THE p-ADIC VALUATION OF EULERIAN NUMBERS: TREES AND BERNOULLI NUMBERS

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ABSTRACT. In this work we explore the p-adic valuation of Eulerian numbers. We construct a tree whose nodes contain information about the p-adic valuation of these numbers. Using this tree, and some classical results for Bernoulli numbers, we compute the exact p divisibility for the Eulerian numbers when the first variable lies in a congruence class and p satisfies some regularity properties.

1. INTRODUCTION

Integer sequences arise in many contexts of combinatorics and number theory. Many of these sequences are very old and very well-known, for instance, the classical sequence of the Fibonacci numbers and the Binomial coefficients. However, as it is expected, new integer sequences continue to arise in the literature. Today, there is a big catalog of these sequences and many of them can be found in The On-Line Encyclopedia of Integer Sequences (OEIS) [12].

A natural question, and an active area of research, is to study divisibility properties of this type of sequences. For example, in [9, 13], the p-divisibility of the Fibonacci numbers is considered. In [2], a study of the p-divisibility of k-central binomial coefficients is presented. Finally, in [1, 14], the 2-divisibility of the Stirling numbers of the second kind is considered.

Nowadays, divisibility properties are discussed within the framework of p-adic valuations. Let p be a prime and n a non-zero integer. The p-adic valuation of n, denoted by $\nu_p(n)$, is defined by

$$n = p^{\nu_p(n)} a, \tag{1.1}$$

where $a \in \mathbb{Z}$ and p does not divide a. In other words, $\nu_p(n)$ is the exponent of the highest power of p that divides n. The value $\nu_p(0)$ is defined (naturally) to be ∞. In this article, we study the p-adic valuation of the combinatorial sequence of Eulerian numbers.

The Eulerian number $A(n,k)$ is the number of permutations of the numbers 1 to n in which exactly k elements are greater than the previous element. The literature about these numbers is very rich and one can find many properties about them. For instance, they satisfy the recurrence

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k) \tag{1.2}$$

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with $A(n,0)=1$, their exponential generating function is given by

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k)\frac{x^n}{n!}\frac{z^k}{k!} = \frac{(z-1)e^x}{ze^x - e^{xz}}, \tag{1.3}$$

they are explicitly defined by

$$A(n,k) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j+1)^n, \tag{1.4}$$

and are related to the Bernoulli numbers B_n via the identities

$$\sum_{m=0}^{n}(-1)^m A(n,m) = 2^{n+1}(2^{n+1}-1)\frac{B_{n+1}}{n+1} \tag{1.5}$$

and

$$\sum_{m=0}^{n}(-1)^m A(n,m)\binom{n}{m}^{-1} = (n+1)B_n. \tag{1.6}$$

The Bernoulli numbers B_n are used throughout this article. They are defined via the generating function

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{1.7}$$

The first few Bernoulli numbers are given by

$$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, \cdots \tag{1.8}$$

It is not hard to show that $B_m = 0$ for m an odd number bigger than or equal to 3. The literature of Bernoulli numbers includes plenty of number-theoretical results. Many of these results are motivated by the relation between Bernoulli numbers and Fermat's last theorem. See [8] and [11] for details. Some of these results will be presented in this article as soon as they are needed.

As mentioned before, the main focus of this article is the p-adic valuation of the Eulerian numbers. To be more specific, we study $\nu_p(A(n,k))$ when k is a fixed natural number and n varies among all integers greater than k. For example, when $k=3$ we have the integer sequence

$$1, 26, 302, 2416, 15619, 88234, 455192, 2203488, 10187685, 45533450, \cdots \tag{1.9}$$

This is entry `A000498` in OEIS. In Figures 1 and 2 you can see a graphical representation of the 2-adic and 3-adic valuations of this sequence.

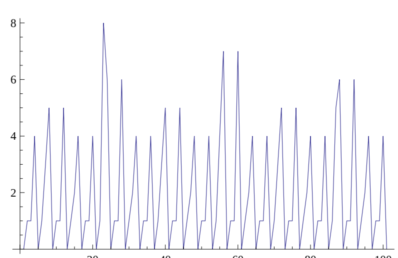

FIGURE 1. 2-adic valuation of $A(n,3)$ for $4 \le n \le 104$.

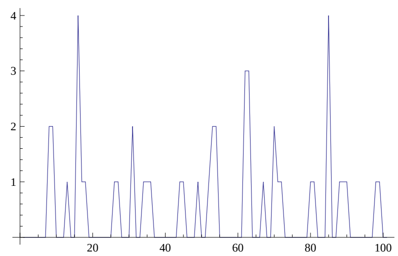

FIGURE 2. 3-adic valuation of $A(n, 3)$ for $4 \leq n \leq 104$.

The problem is to describe patterns (if any) inside this type of graphs. In the next section, we use trees to describe the p-adic valuation of the Eulerian numbers. In sections 3 and 4, we observe some patterns in these trees. Moreover, we show that these patterns are connected to some classical number-theoretical results about Bernoulli numbers. This connection allows us to calculate the exact p-divisibility of the Eulerian numbers when the index n lies in a particular modular class.

2. The p-adic trees

One approach to study the p-divisibility of an integer sequence is to construct a tree (as it was done in [1, 3]) whose nodes give certain information about the p-adic valuation of the sequence. In this section, we present a construction that is completely analogous to the one presented in [1] and [3] for the p-adic valuation of the Stirling numbers of the second kind. Our construction uses the fact that Eulerian numbers are periodic modulo p^m for p prime and m a natural number. More precisely, the sequence $A(n, k) \bmod p^m$, for k fixed, is periodic with period $L_m(p, k) = (p-1)p^{\lfloor \log_p(k) \rfloor + m}$. This is a result of Carlitz and Riordan [4].

We present the construction of the tree for $k = 3$ and $p = 2$ instead of the general case. This was done in order to facilitate the reading of the manuscript and the understanding of the general construction. Since $k = 3$ and $p = 2$, then we are interested in the sequence $\{\nu_2(A(n, 3))\}_{n>3}$. To start the construction, consider a root vertex that represents all natural numbers greater than k, i.e. greater than 3. The next step is to verify if the 2-adic valuation of $A(n, 3)$ is the same constant for every n in this vertex. If it is constant, then we stop the construction, since this implies that we know the 2-adic valuation of the sequence. As expected, the valuation is not the same constant for all n in this vertex. In fact, a simple calculation tells us that the sequence of 2-adic valutions in this vertex is given by

$$0, 1, 1, 4, 0, 1, 3, 5, 0, 1, 1, \cdots .$$

Since the 2-adic valuation of $A(n, 3)$ is not constant for n in this vertex, then we split the vertex. It is here where we use the fact that the sequence $\{A(n, 3)\}_{n>3}$ is periodic mod 2^m with period $L_m(2, 3) = 2^{m+1}$. Split this vertex (the natural numbers bigger than 3) into equivalence classes modulo $L_1(2, 3) = 4$. We call this set of classes the first level (or level one) of the tree. The reason for choosing to split modulo $L_1(2, 3) = 4$ comes from the fact that $n_1 \equiv n_2 \mod 4$ implies $A(n_1, 3) \equiv A(n_2, 3) \mod 2$. In other words, the sequence $\{A(n, 3) \mod 2\}$ is constant on each of these vertices.

In Figure 3 you can see the graphical representation of the tree up to level one. In the picture, the node on the left represents the class of 0 mod 4. Similarly, the second to the left represents the class of 1, the third the class of 2, and the fourth the class of 3. We use the notation $v_{0,1}, v_{1,1}, v_{2,1}$, and $v_{3,1}$ to represent these vertices.

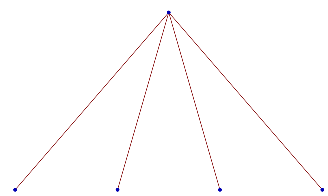

FIGURE 3. The first level for $p = 2$ and $k = 3$.

Definition 2.1. Given any vertex v in the tree, we say that the 2-adic valuation is constant at v if $\nu_2(A(n, 3))$ is the same constant for every $n \in v$.

To continue with the construction, we verify if the 2-adic valuation is constant at each of the vertices in level one. If the valuation is constant at $v_{i,1}$, say the constant is c, then we stop at $v_{i,1}$ (since this implies that we know the valuation for this particular class) and label the edge that connects $v_{i,1}$ with the previous level with the constant c. However, if the valuation is not constant at the vertex $v_{i,1}$, then we split $v_{i,1}$ into the corresponding classes modulo $L_2(2, 3) = 8$ and label these new vertices by $v_{i,2}$ and $v_{i+4,2}$. The set of all the classes coming from the splitting vertices in level one is called the second level. In Figure 4 you can find a graphical representation of the tree up to level 2. The left node at level 2 corresponds to $v_{2,2}$,

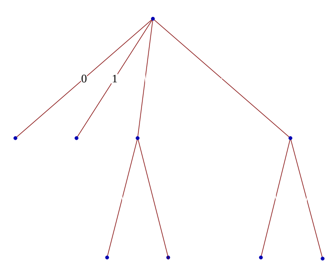

FIGURE 4. The second level for $p = 2$ and $k = 3$.

the second node to $v_{6,2}$, the third to $v_{3,2}$, and the fourth to $v_{7,2}$.

Continue in this manner to construct the third level, the fourth level, and so on. In Figure 5 you can find the tree up to level 8.

The construction of the tree for any k and p is analogous. If the root vertex does not have constant p-adic valuation, then we split it into the classes modulo $L_1(p, k)$ and call them the first level. Then we verify if the p-adic valuation is constant at each of these vertices. If the answer is yes at the vertex $v_{i,1}$, then we stop at it and mark the corresponging edge with the constant valuation. On the other hand, if the

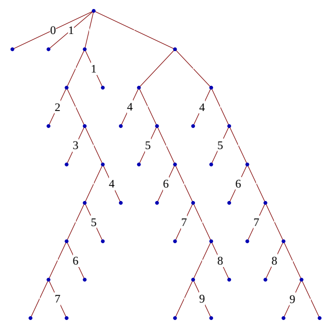

FIGURE 5. The eight level for $p = 2$ and $k = 3$.

valuation at the vertex $v_{i,1}$ is not constant, then we split it into the corresponding classes modulo $L_2(p, k)$ and continue.

In the next two sections we present a study of these trees. We decipher a pattern inside them when $k \equiv -1 \mod p$. This, along with some classical number-theoretical results about Bernoulli numbers, allows us to calculate the exact p-divisibility for $A(n, k)$ when n lies in a particular modular class.

***Remark* 2.2.** The picture in Figure 5 is a guess of the actual tree. The same holds for the picture of every tree on this article. They were generated on *Mathematica* and you can find the implementation here:

`http://emmy.uprrp.edu/lmedina/software/`

Nevertheless, even though these trees are a guess, the information they provide is used to find and prove concrete results.

3. A PATTERN IN THE TREES AND EXPLICIT p-ADIC VALUATIONS

In this section we study the trees constructed in the previous section. It turns out that there is a pattern when $k \equiv -1 \mod p$.

To start the study, consider the trees in Figures 5, 6 and 7. Observe the right-

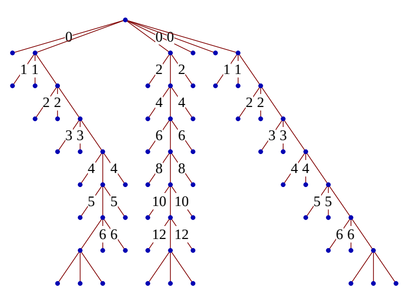

FIGURE 6. p-adic tree when $k = 2$ and $p = 3$.

most branch in all three cases. Note that after the second level, these branches split

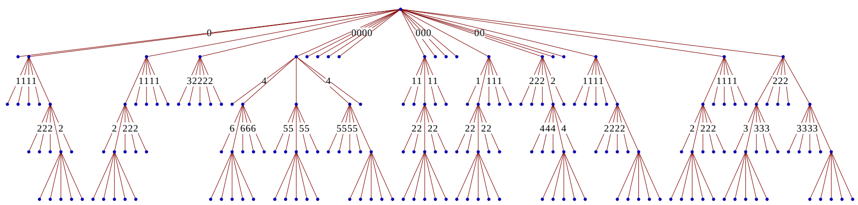

FIGURE 7. p-adic tree when $k = 4$ and $p = 5$.

in the same manner as the p-adic valuation of the positive integers, i.e. the right sibling continues branching while the other ones die. This pattern in the rightmost branch seems to hold when $k \equiv -1 \mod p$. In other words, computer experiments suggest the following observation:

Observation: The p-adic valuation of the sequence $A((p-1)p^{a+1}m - 1, k)$ for $k \equiv -1 \mod p$, $a \geq \lfloor \log_p(k+1) \rfloor$, and m any positive integer is given by

$$\nu_p(A((p-1)p^{a+1}m - 1, k)) = \begin{cases} \nu_p(m) + \nu_p(k+1) + a, & p = 2, 3 \\ \nu_p(m) + \nu_p(k+1) + a + 1, & p > 3. \end{cases} \tag{3.1}$$

We partially prove the observation when p is not a Wolstenholme prime. Remarkably, our proof is elementary.

Definition 3.1. A prime p is a Wolstenholme prime if p divides the numerator of the Bernoulli number B_{p-3}.

***Remark* 3.2.** The only known Wolstenholme primes are 16843 and 2124679. There are no other Wolstenholme primes less than 10^9, see [10].

Our proof depends on Lemma 3.4 (see below). However, the proof of this lemma uses the following classical result.

Theorem 3.3 (von Staudt-Clausen)**.** *Let B_{2n} be the $2n$-th Bernoulli number. Then*

$$B_{2n} + \sum_{(p-1)\,|\,2n} \frac{1}{p} \in \mathbb{Z}. \tag{3.2}$$

Here, p runs over all primes with the property $(p-1)\,|\,2n$. In particular, the denominator of B_{2n} is square-free and divisible by 6.

Lemma 3.4. *Suppose that p is prime, r is an integer such that $0 < r < p$, and m and n are natural numbers with $m > 2$. Let*

$$H_{p,r}(n,m) = \sum_{i=0}^{n-1} \frac{(pn - pi - r)^m}{pi + r}. \tag{3.3}$$

Define

$$\begin{aligned} C(n,m) &:= (m-1)\frac{n-1}{2} - mn, \quad \text{and} \\ D(n,m) &:= \binom{m-1}{2} \frac{(n-1)n(2n-1)}{6}. \end{aligned} \tag{3.4}$$

If $p>3$, then we have

$$H_{p,r}(n,m) \equiv (-1)^m n r^{m-1} + (-1)^m pnC(n,m) r^{m-2} \mod p^{\nu_p(n)+2}. \tag{3.5}$$

If $p=3$, then

$$\begin{aligned} H_{3,r}(n,m) &\equiv (-1)^m n r^{m-1} + (-1)^m 3nC(n,m) r^{m-2} \\ &\quad + (-1)^m 9n^2 D(n,m) r^{m-3} \mod 3^{\nu_3(n)+2}. \end{aligned} \tag{3.6}$$

In particular,

$$\nu_p(H_{p,r}(n,m)) = \nu_p(n) \tag{3.7}$$

for all odd primes. Finally, if m is odd, then

$$\nu_2(H_{2,1}(n,m)) = \nu_2(n). \tag{3.8}$$

Proof. The case when $p=2$ and m is odd can be proved using induction. As a result, we decided not to present the proof of this case.

Suppose that p is an odd prime. Suppose that n is a positive integer. Expand $(pn-pi-r)^m$ to get

$$\begin{aligned} H_{p,r}(n,m) &= \sum_{i=0}^{n-1} \frac{(pn-pi-r)^m}{pi+r} \\ &\equiv \sum_{i=0}^{n-1} \frac{(-pi-r)^m}{pi+r} + mpn \sum_{i=0}^{n-1} \frac{(-pi-r)^{m-1}}{pi+r} \mod p^{\nu_p(n)+2} \qquad (3.9) \\ &\equiv (-1)^m \sum_{i=0}^{n-1} (pi+r)^{m-1} + (-1)^{m-1} mpn \sum_{i=0}^{n-1} (pi+r)^{m-2} \mod p^{\nu_p(n)+2}. \end{aligned}$$

Consider the second sum in (3.9). Note that

$$\begin{aligned} (-1)^{m-1} mpn \sum_{i=0}^{n-1} (pi+r)^{m-2} &= (-1)^{m-1} mpn \sum_{i=0}^{n-1} \sum_{j=0}^{m-2} \binom{m-2}{j} r^{m-2-j} (pin)^j \\ &\equiv (-1)^{m-1} mpn r^{m-2} \mod p^{2+\nu_p(n)}. \qquad (3.10) \end{aligned}$$

Consider now the first sum in (3.9). Observe that

$$\begin{aligned} (-1)^m \sum_{i=0}^{n-1} (pi+r)^{m-1} &= (-1)^m \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \binom{m-1}{j} r^{m-1-j} (pin)^j \\ &= (-1)^m \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \binom{m-1}{j} r^{m-1-j} (pin)^j. \end{aligned}$$

Suppose that $j \geq 3$. In this case, we have the sum

$$(-1)^m \sum_{i=0}^{n-1} \binom{m-1}{j} r^{m-1-j} (pin)^j. \tag{3.11}$$

Apply the classical identity

$$\sum_{k=0}^{x} k^m = \frac{B_{m+1}(x+1) - B_{m+1}}{m+1}, \tag{3.12}$$

where $B_m(x)$ represents the m-th Bernoulli polynomial and B_m the m-th Bernoulli number, to see that (3.11) equals

$$(-1)^m p^j n^j \binom{m-1}{j} r^{m-1-j} \left(\frac{B_{j+1}(n) - B_{j+1}}{j+1} \right). \tag{3.13}$$

Now, the Bernoulli polynomial $B_{j+1}(n)$ can be expressed as

$$B_{j+1}(n) = \sum_{k=0}^{j+1} \binom{j+1}{k} B_{j+1-k} n^k. \tag{3.14}$$

By Theorem 3.3 we know that if p appears in the denominator of any of the B_{j+1-k}'s, for $k = 0, 1, \cdots, j+1$, then it will have multiplicity one. This implies that

$$\nu_p \left(\frac{B_{j+1}(n) - B_{j+1}}{j+1} \right) \geq \nu_p(n) - \nu_p(j+1) - 1. \tag{3.15}$$

Since $j - \nu_p(j+1) - 1 \geq 2$ for $j \geq 3$, then it follows that (3.13) is congruent to 0 modulo $p^{2+\nu_p(n)}$. Thus,

$$(-1)^m \sum_{i=0}^{n-1} (pi + r)^{m-1} \equiv (-1)^m \sum_{j=0}^{2} \sum_{i=0}^{n-1} \binom{m-1}{j} r^{m-1-j} (pin)^j \mod p^{2+\nu_p(n)}. \tag{3.16}$$

Congruences (3.9), (3.10), and (3.16) yield the result. □

Our proof of the observation also uses the next result of Glaisher [5].

Theorem 3.5. *If m is a positive integer and p a prime such that $p \geq m+3$, then*

$$\sum_{j=1}^{p-1} \frac{1}{j^m} \equiv \begin{cases} \frac{m}{m+1} p B_{p-1-m} \mod p^2, & \text{if } m \text{ is even,} \\ -\frac{m(m+1)}{2(m+2)} p^2 B_{p-2-m} \mod p^3, & \text{if } m \text{ is odd} \end{cases} \tag{3.17}$$

Lemma 3.4 may be used to prove the observation for general k with $k \equiv -1 \mod p$. However, we relax the hypothesis of the observation (we now require $a \geq 2\lfloor \log_p(k+1) \rfloor$) in order to provide a simpler elementary proof.

Theorem 3.6. *Suppose that p is a prime that is not Wolstenholme. Suppose that k is an integer such that $k \equiv -1 \mod p$. Let $r = \lfloor \log_p(k+1) \rfloor$ and suppose that a is an integer such that $a \geq 2r$. Then,*

$$\nu_p(A((p-1)p^{a+1}m - 1, k)) = \begin{cases} \nu_p(m) + \nu_p(k+1) + a, & p = 2, 3 \\ \nu_p(m) + \nu_p(k+1) + a + 1, & p > 3. \end{cases} \tag{3.18}$$

Proof. We present the proof for $p > 3$. The proofs for $p = 2$ and $p = 3$ follow in a similar manner. Suppose that $k = pb - 1$ and let $r = \lfloor \log_2(k+1) \rfloor$. Suppose that a is an integer such that $a \geq 2r$. By (1.4), we have

$$\begin{aligned} A((p-1)p^{a+1}m - 1, k) &= \sum_{j=0}^{k+1} (-1)^j \binom{(p-1)p^{a+1}m}{j} (k+1-j)^{(p-1)p^{a+1}m-1} \\ &= \sum_{j=0}^{pb} (-1)^j \binom{(p-1)p^{a+1}m}{j} (pb-j)^{(p-1)p^{a+1}m-1}. \end{aligned} \tag{3.19}$$

The p-adic valuation of $A((p-1)p^{a+1}m-1,k)$ is given by the p-adic valuation of

$$\sum_{j=1, p\nmid j}^{pb} (-1)^j \binom{(p-1)p^{a+1}m}{j} (pb-j)^{(p-1)p^{a+1}m-1}. \tag{3.20}$$

This last sum can be written as

$$\sum_{r=1}^{p-1}(-1)^r \sum_{i=0}^{b-1}(-1)^i \binom{(p-1)p^{a+1}m}{pi+r} (pb-pi-r)^{(p-1)p^{a+1}m-1}. \tag{3.21}$$

Consider the sum

$$\sum_{i=0}^{b-1}(-1)^i \binom{(p-1)p^{a+1}m}{pi+r} (pb-pi-r)^{(p-1)p^{a+1}m-1}. \tag{3.22}$$

Expand the binomials to obtain

$$\begin{aligned} &\frac{(p-1)p^{a+1}m}{r}\left(\prod_{l=1}^{r-1}\frac{(p-1)p^{a+1}m-l}{l}\right)(pb-r)^{(p-1)p^{a+1}m-1} \\ -\quad &\frac{(p-1)p^{a+1}m}{p+r}\left(\prod_{l=1}^{p+r-1}\frac{(p-1)p^{a+1}m-l}{l}\right)(pb-p-r)^{(p-1)p^{a+1}m-1} \\ \vdots\quad & \\ +\quad &(-1)^{b-1}\frac{(p-1)p^{a+1}m}{(b-1)p+r}\left(\prod_{l=1}^{(b-1)b+r-1}\frac{(p-1)p^{a+1}m-l}{l}\right)(p-r)^{(p-1)p^{a+1}m-1}. \end{aligned} \tag{3.23}$$

Now, further expand this expression to to get

$$(-1)^{r-1}(p-1)p^{a+1}H_{p,r}\left(b,(p-1)p^{a+1}m-1\right)m+B_2(r)m^2+B_3(r)m^3+\cdots. \tag{3.24}$$

Thus, the p-adic valuation of $A((p-1)p^{a+1}m-1,k)$ is given by the p-adic valuation of the sum

$$-(p-1)p^{a+1}\sum_{r=1}^{p-1}H_{p,r}\left(b,(p-1)p^{a+1}m-1\right)m+\sum_{r=1}^{p-1}B_2(r)m^2+\cdots. \tag{3.25}$$

We study each term of (3.25) individually.

We start with $H_{p,r}\left(b,(p-1)p^{a+1}m-1\right)$. Note that Lemma 3.4 implies that this number is equivalent to

$$-br^{(p-1)p^{a+1}m-2}-pbC(b,(p-1)p^{a+1}m-1)r^{(p-1)p^{a+1}m-3} \tag{3.26}$$

modulo $p^{\nu_p(b)+2}$. Thus, the p-adic valuation of (3.26) is $\nu_p(b)=\nu_p(k+1)-1$. This implies that

$$\nu_p\left((p-1)p^{a+1}H_{p,r}\left(b,(p-1)p^{a+1}m-1\right)m\right)=a+\nu_p(k+1)+\nu_p(m). \tag{3.27}$$

However, note that in (3.25) we have a sum involving the $H_{p,r}$'s, thus we must consider the contribution of the "free from p" part of them. To find such a contribution, consider the following sum

$$\sum_{r=1}^{p-1} r^{(p-1)p^{a+1}m-2}. \tag{3.28}$$

Note that

$$\sum_{r=1}^{p-1} r^{(p-1)p^{a+1}m-2} \equiv \sum_{r=1}^{p-1} r^{-2} \mod p^2. \tag{3.29}$$

***Remark* 3.7.** Observe that if $p=3$, then this number is not divisible by 3. This helps to explain why the formula (3.18) does not have the $+1$ when $p=3$.

If $p>3$, then Theorem 3.5 implies that

$$\sum_{r=1}^{p-1} r^{-2} \equiv \frac{2}{3}pB_{p-3} \mod p^2. \tag{3.30}$$

Since p is not a Wolstenholme prime, then p divides (3.28), but p^2 does not divide it. On the other hand, note that

$$\begin{aligned}\sum_{r=1}^{p-1} r^{(p-1)p^{a+1}m-3} &\equiv \sum_{r=1}^{p-1} r^{-3} \mod p \\ &\equiv \sum_{r=1}^{p-1} r^{3} \mod p \\ &\equiv 0 \mod p.\end{aligned} \tag{3.31}$$

We conclude that

$$\nu_p\left(-(p-1)p^{a+1}\sum_{r=1}^{p-1} H_{p,r}\left(b,(p-1)p^{a+1}m-1\right)m\right) \tag{3.32}$$

is given by

$$\nu_p(m)+\nu_p(k+1)+a+1. \tag{3.33}$$

Consider now the other terms of (3.24), i.e. the terms $B_t(r)m^t$ for $t\geq 2$. Let us start with $B_2(r)m^2$. Observe that the general term in (3.23) has the form

$$(-1)^i\frac{(p-1)p^{a+1}m}{pi+r}\left(\prod_{l=1}^{pi+r-1}\frac{(p-1)p^{a+1}m-l}{l}\right)(pb-pi-r)^{(p-1)p^{a+1}m-1}, \tag{3.34}$$

where $0\leq i\leq b-1$. This implies that

$$\sum_{r=1}^{p-1} B_2(r) = \sum_{r=1}^{p-1}\sum_{i=0}^{b-1}\sum_{l=1}^{pi+r-1}\frac{(p-1)^2p^{2a+2}}{(pi+r)l}(pb-pi-r)^{(p-1)p^{a+1}m-1}. \tag{3.35}$$

Our choice of a leads to

$$\nu_p\left(\frac{(p-1)p^{2a+2}}{(pi+r)l}(pb-pi-r)^{(p-1)p^{a+1}m-1}\right)\geq 2a+2-r > a+\nu_p(k+1)+1, \tag{3.36}$$

and thus,

$$\nu_2(B_2(r)) > a+\nu_p(k+1)+1. \tag{3.37}$$

This yields

$$\nu_p\left(\sum_{r=1}^{p-1} B_2(r)m^2\right) = \nu_p\left(\sum_{r=1}^{p-1} B_2(r)\right) + 2\nu_p(m) \quad (3.38)$$
$$> 1 + a + \nu_p(k+1) + 2\nu_p(m)$$
$$\geq 1 + a + \nu_p(k+1) + \nu_p(m),$$

which implies that the p-adic valuation of the second term in (3.24) is bigger than the p-adic valuation of the first term. A similar argument shows that

$$\nu_p(B_t m^t) > 1 + a + \nu_p(k+1) + \nu_p(m), \quad (3.39)$$

for all $t \geq 2$. The result now follows from the ultrametric property of the p-adic valuation. This concludes the proof. □

We point out that the above proof can be carried over to the Wolstenholme primes $p = 16843$ and $p = 2124679$. In both cases we have $p|B_{p-3}$, but $p^2 \nmid B_{p-3}$. In view of Theorem 3.5, we have,

$$\nu_p(A((p-1)p^{a+1}m - 1, k)) = \nu_p(m) + \nu_p(k+1) + a + 2, \quad (3.40)$$

for $k \equiv -1 \mod p$ and $a > 2\lfloor \log_p(k+1) \rfloor$. If there is any other Wolstenholme prime p, then

$$\nu_p(A((p-1)p^{a+1}m - 1, k)) \geq \nu_p(m) + \nu_p(k+1) + a + 2, \quad (3.41)$$

for $k \equiv -1 \mod p$ and $a > 2\lfloor \log_p(k+1) \rfloor$.

4. Other patterns when $k \equiv -1 \mod p$ and a connection to regular primes

The pattern described in the previous section is not unique. In fact, it seems that there are other patterns in these trees, even for $k \not\equiv -1 \mod p$. However, we continue with the case $k \equiv -1 \mod p$ and present some patterns and some proofs. Perhaps this study can be extended to the case when $k \not\equiv -1 \mod p$.

Let us consider the tree when $k = 2$ and $p = 3$ and the tree when $k = 4$ and $p = 5$. These are the trees in Figures 6 and 7. However, to facilitate the reading, we include them in Figures 8 and 9.

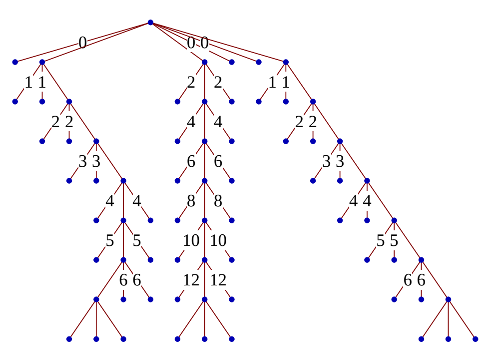

FIGURE 8. p-adic tree when $k = 2$ and $p = 3$.

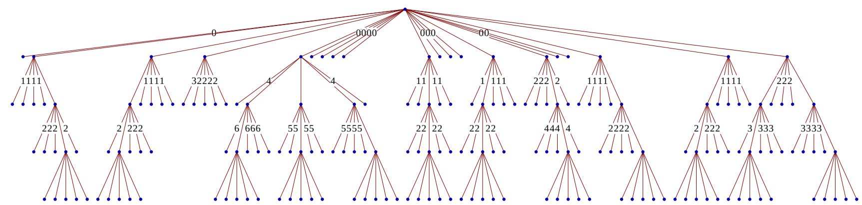

FIGURE 9. p-adic tree when $k = 4$ and $p = 5$.

Observe first the tree for $k = 2$ and $p = 3$. Note that the node that corresponds to $2 \mod 6$ splits very similar to the node $5 \mod 6$. The node $5 \mod 6$ belongs to the case considered in the previous section. Observe now the tree for $k = 4$ and $p = 5$. In Figure 10 you can see the nodes $4 \mod 20$, $9 \mod 20$, and $14 \mod 20$. Note that these nodes split in a similar manner as the node $19 \mod 20$.

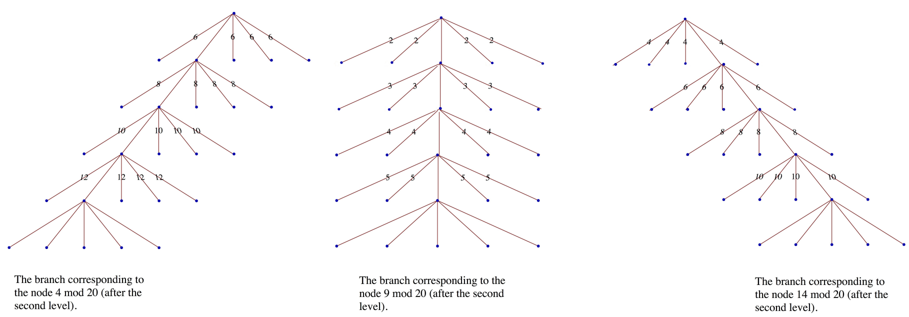

FIGURE 10. The nodes $4 \mod 20$, $9 \mod 20$, and $14 \mod 20$

Again, the node $19 \mod 20$ was considered in the previous section. If we continue analyzing these trees, then a pattern emerges. It appears that the p-adic valuation of $A((p-i)p^{a+1}m - 1, k)$, for $i = 2, \cdots, p-1$, behaves in a manner that is very similar to the p-adic valuation of $A((p-1)p^{a+1}m - 1, k)$.

Let us explore this problem. For simplicity, we assume that $p > 3$. If we follow the proof of Theorem 3.6, then we see that if a is big enough, then the p-adic valuation of $A((p-i)p^{a+1}m - 1, k)$ is given by the p-adic valuation of

$$-(p-i)p^{a+1} \sum_{r=1}^{p-1} H_{p,r}\left(b, (p-i)p^{a+1}m - 1\right) m, \tag{4.1}$$

where $k = pb - 1$. By Lemma 3.4 we know that

$$\begin{aligned} H_{p,r}\left(b, (p-i)p^{a+1}m - 1\right) \equiv\ & (-1)^{m(i-1)+1} b r^{(p-i)p^{a+1}m-2} + \\ & (-1)^{m(i-1)+1} pbC(b, (p-i)p^{a+1}m - 1) r^{(p-i)p^{a+1}m-3} \end{aligned} \tag{4.2}$$

modulo $p^{\nu_p(b)+2}$. As before, since we are considering a sum involving the $H_{p,r}$'s, then we must account for the contribution of the "free from p" part of them. In other words, we must consider the contribution of the sums

$$\sum_{r=1}^{p-1} r^{(p-i)p^{a+1}m-2} \text{ and } \sum_{r=1}^{p-1} r^{(p-i)p^{a+1}m-3}. \tag{4.3}$$

Moreover, in some cases it may be necessary to consider the contribution of

$$\sum_{r=1}^{p-1} r^{(p-i)p^{a+1}m-l} \tag{4.4}$$

for $l > 3$, but we do not consider these cases in this work.

4.1. **A connection to regular primes.** We now present a study for the case when $i = 3$. In other words, we consider the p-adic valuation of

$$A\left((p-3)p^{a+1}m-1, k\right). \tag{4.5}$$

We start with the analysis of the contribution of the sums (4.3) when $i = 3$. This information is summarized in the next theorem.

Theorem 4.1. *Let $p > 3$ be prime and suppose that $a \geq 2$. Suppose that m is positive. Then*

$$\sum_{r=1}^{p-1} r^{(p-3)p^{a+1}m-2} \equiv pB_{(p-3)p^{a+1}m-2} \mod p^2 \tag{4.6}$$

and

$$\sum_{r=1}^{p-1} r^{(p-3)p^{a+1}m-3} \equiv -\frac{3}{2}p^2 B_{(p-3)p^{a+1}m-4} \mod p^3. \tag{4.7}$$

Proof. We present the proof for the first sum. Use the classical identity (3.12) to obtain

$$\sum_{r=1}^{p-1} r^{(p-3)p^{a+1}m-2} = \frac{B_{(p-3)p^{a+1}m-1}(p) - B_{(p-3)p^{a+1}m-1}}{(p-3)p^{a+1}m-1}. \tag{4.8}$$

After writing $B_{(p-3)p^{a+1}m-1}(p)$ in terms of the Bernoulli numbers, we see that

$$\sum_{r=1}^{p-1} r^{(p-3)p^{a+1}m-2} \equiv pB_{(p-3)p^{a+1}m-2} \mod p^2. \tag{4.9}$$

This is guaranteed because the coefficient of p^2 contains $B_{(p-3)p^{a+1}m-3}$ and since $(p-3)p^{a+1}m-3$ is odd, then this Bernoulli number is 0. On the other hand, the coefficient of p^3 contains $B_{(p-3)p^{a+1}m-4}$. However, if p appears in the denominator of this Bernoulli number, then it will have multiplicity one, since the denominator is square-free. Thus, the contribution of p^3 times its coefficient is divisible by p^2 and the result holds. □

Now that we have the contributions of these sums, we move on to the calculation of the p-adic valuation of (4.5). First note that Theorem 4.1 implies that p always divides the second sum in (4.3). Thus, we must analyze the contribution of the first sum in (4.3).

Write m as

$$m = \left(\frac{p-1}{2}\right) l + b, \tag{4.10}$$

where $0 \leq b < (p-1)/2$. Observe that

$$\begin{aligned} \sum_{r=1}^{p-1} r^{(p-3)p^{a+1}m-2} &= \sum_{r=1}^{p-1} r^{(p-3)p^{a+1}\left(\left(\frac{p-1}{2}\right)+b\right)-2} \\ &\equiv \sum_{r=1}^{p-1} r^{(p-3)p^{a+1}b-2} \mod p^2 \\ &\equiv pB_{(p-3)p^{a+1}b-2} \mod p^2. \end{aligned} \tag{4.11}$$

We explore the contribution of (4.11) for each of these residues. Suppose first that $b = (p-3)/2 \equiv -1 \mod (p-1)/2$. Then

$$(p-1) \mid (p-3)p^{a+1}b - 2.$$

Theorem 3.3 implies that p appears in the denominator of $B_{(p-3)p^{a+1}b-2}$, and so p does not divide the sum in (4.11). Thus, the contribution of the first sum in (4.3) when $i = 3$ is zero for this case.

Suppose now that $b \not\equiv -1 \mod (p-1)/2$. Use the p-adic continuity of Bernoulli numbers [7, p. 44]

$$(1-p^{m-1})\frac{B_m}{m} \equiv (1-p^{n-1})\frac{B_n}{n} \mod p^b, \tag{4.12}$$

which holds when b, m and n are positive integers such that m and n are not divisible by $p-1$ and $m \equiv n \pmod{p^{b-1}(p-1)}$, and the fact that

$$(p-3)p^{a+1}b - 2 \equiv (p-3)b - 2 \equiv p - 2b - 3 \mod p-1, \tag{4.13}$$

to obtain

$$B_{(p-3)p^{a+1}b-2} \equiv \frac{2}{2b+3} B_{p-2b-3} \mod p. \tag{4.14}$$

Thus, if $p \nmid B_{p-2b-3}$, then p divides (4.11), but p^2 does not. However, since $b \not\equiv -1$ mod $(p-1)/2$, then b takes on the values $b = 0, 1, 2, \cdots, (p-5)/2$. This implies that we require that

$$\begin{aligned} p &\nmid B_{p-3}, && \text{when } b = 0, \\ p &\nmid B_{p-5}, && \text{when } b = 1, \\ &\vdots \\ p &\nmid B_2, && \text{when } b = (p-5)/2. \end{aligned} \tag{4.15}$$

These primes are special and have a name.

Definition 4.2. We say that an odd prime p is regular if p does not divide any of the Bernoulli numbers B_n for $n = 2, 4, 6, \cdots, p-3$. An odd prime that is not regular is called irregular. Regular primes were introduced by Kummer in his work on Fermat's Last Theorem.

***Remark* 4.3.** The first few irregular primes are

$$37, 59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, 307, \cdots$$

This is sequence `A000367` in OEIS. It is known that there is an infinite amount of irregular primes (Jensen [6]).

The next theorem summarizes what we just found.

Theorem 4.4. *Suppose that $p > 3$ is a regular prime. Suppose that $k \equiv -1 \mod p$ and that $a \geq 2\lfloor \log_p(k+1) \rfloor$. Define*

$$\delta(m,p) = \begin{cases} 0, & \text{if } m \equiv -1 \mod (p-1)/2, \\ 1, & \text{if } m \not\equiv -1 \mod (p-1)/2. \end{cases} \tag{4.16}$$

Then,

$$\nu_p(A(p-3)p^{a+1}m-1,k) = a + \nu_p(m) + \nu_p(k+1) + \delta(m,p). \tag{4.17}$$

Here is an example that shows why regularity of the prime is important.

Example 4.5. Consider the prime $p = 37$ (the first irregular prime), $k = 36$ and $a = 2$. The first few values of the sequence $\nu_{37}(A(34 \times 37^3 m - 1, 36))$ are given by:

$$\begin{gathered} 5,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,4,5,4, \\ 4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,4,6,\cdots \end{gathered} \tag{4.18}$$

while the first few values of $2 + \nu_{37}(m) + \nu_{37}(37) + \delta(m,37)$ are:

$$\begin{gathered} 4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,4,4,4,4, \\ 4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,4,5,\cdots. \end{gathered} \tag{4.19}$$

Observe that they differ in the positions $1, 19, 37, \cdots$. Note that $(p-1)/2 = 18$ and $1, 19, 37, \cdots$ are congruent to $1 \mod 18$. By previous discussion, when $b \equiv 1 \mod (p-1)/2$, we require that p does not divide B_{p-5} for our formula to work. However, $B_{p-5} = B_{32}$ and $p = 37$ is an irregular prime precisely because $37 \mid B_{32}$.

The same technique used in the proof of Theorem 4.4 may be applied to the case when i is odd (the case when i is even is different, but it is not considered in this work). For example, for $i = 5$ we have the following theorem.

Theorem 4.6. *Suppose that $p > 7$ is prime, $k \equiv - \mod p$, and $a \geq 2\lfloor \log_p(k+1) \rfloor$. If $p \equiv 1 \mod 4$ and p does not divide any of the Bernoulli numbers*

$$B_2, B_6, \cdots, B_{p-7}, B_{p-3},$$

then

$$\nu_p(A((p-5)p^{a+1}m-1,k)) = 1 + a + \nu_p(m) + \nu_p(k+1). \tag{4.20}$$

On the other hand, if $p \equiv 3 \mod 4$, define

$$\gamma(m,p) = \begin{cases} 0, & \text{if } m \equiv \frac{p-3}{4} \mod (p-1)/2, \\ 1, & \text{if } m \not\equiv \frac{p-3}{4} \mod (p-1)/2. \end{cases} \tag{4.21}$$

If p is regular, then

$$\nu_p(A((p-5)p^{a+1}m-1,k)) = a + \nu_p(m) + \nu_p(k+1) + \gamma(m,p). \tag{4.22}$$

However, since this does not bring any new techniques, then we decided not to include the study for i odd in general. Nevertheless, we present one last case that is related to the Legendre symbol.

4.2. **Bernoulli, Euler and Legendre.** We now present the last study of this article. We consider the case $i = (p+1)/2$ when $p > 3$. In other words, we are interested in the p-adic valuation of

$$A\left(\left(\frac{p-1}{2}\right)p^{a+1}m-1,k\right). \tag{4.23}$$

It turns out that, as in the previous two cases, the divisibility of the Eulerian numbers is related to the divisibility of the Bernoulli numbers.

We start with the following result.

Theorem 4.7. *Let $\left(\frac{a}{p}\right)$ denote the Legendre symbol. Suppose that m is a positive integer and p a prime such that $p(p-1)/2 \geq m+3$. If m is even, then*

$$\sum_{r=1}^{p-1}\frac{\left(\frac{r}{p}\right)}{r^m} \equiv \begin{cases} pB_{\left(\frac{p-1}{2}\right)p-m} \mod p^2, & \text{if } p\equiv 1 \mod 4,\\ -\frac{m}{2}p^2B_{\left(\frac{p-1}{2}\right)p^2-m-1} \mod p^3, & \text{if } p\equiv 3 \mod 4.\end{cases}$$

However, if m is odd, then

$$\sum_{r=1}^{p-1}\frac{\left(\frac{r}{p}\right)}{r^m} \equiv \begin{cases} -\frac{m}{2}p^2B_{\left(\frac{p-1}{2}\right)p^2-m-1} \mod p^3, & \text{if } p\equiv 1 \mod 4,\\ pB_{\left(\frac{p-1}{2}\right)p-m} \mod p^2, & \text{if } p\equiv 3 \mod 4.\end{cases}$$

Proof. The proof of this theorem is very similar to the one of Theorem 4.1. We present the idea of the proof when m is even. Suppose that p is a prime such that $p \equiv 1 \mod 4$. Note that

$$\sum_{r=1}^{p-1}\frac{\left(\frac{r}{p}\right)}{r^m} \equiv \sum_{r=1}^{p-1} r^{\left(\frac{p-1}{2}\right)p-m} \mod p^2. \tag{4.24}$$

Now use the identity (3.12) and proceed as in Theorem 4.1.

If $p \equiv 3 \mod 4$, then use the fact that

$$\sum_{r=1}^{p-1}\frac{\left(\frac{r}{p}\right)}{r^m} \equiv \sum_{r=1}^{p-1} r^{\left(\frac{p-1}{2}\right)p^2-m} \mod p^3. \tag{4.25}$$

Apply (3.12) and proceed as before to obtain the result. □

Let us explore the problem of finding the p-adic valuation of

$$A\left(\left(\frac{p-1}{2}\right)p^{a+1}m-1,k\right). \tag{4.26}$$

Note that if m is even, i.e. $m = 2l$, then the problem of computing the p-adic valuation of (4.26) is reduced to the calculation of the p-adic valuation of

$$A\left((p-1)p^{a+1}l-1,k\right). \tag{4.27}$$

By Theorem 3.6 we know that if p is not a Wolstenholme prime, then

$$\begin{aligned}\nu_p\left(A\left(\left(\frac{p-1}{2}\right)p^{a+1}m-1,k\right)\right) &= \nu_p(A\left((p-1)p^{a+1}l-1,k\right))\\ &= a+1+\nu_p(k+1)+\nu_p(l)\\ &= a+1+\nu_p(k+1)+\nu_p(m).\end{aligned} \tag{4.28}$$

Thus, suppose that m is odd and consider the sums

$$\sum_{r=1}^{p-1} r^{((p-1)/2)p^{a+1}m-2} \text{ and } \sum_{r=1}^{p-1} r^{((p-1)/2)p^{a+1}m-3}. \tag{4.29}$$

Write $m = 2l+1$ with l a non-negative integer. Note that

$$\begin{aligned} \sum_{r=1}^{p-1} r^{((p-1)/2)p^{a+1}(2l+1)-2} &\equiv \sum_{r=1}^{p-1} r^{((p-1)/2)p^{a+1}-2} \mod p^2 \\ &\equiv \sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^2} \mod p^2. \end{aligned} \tag{4.30}$$

Similarly,

$$\sum_{r=1}^{p-1} r^{((p-1)/2)p^{a+1}(2l+1)-2} \equiv \sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^3} \mod p^2. \tag{4.31}$$

Theorem 4.7 provides information about the divisibility of these two sums.

Theorem 4.8. *Suppose that $p \geq 13$ and $p \equiv 1 \mod 4$. Suppose that p is not a Wolstenholme prime and that*

$$p \nmid B_{\frac{p-5}{2}}.$$

Suppose that $k \equiv -1 \mod p$ and $a \geq 2\lfloor \log_p(k+1) \rfloor$. Then,

$$\nu_p\left(A\left(\frac{p-1}{2}\right)p^{a+1}m-1, k\right) = \nu_p(m) + \nu_p(k+1) + a + 1 \tag{4.32}$$

In particular, the result holds for all regular primes bigger than or equal to 13 that are congruent to 1 modulo 4.

Proof. Note that by the previous discussion, to prove this theorem, it is sufficient to show that p divides

$$\sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^2} \quad \text{and} \quad \sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^3}, \tag{4.33}$$

but p^2 does not divide the first sum. Theorems 3.3 and 4.7 imply that p divides the second sum in (4.33), so it remains to show that p divides the first sum in (4.33), but p^2 does not.

By Theorem 4.7 we know that

$$\sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^2} \equiv pB_{\left(\frac{p-1}{2}\right)p-2} \mod p^2. \tag{4.34}$$

Observe that p does not divide this number if and only if p appears in the denominator of $B_{\left(\frac{p-1}{2}\right)p-2}$. By Theorem 3.3, this would be true if

$$(p-1) \Big| \left(\frac{p-1}{2}\right)p-2. \tag{4.35}$$

However, the only solution to (4.35) is $p = 5$ and we are assuming that $p \geq 13$, thus we know that p divides (4.34).

Now we show that p^2 does not divide (4.34). Note that this is equivalent to showing that p does not divide $B_{(\frac{p-1}{2})p-2}$. Use Kummer's congruence [7, p. 44]

$$\frac{B_{j(p-1)+i}}{j(p-1)+i} \equiv \frac{B_i}{i} \mod p, \tag{4.36}$$

which holds for p odd and i an even number such that $(p-1) \nmid i$, to obtain

$$B_{(\frac{p-1}{2})p-2} \equiv \frac{4}{5} B_{\frac{p-5}{2}} \mod p. \tag{4.37}$$

However, by hypothesis, we know that p does not divide $B_{\frac{p-5}{2}}$. This concludes the proof. □

***Remark* 4.9.** There is no prime $13 \leq p \leq 3.3 \times 10^5$ such that $p \equiv 1 \mod 4$ and $p \mid B_{(p-5)/2}$. Note that the existence of such a prime p is special, because it will be congruent to 1 modulo 4 and

$$p^2 \,\Big|\, \sum_{r=1}^{p-1} \frac{\left(\frac{r}{p}\right)}{r^2}. \tag{4.38}$$

***Remark* 4.10.** Observe that if $k \equiv -1 \mod 5$ and $a \geq 2\lfloor \log_5(k+1) \rfloor$, then, following the proof of the above theorem, we arrive to the conclusion

$$\nu_5(A(2 \times 5^{a+1}m - 1, k) = a + \nu_5(m) + \nu_5(k+1) + \frac{1 + (-1)^m}{2}. \tag{4.39}$$

This coincides with Theorem 4.4 because $\delta(m, 5) = (1 + (-1)^m)/2$.

In conclusion, we showed that a simple construction of a tree for the p-adic valuation of the Eulerian number $A(n, k)$ contains some interesting patterns. Moreover, these patterns are connected to classical number-theoretical results involving Bernoulli numbers. It would be interesting to know if something similar occurs to the p-adic valuation of other combinatorial sequences.

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