{2n} = 0$ we find that

$$\begin{aligned} s\overline{s}\ldots sr\ldots &< s\overline{s}\ldots s\overline{s}\ldots \\ \Longrightarrow r &\leq \overline{s}. \end{aligned}$$

Therefore $r = \overline{s}$.

It follows by induction that $(\varepsilon_i)_{i\geq 1}$ is periodic with the period $s\overline{s}$. ■

Now we can proof the

Lemma 6.17 *Let $(\varepsilon_i)_{i\geq 1}$ be an admissible sequence. Then $(\varepsilon_i)_{i\geq 1} \geq (\delta_i)_{i\geq 1}$.*

Proof. First we show that $\varepsilon_1 = 1 (= \delta_1)$. Indeed, otherwise we could conclude from condition (6.5) that

$$\varepsilon_2\varepsilon_3 \cdots < \varepsilon_1\varepsilon_2 \ldots$$

implying $\varepsilon_2 = 0$. Repeating this argument we could obtain that $\varepsilon_i = 0$ for all i. But this is impossible: this sequence does not satisfy condtion (6.5).

Now let us assume on the contrary that

$$(\varepsilon_i)_{i\geq 1} < (\delta_i)_{i\geq 1}.$$

Then there is an integer $n \geq 0$ such that

$$\varepsilon_1 \dots \varepsilon_{2^n} = \delta_1 \dots \delta_{2^n} \tag{6.9}$$

and

$$\varepsilon_{2^n+1} \dots \varepsilon_{2^{n+1}} < \delta_{2^n+1} \dots \delta_{2^{n+1}}.$$

Since $\delta_{2^n} = 1$ by definition, (6.9) implies in particular that $\varepsilon_{2^n} = 1$.

Using (6.9) and the definition of $(\delta_i)_{i\geq 1}$ we deduce from the latter inequality that

$$\varepsilon_{2^n+1} \dots \varepsilon_{2^{n+1}} \leq \overline{\delta_1} \dots \overline{\delta_{2^n-1}} 0 = \overline{\varepsilon_1 \dots \varepsilon_{2^n}}. \tag{6.10}$$

On the other hand, applying (6.6) it follows that

$$\overline{\varepsilon_{2^n+1} \dots \varepsilon_{2^{n+1}}} \leq \varepsilon_1 \dots \varepsilon_{2^n}. \tag{6.11}$$

Now (6.10) and (6.11) imply the equality

$$\overline{\varepsilon_{2^n+1} \dots \varepsilon_{2^{n+1}}} = \varepsilon_1 \dots \varepsilon_{2^n}$$

which contradicts the preceding lemma. ■

Theorem 6.14 now follows from 6.15 and 6.17. ■

Having proven the existence and uniqueness of the number $q = 1.787231650\dots$ we will now go on to prove that it is not only irrational, but also transcendental, proven as a simple consequence of a result of Mahler.

Theorem 6.18 *The number $q = 1.787231650\dots$ defined as the smallest number in $(1, 2)$ for which there exists a unique expansion of 1 as $1 = \sum_{n=1}^{\infty} \delta_n q^{-n}$, with $\delta_n \in \{0, 1\}$, is transcendental.*

Proof. Mahler proved in [48] that $F(a)$ is transcendental for every algebraic number a, with $0 < |a| < 1$, where F is defined for $|z| < 1$ by

$$F(z) = \prod_{n=0}^{\infty} (1 - z^{2^n}) \ .$$

It is easy to see that

$$F(z) = 1 + \sum_{n=1}^{\infty} (-1)^{\delta_n} z^n$$

where $(\delta_n)_{n\geq 1}$ is the sequence in Theorem 6.11. Since δ_n takes values 0 and 1, we have $(-1)^{\delta_n} = 1 - 2\delta_n$. Hence

$$F(z) = 1 + \frac{z}{1-z} - 2\sum_{n=1}^{\infty} \delta_n z^n \ .$$

Taking $z = q^{-1}$, we obtain

$$F\left(\frac{1}{q}\right) = 1 + \frac{1}{q-1} - 2\sum_{n=1}^{\infty} \delta_n q^{-n} = 1 + \frac{1}{q-1} - 2 = \frac{2-q}{q-1} \ . \tag{6.12}$$

If q were algebraic, this would imply that the left-hand side of (6.12) is transcendental (from Mahler's result), although the right-hand side would be algebraic. ■

The Thue-Morse number

Now we will show that the Thue-Morse number is transcendental. The proof will use the following theorem on analytic functions, which states that zeros of a non-zero function are always isolated.

Theorem 6.19 *Suppose f is an analytic function on some nonempty connected open subset Ω of $\mathbb{C}$. Let*

$$Z(f) = \{z \in \Omega : f(z) = 0\} \ .$$

Then either $Z(f) = \Omega$, or $Z(f)$ has no limit point in Ω.

And now on to the transcendency of the Thue-Morse number:

Theorem 6.20 *Let $(a_n)_{n\geq 0}$ be the Thue-Morse sequence with values 0 and 1. Then the Thue-Morse number $\mathcal{T} = \sum_{n\geq 0} a_n 2^{-n}$ is transcendental.*

Proof. First, we replace the sequence $(a_n)_{n\geq 0}$ by the sequence $(b_n)_{n\geq 0}$, where $b_n = 1-2a_n$. Now the sequence $(b_n)_{n\geq 0}$ takes values ± 1, and it suffices to show that the number $\sum_{n\geq 0} b_n 2^{-n} = 2 - 2\sum_{n\geq 0} a_n 2^{-n}$ is transcendental. For $|z| < 1$ we define

$$B(z) = \sum_{n\geq 0} b_n z^n \ , \tag{6.13}$$

so that the number $\sum_{n\geq 0} b_n 2^{-n}$ is equal to $B\left(\frac{1}{2}\right)$.

The proof now consists of three steps. First, we show that the function B satisfies a functional equation, and can be expressed as an infinite product.

Next, we show that B is a transcendental function over $\mathbb{Q}(z)$. Finally, we show that the number $B\left(\frac{1}{2}\right)$ is transcendental.

So, let's first show that the function B satisfies a functional equation. Since the Thue-Morse sequence $(a_n)_{n\geq 0}$ satisfies, for all $n \geq 0$, the relations $a_{2n} = a_n$ and $a_{2n+1} = 1 - a_n$, we have, for all $n \geq 0$, that $b_{2n} = b_n$ and $b_{2n+1} = -b_n$. Hence

$$B(z) = \sum_{n\geq 0} b_n z^n = \sum_{n\geq 0} b_{2n} z^{2n} + z \sum_{n\geq 0} b_{2n+1} z^{2n} = \sum_{n\geq 0} b_n z^{2n} - z \sum_{n\geq 0} b_n z^{2n} .$$

Hence

$$B(z) = (1-z)B(z^2) . \tag{6.14}$$

Now we define, for $m \geq 1$ and $|z| < 1$,

$$W_m(z) = \prod_{0\leq j\leq m-1} (1 - z^{2^j}) \tag{6.15}$$

Then, iterating Equation (6.14), we have

$$B(z) = W_m(z)B(z^{2^m}) \tag{6.16}$$

for all $m \geq 1$, and for all z with $|z| < 1$. Since $|z| < 1$, $\lim_{m\to\infty} z^{2^m} = 0$, and since B is continuous we have $\lim_{m\to\infty} B(z^{2^m}) = B(0) = 1$. So we get

$$B(z) = \lim_{m\to\infty} W_m(z) = \prod_{j\geq 0}(1 - z^{2^j}) \tag{6.17}$$

for all z with $|z| < 1$. In particular,

$$B(\frac{1}{2}) = \lim_{m\to\infty} W_m(\frac{1}{2}) = \prod_{j\geq 0}(1 - 2^{-2^j}) \tag{6.18}$$

Now we prove that the function $B(z)$ is transcendental over $\mathbb{Q}(z)$. We give a direct elementary proof based upon the functional Equation (6.14) satisfied by B.

Suppose that B is algebraic over $\mathbb{Q}(z)$. Then there exists an integer $d \geq 1$ and $d+1$ polynomials $Q_0, Q_1, \ldots, Q_d$, not all zero, such that

$$\sum_{0\leq k\leq d} Q_k(z)B^k(z) = 0 \tag{6.19}$$

for all z with $|z| < 1$. We can suppose that d is minimal, which implies that $Q_0 \neq 0$. Now, by replacing z by z^2 in Equation (6.19) above and using Equation (6.14) gives

$$\sum_{0\leq k\leq d} Q_k(z^2)B^k(z^2) = \sum_{0\leq k\leq d} Q_k(z^2)(1-z)^{-k}B^k(z) \tag{6.20}$$

and so, by multiplying by $(1-z)^d$, we get

$$\sum_{0\le k\le d} Q_k(z^2)(1-z)^{d-k}B^k(z) = 0 \tag{6.21}$$

for all z with $|z| < 1$. Now, multiplying Equation (6.19) by $Q_d(z^2)$ and Equation (6.21) by $Q_d(z)$, and substracting, we obtain

$$\sum_{0\le k\le d-1} (Q_d(z)Q_k(z^2)(1-z)^{d-k} - Q_d(z^2)Q_k(z))B^k(z) = 0 \tag{6.22}$$

for all z with $|z| < 1$. Since d was chosen to be minimal, this implies that all the coefficients in the sum (6.22) are in fact 0, and in particular, setting $k = 0$, we get

$$Q_d(z)Q_0(z^2)(1-z)^d = Q_d(z^2)Q_0(z) \tag{6.23}$$

for all z with $|z| < 1$. If we define the non-negative integers u and v and the polynomials P_0 and P_d by $Q_0(z) = (1-z)^u P_0(z)$, $Q_d(z) = (1-z)^v P_d(z)$ and $P_0(1) \neq 0$, $P_d(1) \neq 0$, then Equation (6.23) implies that

$$(1-z)^{u+v+d}P_d(z)(1+z)^u P_0(z^2) = (1-z)^{u+v}(1+z)^v P_d(z^2)P_0(z) \tag{6.24}$$

giving a contradiction when we divide this identity by $(1-z)^{u+v}$ and set $z = 1$.

Hence it is proven that B is transcendental over $\mathbb{Q}(z)$ and we are ready to show that the number $B(\frac{1}{2})$ is transcendental.

So let us suppose that $B(\frac{1}{2})$ is algebraic of degree g. Let N be a fixed integer such that $N > 2g$. We claim that it is possible to find $N+1$ polynomials $P_0, P_1, \ldots, P_N$, with integer coefficients and not all 0, such that $\deg P_k \le N$ for all $k \le N$ and

$$\sum_{0\le k\le N} P_k(z)B(z)^k = R(z)$$

for all z with $|z| < 1$, where the formal power series R can be written $R(z) = z^{N^2}\sum_{k\ge 0} r_k z^k$, i.e., the first N^2 coefficients of R are zero. This is indeed possible, since the coefficients of the P_k's are $(N+1)^2$ unknowns, and the condition on R gives rise to N^2 linear homogeneous equations with integer coefficients.

Now we define polynomials $P_{m,k}$ for $m \ge 1$ and $0 \le k \le N$, by

$$P_{m,k}(z) = P_k(z^{2^m})$$

The polynomials $P_{m,k}$ have integer coefficients, and from Equation (6.16) we have, for all $m \ge 1$ ad for all z such that $|z| < 1$,

$$\sum_{0\le k\le N} P_{m,k}(z)W_m(z)^{-k}B(z)^k = \sum_{0\le k\le N} P_{m,k}(z)B(z^{2^m})^k = \sum_{0\le k\le N} P_k(z^{2^m})B(z^{2^m})^k = R(z^{2^m}) \tag{6.25}$$

Now as for a polynomial P, define the norm $||P||$ as follows

$$||a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d|| = \max_j |a_j|$$

Also define the number M by

$$M = \max_{0 \leq k \leq N} \max_{0 \leq x \leq \frac{1}{2}} |P_k(x)|$$

If for $m \geq 1$ we define the polynomials

$$\tilde{P}_m(z) = 2^{N2^{m+1}} \sum_{0 \leq k \leq N} P_{m,k}(\frac{1}{2}) W_m^{N-k}(\frac{1}{2}) z^k \tag{6.26}$$

then these polynomials $\tilde{P}_m$ have their coefficients in $\mathbb{Z}$, since $P_{m,k}$ is a polynomial with integer coefficients of degree $\leq N2^{2^m}$, and W_m is a polynomial with integer coefficients of degree $2^m - 1$. Hence the product $P_{m,k} W_m^{N-k}$ is a polynomial with integer coefficients of degree $< N2^{2^{m+1}}$. Furthermore the polynomials $\tilde{P}_m$ satisfy $||\tilde{P}_m|| \leq M 2^{N2^{m+1}}$ for $m \geq 1$.

Then, defining $\beta = B(\frac{1}{2})$, and putting $z = \frac{1}{2}$ in Equation (6.25) and $z = B(\frac{1}{2}) = \beta$ in Equation (6.26), we have

$$\tilde{P}_m(\beta) = 2^{N2^{m+1}} \sum_{0 \leq k \leq N} P_{m,k}(\frac{1}{2}) W_m^{N-k}(\frac{1}{2}) B(\frac{1}{2})^k = 2^{N2^{m+1}} W_m^N(\frac{1}{2}) R(2^{-2^m}) \tag{6.27}$$

Since the formal power series $R(z)$ begins with a term in z^{N^2}, we can define

$$M' = \max_{0 \leq x \leq \frac{1}{2}} \frac{|R(x)|}{x^{N^2}}$$

Then we have

$$||\tilde{P}_m||^{g-1} |\tilde{P}_m(\beta)| \leq (M 2^{N2^{m+1}})^{g-1} 2^{N2^{m+1}} W_m^N(\frac{1}{2}) |R(2^{-2^m})| \leq M^{g-1} M' 2^{gN2^{m+1}} 2^{-2^m N^2}$$

Hence

$$||\tilde{P}_m||^{g-1} |\tilde{P}_m(\beta)| \leq M^{g-1} M' 2^{(2g-N)N2^m} \tag{6.28}$$

Since we chose $N > 2g$, Equation (6.28) shows that

$$\lim_{m \to \infty} ||\tilde{P}_m||^{g-1} |\tilde{P}_m(\beta)| = 0 \tag{6.29}$$

But $\tilde{P}_m(\beta) \neq 0$ for m large enough. For if $\tilde{P}_m(\beta)$ were equal to zero for infinitely many m, then Equation (6.27) would imply that $R(2^{-2^m})$ was zero for infinitely many values of m. Then, by Theorem 6.19, R would be zero, which, in view of the definition of R, would contradict the transcendence of the function B. This fact, the fact that $\deg \tilde{P}_m = N$, Equation (6.29) and Lemma 2.8 together proof that β cannot be an algebraic number of degree g.

■

Now, after looking at some examples of transcendent automatic real numbers, we will start with a more general approach.

6.3.2 Algebraic Irrational Binary Numbers and Morphisms

Now we will show that an algebraic irrational number cannot be a fixed point of a morphism of non-trivial constant length or one that is primitive. This will be done by proving that a positive real number whose binary expansion is a fixed point of a morphism on the alphabet $\{0, 1\}$ that is either of constant length ≥ 2 or primitive is either rational or transcendental.

Theorem 6.21 *Let x be a positive real number whose binary expansion is a fixed point of a morphism on the alphabet $\Sigma_2 = \{0, 1\}$. If the morphism is either of constant length ≥ 2 or primitive, then the number x is either rational or transcendental.*

Proof. We begin with a nice result due to Séébold.

Theorem 6.22 *The only overlap-free fixed points of non-trivial binary morphisms are the Thue-Morse sequence beginning in 0 and the Thue-Morse sequence beginning in 1, i.e., the two fixed points of the morphism $\mu : \Sigma_2 \to \Sigma_2$ defined as follows (cf. 4.2):*

$$\begin{aligned} \mu(0) &= 01 \\ \mu(1) &= 10 \end{aligned}$$

A proof of this theorem can be found in [15]. Define a morphism to be *overlap-free* if and only if the image of any overlap-free word is itself overlap-free. Thue proved in 1912 that the only binary overlap-free morphisms are μ^k and $E \circ \mu^k$, with $k \geq 0$, where μ is defined as above and E the morphism defined by $E(0) = 1$ and $E(1) = 0$. The proof of Berstel and Séébold in [15] consists of first showing that if a morphism h has the property that the word $h(01101001)$ is overlap-free, then the morphism is either equal to μ^k

or to $E \circ \mu^k$ for some $k \geq 0$. Thue's result can easily be deduced from this theorem. Two other corollaries are, first that a morphism h over a two-letter alphabet $\{\mathtt{a}, \mathtt{b}\}$ is overlap-free, if and only if the word $h(abbabaab)$ is overlap-free, second Séébold's result, i.e. Theorem 6.22. So let's start with the actual proof of Theorem 6.21.

If the binary expansion of x does not contain an overlap, then, applying Theorem 6.22, the sequence of binary digits of x must be either the Thue-Morse sequence beginning in 0 or the Thue-Morse sequence beginning in 1. Replacing, if necessary, x by $1 - x$, we can suppose that its binary digits are given by the Thue-Morse sequence beginning in 0, namely $x = 0.110100110010110\ldots$ We have proven the transcendence of this number in theorem 6.20, see page 50.

If the binary expansion of x contains an overlap, we will prove that x is either rational or transcendental, using the following three theorems.

The first one is a combinatorial translation of Ridout's theorem, given by Ferenczi and Mauduit in [34]. It is Theorem 5.2 that we have proven in the last chapter. We shortly repeat it here to have it at hand.

Theorem 6.23 *Let θ be an irrational number, such that its k-ary expansion begins, for every integer $n \in \mathbb{N}$, in $0.U_n V_n V_n V_n'$, where U_n is a possibly empty word and where V_n is a non-empty word admitting V_n' as a prefix. If $|V_n|$ tends to infinity, $\limsup \left(\frac{|U_n|}{|V_n|}\right) < +\infty$, and $\liminf \left(\frac{|V_n'|}{|V_n|}\right) > 0$, then θ is a transcendental number.*

The second one is a immediate consequence of Theorem 6.23.

Theorem 6.24 *If the k-ary expansion of a real number θ is a non-ultimately periodic fixed point of a primitive morphism σ, and contains a word of the form $V^{2+\beta}$, with $\beta > 0$, then the number θ is transcendental.*

This follows from Theorem 6.23 by setting $V_n = \sigma^n(V)$. To ensure that $|V_n|$ tends to infinity and that the conditions on lim sup and lim inf above are satisfied, it suffices to know that there exists a real number $\lambda > 1$ such that, for every word w,

$$\lim_{n\to\infty} \frac{|\sigma^n(w)|}{\lambda^n} = c(w) > 0,$$

where $c(w)$ may depend on w. This condition is well known to hold for primitive morphisms with λ equal to the Perron-Frobenius eigenvalue of the incidence matrix of σ, the condition $\lambda > 1$ being satisfied for the dominant eigenvalue of a primitive matrix with coefficients in $\mathbb{N}$.

On the other hand, this expression also holds for morphisms of constant length ≥ 2. In this case λ is the length of the morphism, $c(w) = |w|$, and the morphism need not be primitive.

Obviously the existence of a word $V^{2+\beta}$ for some $\beta > 0$ is equivalent to the existence of the overlap VVa where a is the first letter of V, which finally yields the following third theorem.

Theorem 6.25 *Let θ be a positive real number whose k-ary expansion is a non-ultimately periodic fixed point of a morphism on $\{0, 1, \ldots, k-1\}$ which is either primitive or of constant length ≥ 2. If the expansion contains an overlap, then θ is transcendental.*

This last theorem shows that in this case where the binary expansion of x contains an overlap x is either rational or transcendental, and Theorem 6.21 is finally proved. ■

So now we have looked at the connection between algebraic irrational numbers and some morphisms. In the next section we will examine the complexity of those numbers.

6.3.3 Irrational Automatic Numbers

In this section we will prove that the b-adic ($b \geq 2$) expansion of any irrational algebraic number cannot have low complexity. And we will also show that irrational automatic numbers are transcendental.

The transcendence criterion used is Theorem 6.28.

First, we define a property of sequences that will be used in the transcendence criterion.

Definition 6.26 (Condition $(*)$) *A sequence* $\mathbf{a} = (a_n)_{n\geq 1}$ satisfies Condition $(*)$ *if* $\mathbf{a}$ *is not eventually periodic and if there exists a real number $w > 1$ and two sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ such that:*

1. *For any $n \geq 1$, the word $U_n V_n^w$ is a prefix of the word* $\mathbf{a}$*;*
2. *The sequence$(|U_n|/|V_n|)_{n\geq 1}$ is bounded from above;*
3. *The sequence $(|V_n|)_{n\geq 1}$ is strictly increasing.*

A sequence satisfying Condition $(*)$ for some $w > 1$ may be called a *stammering sequence*.

And now follows the definition of two kind of numbers also used in the transcendence criterion.

Definition 6.27 (Pisot and Salem numbers) *A real algebraic integer $\alpha > 1$ is a* Salem number *if all its conjugate roots have absolute value no greater than* 1*, and at least one has absolute value exactly* 1*.*

A Pisot number*, or* Pisot-Vijayaraghavan number*, is an algebraic integer a which is real and exceeds* 1*, but such that its conjugate elements are all less than* 1 *in absolute value.*

Theorem 6.28 (transcendence criterion) *Let β be a Pisot or a Salem number. Let k be an integer. Let $\mathbf{a} = (a_n)_{n\geq 1}$ be a sequence taking its values in $\{-k, \ldots, -1, 0, 1, \ldots, k\}$. If $\mathbf{a}$ satisfies Condition ($*$), then the real number*

$$\alpha := \sum_{i=1}^{+\infty} \frac{a_i}{\beta^i}$$

is transcendental.

Proof.

The proof rests on the p-adic generalization of the Schmidt Subspace Theorem 2.13. Recall the definitions of absolute value $|x|_v$ and height of $\mathbf{x}$ from page 22 and 23.

In the sequel, we assume that the algebraic closure of $\mathbf{K}$ is $\overline{\mathbb{Q}}$. We choose for every place v in $M(\mathbf{K})$ a continuation of $|\cdot|_v$ to $\overline{\mathbb{Q}}$, that we denote also by $|\cdot|_v$.

Now we will continue using the notation and the hypothesis of Theorem 6.28. Assume that the parameter w is fixed, as well as the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$, occurring in the definition of Condition ($*$) (6.26). Set also $r_n = |U_n|$ and $s_n = |V_n|$, for any $n \geq 1$. We want to proof that the real number $\alpha := \sum_{i=1}^{+\infty} \frac{a_i}{\beta^i}$ is transcendental. The key fact is the observation that α admits infinitely many good approximants in the number field $\mathbb{Q}(\beta)$ obtained by truncating its expansion and completing by periodicity. Precisely, for any positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$\begin{aligned} b_k^{(n)} &= a_k \quad \text{for } 1 \leq k \leq r_n + s_n, \\ b_{r_n+k+js_n}^{(n)} &= a_{r_n+k} \quad \text{for } 1 \leq k \leq s_n \text{ and } j \geq 0. \end{aligned}$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is eventually periodic, with preperiod U_n and with period V_n. Set

$$\alpha_n = \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k}$$

and observe that

$$\alpha - \alpha_n = \sum_{k=r_n+s_n+1}^{+\infty} \frac{a_k - b_k^{(n)}}{\beta^k}. \tag{6.30}$$

Lemma 6.29 *For any integer n, there exists an integer polynomial $P_n(X)$ of degree at most $r_n + s_n$ such that*

$$\alpha_n = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}.$$

Furthermore, the coefficients of $P_n(X)$ are bounded in absolute value by 3ℓ.

Proof. By definition of α_n, we get

$$\begin{aligned}
\alpha_n &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=r_n+1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} = \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{+\infty} \frac{b_{k+r_n}^{(n)}}{\beta^k} \\
&= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^k} \left(\sum_{j=0}^{+\infty} \frac{1}{\beta^{js_n}} \right) \\
&= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^{r_n+k-s_n}(\beta^{s_n} - 1)} = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)},
\end{aligned}$$

where we have set

$$P_n(X) = \sum_{k=1}^{r_n} a_k X^{r_n-k}(X^{s_n} - 1) + \sum_{k=1}^{s_n} a_{r_n+k} X^{s_n-k}.$$

The last assertion of the lemma is clear. ∎

So now we turn to the proof of Theorem 6.28. Set $\mathbf{K} = \mathbb{Q}(\beta)$ and denote by d the degree of $\mathbf{K}$. We assume that α is algebraic, and we consider the following linear forms, in three variables and with algebraic coefficients. For the place v corresponding to the embedding of $\mathbf{K}$ defined by $\beta \to \beta$, set $L_{1,v}(x,y,z) = x$, $L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = \alpha x + \alpha y + z$. It follows from Equation (6.30) and Lemma 6.29 that

$$|L_{3,v}(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))|_v = |\alpha(\beta^{r_n}(\beta^{s_n} - 1)) - P_n(\beta)|^{\delta} \ll \frac{1}{\beta^{\delta(w-1)s_n}}, \tag{6.31}$$

for a positive real number δ which only depends on our choice of the continuation of $|\cdot|_v$ to $\overline{\mathbb{Q}}$. Here and throughout this section, the constants implied by the Vinogradov symbol $\ll$ depend (at most) on α, β and ℓ, but are independent of n.

Denote by S'_∞ the set of all other infinite places on $\mathbf{K}$ and by S_0 the set of all finite places on $\mathbf{K}$ dividing β. Observe that S_0 is empty if β is an algebraic unit. For any v in $S_0 \cup S'_\infty$, set $L_{1,v}(x,y,z) = x, L_{2,v}(x,y,z) = y$, and $L_{3,v}(x,y,z) = z$. Denote by S the union of S_0 and the infinite places on $\mathbf{K}$. Clearly, for any v in S, the forms $L_{1,v}, L_{2,v}$ and $L_{3,v}$ are linearly independent.

To simplify the exposition, set

$$\mathbf{x}_n = (\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta)).$$

We want to estimate the product

$$\Pi := \prod_{v\in S}\prod_{i=1}^{3} \frac{|L_{i,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v} = \prod_{v\in S} |\beta^{r_n+s_n}|_v |\beta^{r_n}|_v \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3}$$

from above. By the product formula and the definition of S, we immediately get that

$$\Pi = \prod_{v\in S} \frac{|L_{3,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v^3} \tag{6.32}$$

Since the polynomial $P_n(X)$ has integer coefficients and since β is an algebraic integer, we have $|L_{3,v}(\mathbf{x}_n)|_v = |P_n(\beta)|_v \leq 1$ for any place v in S_0. Furthermore, as the conjugates of β have moduli at most 1, we have for any infinite place v in S'_∞

$$|L_{3,v}(\mathbf{x}_n)|_v \ll (r_n + s_n)^{\frac{d_v}{d}},$$

where $d_v = 1$ or 2 according as v is real infinite or complex infinite, respectively. Together with (6.31) and (6.32), this gives

$$\begin{aligned} \Pi &\ll (r_n+s_n)^{\frac{d-1}{d}} \beta^{-\delta(w-1)s_n} \prod_{v\in S} |\mathbf{x}_n|_v^{-3} \\ &\ll (r_n+s_n)^{\frac{d-1}{d}} \beta^{-\delta(w-1)s_n} H(\mathbf{x}_n)^{-3}, \end{aligned}$$

since $|\mathbf{x}_n|_v = 1$ if v does not belong to S.

Furthermore, it follows from Lemma 6.29 and from the fact that the moduli of the complex conjugates of β are at most 1 that

$$H(\mathbf{x}_n) \ll (r_n + s_n)\beta^{\frac{r_n+s_n}{d}}.$$

Consequently, we infer from Condition $(*)$ that

$$\begin{aligned} \Pi := \prod_{v\in S}\prod_{i=1}^{3} \frac{|L_{i,v}(\mathbf{x}_n)|_v}{|\mathbf{x}_n|_v} &\ll (r_n+s_n)^{\kappa} H(\mathbf{x}_n)^{\frac{-d\delta(w-1)s_n}{r_n+s_n}} H(\mathbf{x}_n)^{-3} \\ &\ll H(\mathbf{x}_n)^{-3-\varepsilon}, \end{aligned}$$

for some positive real numbers κ and ε. If $r_n \leq Cs_n$ for any $n \geq 1$, one can take $\varepsilon = \frac{d\delta(w-1)}{2(C+1)}$. It then follows from Theorem 2.13 that the points $(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))$ lie in a finite number of proper subspaces of $\mathbf{K}^3$.

Thus, there exist a triple $(x_0, y_0, z_0) \in \mathbf{K}^3$ and infinitely many integers n such that

$$x_0 - y_0 \frac{\beta^{r_n}}{\beta^{r_n+s_n}} - z_0 \frac{P_n(\beta)}{\beta^{r_n+s_n}} = 0.$$

Taking the limit along this sub-sequence and noting that $(s_n)_{n\geq 1}$ tends to infinity, we get $x_0 = z_0\alpha$ and α belongs to $\mathbf{K} = \mathbb{Q}(\beta)$. Write then

$$\alpha = \frac{Q(\beta)}{q}, \text{ with } Q(X) \in \mathbb{Z}[X] \text{ and } q \in \mathbb{Z}.$$

For any $n \geq 1$, set $Q_n(\beta) := \beta^{r_n}(\beta^{s_n} - 1)Q(\beta) - qP_n(\beta)$. For any embedding $\sigma : \mathbf{K} \to \mathbb{C}$ with $\sigma(\beta) \neq \beta$, we have

$$|Q_n(\sigma(\beta))| \ll L(Q) + q(r_n + s_n),$$

where $L(Q)$ denotes the sum of the absolute values of the coefficients of the polynomial $Q(X)$. Since $Q_n(\beta)$ is a non-zero algebraic integer, its norm is at least equal to 1, thus

$$|Q_n(\beta)| \gg (L(Q) + q(r_n + s_n))^{-d+1}.$$

Furthermore, we infer from (6.30) and 1 of Condition $(*)$ that

$$|Q_n(\beta)| \ll \beta^{-(w-1)s_n}.$$

Since $(s_n)_{n\geq 1}$ tends to infinity, the two last estimates yield a contradiction if n is large enough. Consequently, the real number α must be transcendental. ■

Now that the transcendence criterion has been proven we will deal with the complexity of the numbers we are interested in. Recall that the complexity of a real number written in some integer base $b \geq 2$ is measured by counting, for any positive integer n, the number $p(n)$ of distinct blocks of n digits (on the alphabet $\{0, 1, \ldots, b-1\}$) occurring in its b-adic expansion, where the function p is called *complexity function*. It follows from results from Ferenczi and Mauduit [34] that the complexity function p of any irrational algebraic number satisfies

$$\liminf_{n\to\infty}(p(n) - n) = +\infty \tag{6.33}$$

The following theorem is an improvement of (6.33).

Theorem 6.30 *Let $b \geq 2$ be an integer. The complexity function of the b-adic expansion of an irrational algebraic number satisfies*

$$\liminf_{n\to\infty} \frac{p(n)}{n} = +\infty$$

Proof. Let α be an irrational number. Without loss of generality, we assume that α is in $(0,1)$ and we denote by $0.u_1u_2\ldots u_n\ldots$ its b-adic expansion. The sequence $(u_n)_{n\geq 1}$ takes its values in $\{0,1,\ldots,b-1\}$ and is not ultimately periodic. We assume that there exists an integer $\kappa \geq 2$ such that the complexity function p of $(u_n)_{n\geq 1}$ satisfies

$$p(n) \leq \kappa n \text{ for infinitely many integers } n \geq 1$$

and we shall derive that Condition $(*)$ is then fulfilled by the sequence $(u_n)_{n\geq 1}$. By Theorem 6.28, this will imply that α is transcendental.

Let n_k be an integer with $p(n_k) \leq \kappa n_k$. Denote by $U(\ell)$ the prefix of $\mathbf{u} := u_1u_2\ldots$ of length ℓ. By the Dirichlet *Schubfachprinzip*, there exists (at least) one word M_k of length n_k which has (at least) two occurrences in $U((\kappa+1)n_k)$. Thus, there are (possibly empty) words A_k, B_k, C_k and D_k, such that

$$U((\kappa+1)n_k) = A_kM_kC_kD_k = A_kB_kM_kD_k \text{ and } |B_k| \geq 1.$$

We observe that $|A_k| \leq \kappa n_k$. We have to distinguish three cases:

1. $|B_k| > |M_k|$;
2. $\lceil \frac{|M_k|}{3} \rceil \leq |B_k| \leq |M_k|$;
3. $1 \leq |B_k| < \lceil \frac{|M_k|}{3} \rceil$.

So now we will look at each case seperately:

1. Under this assumption, there exists a word E_k such that

$$U((\kappa+1)n_k) = A_kM_kE_kM_kD_k$$

Since $|E_k| \leq (\kappa-1)|M_k|$, the word $A_k(M_kE_k)^s$ with $S = 1 + \frac{1}{\kappa}$ is a prefix of $\mathbf{u}$ Furthermore, we observe that

$$|M_kE_k| \geq |M_k| \geq \frac{|A_k|}{\kappa}.$$

2. Under this assumption, there exist two words E_k and F_k such that
$$U((\kappa+1)n_k) = A_k M_k^{\frac{1}{3}} E_k M_k^{\frac{1}{3}} E_k F_k.$$
Thus, the word $A_k(M_k^{\frac{1}{3}} E_k)^2$ is a prefix of $\mathbf{u}$. Furthermore, we observe that
$$|M_k^{\frac{1}{3}} E_k| \geq \frac{|M_k|}{3} \geq \frac{|A_k|}{3\kappa}.$$

3. In the present case, B_k is clearly a prefix of M_k, and we infer from $B_k M_k = M_k C_k$ that B_k^t is a prefix of M_k, where t is the integer part of $\frac{|M_k|}{|B_k|}$. Observe that $t \geq 3$. Setting $s = \lfloor \frac{t}{2} \rfloor$, we see that $A_k(B_k^s)^2$ is a prefix of $\mathbf{u}$ and
$$|B_k^s| \geq \frac{|M_k|}{4} \geq \frac{|A_k|}{4\kappa}.$$

In each of the three cases above, we have proved that there are finite words U_k, V_k and an absolute (i.e., independent of k) real number $w > 1$ such that $U_k V_k^w$ is a prefix of $\mathbf{u}$ and:

- $|U_k| \leq \kappa n_k$;
- $|V_k| \geq \frac{n_k}{4}$;
- $w \geq 1 + \frac{1}{\kappa} > 1$.

Consequently, the sequence $\left(\frac{|U_k|}{|V_k|}\right)_{k\geq 1}$ is bounded from above by 4κ. Furhtermore, it follows from the lower bound $|V_k| \geq \frac{n_k}{4}$ that we may assume that the sequence $(|V_k|)_{k\geq 1}$ is strictly increasing. This implies that the sequence $\mathbf{u}$ satisfies Condition $(*)$. By applying Theorem 6.28 with $\beta = b$ and $\ell = b-1$, we conclude that α is transcendental. ■

It immediately follows from Theorem 6.30 that any irrational real number with sub-linear complexity (i.e., such that $p(n) = O(n)$) is transcendental.

Since the complexity function p of any autmatic sequence satisfies $p(n) = O(n)$ (see [22]), Theorem 6.30 implies straightforwardely the following result.

Theorem 6.31 *Irrational automatic numbers are transcendental.*

Even though Theorem 6.31 is a direct consequence of Theorem 6.30, we will now give a short proof of it, resting on a result of [22].

Proof. Let $\mathbf{a} = (a_n)_{n\geq 1}$ be a non-eventually periodic automatic sequence defined on a finite alphabet Σ. Recall that a morphism is called k-uniform or just uniform if the images of each letter have the same length. Following [22], there exist a letter-to-letter morphism φ from an alphabet $\Delta = \{1, 2, \ldots, r\}$ to the alphabet Σ and an uniform morphism σ from Δ into itself such that $\mathbf{a} = \varphi(\mathbf{u})$, where $\mathbf{u}$ is a fixed point for σ. Observe first that the sequence $\mathbf{a}$ satisfies Condition $(*)$ if this is the case for $\mathbf{u}$. Further, the prefix of length $r+1$ of $\mathbf{u}$ can be written under the form $W_1 u W_2 u W_3$, where u is a letter and W_1, W_2, W_3 are (possibly empty) finite words. We check that the assumptions of Theorem 6.30 are satisfied by $\mathbf{u}$ with the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ defined for any $n \geq 1$ by $U_n = \sigma^n(W_1)$ and $V_n = \sigma^n(uW_2)$. Indeed, since σ is a morphism of constant length, we get, on the one hand, that

$$\frac{|U_n|}{|V_n|} \leq \frac{|W_1|}{1 + |W_2|} \leq r - 1$$

and, on the other hand, that $\sigma^n(u)$ is a prefix of V_n of length at least $\frac{1}{r}$ times the length of V_n. It follows that Condition $(*)$ with $w = 1 + \frac{1}{r}$ is satisfied by the sequence $\mathbf{u}$, and thus by our sequence $\mathbf{a}$. Let $b \geq 2$ be an integer. By applying Theorem 6.28 with $\beta = b$, we conclude that the automatic number $\sum_{k=1}^{+\infty} a_k b^{-k}$ is transcendental. ■

So the irrational automatic reals are transcendental. It is a widely believed conjecture that a wider class of numbers, the irrational morphic numbers, are transcendental. This has not yet been proven in all its generality, but there are results for a wide class of morphisms. For example, as we have show in Section 6.3.2, an algebraic irrational number cannot be a fixed point of a morphism of non-trivial constant length or one that is primitive. The following theorem is a more general result.

Theorem 6.32 *Binary algebraic irrational numbers cannot be generated by a morphism.*

Proof. Let $\mathbf{a}$ be a sequence generated by a morphism φ defined on a finite alphabet Σ. For any positive integer n, there exists a letter a_n satisfying

$$|\varphi^n(a_n)| = \max\{|\varphi^n(j)| : j \in \Sigma\}.$$

This implies the existence of a letter $a \in \Sigma$ and of a strictly increasing sequence of positive integers $(n_k)_{k\geq 1}$ such that for every $k \geq 1$ we have

$$|\varphi^{n_k}(a)| = \max\{|\varphi^{n_k}(j)| : j \in \Sigma\}.$$

Assume from now on that Σ has two elements. Since the sequence $\mathbf{a}$ is not eventually periodic there exist at least two occurrences in $\mathbf{a}$ of the two elements of Σ. In particular, there exist at least two occurrences of the letter

a in the sequence $\mathbf{a}$. We can thus find two (possibly empty) finite words W_1 and W_2 such that $W_1 a W_2 a$ is a prefix of $\mathbf{a}$. We check that the assumptions of Theorem 6.28 are satisfied by $\mathbf{a}$ with the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ defined by $U_n = \varphi^n(W_1)$ and $V_n = \varphi^n(aW_2)$ for any $n \geq 1$. Indeed, by definition of a, we have

$$\frac{|U_n|}{|V_n|} \leq |W_1|$$

and $\varphi^n(a)$ is a prefix of V_n of length at least $\frac{1}{|W_2|+1}$ times the length of V_n. It follows that Condition $(*)$ is satisfied by the sequence $\mathbf{a}$ with $w = 1 + \frac{1}{|W_2|+1}$. We conclude by applying Theorem 6.28. ∎

Chapter 7

Continued Fractions Expansions

In this chapter we will take a look at continued fractions. It is widely believed that the continued fraction expansion of every irrational algebraic number α either is eventually periodic (and we know that this is the case if and only if α is a quadratic irrational), or it contains arbitrarily large partial quotients. But we seem to be very far from a proof (or a disproof, for that matter).

Although we are still lacking such general results, there are some results for more modest approaches. We may expect that if the sequence of partial quotients of an irrational number α is, in some sense, 'simple', then α is either quadratic or transcendental. The term 'simple' can of course lead to many interpretations. It may denote real numbers whose continued fraction expansion has some regularity, or can be produced by a simple algorithm (by a simple Turing machine, for example), or arises from a simple dynamical system etc.

This chapter is organized as follows.

We prove, using a theorem of W.M. Schmidt which is a generalization of the Thue-Siegel-Roth Theorem, that if the sequence of partial quotients of the continued fraction expansion of a positive irrational real number takes only two values, and begins with arbitrary long blocks which are almost squares, then this number is either quadratic or transcendental. This result applies in particular to real numbers whose partial quotients form a Sturmian (or quasi-Sturmian) sequence, or are given by the sequence $(1 + (\lfloor n\alpha \rfloor \bmod 2))_{n\geq 0}$, or are a 'repetitive' fixed point of a binary morphism satisfying some technical conditions.

Furthermore, we establish two transcendence criteria for numbers with a 'stammering' continued fraction expansion.

7.1 Introduction

It is generally believed that the partial quotients in the continued fraction expansion of an algebraic real number of degree > 2 are, in some way, random. In particular it is conjectured that an irrational real number whose continued fraction expansion has bounded partial quotients is either quadratic—in which case the sequence of partial quotients is ultimately periodic—or transcendental.

Although this conjecture seems quite out of reach, some particular cases of it are known to be true, if the sequence of partial quotients has extra properties. In particular M. Queffélec proved the following theorem in [57].

Theorem 7.1 (Queffélec) *Let a and b be two distinct positive integers, and let $(u_n)_{n\geq 0}$ be the Thue-Morse sequence on the alphabet $\{a, b\}$. Then the number $x = [0, u_0, u_1, u_2, \ldots]$ is transcendental.*

One of the key ideas in the proof is the use of a theorem of Schmidt in [65], which roughly says that an algebraic nonquadratic real number cannot be approximated too closely by quadratic numbers. The methods used in the proof where inspired by a paper of Davison [25],where the following result is proved.

Theorem 7.2 (Davison) *Let α be a positive irrational number. Define the sequence of integers $(u_n)_{n\geq 0}$ on $\{1, 2\}$ by $u_n = 1 + (\lfloor n\alpha \rfloor \bmod 2)$. If the continued fraction expansion of α has an infinite number of even-indexed convergents that have even numerators, then the real number $x(\alpha) := [0, u_0, u_1, u_2, \ldots]$ is transcendental.*

In [25] it is asked whether the condition "to have an infinite number of even-indexed convergents that have even numerators" is verified by almost all irrational numbers α. This question was answered positively in [35]. The main theorem of the next section shows in particular that *all* numbers $x(\alpha)$, for α irrational, are transcendental. Before stating this theorem we recall some more definitions and notations.

- Recall that the block-complexity p of a finitely-valued sequence is defined by: for all $n \geq 1$, $p(n)$ is the number of blocks of length n that occur in the sequence.

- A finitely-valued sequence is called *quasi-Sturmian* if it is not ultimately periodic, and if its block-complexity p has the property that there exist two positive integers n_0 and c such that $p(n) \leq n + c$ for $n \geq n_0$.

- Here we will use the characterization of Sturmian sequences mentioned on page 38: A sequence $(t_n)_{n\geq 0}$ is called Sturmian if and only if it satisfies, for some irrational $\alpha \in (0,1)$ and some number $\beta \in \mathbb{R}$, either $t_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$ for all $n \geq 0$ or $t_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$ for all $n \geq 0$. (as usual, $\lfloor y \rfloor$ and $\lceil y \rceil$ denote the lower and upper integer parts of y.)

- If the *partial quotients* of a positive real number x are $0, a_1, a_2, \ldots$, we note as usual $x = [0, a_1, a_2, \ldots]$. We denote the *convergents* by $\frac{p_n}{q_n}$, i.e. $\frac{p_n}{q_n} := [0, a_1, a_2, \ldots, a_n]$.

7.2 Transcendence of Sturmian Continued Fractions

Now follows the main theorem of this section, summarizing our conclusions on Sturmian sequences and continued fractions. The rest of this section is dedicated to the proof of this theorem.

Theorem 7.3 *Let a and b be two distinct positive integers. Let $\boldsymbol{u} = (u_n)_{n\geq 0}$ be a sequence on $\{a, b\}$. Then the number $y := [0, u_0, u_1, u_2, \ldots]$ is transcendental if one of the following conditions holds:*

- *the sequence $\boldsymbol{u}$ is quasi-Sturmian. This is in particular the case if the sequence $\boldsymbol{u}$ is obtained from a Sturmian sequence by replacing 0's by a's and 1's by b's,*

- *the sequence $\boldsymbol{u}$ is obtained from the sequence $(1 + (\lfloor n\alpha \rfloor \bmod 2))_{n\geq 0}$, where α is* any *irrational number, by replacing 1's by a's and 2's by b's,*

- *the sequence $\boldsymbol{u}$ is a non-ultimately periodic fixed point of a (not necessarily primitive) morphism, such that the frequencies of a and b in $\boldsymbol{u}$ exist, and such that $\boldsymbol{u}$ begins in a square UU,*

- *the sequence $\boldsymbol{u}$ is a non-ultimately periodic fixed point of a (not necessarily primitive) morphism of constant length, such that the frequencies of a and b in $\boldsymbol{u}$ exist, and such that $\boldsymbol{u}$ begins with a word UV, where V is a prefix of U, and either* $\inf\{a, b\} \geq 2$ *and* $|V| > 0.64803|U|$, *or* $\inf\{a, b\} = 1$ *and* $|V| \geq 0.7|U|$.

7.2.1 A Combinatorial Consequence of Schmidt's Theorem

In this section we give a combinatorial condition on the continued fraction expansion of an irrational number ξ which, via a nice theorem of Schmidt in [65], implies the transcendence of ξ. Before stating Schmidt's theorem [65], we first recall a definition.

Definition 7.4 *Let ξ be a root of the minimal equation $a\xi^2 + b\xi + c = 0$, with $a, b, c \in \mathbb{Z}$, and $\gcd(|a|, |b|, |c|) = 1$. The* height *of ξ, denoted by $H(\xi)$, is defined by $H(\xi) = \max(|a|, |b|, |c|)$.*

The following theorem of Schmidt is a generalization of the Thue-Siegel-Roth Theorem about the approximation to algebraic numbers by rational numbers. In that theorem the number ξ is approximated by rational numbers and the constant B must be > 2. Here, ξ is approximated by quadratic irrational numbers and $B > 3$.

Theorem 7.5 (W.M. Schmidt) *Let ξ be a real number in $(0, 1)$. We suppose that ξ is neither rational, nor quadratic irrational. If there exists a real number $B > 3$, and infinitely many quadratic irrational numbers ξ_k such that*

$$|\xi - \xi_k| < H(\xi_k)^{-B}$$

then ξ is transcendental.

We will need the following lemma on continued fraction expansions.

Lemma 7.6

1. *Let $\xi \in (0, 1)$ be a number with periodic continued fraction expansion*

   $$\xi = [0, a_1, a_2, \ldots, a_k, a_1, a_2, \ldots, a_k, \ldots]$$

   and convergents $\frac{p_n}{q_n}$. Then the (quadratic irrational) number ξ satisfies $H(\xi) \le q_k$.

2. *If $x, y \in (0, 1]$ have the same first k partial quotients $a_1 a_2 \cdots a_k$ (hence the same first k convergents), then*

   $$|x - y| \le \frac{1}{q_k^2}$$

Proof.

1. We have $\xi = [0, a_1, a_2, \ldots, a_k, \frac{1}{\xi}]$. Hence $\xi = \frac{\frac{1}{\xi}p_k + p_{k-1}}{\frac{1}{\xi}q_k + q_{k-1}}$, which gives
$$q_{k-1}\xi^2 + \xi(q_k - p_{k-1}) - p_k = 0$$
Since $\xi \in (0, 1]$, we have $p_n \leq q_n$ for every $n \geq 1$, hence, since the sequence $(q_n)_{n\geq 0}$ is nondecreasing,
$$H(\xi) \leq \max(q_{k-1}, |q_k - p_{k-1}|, p_k) \leq q_k.$$

2. Since $\frac{p_k}{q_k} = [0, a_1, a_2, \ldots, a_k]$, we have, denoting by $\frac{p'_{k+1}}{q'_{k+1}}$ the next convergent of x and by $\frac{p''_{k+1}}{q''_{k+1}}$ the next convergent of y,
$$\left|x - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q'_{k+1}} \leq \frac{1}{q_k^2} \quad \text{and} \quad \left|y - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k q''_{k+1}} \leq \frac{1}{q_k^2}.$$
Furthermore $x - \frac{p_k}{q_k}$ and $y - \frac{p_k}{q_k}$ have same sign and this sign depends only on k. Hence
$$|x - y| = \left|\left|x - \frac{p_k}{q_k}\right| - \left|y - \frac{p_k}{q_k}\right|\right| \leq \frac{1}{q_k^2}.$$

■

Lemma 7.6 and Schmidt's theorem will permit us to prove the following result.

Theorem 7.7 *Let $\xi \in (0, 1)$ be an irrational number whose continued fraction expansion $\xi = [0, a_1, a_2, \ldots, a_n, \ldots]$ is* non ultimately periodic. *Let $\frac{p_n}{q_n}$ be the convergents of ξ. We suppose that, for an infinite number of k's, the sequence $(a_n)_{n\geq 1}$ begins with the word U_kV_k, where*

- $\lim_{n\to\infty} |U_k| = +\infty$
- *the word V_k is a prefix of U_k.*

Let $\gamma = \liminf_{k\to\infty} \frac{|U_k|+|V_k|}{|U_k|}$, let $M = \limsup_{k\to\infty} q_{|U_k|}^{\frac{1}{|U_k|}}$ and $m = \liminf_{k\to\infty} q_{|U_kV_k|}^{\frac{1}{|U_kV_k|}}$. If the inequality $\gamma > \frac{3\log M}{2\log m}$ holds, then the number ξ is transcendental.

Proof. Let ξ_k have the periodic continued fraction expansion
$$\xi_k = [0, a_1, a_2, \ldots, a_{|U_k|}, a_1, a_2, \ldots, a_{|U_k|}, \ldots].$$

Then, from assertion 1 in Lemma 7.6, we have $H(\xi_k) \leq q_{|U_k|}$. Noting that the continued fraction expansion of ξ_k begins with $[0, a_1, a_2, \ldots, a_{|U_k|}, a_1, a_2, \ldots, a_{|V_k|}, \ldots]$, as does the continued fraction expansion of ξ, we obtain from assertion 2 in Lemma 7.6,

$$|\xi - \xi_k| \leq \frac{1}{q^2_{|U_k V_k|}}.$$

In order to apply Schmidt's theorem, it suffices to prove that there exists a real number $B > 3$, such that $q^B_{|U_k|} < q^2_{|U_k V_k|}$. In fact this will then give

$$|\xi - \xi_k| \leq \frac{1}{q^2_{|U_k V_k|}} < \frac{1}{q^B_{|U_k|}} \leq H(\xi_k)^{-B}.$$

Now, in order to prove the existence of such a B, it suffices to prove that the inequality $3 \log q_{|U_k|} < 2 \log q_{|U_k V_k|}$ holds: we have

$$2 \liminf_{k\to\infty} \frac{\log q_{|U_k V_k|}}{\log q_{|U_k|}} = 2 \liminf_{k\to\infty} \frac{|U_k V_k|}{|U_k|} \frac{\log q_{|U_k V_k|}}{|U_k V_k|} \frac{|U_k|}{\log q_{|U_k|}} \geq \frac{2\gamma \log m}{\log M} > 3.$$

■

7.2.2 A Semigroup of Matrices

In this section we study a semigroup of matrices.

Definition 7.8 *Let a and b be positive integers with $b > a \geq 1$. Let A and B be the 2×2 matrices given by*

$$A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote by $\mathrm{S}(A, B)$ the semigroup under matrix multiplication generated by A and B. We also define

$$\begin{aligned} \mathrm{S}^+(A,B) &:= \{X \in \mathrm{S}(A,B) : \det X = +1\} \\ \mathrm{S}^-(A,B) &:= \{X \in \mathrm{S}(A,B) : \det X = -1\}. \end{aligned}$$

Finally, we denote by Φ_X the map defined by

$$\textit{if} \quad X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{S}(A,B), \qquad \textit{then} \quad \Phi_X(x) = \frac{\delta x + \gamma}{\beta x + \alpha}.$$

Some remarks before we continue.

- We denote the trace of a matrix M by $\mathrm{tr}(M)$, its spectral radius by $\rho(M)$, and its L^2-Norm by $\|M\|$. Note that for any real square matrix we have $\|M\| = \sqrt{\rho(M^t M)}$. Hence, since the matrices A and B are symmetric, we have $\|A\| = \rho(A)$ and $\|B\| = \rho(B)$.

- We have $\mathrm{S}(A,B) = \mathrm{S}^+(A,B) \cup \mathrm{S}^-(A,B)$ (the determinant of every matrix X in $\mathrm{S}(A,B)$ is equal to ± 1).

- $X \in \mathrm{S}(A,B)$ if and only if X can be expressed as the product of elements of a word on the alphabet $\{A,B\}$ with even length ≥ 2.

- If $X \in \mathrm{S}^+(A,B)$, then X has two distinct real eigenvalues (a matrix $X \in \mathrm{S}^+(A,B)$ necessarily satisfies $\mathrm{tr}(X) > 2$).

- For any $X \in \mathrm{S}^+(A,B)$, the equation $\Phi_X(x) = x$ has two distinct solutions that we denote by x_X and y_X, with $x_X < y_X$. It is clear that, if $x_X < x < y_X$, then $x < \Phi_X(x) < y_X$, and if $x > y_X$, then $\Phi_X(x) < x$.

- $\Phi_{MN} = \Phi_M \circ \Phi_N$, for any $M, N \in \mathrm{S}(A,B)$.

We now state the main theorem of this section.

Theorem 7.9 *Let a and b be two integers with $b > a \geq 1$. Let $(u_n)_{n\geq 1}$ be a sequence with values in $\{a,b\}$ such that $\lim_{N\to\infty} \frac{\sharp\{n\in\{1,2,\ldots,N\}, u_n = a\}}{N} = \delta$. Let $(q_n)_{n\geq -1}$ be the sequence defined by $q_{-1} := 0, q_0 := 1$, and for $n \geq 1, q_n := u_n q_{n-1} + q_{n-2}$. Then, if $A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$, we have*

$$\text{if } 0 \leq \delta \leq \frac{1}{2}, \text{ then } \rho(AB)^{\delta}\rho(B)^{1-2\delta} \leq \liminf_{n\to\infty} q_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} q_n^{\frac{1}{n}} \leq \rho(A)^{\delta}\rho(B)^{1-\delta},$$

and

$$\text{if } \frac{1}{2} \leq \delta \leq 1, \text{ then } \rho(AB)^{1-\delta}\rho(B)^{2\delta-1} \leq \liminf_{n\to\infty} q_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} q_n^{\frac{1}{n}} \leq \rho(A)^{\delta}\rho(B)^{1-\delta}.$$

Before proving this theorem (we start with the proof on page 75) we need six lemmata describing properties of the semigroup $\mathrm{S}(A,B)$.

Lemma 7.10 *If $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{S}(A,B)$, then the following inequalities hold*

$$\alpha \geq \beta, \qquad \alpha \geq \delta, \qquad \alpha + \gamma \geq \beta + \delta.$$

Proof. We write X as a word of length say n on the alphabet $\{A, B\}$. The inequalities clearly hold if $n = 1$, since then $X = A$ or $X = B$. The general case is provided by induction on n. ■

Lemma 7.11 *Let* $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{S}(A, B)$ *and* $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \in \mathrm{S}^+(A, B)$. *Suppose that*

1. $\beta x_M + \alpha < 0$,
2. $\Phi_X(x_M) < y_M$.

Then $\mathrm{tr}(MX) > \rho(M)\mathrm{tr}(X)$.

Proof. As noted, the matrix M has real distinct eigenvalues. It is easy to see that

$$x_M + y_M = \frac{m_4 - m_1}{m_2}, \qquad x_M y_M = -\frac{m_3}{m_2}, \text{ and } \qquad \rho(M) = m_1 + m_2 y_M.$$

Now $\mathrm{tr}(X) = \alpha + \delta$, and $\mathrm{tr}(MX) = \alpha m_1 + \beta m_3 + \gamma m_2 + \delta m_4$. Hence

$$\mathrm{tr}(MX) - \rho(M)\mathrm{tr}(X) = m_2(\beta x_M + \alpha)(\Phi_X(x_M) - y_M) > 0.$$

■

Lemma 7.12 *We have the following inequalities*

- $x_{AB} < x_B < x_A < x_{BA} < 0 < y_{BA} < y_B < y_A < y_{AB}$
- $\Phi_B(x_{AB}) = x_{BA}$
- $x_{BA} < \Phi_A(x_{AB}) < 0$
- $x_{BA} > -(a + \frac{1}{b})$.

Proof. We have: $x_A = \frac{-a-\sqrt{a^2+4}}{2}, x_B = \frac{-b-\sqrt{b^2+4}}{2}, x_{AB} = \frac{-b-\sqrt{b^2+\frac{4b}{a}}}{2}, x_{BA} = \frac{-a-\sqrt{a^2+\frac{4a}{b}}}{2}$.

- Since $1 \leq a < b$, we have $x_{AB} < x_B < x_A < x_{BA} < 0$.
- Similarly $0 < y_{BA} < y_B < y_A < y_{AB}$. (to prove for example the inequality $y_{BA} < y_B$, consider the function $x \to \sqrt{x^2 + \frac{4x}{b}} - x$.)

- The equality $\Phi_B(x_{AB}) = x_{BA}$ and the inequality $x_{BA} > -(a + \frac{1}{b})$ are straightforward.
- We have $\Phi_A(x_{AB}) = \frac{1}{a+x_{AB}} > \frac{1}{b+x_{AB}} = x_{BA}$. Furhtermore $a + x_{AB} < a + x_A = \frac{a-\sqrt{a^2+4}}{2} < 0$, hence $\Phi_A(x_{AB}) = \frac{1}{a+x_{AB}} < 0$.

■

Lemma 7.13 *For all* $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{S}^-(A, B)$ *we have*

1. $\beta x_{AB} + \alpha < 0$,
2. $x_{BA} \le \Phi_X(x_{AB}) \le y_{AB}$.

Proof. The proof is by induction on the (odd) length of X as a word on the alphabet $\{A, B\}$.

- If $X = A$ (resp.B), condition 1 of the lemma reads $x_{AB} < -a$ (resp. $x_{AB} < -b$) which is true since

$$x_{AB} = \frac{-b - \sqrt{b^2 + \frac{4b}{a}}}{2} < x_B < -b < -a.$$

Now condition 2 reads $x_{BA} \le \Phi_A(x_{AB}) \le y_{AB}$ (resp. $x_{BA} \le \Phi_B(x_{AB}) \le y_{AB}$). This is a consequence of lemma 7.12 above which gives $x_{BA} < \Phi_A(x_{AB}) < 0 < y_{AB}$ (resp. $\Phi_B(x_{AB}) = x_{BA}$).

Suppose now that $X^* = MX$, where $M \in \{A^2, B^2, AB, BA\}$. Write $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. We have

$$\beta^* x_{AB} + \alpha^* = (\beta x_{AB} + \alpha)(m_1 + m_2 \Phi_X(x_{AB})).$$

The possible values of $(m_1 + m_2\Phi_X(x_{AB}))$ are $(a^2+1) + a\Phi_X(x_{AB})$, $(b^2+1) + b\Phi_X(x_{AB})$, $(ab+1)+a\Phi_X(x_{AB})$, $(ab+1)+b\Phi_X(x_{AB})$. These four numbers are positive: namely, using the induction hypothesis we have $\Phi_X(x_{AB}) \ge x_{BA}$, and the claim is a consequence of the inequality (see lemma 7.12)

$$x_{BA} > -(a + \frac{1}{b}) = \min\{a + \frac{1}{a}, a + \frac{1}{b}, b + \frac{1}{a}, b + \frac{1}{b}\}.$$

Hence $\beta^* x_{AB} + \alpha^*$ and $\beta x_{AB} + \alpha$ have the same sign, which gives $\beta^* x_{AB} + \alpha^* < 0$.

Now, $\Phi^*_X(x_{AB}) = \Phi_{MX}(x_{AB}) = \Phi_M(\Phi_X(x_{AB}))$. It is easily seen that $x_M \leq x_{BA}$. Hence, using Lemma 7.12, the remark that $x_{A^2} = x_A$ and $x_{B^2} = x_B$, and the induction hypothesis, we have

$$x_M \leq x_{BA} \leq \Phi_X(x_{AB}) \leq y_{AB}. \tag{7.1}$$

Since Φ_X is increasing (M belongs to $\mathrm{S}^+(A,B)$), we have

$$\Phi_M(\Phi_X(x_{AB})) \leq \Phi_M(y_{AB}).$$

But $M \in \mathrm{S}^+(A,B)$ and $y_{AB} \geq y_M$ (Lemma 7.12). Hence, $\Phi_M(y_{AB}) \leq y_{AB}$ (see the second last remark on page 71), which implies

$$\Phi_{X^*}(x_{AB}) = \Phi_M(\Phi_X(x_{AB})) \leq y_{AB}$$

Now, using Equation (7.1) again, we have two cases

- if $x_M \leq \Phi_X(x_{AB}) \leq y_M$, then (from the second last remark on page 71) $\Phi_M(\Phi_X(x_{AB})) \geq \Phi_X(x_{AB}) \geq x_{BA}$;
- if $\Phi_X(x_{AB}) > y_M$, then $\Phi_M(\Phi_X(x_{AB})) > \Phi_M(y_M) = y_M > x_{BA}$.

Hence, finally,

$$\Phi_{X^*}(x_{AB}) = \Phi_M(\Phi_X(x_{AB})) \geq x_{BA}$$

and the lemma is proved. ■

Lemmata 7.11 and 7.13 easily give the following lemma.

Lemma 7.14 *For all $X \in \mathrm{S}^-(A,B)$, we have $\mathrm{tr}(ABX) > \rho(AB)\mathrm{tr}(X)$.*

The previous lemma will imply the following result.

Lemma 7.15 *Let n be an odd integer ≥ 3. Let $W_n(A,B)$ be an element in $\mathrm{S}(A,B)$ given by a word on $\{A,B\}$ of length n, and where the matrix A occurs k times. We have*

- *if $0 \leq k < \frac{n}{2}$, then $\mathrm{tr}(W_n(A,B)) \geq \rho(AB)^k \mathrm{tr}(B^{n-2k})$;*
- *if $\frac{n}{2} < k \leq n$, then $\mathrm{tr}(W_n(A,B)) \geq \rho(AB)^{n-k} \mathrm{tr}(A^{2k-n})$.*

Proof. If $k = 0$, then $W_n(A,B) = B^n$, and if $k = n$, then $W_n(A,B) = A^n$. In either cases the result is clear.

If $k \geq 1$, we can write $\mathrm{tr}(W_n(A,B)) = \mathrm{tr}(ABW_{n-2}(A,B))$, where $W_{n-2}(A,B) \in \mathrm{S}(A,B)$ is given by a word of length $n-2$ where the matrix A occurs $k-1$ times (the trace of a product of matrices is invariant when the matrices are cyclically permuted). Hence, using Lemma 7.14, we have $\mathrm{tr}(W_n(A,B)) > \rho(AB)\mathrm{tr}(W_{n-2}(A,B))$. An easy recurrence on n odd ends the proof. ∎

We are now ready for the proof of Theorem 7.9:

Proof. We first write

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} u_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix} = \cdots = W_n(A,B) \begin{pmatrix} q_0 \\ q_{-1} \end{pmatrix} = W_n(A,B) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $W_n(A,B)$ is an element of $\mathrm{S}(A,B)$ of length n.

We now prove the upper bound. Let k be the number of A's in $W_n(A,B)$, we have

$$q_n^{\frac{1}{n}} \leq \left\| \begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} \right\|^{\frac{1}{n}} \leq \left(\|W_n(A,B)\| \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \right)^{\frac{1}{n}} \leq \|A\|^{\frac{k}{n}} \|B\|^{1-\frac{k}{n}}.$$

Hence

$$\limsup_{n\to\infty} q_n^{\frac{1}{n}} \leq \|A\|^{\delta} \|B\|^{1-\delta} = \rho(A)^{\delta} \rho(B)^{1-\delta}.$$

Let us prove the lower bound, i.e., if $0 \leq \delta < \frac{1}{2}$, then $\rho(AB)^{\delta}\rho(B)^{1-2\delta} \leq \liminf_{n\to\infty} q_n^{\frac{1}{n}}$ (the cases $\delta = \frac{1}{2}$ and $\frac{1}{2} < \delta \leq 1$ are similar). Since the sequence $(q_n)_{n\geq 0}$ is nondecreasing we may take the lim inf over odd integers. Let n be a large odd integer. Suppose that $\sharp\{j \in \{0,1,\ldots,n\}, u_j = a\} = k < \frac{n}{2}$. We have

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = W_n(A,B) \begin{pmatrix} 1 \\ 0 \end{pmatrix} := \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

hence, using Lemma 7.10 and 7.15,

$$q_n = \alpha_n \geq \frac{\alpha_n + \delta_n}{2} = \frac{\mathrm{tr}(W_n(A,B))}{2} \leq \frac{\rho(AB)^k \mathrm{tr}(B^{n-2k})}{2}.$$

Let $\lambda > \mu$ be the (real) eigenvalues of the matrix B. We have $\lambda > 1$ and $|\mu| < 1$. Hence

$$q_n^{\frac{1}{n}} \geq 2^{-\frac{1}{n}} \rho(AB)^{\frac{k}{n}} (\lambda^{n-2k} + \mu^{n-2k})^{\frac{1}{n}} = \rho(AB)^{\frac{k}{n}} \lambda^{1-\frac{2k}{n}} \theta_n,$$

where

$$\theta_n := 2^{-\frac{1}{n}} \left(1 + \left(\frac{\mu}{\lambda}\right)^{n-2k} \right)^{\frac{1}{n}}.$$

Hence $2^{-\frac{1}{n}} \leq \theta \leq 1$, and $\theta_n \to 1$ as $n \to \infty$, which gives

$$\liminf_{n \,\mathrm{odd} \to\infty} q_n^{\frac{1}{n}} \geq \rho(AB)^{\delta} \rho(B)^{1-2\delta}.$$

∎

7.2.3 Transcendence of Binary Continued Fractions

In this section we combine Theorem 7.7 and 7.9 to obtain transcendence results for real numbers whose infinite continued fraction expansion contains only two different partial quotients.

A General Theorem

We begin with a general theorem.

Theorem 7.16 *Let a and b be two integers with $b > a \geq 1$. Let A and B be the 2×2 matrices defined by $A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}$. Let x be an irrational real number in $(0,1)$ such that the partial quotients in its continued fraction expansion $x = [0, u_0, u_1, u_2, \ldots]$ only take values a and b. Suppose that the frequencies of a's and b's in the sequence $(u_n)_{n\geq 0}$ exist. Suppose also that, for an infinite number of k's, the sequence $(u_n)_{n\geq 0}$ begins with the word $U_k V_k$, where*

- $\lim_{k\to\infty} |U_k| = +\infty$,
- *the word V_k is a prefix of U_k.*

Let $\gamma = \liminf_{k\to\infty} \frac{|U_k|+|V_k|}{|U_k|}$. If the inequality $\gamma > \frac{3}{2} \frac{\log(\rho(A)\rho(B))}{\log \rho(AB)}$ holds, then the number x is either quadratic (if the sequence $(u_n)_{n\geq 0}$ is ultimately periodic) or transcendental.

This inequality is in particular satisfied, if either $a \geq 2$ and $\gamma > 1.64803$, or $a = 1$ and $\gamma \geq 1.7$.

Proof. Let δ be the frequency of a's in the sequence $(u_n)_{n\geq 0}$. Using Theorems 7.7 and 7.9, we see that if suffices to prove that

$$\text{if} \qquad 0 \leq \delta \leq \frac{1}{2}, \qquad \text{then} \qquad \gamma > \frac{3\log(\rho(A)^\delta \rho(B)^{1-\delta})}{2\log(\rho(AB)^\delta \rho(B)^{1-2\delta})} = \frac{3}{2}H(\delta)$$

and

$$\text{if} \qquad \frac{1}{2} \leq \delta \leq 1, \qquad \text{then} \qquad \gamma > \frac{3\log(\rho(A)^\delta \rho(B)^{1-\delta})}{2\log(\rho(AB)^{1-\delta} \rho(A)^{2\delta-1})} = \frac{3}{2}H(\delta),$$

where $H(\delta)$ is defined for $0 \leq \delta \leq 1$ by

$$\text{if} \qquad 0 \leq \delta \leq \frac{1}{2}, \qquad \text{then} \qquad H(\delta) = \frac{r\delta + s(1-\delta)}{t\delta + s(1-2\delta)}$$

and

$$\text{if} \qquad \frac{1}{2} \leq \delta \leq 1, \qquad \text{then} \qquad H(\delta) = \frac{r\delta + s(1-\delta)}{t(1-\delta) + r(2\delta - 1)}$$

with $r = \log \rho(A), s = \log \rho(B)$, and $t = \log \rho(AB)$.

Using that $t = \log \rho(AB) \leq \log \|AB\| \leq \log(\|A\|\|B\|) = \log(\rho(A)\rho(B)) = r + s$, and that $r < s$ (since $a < b$), we see that $H(\delta)$ is nondecreasing for $0 \leq \delta \leq \frac{1}{2}$ and nonincreasing for $\frac{1}{2} \leq \delta \leq 1$. Hence

$$H(\delta) \leq H\left(\frac{1}{2}\right) = \frac{r+s}{t} = \frac{\log(\rho(A)\rho(B))}{\log \rho(AB)}.$$

Now it is easy to compute the spectral radii of A, B and AB,

$$\rho(A) = \frac{a + \sqrt{a^2+4}}{2}, \qquad \rho(B) = \frac{b + \sqrt{b^2+4}}{2}, \qquad \rho(AB) = \frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2}.$$

We thus have $\gamma > \frac{3}{2}\frac{\log(\rho(A)\rho(B))}{\log \rho(AB)}$ if and only if

$$2\gamma \log\left(\frac{ab+2+\sqrt{a^2b^2+4ab}}{2}\right) - 3\log\left(\frac{a+\sqrt{a^2+4}}{2}\right) - 3\log\left(\frac{b+\sqrt{b^2+4}}{2}\right) > 0.$$

Define the function $\Phi(x)$ for $a \leq x < +\infty$ by

$$\Phi(x) := 2\gamma \log\left(\frac{ax+2+\sqrt{a^2x^2+4ax}}{2}\right) - 3\log\left(\frac{a+\sqrt{a^2+4}}{2}\right) - 3\log\left(\frac{x+\sqrt{x^2+4}}{2}\right).$$

Clearly $\Phi(a) = (2\gamma - 3)\log\left(\frac{a^2+2+a\sqrt{a^2+4}}{2}\right)$, and $\Phi(x) \sim (2\gamma - 3)x$ as $x \to \infty$. Furthermore computing $\Phi'(x)$ shows that it has the same sign as $x^2a^2(4\gamma^2 - 9) - 36ax + 16a^2\gamma^2$ We then distinguish two cases:

- If $a \geq 2$, we suppose that $\gamma > 1.64803$.

  The discriminant of the trinomial $x^2a^2(4\gamma^2-9) - 36ax + 16a^2\gamma^2$ is equal to $4a^2(81 - 4a^2\gamma^2(4\gamma^2 - 9))$. Hence it is negative provided $4a^2\gamma^2(4\gamma^2 - 9) - 81 > 0$. This is true if $16\gamma^2(4\gamma^2 - 9) - 81 > 0$. Now the trinomial $64z^2 - 144z - 81$ is positive if $z > \frac{9}{8}(1 + \sqrt{2})$. This finally gives that the trinomial $x^2a^2(4\gamma^2 - 9) - 36ax + 16a^2\gamma^2$ is positive, and hence that $\Phi(x)$ is increasing for $a \leq x < \infty$ as soon as $\gamma > \frac{3}{4}\sqrt{2 + 2\sqrt{2}} = 1.6480262\ldots$. Since we have $\Phi(a) > 0$ and $\Phi(+\infty) > 0$ for $\gamma > \frac{3}{2}$, we thus have $\Phi(x) > 0$ for $a \leq x < +\infty$.

- If $a = 1$ (hence $b \geq 2$), we suppose that $\gamma \geq 1.7$.

  Since $\Phi(x)$ is of the form $\gamma\Phi_1(x) + \Phi_2(x)$, with $\Phi_1(x) > 0$, it suffices to show that $\Phi(x)$ is positive for $\gamma = 1.7$ and $1 \leq x < +\infty$. We still have $\Phi(1) > 0$ and $\Phi(+\infty) > 0$. The derivative $\Phi'(x)$ now has the same sign as $16x^2 - 225x + 289$, the roots of which are $\frac{225 \pm \sqrt{32129}}{32}$. The smaller root belongs to $(1, 2)$ and the larger to $(12, 13)$. It then suffices to show that $\Phi(12)$ and $\Phi(13)$ are positive, which is straightforward.

■

A remark: The constants given in Theorem 7.16 are not optimal. A more precise computation shows for example that the constant 1.7 in the second case can be improved to 1.69333.

Sturmian Sequences and Binary Sequences of Subaffine Block-Complexity

Recall the characterization of Sturmian sequences on two letters mentioned in the introduction of this chapter on page 67. A sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is called *characteristic Sturmian* (or *homogeneous Sturmian*) if it is Sturmian with $\beta = \alpha$. It can be proved that the sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is characteristic Sturmian if and only if there exists an irrational number $\alpha \in (0, 1)$ such that

$$\mathbf{u} = (\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor)_{n\geq 1} = (\lceil \alpha(n+1) \rceil - \lceil \alpha n \rceil)_{n\geq 1}.$$

As mentioned earlier, Sturmian sequences on two letters are exactly the sequences whose *complexity function* satisfies $p(n) = n + 1$ for all $n \geq 1$ (i.e., the sequences containing exactly $n + 1$ subwords of length n for all $n \geq 1$). Recall that Sturmian sequences are the "simplest" sequences on two letters that are not ultimately periodic.

The following theorem yields some propositions we will need in the proof of Theorem 7.18.

Theorem 7.17 *Let $\alpha \in (0, 1)$ be an irrational real number with continued fraction expansion $\alpha = [0, a_1, a_2, a_3, \ldots]$. Let $(X_n)_{n\geq -1}$ be the sequence of words on $\{0, 1\}$ defined by $X_{-1} = 1, X_0 = 0, X_1 = 0^{a_1-1}1$, and, for $n \geq 2, X_n = X_{n-1}^{a_n} X_{n-2}$. We further define, for $n \geq 2$, the words Z_n and T_n by $X_n = Z_n T_n$ and $|T_n| = 2$. We then have the following properties.*

1. *For $n \geq 1$, we have $T_{2n} = 10$ and $T_{2n+1} = 01$.*

2. *For $n \geq 3$, we have*
$$X_n Z_{n-1} = X_{n-1} Z_n.$$

3. *For $n \geq 3$, the word X_{n+2} begins with $X_n^{1+a_{n+1}} Z_{n-1}$.*

4. *For every $n \geq 1$, the word X_n is the prefix of length q_n (q_n is the denominator of the n-th partial quotient of α) of the characteristic Sturmian sequence $(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)_{n\geq 1}$.*

5. *For every $n \geq 5$ the characteristic Sturmian sequence $(\lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor)_{n\geq 1}$ begins with the square $X_n X_n$ whose length is arbitrarily large as $n \to \infty$.*

Proof.

1. This is clear from the definition of X_n.

2. (cf. [41].) We prove this relation by induction. We first check it for $n = 3$:

$$\begin{aligned} X_3 Z_2 &= ((0^{a_1-1}1)^{a_2}0)^{a_3} 0^{a_1-1} 1 (0^{a_1-1}1)^{a_2-1} 0^{a_1-1} \\ &= ((0^{a_1-1}1)^{a_2}0)^{a_3} (0^{a_1-1}1)^{a_2} 0^{a_1-1} \\ &= \begin{cases} ((0^{a_1-1}1)^{a_2}0)^{a_3+1} 0^{a_1-2} & \text{if} \quad a_1 \neq 1, \\ (1^{a_2}0)^{a_3} 1^{a_2} & \text{if} \quad a_1 = 1, \end{cases} \end{aligned}$$

which is exactly $X_2 Z_3$.

Now we suppose that, for some $n \geq 3$,

$$X_n Z_{n-1} = X_{n-1} Z_n.$$

We note that

$$Z_{n+1} T_{n+1} = X_{n+1} = X_n^{a_{n+1}} X_{n-1} = X_n^{a_{n+1}} Z_{n-1} T_{n-1};$$

hence, since $T_{n+1} = T_{n-1}$,

$$Z_{n+1} = X_n^{a_{n+1}} Z_{n-1}.$$

We thus have

$$\begin{aligned} X_n Z_{n+1} = X_n (X_n^{a_{n+1}} Z_{n-1}) &= X_n^{a_{n+1}} (X_n Z_{n-1}) \\ &= X_n^{a_{n+1}} (X_{n-1} Z_n) = (X_n^{a_{n+1}} X_{n-1}) Z_n, \end{aligned}$$

i.e.,

$$X_n Z_{n+1} = X_{n+1} Z_n.$$

3. We have

$$X_{n+2} = X_{n+1}^{a_{n+2}} X_n = (X_n^{a_{n+1}} X_{n-1})^{a_{n+2}} X_n.$$

Hence X_{n+2} begins with $X_n^{a_{n+1}} X_{n-1} X_n$ (remember a_{n+2} and a_{n+1} are both ≥ 1), i.e., with $X_n^{a_{n+1}} X_{n-1} Z_n$ which, from the previous assertion, is equal to $X_n^{a_{n+1}} X_n Z_{n-1}$ and we are done.

4. This is very classical and goes back to Smith [67].

5. This assertion is an easy consequence of assertions 3 and 4.

■

So now we have proven that a characteristic Sturmian sequence begins in arbitrarily long squares. We prove now that this repetition property is actually true for all Sturmian sequences.

Theorem 7.18 *Any Sturmian sequence* $\mathbf{u}$ *begins in arbitrarily long squares.*

Proof. Let $\mathbf{u}$ be a Sturmian sequence. We can assume that $\mathbf{u}$ takes values 0 and 1. We recall that a subword of w of the sequence $\mathbf{u}$ is called *right special* if both $w0$ and $w1$ are subwords of $\mathbf{u}$. It is called *left special* if both $0w$ and $1w$ are subwords of $\mathbf{u}$. It is called *bispecial* if it is both right and left special. Since the complexity of our sequence $\mathbf{u}$ satisfies $p(n+1) - p(n) = 1$, there is for each n exactly one right special subword of length n and exactly one left special subword of length n. Hence for each left special subword w of the sequence $\mathbf{u}$ there exists a unique letter $a \in \{0, 1\}$ such that wa is also a left special subword.

If $\mathbf{u}$ is a characteristic Sturmian sequence the result holds from Theorem 7.17. If the sequence $\mathbf{u}$ is not characteristic, then it begins in only finitely many left special subwords. (if infinitely many prefixes were left special subwords, then both sequences $0\mathbf{u}$ and $1\mathbf{u}$ would be Sturmian, and it is easy to see that this implies that the sequence $\mathbf{u}$ is a characteristic sequence).

Since $\mathbf{u}$ is not ultimately periodic and since it is uniformly recurrent (each subword occurring in $\mathbf{u}$ occurs infinitely many times and with bounded gaps), it follows that $\mathbf{u}$ begins in an infinite number of right special subwords. Thus $\mathbf{u}$ begins in infinitely many right special subwords (w_n) which are not left special. Let w be a right special subword of $\mathbf{u}$. By a *first return* to w we mean a subword z of $\mathbf{u}$ with exactly two occurrences of w, one at the beginning of the word and one at the end of the word. We will need the following lemma.

Lemma 7.19 *Let* w *be right special and let* z *be a first return to* w*. Then* $|z| \leq 2|w| + 1$*.*

Proof. Since $\mathbf{u}$ contains exactly $|w| + 1$ many subwords of length $|w|$, if $|z| > 2|w| + 1$ there would exist a subword $v \neq u$ of $\mathbf{u}$ of length $|w|$ which occurs at least twice in z. Hence z contains a subword t which is a first return to v. It follows that w does not occur in t and hence no prefix of t of length greater than or equal to the length of v is right special. But then every occurence of v in $\mathbf{u}$ is an occurence of t in $\mathbf{u}$. Since v is a suffix of t it follows that $\mathbf{u}$ is ultimately periodic, a contradiction. Hence $|z| \leq 2|w| + 1$. ∎

It follows from the above lemma that a first return to w is either of the form wv for some subword v of length $|v| \leq |w|$ or of the form waw for some letter $a \in \{0, 1\}$. Each right special subword w has exactly two first returns, and exactly one is right special.

Now suppose w is a prefix of $\mathbf{u}$ which is right special but not left special. Then $\mathbf{u}$ begins in a first return to w. If $\mathbf{u}$ begins in a first return to w of length $\leq 2|w|$ then $\mathbf{u}$ begins in a square. Otherwise $\mathbf{u}$ begins in a first return to w of the form waw for some letter $a \in \{0, 1\}$. Note in this case the other first return to w is of the form wv with $|v| \leq |w|$. In fact, if the other first return to w were of the form wbw with $b \neq a$, then w would be left special contrary to hypothesis.

Case 1 waw is right special. If $\mathbf{u}$ begins in a first return to waw of length $\leq 2|waw|$ then $\mathbf{u}$ begins in a square. Otherwise $\mathbf{u}$ begins in a first return to waw of the form $wawbwaw$. Since w is not left special, $a = b$ and hence $\mathbf{u}$ begins in the square $wawa$.

Case 2 waw is not right special. In this case wv (the other first return to w) is right special, every occurrence of waw is an occurrence of $wawv$ (otherwise $\mathbf{u}$ would be ultimately periodic) and there exists $n \geq 1$ so that $wawv^n$ is right special and $\mathbf{u}$ begins in $wawv^n$.

In this case we claim that each first return to $wawv^n$ is of length $\leq 2|wawv^n|$. In fact suppose to the contrary that a first return to $wawv^n$ is of the form $wawv^n bwawv^n$. Since w is not left special, $a = b$ and so we have that $\mathbf{u}$ contains the subword $wawv^n awawv^n$. Since w is a suffix of wv^n (in fact w is a suffix of wv), the word $wawv^n awawv^n$ contains $wawaw$ as a subword. But since we are in the case where waw is not right special, the word $wawaw$ cannot occur in $\mathbf{u}$ for otherwise $\mathbf{u}$ would be ultimately periodic. Thus $\mathbf{u}$ begins in a first return to $wawv^n$ of length $\leq 2|wawv^n|$ and hence $\mathbf{u}$ begins in a square.

Thus we proved that $\mathbf{u}$ always begins in a square. But since the right special subwords w can be taken to be arbitrarily long, it follows that the squares generated at the beginning of $\mathbf{u}$ are also arbitrarily long. This concludes the proof of Theorem 7.18. ∎

A remark to Theorem 7.18: For squares *anywhere* in Sturmian sequences see [54]. In Chapter 5 we discussed “almost cubes” *anywhere* in Sturmian sequences in relation with a transcendence result.

The next theorem states that a quasi-Sturmian sequence is a simple transformation of a Sturmian sequence. First it is not hard to see that a sequence is quasi-Sturmian if and only if there exist two positive integers n_1 and c' such that $p(n) = n + c'$ for $n \geq n_1$ (see [23], Lemma 1.3). The following result is due to Paul [52] in the uniformly recurrent case, and to Coven [23] in the general case (see also [34] and [38]).

Theorem 7.20 *A sequence $\boldsymbol{u}$ on the alphabet Σ is quasi-Sturmian if and only if it can be written as $\boldsymbol{u} = w\psi(\boldsymbol{v})$, where w is a finite word, $\boldsymbol{v}$ is a Sturmian sequence on the binary alphabet $\{0,1\}$, and ψ is a morphism from $\{0,1\}^*$ to Σ^*, such that $\psi(01) \neq \psi(10)$.*

We can now state a theorem that addresses the first assertion in the main Theorem 7.3 of this section.

Theorem 7.21 *Let a and b be two distinct positive integers. Let $(u_n)_{n\geq 0}$ be a quasi-Sturmian sequence on the alphabet $\{a,b\}$. Then the real number $x := [0, u_0, u_1, \ldots]$ is transcendental. The result holds in particular if the sequence $(u_n)_{n\geq 0}$ is Sturmian.*

Proof. The theorem is a consequence of Theorem 7.16, 7.18 and 7.20. ∎

Sequences $(1 + ([n\alpha] \bmod 2))_{n\geq 1}$

We prove here the second assertion in this section's main Theorem 7.3. We first need the following theorem.

Theorem 7.22 *Let α be an irrational real number. Let $(v_n)_{n\geq 0}$ be the sequence defined by $v_n = (\lfloor n\alpha \rfloor \bmod 2)$. This sequence begins with infinitely many words U_kV_k, such that: $|U_k| \to \infty$, the word V_k is a prefix of the word U_k, and $\liminf_{k\to\infty} \frac{|U_kV_k|}{|U_k|} \geq \frac{2+\sqrt{2}}{2} = 1.70710\ldots$*

Proof. First note that, if $\alpha = 2d + \beta$ with d integer, then $(\lfloor n\alpha \rfloor \bmod 2) = (\lfloor n\beta \rfloor \bmod 2)$. We thus can restrict ourselves to $0 < \alpha < 2$. We distinguish two cases, $0 < \alpha < 1$ and $1 < \alpha < 2$.

- If $0 < \alpha < 1$, define the sequence $(u_n)_{n\geq 1}$ by $u_n := \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$. Then clearly
$$\sum_{k=1}^{n-1} u_k \equiv v_n \bmod 2.$$
If the sequence $\mathbf{u} = (u_n)_{n\geq 1}$ begins with a square XX, and if X contains an even number of 1's, then the sequence $\mathbf{v} = (v_n)_{n\geq 1}$ also begins with a square of same length. Taking the notations of Theorem 7.17 above,

we have that $\mathbf{u}$ begins with $X_n X_n$. The number of 1's in X_n, say r_n, satisfies a recurrence relation coming from the recurrence relation of X_n, namely

$$\forall n \geq 2, r_{n+2} = a_{n+2} r_{n+1} + r_n$$

We thus see that if all r_n are odd for n large enough, then all a_n are even for n large enough. Hence we have two cases:

- either there are infinitely many n's such that r_n is even. For these n's the square $X_n X_n$ in the sequence $\mathbf{u}$ gives rise to a square of same length in the sequence $\mathbf{v}$, and we are done.
- or all r_n are odd from some point on. This implies that all a_n are even for n large enough. Hence either the a_n's are all equal to 2 from some point on, or there are infinitely many n's such that a_n is larger than or equal to 4.
  - Suppose there are infinitely many n's such that a_n is larger or equal to 4. For these n's, the sequence $\mathbf{u}$ begins with $X_n X_n X_n X_n$ (actually $X_n X_n X_n X_n X_n$, from Theorem 7.17). We proceed as above, replacing X_n in the previous case by $(X_n X_n)$ that *does* contain an even number of 1's.
  - Suppose that all a_n are equal to 2 for n large enough. From Theorem 7.17 we know that the sequence $\mathbf{u}$ begins with $X_n^3 Z_{n-1}$ for n large enough. Write $X_n^3 Z_{n-1} = (X_n X_n)(X_n Z_{n-1})$. The word Z_{n-1} is a prefix of X_{n-1} (hence of X_n) of length $q_{n-1} - 2$. Since $(X_n X_n)$ contains an even number of 1's, the word $(X_n X_n)(X_n Z_{n-1})$ gives rise to a word $V_n W_n$ in the sequence $\mathbf{v}$ such that W_n is a prefix of V_n, and

    $$\frac{|V_n W_n|}{|V_n|} = \frac{|X_n X_n X_n Z_{n-1}|}{|X_n X_n|} = \frac{3q_n + q_{n-1} - 1}{2q_n}.$$

    Since all a_n's are equal to 2 from some point on, this quantity converges to $\frac{2+\sqrt{2}}{2} = 1.70710\ldots > 1.7$ (remember that $q_{n+2} = a_{n+2} q_{n+1} + q_n = 2q_{n+1} + q_n$ for n large enough) and we can apply Theorem 7.16.

- If $1 < \alpha < 2$, let $\alpha' = \alpha - 1 \in (0, 1)$. Define the sequence $(u_n)_{n \geq 1}$ by $u_n := \lfloor (n+1)\alpha' \rfloor - \lfloor n\alpha' \rfloor$. Then,

$$\sum_{k=1}^{n-1} u_k + n \equiv \lfloor n\alpha \rfloor \bmod 2.$$

In order to ensure that a square (or an "almost-square") in the sequence **u** gives rise to a square (or an "almost-square") in the sequence **v**, it suffices that the number of 1's and the length of the word that is (almost) repeated have the same parity. Taking the notations of Theorem 7.17 above, we have that **u** begins with $X_n X_n$. Let again r_n be the number of 1's in the word X_n, and let ℓ_n be the length of X_n. These quantities satisfy the (same) recurrence relations

$$\begin{aligned} \forall n \geq 2, r_{n+2} &= a_{n+2} r_{n+1} + r_n, \\ \forall n \geq 2, \ell_{n+2} &= a_{n+2} \ell_{n+1} + \ell_n. \end{aligned}$$

Hence their sum $R = r + \ell$ also satisfies the same recurrence relation

$$R_{n+2} = a_{n+2} R_{n+1} + R_n.$$

Thus we can argue exactly as above, replacing r_n by R_n.

■

We are now ready for the following theorem.

Theorem 7.23 *Let α be a positive irrational number and let $(u_n)_{n\geq 1}$ be the sequence defined by $u_n := 1 + (\lfloor n\alpha \rfloor \bmod 2)$. Then the number*

$$x(\alpha) := [0, u_1, u_2, \ldots]$$

is transcendental. Furthermore the same result holds for the sequence obtained by replacing each 1 by a and each 2 by b in the sequence $(u_n)_{n\geq 1}$, where a and b are any two distinct positive integers.

Proof. The theorem is a consequence of the Theorems 7.16 and 7.22. ■

Some Sequences Generated by Morphisms

Theorem 7.16 can be easily applied to infinite sequences that are fixed points of binary morphisms provided frequencies of letters exist and provided the conditions on U_k, V_k and γ are satisfied. Note that, if a sequence $(u_n)_{n\geq 0}$ on two letters a and b is a fixed point of a morphism σ, and if this sequence begins with a word UV where V is a prefix of U, then the sequence $(u_n)_{n\geq 0}$ begins with $\sigma^k(U)\sigma^k(V)$, where $\sigma^k(V)$ is a prefix of $\sigma^k(U)$. This gives immediately the following result as a corollary of Theorem 7.16.

Theorem 7.24 *Let a and b be two integers such that $b > a \geq 1$. Let $(u_n)_{n\geq 0}$ be a sequence on $\{a, b\}$, such that the frequencies of a and b in this sequence exist. Suppose that the sequence $(u_n)_{n\geq 0}$ is a fixed point of a (not necessarily primitive) morphism σ, and that*

- *either the sequence* $(u_n)_{n\geq 0}$ *begins in a square* UU *where* U *is a word on* $\{a, b\}$,
- *or the morphism* σ *has constant length, and the sequence* $(u_n)_{n\geq 0}$ *begins with a word* UV *such that* V *is a prefix of* U, *with either* $a \geq 2$ *and* $\frac{|UV|}{|U|} > 1.64803$, *or* $a = 1$ *and* $\frac{|UV|}{|U|} \geq 1.7$.

Then the real number $x := [0, u_0, u_1, u_2, \ldots]$ *is quadratic or transcendental.*

Some remarks to this theorem.

- As noted in the remark after the proof of Theorem 7.16 on page 78, the constant 1.7 in Theorem 7.24 above is not optimal. In particular it can be replaced by 1.69333.
- This theorem applies for example to the *doubling period sequence* defined as the fixed point of the morphism $1 \to 12, 2 \to 11$, where 1 and 2 can be replaced by any two distinct integers. Namely, this sequence begins with (1211)(121) and $\frac{7}{4} = 1.75 > 1.7$.
- This theorem applies to the Thue-Morse sequence on any alphabet $\{a, b\}$ **provided** $\inf\{a, b\} \geq 2$. Recall that the Thue-Morse sequence on $\{a, b\}$ is the fixed point beginning in a of the morphism $a \to ab, b \to ba$. Hence it begins with $(abb)(ab)$ and $\frac{5}{3} > 1.64803$. But this theorem does **not** apply to the Thue-Morse sequence on $\{1, b\}$ for any $b > 1$. In this case a more precise computation is needed (see [57], for example).
- In the example above, the existence of frequencies is a consequence of the *primitivity* of the morphisms. But the primitivity is not needed. It is easy to slightly modify the arguments above to prove that the result holds for the *Chacon sequence* that is the infinite fixed point of the *non-primitive* morphism $1 \to 1121, 2 \to 2$, where 1 and 2 can be replaced by any two distinct integers. Namely this sequence begins with arbitrarily large squares (since it begins with 11). Furthermore the frequencies exist, this is for example a consequence of a result proved in [33]: the Chacon sequence can be obtained by first taking the fixed point beginning with C of the primitive morphism on the alphabet $\{A, B, C\}$ defined by $A \to AB, B \to CAB, C \to CCAB$, second by taking the pointwise image of this fixed point by the map $A \to 2, B \to 1, C \to 1$.

As proved in [56], the denominators q_n of the convergents of the real number $x := [0, 2, 1, 1, 2, 1, 2, 2, 1, \ldots]$ (whose partial quotients are given by the Thue–Morse sequence on $\{1, 2\}$) have the property that the limit of $q_n^{\frac{1}{n}}$ for $n \to \infty$ exists. Define the Lévy constant of the real number $x := [a, a_0, a_1, a_2, \ldots]$ to be the limit, if it exists, $\lim_{n\to\infty} \frac{\log(q_n)}{n}$, where q_n is the denominator of the n-th convergent of the continued fraction of x. It is known [45] that almost all positive real numbers have a Lévy constant, equal to $\frac{\pi^2}{12 \log 2}$. M. Queffélec proved in [56] the existence of the Lévy constant for any positive real number whose continued fraction expansion is a fixed point of a primitive morphism. Combined with Theorem 7.7, this statement implies the following result.

Theorem 7.25 *Let $x \in (0, 1)$ be a positive real number with continued fraction expansion $x := [0, u_1, u_2, \ldots]$. If the sequence $(u_n)_{n\geq 1}$ is a fixed point of a primitive morphism of constant length, and if this sequence begins with UV where V is a prefix of U, and $\frac{|UV|}{|U|} > \frac{3}{2}$, then the number x is either quadratic or transcendental.*

So in this section we focused on real numbers whose continued fraction expansion is a Sturmian sequence or a fixed point of a morphism. In the next section we will take a more general approach and also improve the results of this section.

7.3 Complexity of Continued Fraction Expansions

The main result of this section are two combinatorial transcendence criteria, which improve considerably upon those from the last section (cf.[8]), [27] and [14].

7.3.1 Transcendence criteria

In this section we give two transcendence criteria for stammering continued fractions expansions which we will prove in the next section. First, we define a condition similar to the one in the last chapter, see page 56.

Definition 7.26 (Condition $(\star)_w$) *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of elements of an alphabet Σ. Let w be a rational number with $w > 1$. We say that $\mathbf{a}$ satisfies* Condition $(\star)_w$ *if $\mathbf{a}$ is not eventually periodic and if there exists a sequence of finite words $(V_n)_{n\geq 1}$ such that:*

1. *For any $n \geq 1$, the word V_n^w is a prefix of the word $\mathbf{a}$;*
2. *The sequence $(|V_n|)_{n\geq 1}$ is increasing.*

Roughly speaking, $\mathbf{a}$ satisfies Condition $(\star)_w$ if $\mathbf{a}$ is not eventually periodic and if there exist infinitely many 'non-trivial' repetitions (the size of which is measured by w) at the beginning of the infinite word $a_1a_2\cdots a_\ell\cdots$

Our transcendence criterion for 'purely' stammering continued fractions can be stated as follows.

Theorem 7.27 *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $\left(\frac{p_\ell}{q_\ell}\right)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0, a_1, a_2, \ldots, a_\ell, \ldots].$$

If there exists a rational number $w \geq 2$ such that $\mathbf{a}$ satisfies Condition $(\star)_w$, then α is transcendental. If there exists a rational number $w > 1$ such that $\mathbf{a}$ satisfies Condition $(\star)_w$, and if the sequence $\left(q_\ell^{\frac{1}{\ell}}\right)_{\ell\geq 1}$ is bounded (which is in particular the case when the sequence $\mathbf{a}$ is bounded), then α is transcendental.

The interesting thing with the first statement of Theorem 7.27 is that there is no condition on the growth of the sequence $(q_\ell)_{\ell\geq 1}$. The second statement improves upon Theorem 7.7 from the last section, which requires, together with some extra rather constraining hypothesis, the stronger assumption $w > \frac{3}{2}$.

The condition that the sequence $\left(q_\ell^{\frac{1}{\ell}}\right)_{\ell\geq 1}$ has to be bounded is in general very easy to check, and is not very restrictive, since it is satisfied by almost all real numbers (in the sense of Lebesgue measure). Apart from this assumption, Theorem 7.27 does not depend on the size of the partial quotients of α. This is in a striking contrast to our results in the previous section, in which, rougly speaking, the size of w of the repetition is required to be all the more large than the partial quotients are big. Unlike these results, Theorem 7.27 can be easily applied even if α has unbounded partial quotients.

Unfortunately, in the statement of Theorem 7.27, the repetitions must appear at the very beginning of $\mathbf{a}$. Results from [27] allow a shift, whose length, however, must be controlled in terms of the size of the repetitions. Similar results cannot be deduced from our Theorem 7.27. However, many ideas from the proof of Theorem 7.27 can be used to deal also with this situation, under some extra assumptions, and to improve upon the transcendence criterion from [27].

The following Condition $(\star\star)_{w,w'}$ is essentially the same as Condition $(*)$ in the last chapter (see 6.26 on p.56). We explicitely use the upper bound w' in the second statement and add this to the name of the condition so it is easier to distinguish it from Condition $(\star)_w$.

Definition 7.28 (Condition $(\star\star)_{w,w'}$) *Let w and w' be non-negative rational numbers with $w > 1$. We say that $\mathbf{a} = (a_n)_{n\geq 1}$ satisfies* Condition $(\star\star)_{w,w'}$ *if $\mathbf{a}$ is not eventually periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}, (V_n)_{n\geq 1}$ such that:*

1. *For any $n \geq 1$, the word $U_n V_n^w$ is a prefix of the word $\mathbf{a}$;*
2. *The sequence $\left(\frac{|U_n|}{|V_n|}\right)_{n\geq 1}$ is bounded from above by w';*
3. *The sequence $(|V_n|)_{n\geq 1}$ is increasing.*

We are now ready to state our transcendence criterion for (general) stammering continued fractions.

Theorem 7.29 *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $\left(\frac{p_\ell}{q_\ell}\right)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0, a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $\left(q_\ell^{\frac{1}{\ell}}\right)_{\ell\geq 1}$ *is bounded and set* $M = \limsup_{\ell\to+\infty} q_\ell^{\frac{1}{\ell}}$ *and* $m = \liminf_{\ell\to+\infty} q_\ell^{\frac{1}{\ell}}$. *Let* w *and* w' *be non-negative real numbers with*

$$w > (2w'+1)\frac{\log M}{\log m} - w'. \tag{7.2}$$

If $\mathbf{a}$ *satisfies Condition* $(\star\star)_{w,w'}$, *then* α *is transcendental.*

Here is a immediate consequence of Theorem 7.29.

Corollary 7.30 *Let* $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ *be a sequence of positive integers. Let* $\left(\frac{p_\ell}{q_\ell}\right)_{\ell\geq 1}$ *denote the sequence of convergents to the real number*

$$\alpha := [0, a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $\left(q_\ell^{\frac{1}{\ell}}\right)_{\ell\geq 1}$ *converges. Let* w *and* w' *be non-negative real numbers with* $w > w'+1$. *If* $\mathbf{a}$ *satisfies Condition* $(\star\star)_{w,w'}$, *then* α *is transcendental.*

The main tool for the proofs of Theorems 7.27 and 7.29, given in the next section, is again the Schmidt Subspace Theorem [63], [64].

7.3.2 Proofs

In this section we prove our transcendence criteria Theorems 7.27 and 7.29. As said before, the proofs rest on the deep result commonly known as the Schmidt Subspace Theorem, i.e. Theorem 2.12 (cf. the p-adic generalization of Evertse [32], used in the last chapter, Theorem 2.13 on p.23).

We further need an easy auxiliary result.

Lemma 7.31 *Let* $\alpha = [a_0, a_1, a_2, \ldots]$ *and* $\beta = [b_0, b_1, b_2, \ldots]$ *be real numbers. Assume that, for some positive integer* m, *we have* $a_j = b_j$ *for any* $j = 0, \ldots, m$. *Then, we have*

$$|\alpha - \beta| < \frac{1}{q_m^2}$$

where q_m *is the denominator of the convergent* $[a_0, a_1, \ldots, a_m]$.

Proof. Since $[a_0, a_1, \ldots, a_m] =: \frac{p_m}{q_m}$ is a convergent to α and β, the real numbers $\alpha - \frac{p_m}{q_m}$ and $\beta - \frac{p_m}{q_m}$ have the same sign and are both in absolute value less than $\frac{1}{q_m^2}$, hence the lemma. ■

Now, we have all the tools to establish Theorems 7.27 and 7.29.

Proof of Theorem 7.27.

Keep the notation and the hypothesis of this theorem. Assume that the parameter $w > 1$ is fixed, as well as the sequence $(V_n)_{n\geq 1}$ occurring in the definition of Condition $(\star)_w$. Set also $s_n = |V_n|$, for any $n \geq 1$. We want to prove that the real number

$$\alpha := [0, a_1, a_2, \ldots]$$

is transcendental. We assume that α is algebraic of degree at least three and we aim at deriving a contradiction. Throughout this section, the constants implied by $\ll$ depend only on α.

Let $\left(\frac{p_\ell}{q_\ell}\right)_{\ell\geq 1}$ denote the sequence of convergents to α. Observe first that we have

$$q_{\ell+1} \ll q_\ell^{1.1}, \qquad \text{for any } \ell \geq 1, \tag{7.3}$$

by Roth's Theorem [60].

The key factor for the proof of Theorem 7.27 is the observation that α admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for any positive integer n, we define the sequence $\left(b_k^{(n)}\right)_{k\geq 1}$ by

$$b_{h+js_n}^{(n)} = a_h \qquad \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0.$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is purely periodic with period V_n. Set

$$\alpha_n = [0, b_1^{(n)}, b_2^{(n)}, \ldots]$$

and observe that α_n is the root of the quadratic polynomial

$$P_n(X) := q_{s_n-1}X^2 + (q_{s_n} - p_{s_n-1})X - p_{s_n}.$$

By Rolle's theorem and Lemma 7.31, for any positive integer n, we have

$$|P_n(\alpha)| = |P_n(\alpha) - P_n(\alpha_n)| \ll q_{s_n}|\alpha - \alpha_n| \ll \frac{q_{s_n}}{q_{[ws_n]}^2}, \tag{7.4}$$

since the first $[ws_n]$ partial quotients of α and α_n are the same. Furthermore, we clearly have

$$|q_{s_n}\alpha - p_{s_n}| \leq \frac{1}{q_{s_n}} \tag{7.5}$$

and we infer from (7.3) that

$$|q_{s_n-1}\alpha - p_{s_n-1}| \le \frac{1}{q_{s_n-1}} \ll \frac{1}{q_{s_n}^{0.9}}. \tag{7.6}$$

Consider now the four linearly independent linear forms:

$$\begin{aligned} L_1(X_1,X_2,X_3,X_4) &= \alpha^2 X_2 + \alpha(X_1 - X_4) - X_3, \\ L_2(X_1,X_2,X_3,X_4) &= \alpha X_1 - X_3, \\ L_3(X_1,X_2,X_3,X_4) &= X_1, \\ L_4(X_1,X_2,X_3,X_4) &= X_2. \end{aligned}$$

Evaluating them on the quadruple $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$, it follows from (7.4) and (7.5) that

$$\prod_{1\le j\le 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| \ll \frac{q_{s_n}^2}{q_{[ws_n]}^2} \tag{7.7}$$

By assumption, there exists a real number M such that $\log q_\ell \le \ell \log M$ for any positive integer ℓ. Furthermore, an immediate induction shows that $q_{\ell+2} \ge 2q_\ell$ holds for any positive integer ℓ. Consequently, for any integer $n \ge 3$, we get

$$\frac{q_{[ws_n]}}{q_{s_n}} \ge \sqrt{2}^{[(w-1)s_n]-1} \ge q_{s_n}^{(w-1-2/s_n)(\log\sqrt{2})/\log M},$$

and we infer from (7.7) and $w > 1$ that

$$\prod_{1\le j\le 4} |L_j(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})| \ll q_{s_n}^{-\varepsilon}$$

holds for some positive real number ε, when n is large enough.

It then follows from Theorem 2.12 that the points $(q_{s_n}, q_{s_n-1}, p_{s_n}, p_{s_n-1})$ lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1 q_{s_n} + x_2 q_{s_n-1} + x_3 p_{s_n} + x_4 p_{s_n-1} = 0, \tag{7.8}$$

for any n in $\mathcal{N}_1$. Observe that $(x_2, x_4) \ne (0,0)$, since, otherwise, by letting n tend to infinity along $\mathcal{N}_1$ in (7.8), we would get that the real number α is rational. Dividing (7.8) by q_{s_n}, we obtain

$$x_1 + x_2\frac{q_{s_n-1}}{q_{s_n}} + x_3\frac{p_{s_n}}{q_{s_n}} + x_4\frac{p_{s_n-1}}{q_{s_n-1}} \cdot \frac{q_{s_n-1}}{q_{s_n}} = 0. \tag{7.9}$$

By letting n tend to infinity along $\mathcal{N}_1$ in (7.9), we get that

$$\beta := \lim_{\mathcal{N}_1 \ni n \to +\infty} \frac{q_{s_n - 1}}{q_{s_n}} = -\frac{x_1 + x_3\alpha}{x_2 + x_4\alpha}.$$

Furthermore, observe that, for any n in $\mathcal{N}_1$, we have

$$\left|\beta - \frac{q_{s_n-1}}{q_{s_n}}\right| = \left|\frac{x_1 + x_3\alpha}{x_2 + x_4\alpha} - \frac{x_1 + x_3\frac{p_{s_n}}{q_{s_n}}}{x_2 + x_4\frac{p_{s_n-1}}{q_{s_n-1}}}\right| \ll \frac{1}{q_{s_n-1}^2} \ll \frac{1}{q_{s_n}^{1.8}}, \tag{7.10}$$

by (7.5) and (7.6). Since q_{s_n-1} and q_{s_n} are coprime and s_n tends to infinity when n tends to infinity along $\mathcal{N}_1$, this implies that β is irrational.

Consider now the three linearly indpendent linear forms:

$$\begin{aligned} L'_1(Y_1, Y_2, Y_3) &= \beta Y_1 - Y_2, \\ L'_2(Y_1, Y_2, Y_3) &= \alpha Y_1 - Y_3, \\ L'_3(Y_1, Y_2, Y_3) &= Y_1. \end{aligned}$$

Evaluating them on the triple $(q_{s_n}, q_{s_n-1}, p_{s_n})$ with $n \in \mathcal{N}_1$, we infer from (7.5) and (7.10) that

$$\prod_{1 \le j \le 3} |L'_j(q_{s_n}, q_{s_n-1}, p_{s_n})| \ll q_{s_n}^{-0.8}.$$

It then follows from Theorem 2.12 that the points $(q_{s_n}, q_{s_n-1}, p_{s_n})$ with $n \in \mathcal{N}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$y_1 q_{s_n} + y_2 q_{s_n-1} + y_3 p_{s_n} = 0, \tag{7.11}$$

for any n in $\mathcal{N}_2$. Dividing (7.11) by q_{s_n} and letting n tend to infinity along $\mathcal{N}_2$, we get

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{7.12}$$

To obtain another equation linking α and β, we consider the three linearly independent linear forms:

$$\begin{aligned} L''_1(Z_1, Z_2, Z_3) &= \beta Z_1 - Z_2, \\ L''_2(Z_1, Z_2, Z_3) &= \alpha Z_2 - Z_3, \\ L''_3(Z_1, Z_2, Z_3) &= Z_1. \end{aligned}$$

Evaluating them on the triple $(q_{s_n}, q_{s_n-1}, p_{s_n-1})$ with $n \in \mathcal{N}_1$, we infer from (7.6) and (7.10) that

$$\prod_{1 \le j \le 3} |L''_j(q_{s_n}, q_{s_n-1}, p_{s_n-1})| \ll q_{s_n}^{-0.7}.$$

It then follows from Theorem 2.12 that the points $(q_{s_n}, q_{s_n-1}, p_{s_n-1})$ with $n \in \mathcal{N}_1$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_3$ such that

$$z_1 q_{s_n} + z_2 q_{s_n-1} + z_3 p_{s_n-1} = 0, \tag{7.13}$$

for any n in $\mathcal{N}_3$. Dividing (7.13) by q_{s_n-1} and letting n tend to infinity along $\mathcal{N}_3$, we get

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0. \tag{7.14}$$

Observe that $y_2 \neq 0$ since α is irrational. We infer from (7.12) and (7.14) that

$$(z_3\alpha + z_2)(y_3\alpha + y_1) = y_2 z_1.$$

If then $y_3 z_3 = 0$, then (7.12) and (7.14) yield that β is rational, which is a contradiction. Consequently, $y_3 z_3 \neq 0$ and α is a quadratic real number, which is again a contradiction. This completes the proof of the second assertion of the theorem.

It then remains for us to explain why we can drop the assumption on the sequence $\left(q_\ell^{\frac{1}{\ell}}\right)_{\ell\geq 1}$ when w is sufficiently large. We return to the beginning of the proof, and we assume that $w \geq 2$. Using well-known facts from the theory of continuants (see e.g. [53]), Inequality (7.4) becomes

$$|P_n(\alpha)| \ll \frac{q_{s_n}}{q_{2s_n}^2} \ll \frac{q_{s_n}}{q_{s_n}^4} \ll \frac{1}{q_{s_n}^3} \ll \frac{1}{H(P_n)^3},$$

where $H(P_n)$ denotes the height of the polynomial P_n, that is, the maximum of the absolute values of its coefficients. By the main result from [65] (or by using Theorem 2.12 with $m = 3$ and the linear forms $\alpha^2 X_2 + \alpha X_1 + X_0, X_2$ and X_1), this immediately implies that α is transcendental. ∎

Proof of Theorem 7.29. Assume that the parameters w and w' are fixed, as well as the sequences $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ occurring in the definition of Condition $(\star\star)_{w,w'}$. Without any loss of generality, we add in the statement of Condition $(\star\star)_{w,w'}$ the following two assumptions:

4. The sequence $(|U_n|)_{n\geq 1}$ is unbounded;

5. For any $n \geq 1$, the last letter of the word U_n differs from the last letter of the word V_n.

We point out that the conditions 4. and 5. do not at all restrict the generality. Indeed, if 4. is not fulfilled by a sequence **a** satisfying 1.–3. of

Condition $(\star\star)_{w,w'}$, then the desired result follows from Theorem 7.27. To see that 5. does not cause any trouble, we make the following observation. Let a be a letter and U and V be two words such that $\mathbf{a}$ begins with $Ua(Va)^w$. Then, $\mathbf{a}$ also begins with $U(aV)^w$ and we have trivially $\frac{|U|}{|aV|} \leq \frac{|Ua|}{|Va|}$.

Set $r_n = |U_n|$ and $s_n = |V_n|$, for any $n \geq 1$. We want to prove that the real number

$$\alpha := [0, a_1, a_2, \ldots]$$

is transcendental. We assume that α is algebraic of degree at least three and we aim at deriving a contradiction. Let $\left(\frac{p_n}{q_n}\right)_{n\geq 1}$ denote the sequence of convergents to α.

Let n be a positive integer. Since $w > 1$ and $r_n \leq w's_n$, we get

$$\frac{2r_n + s_n}{r_n + ws_n} \leq \frac{2w's_n + s_n}{w's_n + ws_n} = \frac{2w' + 1}{w' + w} < \frac{\log m}{\log M},$$

by (7.2). Consequently, there exist positive real numbers η and η' with $\eta < 1$ such that

$$(1+\eta)(1+\eta')(2r_n + s_n)\log M < (1-\eta')(r_n + ws_n)\log m, \tag{7.15}$$

for any $n \geq 1$. Notice that we have

$$q_{\ell+1} \ll q_\ell^{1+\eta}, \qquad \text{for any } \ \ell \geq 1, \tag{7.16}$$

by Roth's Theorem [60].

As for the proof of Theorem 7.27, we observe that α admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for any positive integer n, we define the sequence $\left(b_k^{(n)}\right)_{k\geq 1}$ by

$$\begin{aligned} b_h^{(n)} &= a_h \qquad \text{for } 1 \leq h \leq r_n + s_n, \\ b_{r_n+h+js_n}^{(n)} &= a_{r_n+h} \qquad \text{for } 1 \leq h \leq s_n \text{ and } j \geq 0. \end{aligned}$$

The sequence $\left(b_k^{(n)}\right)_{k\geq 1}$ is eventually periodic, with preperiod U_n and with period V_n. Set

$$\alpha_n = [0, b_1^{(n)}, b_1^{(n)}, \ldots]$$

and observe that α_n is root of the quadratic polynomial

$$\begin{aligned} P_n(X) \;:=\; & (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ - \; & (q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} + p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1})X \\ + \; & (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

For any positive integer n, we infer from Rolle's theorem and Lemma 7.31 that

$$|P_n(\alpha)| = |P_n(\alpha) - P_n(\alpha_n)| \ll q_{r_n} q_{r_n+s_n} |\alpha - \alpha_n| \ll \frac{q_{r_n} q_{r_n+s_n}}{q^2_{r_n+[ws_n]}}, \tag{7.17}$$

since the first $r_n + [ws_n]$ partial quotients of α and α_n are the same. Furthermore, by (7.16), we have

$$|(q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1})\alpha - (q_{r_n-1} p_{r_n+s_n} - q_{r_n} p_{r_n+s_n-1})| \ll q_{r_n} q^{-1+\eta}_{r_n+s_n} \tag{7.18}$$

and

$$|(q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1})\alpha - (p_{r_n-1} q_{r_n+s_n} - p_{r_n} q_{r_n+s_n-1})| \ll q^{-1+\eta}_{r_n} q_{r_n+s_n}. \tag{7.19}$$

We have as well the obvious upper bound

$$|q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1}| \le q_{r_n} q_{r_n+s_n}. \tag{7.20}$$

Consider the four linearly independent linear forms:

$$\begin{aligned}
L_1(X_1, X_2, X_3, X_4) &= \alpha^2 X_1 - \alpha(X_2 + X_3) + X_4,\\
L_2(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2,\\
L_3(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3,\\
L_4(X_1, X_2, X_3, X_4) &= X_1.
\end{aligned}$$

Evaluating them on the quadruple

$$\begin{aligned}
\underline{z}_n := (&q_{r_n-1} q_{r_n+s_n} - q_{r_n} q_{r_n+s_n-1}, q_{r_n-1} p_{r_n+s_n} - q_{r_n} p_{r_n+s_n-1},\\
&p_{r_n-1} q_{r_n+s_n} - p_{r_n} q_{r_n+s_n-1}, p_{r_n-1} p_{r_n+s_n} - p_{r_n} p_{r_n+s_n-1}),
\end{aligned}$$

it follows from (7.17), (7.18), (7.19) and (7.20) that

$$\prod_{1 \le j \le 4} |L_j(\underline{z}_n)| \ll \frac{q^{2+\eta}_{r_n} q^{2+\eta}_{r_n+s_n}}{q^2_{r_n+[ws_n]}} \ll \frac{\left(\frac{q^{1+\eta}_{r_n} q^{1+\eta}_{r_n+s_n}}{q_{r_n+[ws_n]}}\right)^2}{(q_{r_n} q_{r_n+s_n})^\eta}$$

Assuming η sufficiently large, we have

$$q_{r_n} \le M^{(1+\eta')r_n}, \qquad q_{r_n+s_n} \le M^{(1+\eta')(r_n+s_n)}, \qquad \text{and} \quad q_{r_n+[ws_n]} \ge m^{-(1-\eta')(r_n+ws_n)},$$

with η' as in (7.15). Consequently, we get

$$\frac{q^{1+\eta}_{r_n} q^{1+\eta}_{r_n+s_n}}{q_{r_n+[ws_n]}} \le M^{(1+\eta)(1+\eta')(2r_n+s_n)} m^{-(1-\eta')(r_n+ws_n)} \le 1,$$

by (7.15). Thus, we get the upper bound

$$\prod_{1\le j\le 4} |L_j(\underline{z}_n)| \ll \frac{1}{(q_{r_n} q_{r_n+s_n})^{\eta}}$$

for any positive integer n.

It then follows from Theorem 2.12 that the points $\underline{z}_n$ lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$\begin{aligned} x_1(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}) + x_2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}) \\ + x_3(p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}) + x_4(p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}) = 0 \end{aligned} \tag{7.21}$$

for any n in $\mathcal{N}_1$.

Divide (7.21) by $q_{r_n}q_{r_n+s_n-1}$ and observe that $\frac{p_{r_n}}{q_{r_n}}$ and $\frac{p_{r_n+s_n}}{q_{r_n+s_n}}$ tend to α as n tends to infinity along $\mathcal{N}_1$. Taking the limit, we get that either

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0 \tag{7.22}$$

or

$$\frac{q_{r_n-1}q_{r_n+s_n}}{q_{r_n}q_{r_n+s_n-1}} \quad \text{tends to 1 as } n \text{ tends to infinity along } \mathcal{N}_1 \tag{7.23}$$

must hold. In the former case, since α is irrational and not quadratic, we get that $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, x_2 is non-zero and, for any n in $\mathcal{N}_1$, we have $q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1} = p_{r_n-1}q_{r_n+s_n} - p_{r_n}q_{r_n+s_n-1}$. Thus, the polynomial $P_n(X)$ can simply be expressed as

$$\begin{aligned} P_n(X) \;:=\; & (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1})X^2 \\ & -2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1})X + (p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}). \end{aligned}$$

Consider now the three linearly independent linear forms:

$$\begin{aligned} L'_1(Y_1, Y_2, Y_3) &= \alpha Y_1^2 - 2\alpha Y_2 + Y_3, \\ L'_2(Y_1, Y_2, Y_3) &= \alpha Y_1 - Y_2, \\ L'_3(Y_1, Y_2, Y_3) &= Y_1. \end{aligned}$$

Evaluating them on the triple

$$\begin{aligned} \underline{z}'_n \;:=\; & (q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}, q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}, \\ & p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}), \end{aligned}$$

it follows from (7.17), (7.18) and (7.20) that

$$\prod_{1 \le j \le 3} |L'_j(\underline{z}'_n)| \ll \frac{q_{r_n}^3 q_{r_n+s_n}^{1+\eta}}{q_{r_n+[ws_n]}^2} \ll \frac{q_{r_n}^2 q_{r_n+s_n}^{2+\eta}}{q_{r_n+[ws_n]}^2} \ll \frac{1}{(q_{r_n} q_{r_n+s_n})^\eta},$$

by the above computation.

It then follows from Theorem 2.12 that the points $\underline{z}'_n$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (x'_1, x'_2, x'_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ included in $\mathcal{N}_1$ such that

$$\begin{aligned} x'_1(q_{r_n-1}q_{r_n+s_n} - q_{r_n}q_{r_n+s_n-1}) + x'_2(q_{r_n-1}p_{r_n+s_n} - q_{r_n}p_{r_n+s_n-1}) \\ +x'_3(p_{r_n-1}p_{r_n+s_n} - p_{r_n}p_{r_n+s_n-1}) = 0, \end{aligned} \tag{7.24}$$

for any n in $\mathcal{N}_2$.

Divide (7.24) by $q_{r_n}q_{r_n+s_n-1}$ and observe that $\frac{p_{r_n}}{q_{r_n}}$ and $\frac{p_{r_n+s_n}}{q_{r_n+s_n}}$ tend to α as n tends to infinity along $\mathcal{N}_2$. Taking the limit, we get that either

$$x'_1 + x'_2\alpha + x'_3\alpha^2 = 0 \tag{7.25}$$

or

$$\frac{q_{r_n-1}q_{r_n+s_n}}{q_{r_n}q_{r_n+s_n-1}} \quad \text{tends to 1 as } n \text{ tends to infinity along } \mathcal{N}_2 \tag{7.26}$$

must hold. In the former case, we have a contradiction since α is irrational and not quadratic.

Consequently, to conclude the proof of our theorem, it is enough to derive a contradiction from (7.23) (resp. from (7.26)), assuming that (7.22) (resp. from (7.25)) does not hold. To this end, we observe that (7.21) (resp. from (7.24)) allows us to control the speed of convergence of $Q_n := \frac{q_{r_n-1}q_{r_n+s_n}}{q_{r_n}q_{r_n+s_n-1}}$ to 1 along $\mathcal{N}_1$ (resp. along $\mathcal{N}_2$).

Thus, we assume that the quadruple (x_1, x_2, x_3, x_4) obtained after the first application of Theorem 2.12 satisfies $x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 \neq 0$. Dividing (7.21) by $q_{r_n}q_{r_n+s_n-1}$, we get

$$\begin{aligned} x_1(Q_n - 1) + x_2\left(Q_n\frac{p_{r_n+s_n}}{q_{r_n+s_n}} - \frac{p_{r_n+s_n-1}}{q_{r_n+s_n-1}}\right) + x_3\left(Q_n\frac{p_{r_n-1}}{q_{r_n-1}} - \frac{p_{r_n}}{q_{r_n}}\right) \\ +x_4\left(Q_n\frac{p_{r_n-1}}{q_{r_n-1}}\frac{p_{r_n+s_n}}{q_{r_n+s_n}} - \frac{p_{r_n}}{q_{r_n}}\frac{p_{r_n+s_n-1}}{q_{r_n+s_n-1}}\right) = 0, \end{aligned} \tag{7.27}$$

for any n in $\mathcal{N}_1$. To shorten the notation, for any $\ell \ge 1$, we put $R_\ell := \alpha - \frac{p_\ell}{q_\ell}$. We rewrite (7.27) as

$$\begin{aligned} x_1(Q_n - 1) + x_2\left(Q_n(\alpha - R_{r_n+s_n}) - (\alpha - R_{r_n+s_n-1})\right) + x_3\left(Q_n(\alpha - R_{r_n-1}) - (\alpha - R_{r_n})\right) \\ +x_4\left(Q_n(\alpha - R_{r_n-1})(\alpha - R_{r_n+s_n}) - (\alpha - R_{r_n})(\alpha - R_{r_n+s_n-1})\right) = 0, \end{aligned}$$

This yields

$$\begin{aligned}
& (Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2) \\
= \;& x_2 Q_n R_{r_n+s_n} - x_2 R_{r_n+s_n-1} + x_3 Q_n R_{r_n-1} - x_3 R_{r_n} - x_4 Q_n R_{r_n-1} R_{r_n+s_n} \\
& + x_4 R_{r_n} R_{r_n+s_n-1} + \alpha(x_4 Q_n R_{r_n-1} + x_4 Q_n R_{r_n+s_n} - x_4 R_{r_n} - x_4 R_{r_n+s_n-1}).
\end{aligned} \tag{7.28}$$

Observe that $|R_\ell| \leq \frac{1}{q_\ell^2}$ for any $\ell \geq 1$. Furthermore, for n large enough, we have $\frac{1}{2} \leq Q_n \leq 2$, by our assumption (7.23). Consequently, we derive from (7.28) that

$$|(Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2)| \ll |R_{r_n-1}| \ll \frac{1}{q_{r_n-1}^2}.$$

Since we have assumed that (7.22) does not hold, we get

$$|Q_n - 1| \ll \frac{1}{q_{r_n-1}^2}. \tag{7.29}$$

On the other hand, observe that the rational number Q_n is the quotient of the two continued fractions $[a_{r_n+s_n}, a_{r_n+s_n-1}, \ldots, a_1]$ and $[a_{r_n}, a_{r_n-1}, \ldots, a_1]$. By assumption 4. from Condition $(\star\star)_{w,w'}$, we have $a_{r_n+s_n} \neq a_{r_n}$, thus either $a_{r_n+s_n} - a_{r_n} \geq 1$ or $a_{r_n} - a_{r_n+s_n} \geq 1$ holds. A simple calculation then shows that

$$|Q_n - 1| \gg \frac{1}{a_{r_n}} \min\{\frac{1}{a_{r_n+s_n-1}} + \frac{1}{a_{r_n-2}}, \frac{1}{a_{r_n+s_n-2}} + \frac{1}{a_{r_n-1}}\} \gg \frac{1}{a_{r_n} q_{r_n-1}},$$

since $q_{r_n-1} \geq \max\{a_{r_n-1}, a_{r_n-2}\}$. Combined with (7.29), this gives $a_{r_n} \gg q_{r_n-1}$ and

$$q_{r_n} \geq a_{r_n} q_{r_n-1} \gg q_{r_n-1}^2. \tag{7.30}$$

Since $\eta < 1$ and (7.30) holds for infinitely many n, we get a contradiction with (7.16).

We derive a contradiction from (7.26) in an entirely similar way. This completes the proof of our theorem. ■

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