# ONE-WAREHOUSE MULTI-RETAILER PROBLEM UNDER INVENTORY CONTROL AND TRANSPORTATION POLICIES

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## ABSTRACT

### ONE-WAREHOUSE MULTI-RETAILER PROBLEM UNDER INVENTORY CONTROL AND TRANSPORTATION POLICIES

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We consider a one-warehouse multi-retailer system where the warehouse orders or receives from its supplier and replenishes multiple retailers with direct shipping or multi-stop routing over a finite time horizon. The warehouse has the knowledge of external (deterministic) demands at the retailers and manages their inventories while ensuring no stock-out. We consider two problems with direct shipping policy and two problems with routing policy. For the direct shipping policy, the problem is to determine the optimal replenishments for the warehouse and retailers such that the system-wide costs are minimized. In one problem, the warehouse decides about how much and when to ship to the retailers while in the other problem, inventory level of the retailer has to be raised up to a predetermined level whenever replenished. We propose strong mixed integer programming formulations for these problems. Computational experiments show that our formulations are better than their competitors and are very successful in solving the problems to optimality. For the routing policy, the problem is to decide on when and in what sequence to visit the retailers and how much to ship to a retailer so as to minimize system-wide costs. In one problem, the warehouse receives given amounts from its supplier while in the other the warehouse decides on its own replenishments. We propose branch-and-cut algorithms and heuristics based on strong formulations for both problems. Computational results reveal that our procedures perform better than their competitors in the literature for both problems.

Keywords: One-warehouse multi-retailer system, Lot sizing, Inventory-routing, Strong formulation, Branch-and-cut

# ÖZ

## ENVANTER KONTROL VE ULAŞIM POLİTİKALARI ALTINDA BİR TEDARİKÇİ-ÇOKLU PERAKENDECİ PROBLEMİ

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Bu tezde, bir tedarikçinin çok sayıdaki perakendecinin envanterini bir planlama ufku boyunca yönetmesi işlenmiştir. Tedarikçi perakendecilerin tahminlerle belirlenmiş müşteri taleplerini bilmektedir; kendisinin ve perakendecilerin envanter seviyesinin sıfırın altına düşmesine izin vermeden sistemi yönetmektedir. Dağıtımın doğrudan sevkiyatla veya rota marifetiyle yapıldığı ikişer problem ele alınmıştır. Dağıtımın doğrudan sevkiyatla yapıldığı problemlerde, tedarikçinin ne zaman, ne kadar mal sipariş vereceğine ve eldeki miktarın kimlere dağıtılacağına sistemin toplam maliyetini enazlayacak şekilde karar verilmektedir. Doğrudan sevkiyat problemlerinin birinde, perakendecilere ne zaman ne kadar mal verileceğine tamamen tedarikçi karar verirken, diğerinde tedarikçi ziyaret ettiği perakendecinin elindeki envanterini tavan seviyesine çekecek şekilde mal verir. Bu problemler için güçlü karışık tam sayılı programlama formülasyonları geliştirilmiştir. Sayısal sonuçlar, önerilen formülasyonların literatürdeki rakiplerinden daha iyi olduğunu ve büyük ölçekli problemlerin en iyi çözümünü bulmada etkin olduğunu göstermiştir. Rotayla dağıtımın yapıldığı problemlerde, perakendecilerin ne zaman ve hangi sıra ile ziyaret edileceği ve ziyaret sırasında perakendecilere ne kadar mal verileceği kararları sistemdeki toplam maliyeti enazlayacak şekilde verilmektedir. Bu problemlerin birinde tedarikçiye her dönemde sevkiyat yapılırken, diğerinde sevkiyatın sıklığına ve içeriğine tedarikçinin kendisi karar vermektedir. Bu problemler için güçlü formülasyonlara dayalı dal-kesi algoritmaları ve sezgiseller önerilmiştir. Sayısal deneyler, her iki problem için de dal-kesi algoritmasının ve sezgiselin literatürdeki rakiplerinden daha iyi çalıştığını göstermiştir.

Anahtar Kelimeler: Bir tedarikçi-çoklu perakendeci sistemi, Kafile büyüklüğü problemi, Envanter-rotalama, Güçlü formülasyon, Dal-kesi

*To the beloved memory of my grandfather,*

*Oğuz Behaettin Solyalı…*

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## TABLE OF CONTENTS

**LIST OF TABLES**

TABLES

## LIST OF FIGURES

FIGURES

# CHAPTER 1

# INTRODUCTION

Supply chain management is a systems approach where different planning problems (inventory management, production planning, distribution planning, etc.) of several parties (suppliers, manufacturers, warehouses, retailers, etc.) can be integrated and viewed as a whole. Numerous studies such as Chandra (1993), Fumero and Vercellis (1999), and Boudia and Prins (2007) have investigated integration of different functions within the echelons of the system and have reported significant cost savings. Boudia and Prins (2007), for instance, report cost savings ranging from 15% to 30% achieved by an integrated production and distribution planning over an uncoordinated approach (where the output of production planning is the input for distribution planning, or vice versa). A review of such studies can be found in Thomas and Griffin (1996), Sarmiento and Nagi (1999) and Baita et al. (1998). Most of these studies have considered integration of inventory/production and transportation/distribution management issues, which is also the focus of this study.

Traditionally, inventory and transportation management issues are treated separately (see Silver et al., 1998; Toth and Vigo, 2002). According to Timme and Williams-Timme (2003), inventory accounts for around 37%, 56% and 62% of net operating assets in manufacturing, retail and distribution industries, respectively. Transportation sector, on the other hand, accounts for around 10% of gross domestic product (GDP) in European Union (Salani, 2006) and around 16% of GDP in Canada (Canada Research Chair in Distribution Management, 2008). Considering all these figures together, one can say that there is a huge opportunity to obtain significant cost savings by taking an integrated approach to inventory and transportation management since frequent (rare) inventory replenishment decreases (increases) the inventory carrying cost but increases (decreases) the transportation cost. Moreover, a recent survey conducted to analyze the role of inventories from the manager's point of view has revealed that an integrated approach with different functions of the companies is required for an efficient inventory management (Chikan, 2008). In the following, we present an integrated inventory/production and transportation/distribution management system that we address, our motivation and purpose in addressing such a system, and finally the outline of the thesis.

## 1.1 System under study, motivation and purpose

In this study, we consider a basic two-level supply chain structure, namely arborescent or distribution structure, composed of one warehouse and multiple retailers in which the warehouse orders or receives a single commodity from its supplier and replenishes multiple retailers with direct shipping (i.e. visiting a single retailer on a trip from the warehouse) or multi-stop routing (i.e. visiting several retailers on a trip from the warehouse) over a finite time horizon. Such a supply chain frequently occurs, for instance, when (i) the warehouse and retailers are different echelons of the same company or (ii) they do not belong to the same company but the warehouse (vendor) manages the inventories of the retailers in a vendor managed inventory (VMI) setting. VMI is different from the traditional customer (retailer) managed systems in that instead of customer orders in the traditional system, vendor (warehouse) decides on when and how much to replenish its customers' inventories while guaranteeing no stock-out at its customers (assuming a deterministic setting). It provides several benefits to both vendors and customers. For example, customers do not have to allocate their resources to inventory management and would have improved service levels due to the vendor's no stock-out guarantee. Vendor, on the other hand, is able to better utilize its own resources as it has the full authority over the system. To be able to manage the inventories of its customers, vendor needs timely information on the inventory status of its customers and it is met with the help of latest advances in information and communication technologies. Implementation of VMI systems has generated favorable results for the involved parties in areas such as industrial gas industry

(Campbell et al., 2002; Campbell and Savelsbergh, 2004), supermarket chains (Mongelluzo, 1998) and grocery industry (Ross, 1998). For detailed information on benefits of VMI systems and their application areas, we refer the reader to Çetinkaya and Lee (2000), and Campbell et al. (2002). Note that we do not deal with the allocation of (dis)benefits of VMI system to the involved parties and this issue is, therefore, out of scope of this thesis. In addition, even though we consider a single commodity in the system, it is also applicable to systems involving multiple commodities when those commodities do not create any resource conflicts, and thus can be aggregated and treated as a single commodity.

We present an integrated view that simultaneously considers management of the shipments from the warehouse to the retailers and/or from the supplier to the warehouse in addition to inventory management at the system. We specifically address integrated inventory and transportation management problems in one warehouse multi-retailer systems where we assume a periodic review system in that all information and decisions occur in discrete points in time. In addition, we assume that the warehouse has the complete knowledge of time-varying external (deterministic) demand occurring at the retailers and manages the inventories of the retailers while ensuring no stock-out (i.e. backlogging is not allowed) both at the retailers and at itself. We consider two problems with direct shipping policy and two problems with multi-stop routing policy.

*Two problems with direct shipping policy*

In the problems with direct shipping policy, we consider a two-level production-distribution system in which replenishment decisions are given for all the facilities (i.e. both warehouse and retailers) and there are no capacity restrictions on the replenishment quantities from the supplier to the warehouse as well as on the shipment quantities from the warehouse to the retailers. This fundamental structure is not only important in its own right but also arises as a subproblem in many more complex systems. For example, it arises in the one-warehouse multi-retailer systems with multi-stop routing of vehicles for the shipments from the warehouse to the retailers instead of the direct shipping policy (e.g. Fumero and Vercellis, 1999; Bard and Nananukul, 2008). It also arises in one warehouse multi-retailer systems with capacity restrictions imposed on replenishment quantities to the warehouse and/or to the retailers (e.g. Robinson and Lawrence, 2004; Federgruen et al., 2007), or in multi-echelon systems involving production-distribution structure (e.g. Veinott, 1969; Kalymon, 1972; Diaby and Martel, 1993). Thus, contributions to a relatively fundamental system will contribute to the more complex systems as well. Indeed, one-warehouse multi-retailer systems with direct shipping policy can be considered as a generalization of single-level lot sizing problems since production/order decisions at the warehouse level in the former should be given together with the replenishment decisions at the retailers level in the latter. The two problems with direct shipping policy differ from each other only in the inventory control policy at the retailers.

*Problem with endogenous policy*

In one problem, we consider an endogenously defined inventory control policy at the retailers so that the warehouse has the complete control on when and how much to ship to the retailers. The problem, referred to as the one-warehouse multi-retailer problem with endogenous policy, is to decide on how much and when to order for the warehouse as well as how much and when to ship to the retailers so that total system-wide costs composed of variable and fixed order costs, inventory holding costs, and transportation costs are minimized. The variable order cost incurred per unit can be considered as the purchasing cost and the fixed order cost paid whenever an order is placed for the warehouse can be considered as the order processing cost. The inventory holding cost is a linear function of the ending inventory levels, which is the most widely used measure in the literature. Transportation cost is a fixed cost incurred whenever a retailer is replenished and it can be considered as a cost paid to a third party logistics provider shipping the goods from the warehouse to the retailers.

*Problem with order-up-to level policy*

In the other problem, however, we consider an exogenously defined inventory control policy at the retailers where the inventory level of the retailer has to be raised up to its prespecified maximum level whenever replenished by the warehouse (called order-up-to level policy). This problem, referred to as one-warehouse multi-retailer problem with order-up-to level policy, is the same as the one with endogenous policy in other respects. The order-up-to level policy is a kind of deterministic application of the well-known stochastic (*s*, *S*) policy (see e.g. Silver et al., 1998) and it can be observed in practice in the replenishment of tanks of industrial gas dealers, shelf-spaces of supermarkets and vending machines where the tanks, shelf-spaces or vending machines are filled up to their capacities whenever replenished. While the order-up-to level policy is imposed by the retailers (customers) to the warehouse (vendor) in above-described contexts, the warehouse is free to choose the time and quantity of replenishments to the retailers as being in systems with endogenous policy when the retailers do not have an inventory holding capacity and they consent to such a policy. Obviously, the order-up-to level policy is a more restrictive policy than the endogenous policy and this may cause larger total system-wide costs in the former.

*Two problems with multi-stop routing policy*

In the problems with multi-stop routing policy, we address a two-level system in which the warehouse either receives given amounts from its supplier or decides on when and how much to order/produce, and then ships to the retailers using multi-stop routes. Such systems with routing policy arise in the same contexts described above for the direct shipment problems but especially when less-than-truck-load shipments to the retailers are concerned. One-warehouse multi-retailer systems with multi-stop routing policy can also be considered as a generalization of classical vehicle routing problems (see e.g. Toth and Vigo, 2002) in that the two problems with routing policy considered in this thesis involve inventory management issues in addition to vehicle routing issues over a finite horizon. The two problems with routing policy differ from each other in having production/order decisions at the warehouse or not.

*Problem with no order decision at the warehouse*

In one of the settings where a multi-stop routing is desirable, we consider the same problem of one-warehouse multi-retailer problem with order-up-to level policy with routing instead of the direct shipment policy. Also, different from the corresponding direct shipment problem, we consider capacity restrictions on the shipment quantities from the warehouse to the retailers due to vehicle capacity and a limited amount made available at the warehouse in each period. The problem, referred to as the inventory routing problem (i.e. no production/order decisions at the warehouse) with order-up-to level policy, is to decide on the delivery times and quantities to the retailers and routing of vehicles such that the total system-wide costs composed of inventory holding costs and transportation costs are minimized. The inventory holding cost is a linear function of ending inventory levels while the transportation cost is incurred based on the distance travelled by vehicles, which is the most widely used measure in the literature.

*Problem with order decision at the warehouse*

In the other problem, we consider the same problem except that an endogenous policy is allowed instead of the order-up-to level policy and there is a decision problem of how much and when to produce/order at the warehouse instead of a given amount made available at the warehouse in each period. The problem, referred to as the production-distribution-routing problem (i.e. production/order decisions should be given at the warehouse), is to determine how much and when to order for the warehouse, and the replenishment quantities, delivery times and vehicles' routes to the retailers such that total costs comprised of inventory holding costs, order costs and transportation costs are minimized. Inventory holding and transportation costs are the same as described above while order cost is composed of a variable part incurred for each unit ordered for the warehouse (can be considered as the purchasing cost) and a fixed part incurred whenever an order is placed for the warehouse (can be considered as the setup cost or order processing cost). Note that this problem is a generalization of the one-warehouse multi-retailer problem with endogenous policy in that it allows multi-stop routing and there are capacities over the replenishment quantities to retailers.

In this thesis, our aim is to address the above-mentioned problems by proposing effective mathematical programming formulations and solution algorithms. We try to solve these problems to optimality because this not only yields the best solutions but also helps in gauging the quality of solutions attainable by heuristics if necessary. Instead of formulating the problems using standard (mostly weak) mathematical programming formulations, our approach is to develop their strong representations so that exact (i.e. optimal) solutions to the problems can be obtained efficiently. Strong formulations are vital in this regard since they lend themselves to exact solution even using an off-the-shelf optimization solver or they can be used within advanced decomposition/cutting plane algorithms due to their strong bounds. For the two-level problems considered in this thesis, there are very few to a certain extent or even no studies using strong formulations and attempting to obtain exact solutions to the problems. We first investigate the one-warehouse multi-retailer problems with direct shipping policy. We propose strong formulations for these problems which are both theoretically and empirically better than the existing ones. We show that those formulations can be used to solve medium/large size instances (in terms of number of retailers and length of planning horizon) to optimality in reasonable time using an off-the-shelf optimization solver. Second, we consider problems with multi-stop routing policy. Since these problems are much more difficult than their direct shipment counterparts, we propose strong formulations and embed them into advanced cutting plane based algorithms (called branch-and-cut), where some inequalities (cutting planes) are dynamically added to the formulations. Moreover, as multi-stop routing policy adds a significant complexity to the problem and these problems can only be solved exactly up to a certain size, we use our strong formulations in conjunction with an idea to get rid of the complexity due to routing so as to develop heuristics for these problems.

We assume that deterministic external demands occur at the retailers over a planning horizon or in other words, our models accept forecasted demand data as input. As the demand is mostly stochastic in real-life, such deterministic multi-period models are mostly used within a rolling horizon framework (see e.g. Chand et al., 2002), which provide approximations to actual stochastic problems. Bitran and Leong (1992), for instance, have shown that the deterministic approximations under rolling horizon framework are quite satisfactory (3% error) in the context of a multi-period multi-item production planning problem. Dhaenens-Flipo and Finke (2001) have proposed a deterministic multi-period mathematical programming model for a real-life production-distribution problem and solved it to optimality with an off-the-shelf optimization solver. Usage of the model has improved the distribution and lead to the closing of some warehouses. Besides, the studies on one-warehouse multi-retailer systems with stochastic demand ignore many other aspects emerging as the features of real-life such as capacities, fixed (transportation) costs and multi-stop routing (see the references in Doğru, 2006). In this sense, our deterministic approximations via powerful mathematical programming seem promising to deal with real-life problems. Our models with direct shipping policy are for medium-term (tactical) planning and they can be used, for example, in constructing master production schedules while the models with routing policy are aimed at short-term (operational) planning and can be used in detailed production and distribution planning.

Another important issue addressed in this study is about initial inventories at the warehouse. Almost all of the literature related to the problems considered in this thesis assumes zero initial inventories. We consider nonzero initial inventories as an important issue and explicitly incorporate them into our models because they cannot be treated as zero since the models proposed are mostly used within a rolling horizon framework, implying presence of initial inventories. Nonzero initial inventories may affect the complexity of the problem. For instance, the only known easy multi-level lot sizing problem, the one with a serial structure for which Zangwill (1969) proposed a polynomial time recursive algorithm (Pochet and

Wolsey, 2006) assumes zero initial inventories. It is shown that Zangwill’s recursion does not apply in the presence of initial inventories at echelons other than the retailer level and its adaptation accordingly results in an exponential time algorithm (van Hoesel at al., 2005). Therefore, we explicitly incorporate nonzero initial inventories into our models.

## 1.2 Outline

The rest of the thesis is organized as follows. In Chapter 2, we present some preliminaries and a literature review on studies related with the problems considered in this thesis to make it self-contained as much as possible.

In Chapter 3, we consider the one-warehouse multi-retailer problem with endogenous inventory control policy at the retailers. We present two mixed integer programming (MIP) formulations from the literature and propose a new stronger formulation for the problem. We present theoretical and empirical results on the strength of those formulations both in the presence and absence of initial inventory at the warehouse. We also show important theoretical results about the following cases: (i) when there is a single retailer, (ii) when the warehouse does not keep inventory at all. For case (i), our new formulation is tight in that its linear programming (LP) relaxation yields integer optimal solutions. For case (ii), the new formulation and an existing one are LP equivalent. Furthermore, we perform an experimental analysis over a set of randomly generated instances to assess the performance of formulations against each other and to observe whether these formulations are able to solve large instances using an off-the-shelf optimization solver.

Chapter 4 addresses the one-warehouse multi-retailer problem with order-up-to level inventory control policy at the retailers (exogenously defined policy). We show that the problem addressed is ***NP***-hard and propose strong MIP formulations for the problem under zero and nonzero initial inventory at the warehouse. For the case of zero initial inventory at the warehouse and a single retailer, we show that our formulation is tight. We test the performance of our formulations over a set of randomly generated instances against a standard formulation of the problem. We also computationally compare the VMI system (computed by solving MIPs) over the traditional retailer-managed system.

Chapter 5 is devoted to the inventory routing problem with order-up-to level policy where shipments to retailers are performed by multi-stop routing instead of direct shipment. We show that even the feasibility problem is ***NP***-complete in the strong sense. We propose a branch-and-cut algorithm based on a strong MIP formulation for the problem. We also develop a mathematical programming based heuristic algorithm using the strong MIP formulation. We perform benchmarking by comparing the performance of our algorithms (both branch-and-cut and heuristic) with their competitors in the literature over a set of test instances. We discuss how to extend our approach to the problems with different inventory control policies at the retailers.

We consider the production-distribution-routing problem in Chapter 6. We propose a strong MIP formulation for the problem and develop a branch-and-cut algorithm based on the strong formulation. Also, we adapt the mathematical programming based heuristic in Chapter 5 to the production-distribution routing problem. We conduct experimental analysis over several test problems to evaluate the performance of our branch-and-cut and heuristic algorithms against their competitors in the literature.

We conclude the study by briefly stating our contributions and giving future research directions in Chapter 7.

We should note that each chapter of this thesis is self-contained so that the indices, parameters and variables defined in a specific chapter are merely valid for that given chapter.

# CHAPTER 2

# PRELIMINARIES AND LITERATURE REVIEW

In this chapter, we first give some preliminaries necessary for the comprehension and completeness of the thesis. Then, we review the related literature on one-warehouse multi-retailer problem with different inventory control policies, inventory routing problem and production-distribution-routing problem. In addition to these, we give information about some related problems such as single-level and multi-level lot sizing problems as well as vehicle routing problem. In the following chapters, we briefly cite the pertinent studies. We should mention that we use warehouse/supplier/plant as well as retailer/customers mostly interchangeably in the sequel. If they refer to the facilities at different levels, the distinction will be clear from its context.

## 2.1 Preliminaries

A mixed integer (linear) program (MIP) is a mathematical program composed of a linear objective function and linear constraint(s) where some or all of the decision variables are restricted to be integers. Although it is quite powerful in modeling various real-life problems, it is well-known that solving a MIP to optimality is difficult, namely an *NP*-hard problem. Nevertheless, in order to solve a MIP to optimality, a general implicit enumeration technique, called branch-and-bound, has been proposed. The branch-and-bound ensures optimality by actually enumerating all the possible solutions implicitly. Linear programming (LP), which can be solved to optimality efficiently, is an important component of general-purpose branch-and-bound algorithms since LP is solved at each node of the branch-and-bound tree to obtain valid lower bounds (assuming a minimization problem throughout this chapter).

Since the lower bounds are crucial for the success of a branch-and-bound algorithm, one ideally desires to have a MIP formulation with an LP relaxation objective value close to or even equal to the optimal objective value of the MIP formulation. We call a MIP formulation as *strong* when its associated LP relaxation objective value is close to its optimal objective value. For a MIP formulation to be *stronger* than another one, the former should have an LP relaxation objective value closer to the optimal objective value than the latter. A formulation is called *tight* whenever the LP relaxation of that formulation guarantees an integer optimal solution. Whenever a formulation is *tight*, it means that that formulation defines the convex hull of feasible solutions of the corresponding problem or that formulation gives the complete linear description of the corresponding problem.

One way to obtain a stronger formulation is to add valid inequalities (i.e. those inequalities that cut fractional solutions but not integer solutions) into the MIP formulation at hand. Assuming that there exist valid inequalities for a specific formulation, if these valid inequalities are polynomial in number then it is mostly better to add them to the formulation a priori, whereas if valid inequalities are exponential in number then it is better to add them dynamically which is actually referred to as branch-and-cut rather than branch-and-bound. Branch-and-cut algorithms are indeed branch-and-bound algorithms where valid inequalities are usually added dynamically in a cutting plane fashion to each node of the branch-and-bound tree. In addition to the two main decisions in classical branch-and-bound algorithms, namely branching variable and node selection decisions, one has to decide on how often to look for violated inequalities and how to find violated inequalities, called separation problem, if there is any at all in branch-and-cut algorithms.

Another way to obtain a stronger formulation is to try to better represent a problem utilizing its solution or structural features (e.g. knowing the optimality properties). This mostly requires defining new variables (variable redefinition) which are usually larger in number than variables needed in a standard formulation. Obtaining a strong formulation via variable redefinition (also called strong extended formulation), however, has several advantages. First, certain sizes of such strong formulations are even solvable to optimality using off-the-shelf optimization solvers (i.e. MIP software systems) which does not require any advanced programming language knowledge. This also implies that one can easily add several side constraints to the strong formulations and solve a problem with additional issues to optimality by means of a solver. Second, even if it is not possible to obtain exact solutions in reasonable times, one can use these strong formulations within advanced customized cutting plane/decomposition algorithms since they provide lower bounds close to the optimal objective value. One possible disadvantage of strong formulations within a branch-and-bound algorithm may be larger number of variables (may be constraints as well) due to variable redefinition compared to a standard formulation since it will take longer time to solve LP relaxations in the tree.

Although we describe above two ways of obtaining strong formulations, this does not mean that these two ways are mutually exclusive. Of course, one can add valid inequalities to a strong (extended) formulation as well. Branch-and-cut algorithms and strong formulations dominate the current MIP literature as they are very effective and useful in solving a MIP to optimality (Wolsey, 2003; Pochet and Wolsey, 2006). The focus of the book by Pochet and Wolsey (2006), for example, is on recognizing subproblems of production planning problems and better representing these subproblems via either a strong reformulation (variable redefinition) or strong valid inequalities so that the solution of the problem to optimality may be possible.

In the last decade, MIP software systems have dramatically been improved (Bixby et al., 2000; Atamtürk and Savelsbergh, 2005) such that they become powerful tools for solving (mixed) integer programs to optimality. This is mainly due to the improvements in LP solvers and incorporation of effective cutting planes as well as primal heuristics, which significantly improve lower and upper bounds, respectively. The state-of-the-art commercial MIP software systems, such as CPLEX, LINDO and Xpress-MP, are able to solve MIPs with thousands of integer variables within reasonable times (Atamtürk and Savelsbergh, 2005). Moreover, some of these software systems involve environments like Concert Technology and Xpress-Mosel that ease the development of models and algorithms. There are also noncommercial MIP software systems like ABACUS, GLPK, SYMPHONY, etc. but they are still behind the commercial ones with regard to speed and robustness. For a detailed overview on noncommercial MIP software systems, one can refer to Linderoth and Ralphs (2006). In the following, we give brief information about MIP solver of CPLEX 10.1 and Concert Technology in CPLEX, which we use in the succeeding chapters to solve MIPs.

MIP solver of CPLEX 10.1 is basically a branch-and-cut algorithm, default version of which, automatically determines when and how often to look for adding a certain class of valid inequality. Available classes of inequalities in CPLEX 10.1 are clique, cover, disjunctive, flow cover, flow path, Gomory, GUB cover, implied bound and mixed integer rounding inequalities (see Atamtürk and Savelsbergh, 2005 for information on these inequalities), which proved to be effective for general (mixed) integer programs. Concert Technology 2.2 available in CPLEX 10.1 can be used by anyone with some knowledge on C++, C#, Java or Visual Basic to develop customized branch-and bound based algorithms easily. The advantages of using Concert Technology are the chance of rapidly developing an algorithm, not being having to code a branch-and-bound tree structure, which may be difficult, and the opportunity to use CPLEX's inequalities with no coding effort.

## 2.2 Literature review

In the following, we make literature reviews in several areas related with the decision problems we encounter. However, the aim is not to give an exhaustive review, instead we highlight major achievements and describe closely related studies in detail while briefly mentioning different approaches.

### 2.2.1 Single-level lot sizing

One-warehouse multi-retailer systems we consider involve lot sizing issues since there is an associated replenishment problem for each facility (warehouse and retailers). It requires a decision regarding when and how much to order (or ship in terms of transportation) to each facility. Thus, it is pertinent to start with briefly reviewing literature on single-level problems and then to extend the review to multi-level problems, in particular one-warehouse multi-retailer problem.

The most basic problem in the single-level lot sizing literature is the well-known single-item uncapacitated lot sizing problem (ULS) where the problem is to determine when and how much to produce (or order) so as to satisfy external deterministic dynamic demands over a finite time horizon. Since we will propose strong formulations in the following chapters, based on those formulations which are developed for ULS, we would like to give brief information on strong formulations developed for ULS. There are two fundamental strong formulations known for ULS: transportation formulation (*TF*) and shortest path formulation (*SPF*): Krarup and Bilde (1977) develop the *TF* (they called it plant location formulation) and show that it always gives integral solution when its linear programming relaxation is solved. Eppen and Martin (1987) develop *SPF* for a multi-item capacitated lot sizing problem and show that it describes the convex hull of feasible solutions of ULS. Below, we give *TF* and *SPF* formulations.

Consider a finite time horizon $T$ with discrete periods $t = \{1, 2, ..., T\}$. Let $r_t$ be the demand in period $t$, $f_t$ be the fixed order cost in period $t$, $h_t$ be the holding cost incurred for each unit held in stock at the end of period $t$, $p_t$ be the unit order cost in period $t$, and $c_{qt}$ be the unit cost of ordering in period $q$ to satisfy the demand in period *t*. Defining $W_{qt}$ as the quantity ordered in period *q* to satisfy the demand in period *t*, and $z_t$ as 1 if an order occurs in a period *t* and 0 otherwise, the *TF* formulation is as follows.

$$TF\text{: } Min \sum_{q=1}^{T}\sum_{t=q}^{T} c_{qt}W_{qt} + \sum_{t=1}^{T} f_t z_t \tag{2.1}$$

*s.t.*

$$\sum_{q=1}^{t} W_{qt} = r_t \qquad 1 \le t \le T \tag{2.2}$$

$$W_{qt} \le r_t z_q \qquad 1 \le q \le t \le T \tag{2.3}$$

$$W_{qt} \ge 0 \qquad 1 \le q \le t \le T \tag{2.4}$$

$$z_t \in \{0,1\} \qquad 1 \le t \le T \tag{2.5}$$

where $c_{qt} = p_q + \sum_{r=q}^{t-1} h_r$.

Objective function (2.1) is the total of fixed and variable ordering costs and inventory holding cost. Constraints (2.2) ensure that the demand in period *t* is satisfied by ordering through the interval from period 1 to period *t*. Constraints (2.3) stipulate that a fixed order cost is incurred if an order is placed. Constraints (2.4) are for nonnegativity of variables while constraints (2.5) are for integrality of variables.

Additional parameters and variables are defined for *SPF* as follows. Let $R_{tk}$ be the demand from period *t* through period *k*, $e_{tk}$ be the cost of satisfying a fraction of demand from period *t* through *k*. Defining $X_{tk}$ as the fraction of demand from period *t* through *k* that is satisfied in period *t*, *SPF* is as follows.

$$SPF\text{: } Min \sum_{t=1}^{T}\sum_{k=t}^{T} e_{tk}X_{tk} + \sum_{t=1}^{T} f_t z_t \tag{2.6}$$

*s.t.* (2.5) and

$$\sum_{k=1}^{T} X_{1k} = 1 \tag{2.7}$$

$$\sum_{k=t}^{T} X_{tk} - \sum_{k=1}^{t-1} X_{k,t-1} = 0 \qquad 2 \le t \le T \tag{2.8}$$

$$\sum_{k=t}^{T} a_{tk} X_{tk} \le z_t \qquad 1 \le t \le T \tag{2.9}$$

$$X_{tk} \ge 0 \qquad 1 \le q \le t \le T \tag{2.10}$$

where $R_{tk} = \sum_{i=t}^{k} r_i$ , $e_{tk} = p_t R_{tk} + \sum_{l=t}^{k-1} h_l R_{l+1,k}$ and $a_{tk} = \begin{cases} 1 \text{ if } R_{tk} > 0 \\ 0 \text{ otherwise.} \end{cases}$

Objective function (2.6) is the same as (2.1). Constraint (2.7) and (2.8) are flow balance equations of the shortest path network. Constraints (2.9) stipulate that a fixed order cost is incurred if an order is placed. Constraints (2.10) are for nonnegativity of variables. It is shown by Denizel et al. (2008) that both formulations give the same LP relaxation objective values even if more complicated constraints are added to ULS. Thus, these formulations and their equivalence result in the single-level case enables us the opportunity to derive different formulations based on *TF* and/or *SPF* for the two-level one-warehouse multi-retailer problems.

ULS and its variants have been extensively studied in the literature and one can refer to Pochet and Wolsey (2006) for a detailed analysis of these problems.

### 2.2.2 Two-level lot sizing

We now review one-warehouse multi-retailer (OWMR) problem with deterministic demand occurring at retailers and its two special cases. One of these special cases is the single retailer case, referred to as single-warehouse single retailer (SWSR) problem. The other is where the warehouse acts as a crossdocking or transshipment point (i.e. no inventory is kept at the warehouse), referred to as joint replenishment problem (JRP).

*Review on the OWMR problem*

Studies on infinite horizon OWMR problems commonly assume a constant deterministic demand rate and try to minimize the long-run average cost of the system. They consider a specific policy (mainly stationary nested) to approximate the optimal policy. Examples to such studies are Schwarz (1973), Roundy (1985), Gallego and Simchi-Levi (1990), and Yao and Wang (2006). For a recent review on infinite horizon or continuous review OWMR problems, one may refer to Yao and Wang (2006).

Federgruen and Tzur (1999) propose a time-partitioning heuristic for the multi-item OWMR problem, which can be designed to give a certain performance guarantee. The heuristic divides the problem into smaller interval problems that are modeled using echelon stock formulation and solved to optimality by a Lagrangian relaxation based branch-and-bound algorithm. The smaller problems are then merged into a solution to the original problem. Levi et al. (2008) formulate the OWMR problem with a general inventory holding cost structure and stationary fixed order cost at retailers as a MIP, whose LP relaxation is used to develop an approximation algorithm with worst case performance guarantee of 1.8 times the optimal objective value. Chan et al. (2002) address a variant of the OWMR problem where a piecewise linear concave order cost (modified all-units discount order cost) is incurred for the shipments to the warehouse (retailers) acting as a crossdocking point. They show that zero-inventory ordering (ZIO) policy has a worst case performance guarantee of 4/3 times the optimal objective value, and propose an LP-based heuristic which finds a nearly-optimal ZIO policy as finding the optimal ZIO policy itself is an *NP*-hard problem. Jin and Muriel (2006) propose two Lagrangian decomposition algorithms based on a standard MIP formulation for a variant of the OWMR problem where all the cost parameters are constant over time and there are cargo constraints for the replenishment quantities to both the warehouse and the retailers requiring enough number of trucks to be dispatched.

*Review on the JRP*

Contrary to the limited literature on the OWMR problem, there are a vast number of studies for the JRP. Note that the JRP is also called coordinated (replenishment) lot sizing problem or lot sizing with joint (or major) setups in the literature. Zangwill (1966) and Kao (1979) propose dynamic programming algorithms while Erengüç (1988) develops a hybrid algorithm composed of dynamic programming and branch-and-bound. Those mentioned studies, however, can only solve instances with a few number of periods and retailers (or items) to optimality. Remarkable progress in solving larger sized instances is achieved by developing algorithms based on strong mathematical programming formulations for the JRP, which capitalize on tight representations of the uncapacitated single-item lot sizing problem (Wagner and Whitin, 1958). Joneja (1990) is the first to formulate the JRP as a shortest path based formulation. Kırca (1995) proposes a dual ascent based branch-and-bound algorithm using this formulation. Robinson and Gao (1996) is the first to develop a transportation based formulation and propose another dual ascent based branch-and-bound algorithm to solve this formulation. Both algorithms are currently the best exact algorithms for the JRP (Gao et al., 2008). Several studies empirically compare the LP relaxation objective values of shortest path and transportation based formulations and find the same values on a wide set of test instances (Gao et al., 2008; Robinson et al., 2009). There are also several studies considering heuristic solution approaches (see e.g. Joneja, 1990; Boctor et al., 2004; Federgruen et al., 2007). Interested reader can refer to Robinson et al. (2009) for a recent review on the JRP.

*Review on the SWSR problem*

The SWSR problem is also the two-level case of the uncapacitated multi-level lot sizing problem in series (Zangwill, 1969). Zangwill (1969) proposes a DP algorithm that runs in $O(T^3)$ time for the SWSR problem with general concave costs where $T$ is the horizon length. The DP algorithm of van Hoesel et al. (2005) for the two level case runs in $O(T^7)$ under general concave production, inventory and transportation costs with stationary (i.e. constant) production capacity at the warehouse level. Lee et al. (2003) consider a variant of SWSR where a fixed cost plus fixed cost per vehicle dispatched (stepwise cargo cost structure) is associated with the shipments from the warehouse to the retailer. They develop a DP algorithm running in $O(T^6)$ when backlogging is allowed and in $O(T^4)$ when backlogging is not allowed. Jin and Muriel (2006) propose a DP algorithm that runs in $O(T^3)$ time when there is a single retailer in their problem. Solyalı and Süral (2008a) consider a variant of SWSR where the retailer employs order-up-to level policy. They propose a DP algorithm running in $O(T^3)$ time for the problem. Also, they present a pseudo-polynomial DP algorithm to determine the optimal order-up-to level besides the replenishment quantities to the warehouse and retailer.

### 2.2.3 Multi-level lot sizing

Based on the product structure, multi-level lot sizing problems can be classified into four groups: series structure (i.e. each node has only one predecessor and one successor), assembly structure (i.e. each node has only one successor), general structure and production-distribution/arborescent structure (i.e. each node has only one predecessor). Among these, only the uncapacitated multi-level lot sizing problem in series is solvable in polynomial time (Zangwill, 1969). On the other hand, the problem with assembly structure is still an open problem such that neither a polynomial time algorithm nor an NP-hardness result exists for it (Pochet and Wolsey, 2006). For detailed information on these problems, one can refer to Chapter 13 of Pochet and Wolsey (2006).

Zangwill (1969) considers an uncapacitated multi-level (say, $L$ levels) lot sizing problem in series and proposes a DP algorithm for solving the problem, which runs in $O(T^3+(L-2)T^4)$ time (van Hoesel et al., 2005). van Hoesel et al. (2005) extend Zangwill's work to a more general problem in which a stationary capacity on production is considered in the first level. To the best of our knowledge, van Hoesel et al. (2005) is the only study that deals with initial inventories at levels other than last level (retailer level).

For multi-level lot sizing problem with assembly or general structure, echelon stock concept plays an important role such that the strongest MIP formulations up to now are obtained using echelon stock concept. Echelon stock idea enables one to separate lot sizing problem of each item in the product structure and thus each of these lot sizing problems can be represented using transportation or shortest path formulations given in Section 2.2.1 which gives the strongest formulations (Stadtler, 1996 and 1997). These formulations are related with the echelon stock formulation proposed for the OWMR problem by Federgruen and Tzur (1999).

Our interest in multi-level lot sizing problem with arborescent structure is due to its being a generalization of the OWMR problem. For this problem, Veinott (1969) and Kalymon (1972) develop exact dynamic programming and implicit enumeration algorithms, respectively, which are exponential in running time. Diaby and Martel (1993) propose a Lagrangian relaxation algorithm based on a standard formulation to the problem where general piecewise linear costs are incurred for the shipments.

### 2.2.4 Vehicle routing problem

The problems with multi-stop routing policy involve decisions regarding the routing of vehicles such that the sequence of customers to be visited should be decided. The classical vehicle routing problem (VRP), a strongly NP-hard problem, is the problem of finding a collection of routes with each starting from the depot, visiting a subset of customers without exceeding vehicle capacity and returning back to the depot such that total distance is minimized. VRP has a close connection with inventory routing problem and production-distribution-routing problem in that they can be seen as a multi-period extension of VRP with some side constraints. Thus, it is pertinent to provide a review on VRP literature.

VRP has been widely studied by researchers such that many exact and heuristic algorithms have been proposed (Laporte, 2007). The most successful exact algorithms proposed up to now are based on sophisticated branch-and-cut (Naddef and Rinaldi, 2002; Lysgaard et al., 2004; Baldacci et al., 2004) and branch-and-cut-and-price algorithms (Fukasawa et al., 2006; Baldacci et al., 2008). Naddef and Rinaldi (2002) review the exact algorithms proposed in the VRP literature up to 2002 and present the best algorithm using a two-index vehicle flow formulation which can solve instances up to 135 customers to optimality with a variable success rate. Lysgaard et al. (2004) improve the two-index flow formulation by adding effective inequalities such as framed capacity, strengthened comb, generalized multistar, etc. in a cutting plane fashion. Baldacci et al. (2004) propose an exact algorithm based on a two-index two-commodity flow formulation. Fukasawa et al. (2006) and Baldacci et al. (2008) propose a set partitioning formulation (STP) with additional inequalities. They dynamically generate routes (variables in STP) via a pricing problem and add valid inequalities to STP. Currently, the best exact algorithms are due to Fukasawa et al. (2006) and Baldacci et al. (2008), which can solve instances up to 135 customers to optimality with a constant success rate. In the following, we give the two-index vehicle flow formulation for the VRP with homogeneous fleet because we will use this formulation in Chapters 5 and 6.

Let 0 denote the depot, $M$ and $M'$ be the set of customers and the set of facilities respectively where $M' = \{0\} \cup M$, $c_{ij}$ be the cost of traveling from facility $i \in M'$ to $j \in M'$, $d_i$ be the demand of customer $i \in M$, $V$ be the number of vehicles, $Q$ be the capacity of each vehicle. Define $y_{ij}$ as 1 if vehicle visits facility $j \in M'$ immediately after facility $i \in M'$, and 0 otherwise. Then the two-index vehicle flow formulation is as follows.

$$VF: Min \sum_{i \in M'} \sum_{j \in M', i > j} c_{ij} y_{ij} \quad (2.11)$$

*s.t.*

$$\sum_{j \in M} y_{j0} = 2V \tag{2.12}$$

$$\sum_{j \in M', j<i} y_{ij} + \sum_{j \in M', j>i} y_{ji} = 2 \qquad i \in M \tag{2.13}$$

$$\sum_{i \in S} \sum_{j \notin S, j<i} y_{ij} + \sum_{i \notin S} \sum_{j \in S, j<i} y_{ij} \geq 2r(S) \qquad S \subseteq M, |S| \geq 2 \tag{2.14}$$

$$y_{ij} \in \{0,1\} \qquad i \in M, j \in M, j < i \tag{2.15}$$

$$y_{i0} \in \{0,1,2\} \qquad i \in M \tag{2.16}$$

Objective function (2.11) is the total traveling cost. Constraints (2.12) stipulate that $V$ vehicles depart and return back to the depot. Constraints (2.13) are degree constraints ensuring that two edges are incident to customer $i$. Constraints (2.14) ensure that subtours are eliminated and capacity of vehicles are not exceeded. In constraints (2.14), $r(S)$ denotes the minimum number of vehicles required to satisfy demands of customers in $S$. Researchers use a lower bound value, $\left\lceil \sum_{i \in S} d_i / Q \right\rceil$, instead of finding the exact value of $r(S)$ which requires solving an NP-hard bin packing problem. Constraints (2.15) and (2.16) are for the integrality of variables. In constraints (2.16), $y_{i0}$ is allowed to take 2 to account for single retailer trip between customer $i \in M$ and depot. Note that constraints (2.14) can be equivalently rewritten as

$$\sum_{i \in S} \sum_{j \in S, j<i} y_{ij} \leq |S| - r(S) \qquad S \subseteq M, |S| \geq 2 \tag{2.17}$$

There is also a very rich literature on heuristic algorithms applied to VRP. For a recent review of exact and heuristic solution approaches to VRP, one can refer to Laporte (2007). VRP and its variants have been extensively studied in the literature and one can refer to Toth and Vigo (2002) as well as Golden et al. (2008) for a detailed treatment of those problems.

The well-studied traveling salesman problem (TSP) is a special case of the VRP where there is a single vehicle with enough capacity to visit all the customers. Although TSP is a strongly NP-hard problem, its solution is a success story in combinatorial optimization since instances with thousands of customers (up to 2500 customers) can be solved to optimality within reasonable times (<1000 CPU seconds) by means of a solver, called CONCORDE. One can refer to Applegate et al. (2007) for a detailed treatment of TSP and an explanation of the theory and algorithms utilized to develop CONCORDE. We use CONCORDE to solve TSPs arising in inventory routing and production-distribution-routing problems in Chapters 5 and 6 to optimality.

### 2.2.5 Inventory routing problems

The inventory routing problem (IRP) can be defined as the problem of deciding on delivery times, quantities and routes to customers such that a criterion (cost or profit) is optimized. There are numerous studies on different variants of the IRP. Researchers have considered different characteristics such as planning horizon (finite, infinite) and demand process (deterministic, stochastic) under different cost structures or profit. In this subsection, we give brief information on related studies by mentioning the important aspects. We restrict ourselves to the deterministic cases and refer the interested reader to Hvattum and Lokketangen (2008) for stochastic cases.

First studies on IRP appeared in 1980s and attempted to take into account inventory control in addition to vehicle routing on single period models. The seminal work by Federgruen and Zipkin (1984) considers distribution of a limited quantity of a single product available at a supplier to multiple retailers with stochastic demand using a fleet of capacitated vehicles. Their aim is to minimize the expected inventory holding and shortage costs as well as routing costs. Federgruen et al. (1986) extend the former work to the case of perishable products. Chien et al. (1989) address a single period problem with deterministic demand at retailers where the aim is to maximize revenues less delivery costs. They formulate the problem as a MIP, which they use to develop a Lagrangian relaxation algorithm yielding good upper and lower bounds.

Dror et al. (1985) and Dror and Ball (1987) are the first to study a multi-period IRP. They reflect the long-term effect of short-term decisions transforming the multi-period problem into a single period problem where demands at customers are treated as deterministic. While the latter focuses on the analysis of transforming the multi-period problem into a single period problem, the former considers the solution of the single period problem. Campbell and Savelsbergh (2004) develop a two-phase solution approach to the multi-period IRP with constant demand at customers. Delivery quantities and times are determined in the first-phase by solving an integer program, and delivery routes are obtained using heuristics in the second-phase. Bertazzi et al. (2002), Pınar and Süral (2006), and Archetti et al. (2007a) address the multi-period IRP with deterministic dynamic demand at customers. They employ a deterministic order-up-to level inventory control policy at customers (or retailers), which requires the supplier to raise each customer's inventory level to its predetermined maximum level whenever visited. Bertazzi et al. (2002) develop an improvement heuristic to the problem and analyze the impact of different cost structures on the solution. Pınar and Süral (2006) propose a Lagrangian relaxation algorithm which yields upper and lower bounds to the problem. Archetti et al. (2007a), on the other hand, propose the only exact algorithm for the multi-period IRP and analyze the effect of relaxing the order-up-to policy at customers. In contrast to the Dror et al. (1985), Dror and Ball (1987) and Campbell et al. (2004) where a limitless amount of product is assumed to be available at the supplier whenever needed, Bertazzi et al. (2002), Pınar and Süral (2006), and Archetti et al. (2007a) assume that the supplier receives a given amount of product each period and supplier can only distribute the amount in its inventory. Abdelmaguid et al. (2008) consider a multi-period IRP with backlogging allowed at customers and propose construction and improvement heuristics to solve the problem. They also provide a multi-commodity flow based MIP formulation which they use to obtain lower and upper bounds by solving with an off-the-shelf solver. Yugang et al. (2008) also study a multi-period IRP with dynamic demand at customers and bound constraints on the inventory levels at the customers. They propose two Lagrangian relaxation algorithms: one is based on an approximate MIP formulation that provides upper bound while the other is a complete formulation yielding lower bounds.

Anily and Federgruen (1990), Anily (1994) and Viswanathan and Mathur (1997) consider infinite horizon IRPs with constant deterministic demand at customers where the aim is to minimize long-run average costs. Anily and Federgruen (1990) propose a heuristic, which is asymptotically optimal under certain conditions. Anily (1994) generalizes the work of Anily and Federgruen (1990) to the case of retailer-dependent holding costs while the study of Viswanathan and Mathur (1997) generalizes that of Anily and Federgruen (1990) to the multiple products.

Burns et al. (1985), Gallego and Simchi-Levi (1990), and Bertazzi (2008) analyze the performance of direct shipping policy and multi-stop routing policy on infinite horizon. Burns et al. (1985) neglect many details of the system (e.g. spatial density of customers is used instead of their precise locations) and obtain analytical formulas in terms of a few measurable parameters which enable one to make sensitivity analyses and cost trade-off easily and quickly. Their results reveal that optimal shipment size is given by the economic order quantity formula for direct shipping policy whereas it is the full truck for multi-stop routing policy. Gallego and Simchi-Levi (1990) show that direct shipping policy is at most 1.061 of the optimal policy when the minimal economic lot size over all retailers is at least 71% of the vehicle capacity. Bertazzi (2008) analyzes different direct shipping policies in terms of their worst case performance and their empirical performance (on randomly generated instances). In Gallego and Simchi-Levi (1990), shipments can be performed in continuous time whereas shipments can be performed in discrete time in Bertazzi (2008).

There are also some other studies that differ from the standard inventory routing problem in some aspects. Webb and Larson (1995) consider a strategic IRP in that they try to find the best fleet size. Bard et al. (1998) and Jaillet et al. (2002) consider IRP with satellite facilities where the vehicles can be reloaded. Savelsbergh and Song (2008) address an IRP with continuous moves in which customers cannot be served in a single period by out-and-back trips since delivery to customers spans more than a single period, and product pickups occur at different facilities. They develop an optimization algorithm for the problem.

For detailed information on IRP, we refer the readers to the studies themselves or to review papers such as Campbell et al. (1998; 2002), Baita et al. (1998), Schwarz et al. (2004), and Moin and Salhi (2007).

### 2.2.6 Production-distribution routing problem

The Production-distribution-routing (PDR) problem is a generalization of the IRP in that production/order decisions should be given in addition to the IRP decisions. The PDR problem can also be seen as an integrated production planning (lot sizing) and distribution management (vehicle routing) problem. Although there is a vast amount of literature on the IRP, the literature on the PDR problem is rather limited. While the studies on the PDR problem consider different characteristics such as single/multiple items, no production/production capacity, not to/to split deliveries, etc., the majority of them propose two-phase heuristic methods where the solution of the production planning problem at the upper level (i.e. at the warehouse) is an input to the distribution management problem (as in the form of multi-period VRP or IRP) or vice versa. Those studies that develop two-phase heuristics mostly lack a lower bounding procedure. They mainly try to measure the cost savings attainable by a coupled approach over the decoupled/sequential approach.

Chandra (1993) addresses a system in which a warehouse orders multiple items and distributes to the retailers via capacitated vehicles over a finite horizon. The vehicle fleet is unrestricted in size and there is no production/order capacity at the warehouse. It is allowed to split deliveries (i.e. delivering to a retailer with more than one vehicle is possible in any period) which can reduce the routing costs significantly compared to the case where split delivery is not allowed (Archetti et al., 2008). The author investigates the savings obtained by the coupled approach over the decoupled one. Chandra and Fisher (1994) is an extension of Chandra (1993) which consider production capacities at the warehouse. Yugang et al. (2007) show that the MIP formulation proposed in Chandra and Fisher (1994) is not correct in that it may not yield the optimal solution but can be used to find lower bounds. Fumero and Vercellis (1999) consider the same problem as in Chandra and Fisher (1994) but with a restricted fleet size for vehicles and a quantity as well as distance based transportation cost instead of the distance based cost at Chandra and Fisher (1994). Fumero and Vercellis (1999) propose a multi-commodity flow based MIP formulation and develop a Lagrangian relaxation algorithm yielding upper and lower bounds to the problem. It is the first study to propose a lower bounding procedure in the PDR literature.

Bertazzi et al. (2005) address a PDR problem with order-up-to level inventory policy at the retailers. They propose an improvement heuristic in which the initial solution is found by a decoupled approach and then improved by modifying replenishment decisions of retailers taking into account the impact of this modification on the corresponding routing costs and costs at the warehouse. They show that the vendor managed inventory policy (found by their improvement heuristic) reduces total cost significantly compared to the retailer managed policy (found by the decoupled approach). They also partially relax the order-up-to level policy and obtain reduction in the total cost. Solyalı and Süral (2008b) consider a PDR problem with order-up-to level inventory policy at the retailers but have some differences with Bertazzi et al. (2005) in the cost structure. They propose a multi-commodity flow formulation for the problem, which is used to develop a

Lagrangian relaxation algorithm providing upper and lower bounds. The study of Solyalı and Süral (2008b) reveal that it is only possible to solve small instances (8 retailers, 5 time periods and a single vehicle) to optimality with the multi-commodity flow formulation due to its large number of binary variables and constraints.

Lei et al. (2006) address a PDR problem motivated by a real-life problem where there are multi-plants producing a single product subject to production capacities. The product is distributed by heterogeneous vehicles. Each vehicle is allowed to make multiple trips in a period provided that the available time is not exceeded. Unlike Chandra (1993), Chandra and Fisher (1994), and Fumero and Vercellis (1999), Lei et al. (2006) consider inventory bound constraints on the level of inventory carried at the plants and customers. They propose a multi-commodity flow formulation for the problem and develop a two-phase algorithm. In the first phase, routing constraints are removed from the formulation (i.e. direct shipment is assumed between plants and customers) which is solved to give a feasible solution to the problem. Then, in the second phase, the feasible solution of the first phase is tried to be improved by consolidating the shipments into routes involving multiple customers. The algorithm is benchmarked against the best solution found by CPLEX using the complete formulation within 4-hour time limit on small instances (up to 12 customers, 4 time periods, 2 heterogeneous vehicles). Archetti et al. (2007b) consider a plant with no production capacity distributing a single product to multiple retailers with capacitated vehicles. They consider a fleet of homogeneous vehicles unrestricted in size and do not allow split deliveries. Like Lei et al. (2006), they have inventory bound constraints but only on the inventory levels of retailers. They refer to this PDR problem as the PDR with maximum level policy. For the single vehicle case, they propose a branch-and-cut algorithm using a standard formulation, which is the only exact algorithm for a PDR problem we are aware of. They also propose an improvement heuristic for the problem with multi-vehicles. They compare the best solutions found by their branch-and-cut and heuristic algorithms on a set of test problem instances (19 retailers, 6 time periods, single vehicle). Furthermore, they compare the PDR problem with order-up-to level policy and maximum level policy and show that the solution found by order-up-to level policy can be arbitrarily worse than the relaxed one.

Recently, a couple of studies, such as Boudia et al. (2007), Boudia and Prins (2007), Boudia et al. (2008), and Bard and Nananukul (2008), address a PDR problem where a plant produces a single product subject to production capacity and ships to multiple customers using a fleet of capacitated vehicles. They consider inventory bound constraints on the inventory levels of both the plant and customers. They do not allow split delivery. The distinguishing feature of those studies is that they do not consider inventory holding cost at the customers. Boudia and Prins (2007) propose a memetic algorithm combined with population management which outperforms their heuristics in Boudia et al. (2007; 2008). Bard and Nananukul (2008) develop a two-phase algorithm similar to that of Lei et al. (2006) such that in the first phase a standard MIP formulation without routing constraints (they called it as allocation model) is solved to obtain an initial solution and in the second phase, a reactive tabu search is developed that tries to improve the initial solution. Different from Boudia et al. (2007), Boudia and Prins (2007), and Boudia et al. (2008), Bard and Nananukul (2008) propose a lower bounding procedure based on a modification of the allocation model, though it is not so effective.

While above mentioned studies consider dynamic deterministic demands at retailers over a finite horizon, there are also studies like Anily and Federgruen (1993) and Herer and Roundy (1997) that incorporate predictable vehicle routing costs to the basic infinite horizon OWMR problem.

# CHAPTER 3

# THE ONE-WAREHOUSE MULTI-RETAILER PROBLEM WITH ENDOGENOUS POLICY

In this chapter, we consider the one-warehouse multi-retailer (OWMR) problem with endogenous policy in which a warehouse places orders and decides on when and how much to ship to the retailers. OWMR problems have been widely studied in the literature under various settings, as discussed in Chapter 2. The OWMR problem considered here can be thought of as a two-level lot sizing problem generalizing the well-known uncapacitated single-level lot sizing problem (ULS). The OWMR problem also generalizes the joint replenishment problem (JRP) such that the former allows keeping inventory at the warehouse level whereas the latter does not (i.e. the warehouse acts as a crossdocking or transshipment point in JRP). Although there are various studies on strong mixed integer programming formulations and exact algorithms based on such formulations for ULS and JRP, there are only two studies considering strong MIP formulations in the OWMR literature. However, both studies use their strong formulations in developing heuristic algorithms with performance guarantees rather than using them in devising exact algorithms.

One of our aims in this chapter is to devise a stronger formulation than the existing ones for the OWMR problem such that solving certain sizes of OWMR problems to optimality would be possible by means of an off-the-shelf optimization solver. We consider strong formulations as important because the OWMR problem is not only important in its own right, but also arises as a subproblem in many involved settings such as variants with capacities over replenishment quantities and variants with multi-stop routing policy. As the strong formulations lend themselves to an exact solution, they create an opportunity in solving such complex variants. Consider, for instance, the OWMR problem with multi-stop routing policy (called production-disribution routing problem in Chapter 1) where the problem is comprised of two parts: inventory replenishment part (i.e. OWMR problem) and routing part. Although the routing part is well-studied and its shortcomings are well-known, to the best of our knowledge, strong formulations of the inventory replenishment part have not been studied and implemented yet in the literature. For an evidence of the effectiveness of using a strong formulation in that context, one can refer to Chapter 6. Another aim in this chapter is to analyze the impact of nonzero initial inventories at the warehouse, the importance of which is discussed in Chapter 1. Note that the amount of inventory initially available at the warehouse cannot be eliminated simply by deducing that amount from the beginning periods' demands since how much each retailer will demand is not known a priori.

In this chapter, we propose a new shortest path based strong formulation for the OWMR problem. Considering two other formulations, namely, echelon stock (Federgruen and Tzur, 1999) and transportation based (Levi et al., 2008) formulation, we analyze and demonstrate the relation among their LP relaxations. We show that the new formulation gives the complete linear description of the OWMR problem if there is a single retailer, referred to as single warehouse-single retailer (SWSR) problem, whereas the previously proposed formulations do not. As an important consequence of this, we reveal that the new formulation is stronger than the transportation based one which is stronger than the echelon stock formulation. It is contrary to the results in single-level lot sizing where shortest path and transportation based formulations are the LP equivalent (see Nemhauser and Wolsey, 1988 for ULS problem and Denizel et al., 2008 for the capacitated multi-item lot sizing problem with setups). Besides, we resolve the question of whether the empirical results on the equivalence of LP relaxation solution values are valid for all instances of the JRP or are just due to a given sample of instances in the literature, and elucidate that both formulations of the JRP are theoretically equivalent. Also, we explicitly consider nonzero initial inventory at the warehouse, extend all formulations to the case of nonzero initial inventory at the warehouse and present the relation among LP relaxations of those formulations. Finally, we test the computational performance of the MIP formulations on a set of test instances. Computational results reveal that our strong formulation is quite satisfactory to close the integrality gap and to solve large problem instances, using standard MIP solvers.

The rest of this chapter is organized as follows. In Section 3.1, we present the MIP formulations for the OWMR problem and analyze the strength of their LP relaxations with respect to each other. We consider the SWSR problem and the JRP in Sections 3.2 and 3.3, respectively. In Section 3.4, we extend the reviewed formulations to the case of nonzero initial inventory at the warehouse and analyze the strength of their LP relaxations. Section 3.5 is devoted to the computational experiments. Note that the notation and abbreviations defined in this chapter is only valid in this chapter and in Appendix A.

### 3.1 Problem definition and formulations

The OWMR problem is defined as follows. A warehouse replenishes multiple retailers over a finite time horizon $T$. Retailer $i$ ($1 \leq i \leq N$) faces external deterministic dynamic demand $d_{it}$ in period $t$ ($1 \leq t \leq T$) and may keep inventory $I_{it}$ at the end of period $t$ to satisfy demands of future periods $k$, where $t+1 \leq k \leq T$. The warehouse ($i$=0) manages the entire inventories in the system and has to order from its supplier so as to be able to replenish the retailers. The warehouse may keep inventory $I_{0t}$ to satisfy future retailers' demands. There are no capacities over the replenishment quantities in any level. The shipments to the warehouse incur a fixed order cost $f_{0t}$, independent of the size of shipment, and a variable order cost $p_{0t}$, which is charged for each unit ordered in $t$. A fixed order cost $f_{it}$ and a variable order cost $p_{it}$ are also incurred whenever retailer $i$ ($1 \leq i \leq N$) receives a shipment in $t$ ($1 \leq t \leq T$). Both parties incur a linear holding cost for each item carried at the end of a period, $h_{it}$. All the parameters are assumed to be nonnegative. The OWMR problem is to jointly determine lot sizing policy of the warehouse and the retailers such that the total of inventory holding costs and order costs at both levels is minimized. Arkin et al. (1989) show that the OWMR problem is *NP*-hard by reducing the JRP to it.

We assume, without loss of generality, that there is no lead time for the shipments in and between levels. We also assume that there is no initial inventory in any level. Note that having initial inventory at retailers does not have any impact on the problem difficulty since one can simply deduce external demands at the retailers from their initial inventory levels and obtain an equivalent problem with zero initial inventories at the retailers. However, it is not the case when assuming initial inventories at the warehouse as we elaborate later.

A standard formulation for the OWMR problem is given in Appendix A which has $O(NT)$ binary and continuous variables, and $O(NT)$ constraints. It is a small-size weak formulation and there is no study, to the best of our knowledge, which uses it for solving the problem to optimality. There are two alternative formulations proposed in the literature for the OWMR problem: echelon stock formulation and transportation type lot sizing formulation. Besides presenting them, we will propose a new formulation in this section.

In addition to the notation defined before, we define several parameters and variables that will commonly be used in the subsequent subsections. Let $D_{itk}$ be the total demand of facility $i$ $(0 \le i \le N)$ from period $t$ through $k$, $D_{itk} = \sum_{r=t}^{k} d_{ir}$ , and $y_{it}$ be 1 if an order for facility $i$ ( $0 \le i \le N$ ) is placed in period $t$ and 0 otherwise.

### 3.1.1 Echelon stock formulation

The echelon stock (*ES*) formulation given below is proposed by Federgruen and Tzur (1999). Let $d_{0t}$ be the total external demand in period *t*, i.e. $d_{0t} = \sum_{i=1}^{N} d_{it}$ . Let $Q_{it}$ be the quantity ordered for facility *i* ( $0 \le i \le N$ ) in period *t*.

$$ES\text{: } Min \sum_{t=1}^{T} \left\{ \sum_{i=0}^{N} f_{it} y_{it} + \sum_{i=0}^{N} p_{it} Q_{it} + h_{0t} I_{0t} + \sum_{i=1}^{N} (h_{it} - h_{0t}) I_{it} \right\} \quad (3.1)$$

*s.t.*

$$I_{i,t-1} + Q_{it} = d_{it} + I_{it} \qquad 0 \le i \le N,\ 1 \le t \le T \quad (3.2)$$

$$Q_{it} \le D_{itT} y_{it} \qquad 0 \le i \le N, 1 \le t \le T \quad (3.3)$$

$$\sum_{r=1}^{t} Q_{0r} \ge \sum_{i=1}^{N} \sum_{r=1}^{t} Q_{ir} \qquad 1 \le t \le T \quad (3.4)$$

$$y_{it} \in \{0,1\} \qquad 0 \le i \le N,\ 1 \le t \le T \quad (3.5)$$

$$Q_{it} \ge 0 \qquad 0 \le i \le N,\ 1 \le t \le T \quad (3.6)$$

$$I_{it} \ge 0 \qquad 0 \le i \le N,\ 1 \le t \le T \quad (3.7)$$

The objective function (3.1) of the model is the sum of fixed and variable order costs and inventory holding costs at the warehouse and retailers. Constraints (3.2) are the inventory balance constraints for the warehouse and retailers. Constraints (3.3) stipulate that a fixed order cost is incurred at facility *i* ( $0 \le i \le N$ ) if an order is placed for *i* in a period. Constraints (3.4) ensure that the total amount ordered for the warehouse up to and including period *t* must be greater than or equal to the total amount ordered for all of the retailers up to and including *t*. Constraints (3.5) are for integrality of variables while (3.6) and (3.7) are for nonnegativity of variables.

The *ES* separates the lot sizing decisions at the warehouse and retailers, and then associates those decisions via a linking constraint like constraints (3.4). Note that constraints (3.2), (3.3) and (3.5)–(3.7) can be decomposed into *N*+1 facilities, each of which defines an ULS problem, when constraints (3.4) are relaxed. Federgruen and Tzur (1999) dualize constraints (3.4) into the objective function (3.1) to develop a Lagrangian relaxation based branch-and-bound algorithm so as to solve small-size OWMR problems as part of their heuristic. They solve $N$+1 ULS problems to optimality in each iteration of the relaxation. The Lagrangian relaxed problems do not have integrality property; therefore one can find better Lagrangian bound values than the LP relaxation solution values of the *ES* in this framework.

We can obtain a stronger *ES* (*SES*) by replacing constraints (3.2), (3.3) and (3.5)–(3.7) with their strong counterparts of the formulation giving the convex hull of feasible solutions of ULS. For this purpose, we use transportation formulation for developing strong counterparts, and thus show the relation among LP relaxation solution values of different formulations.

Defining $X_{itk}$ as the quantity ordered to facility $i$ ($0 \le i \le N$) in period $t$ to satisfy the demand of $i$ in period $k$ ($1 \le t \le k \le T$), a stronger formulation *SES* is obtained as follows.

$$SES\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=0}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H_{itk}X_{itk} \qquad (3.8)$$

*s.t.* (3.5) and

$$\sum_{t=1}^{k} X_{itk} = d_{ik} \qquad 0 \le i \le N, 1 \le k \le T \qquad (3.9)$$

$$X_{itk} \le d_{ik}y_{it} \qquad 0 \le i \le N, 1 \le t \le k \le T \qquad (3.10)$$

$$\sum_{r=1}^{t}\sum_{k=r}^{T} X_{0rk} \ge \sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T} X_{irk} \qquad 1 \le t \le T \qquad (3.11)$$

$$X_{itk} \ge 0 \qquad 0 \le i \le N, 1 \le t \le k \le T \qquad (3.12)$$

where $H_{itk} = p_{it} + \sum_{l=t}^{k-1}(h_{il} - h_{0l})$ for $1 \le i \le N$, and $H_{0tk} = p_{0t} + \sum_{l=t}^{k-1} h_{0l}$.

Objective function (3.8) is equivalent to (3.1) while constraints (3.10) and (3.11) are used in place of (3.3) and (3.4), respectively. Constraints (3.9) are for demand satisfaction at the warehouse and retailers instead of balance equations (3.2). Constraints (3.12) are for nonnegativity of variables. Note that *SES* has $O(NT)$ binary variables and $O(NT^2)$ constraints.

*SES* is stronger than *ES* since the former describes the convex hull of feasible solutions of ULS problem for each facility *i* ( $0 \le i \le N$ ) whereas the latter does not. One can show that the best Lagrangian bound value attainable in Federgruen and Tzur (1999) would not be better than the LP relaxation solution value of the *SES*.

### 3.1.2 Transportation based formulation

We refer to the next formulation as transportation based formulation (*TP*) since lot sizing problems of both the warehouse and the retailers are modeled using transportation type lot sizing formulation. Let $H'_{itk}$ $(= p_{it} + \sum_{r=t}^{k-1} h_{ir})$ be the unit cost of satisfying demand of facility *i* ( $0 \le i \le N$ ) in period *k* by placing an order in period *t*. Let $W_{iqtk}$ be the quantity ordered by the warehouse in period *q* and sent to the retailer *i* in period *t* to satisfy the demand of *i* in period *k* ( $1 \le q \le t \le k \le T$ ).

$$TP\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} H'_{0qt}W_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk}X_{itk} \tag{3.13}$$

*s.t.* (3.5) and

$$\sum_{q=1}^{t} W_{iqtk} = X_{itk} \qquad 1 \le i \le N,\ 1 \le t \le k \le T \tag{3.14}$$

$$\sum_{t=q}^{k} W_{iqtk} \le d_{ik}y_{0q} \qquad 1 \le i \le N,\ 1 \le q \le k \le T \tag{3.15}$$

$$\sum_{t=1}^{k} X_{itk} = d_{ik} \qquad 1 \le i \le N, 1 \le k \le T \tag{3.16}$$

$$X_{itk} \le d_{ik}y_{it} \qquad 1 \le i \le N,\ 1 \le t \le k \le T \tag{3.17}$$

$$X_{itk} \ge 0 \qquad 1 \le i \le N,\ 1 \le t \le k \le T \tag{3.18}$$

$$W_{iqtk} \geq 0 \qquad 1 \leq i \leq N, 1 \leq q \leq t \leq k \leq T \qquad (3.19)$$

The objective function (3.13) is the sum of fixed and variable order costs and inventory holding costs at the warehouse and retailers. Constraints (3.14) ensure that if retailer $i$ places an order in period $t$ then it is satisfied by placing an order for the warehouse prior to or at $t$. Constraints (3.15) guarantee that a fixed order cost is incurred at the warehouse if an order is placed by the warehouse in a period. Constraints (3.16) ensure that the total amount received by the retailer $i$ from period 1 through $k$ is equal to the demand of $i$ in $k$. Constraints (3.17) stipulate that a fixed order cost is incurred at retailer $i$ if $i$ places an order in a period. Constraints (3.18) and (3.19) are for nonnegativity of variables.

Note that all $X_{itk}$ variables in *TP* can be eliminated using (3.14) so that the formulation becomes more compact. The resulting formulation, referred to as *TP-c*, is as follows.

$$TP\text{-}c\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} \hat{H}_{iqtk}W_{iqtk} \qquad (3.20)$$

*s.t.* (3.5), (3.15), (3.19) and

$$\sum_{q=1}^{t}\sum_{t=1}^{k} W_{iqtk} = d_{ik} \qquad 1 \leq i \leq N, 1 \leq k \leq T \qquad (3.21)$$

$$\sum_{q=1}^{t} W_{iqtk} \leq d_{ik}y_{it} \qquad 1 \leq i \leq N, 1 \leq t \leq k \leq T \qquad (3.22)$$

where $\hat{H}_{iqtk} = H'_{0qt} + H'_{itk}$.

*TP-c* is the same as in Levi et al. (2008) except that they consider a more general cost term $\hat{H}_{iqtk}$ than ours and use $W'_{iqtk}$ in place of $W_{iqtk}$ variables where $W'_{iqtk} = W_{iqtk} / d_{ik}$. We next show that $y_{it}$ variables ($1 \leq i \leq N, 1 \leq t \leq T$) of *TP-c* can be treated as continuous variables.

**Proposition 3.1.** Given the integral values of $y_{0t}$ variables, the optimal solution of the *TP-c* formulation yields integral values of continuous $y_{it}$ variables ($1 \leq i \leq N, 1 \leq t \leq T$).

**Proof.** Given the integral values of $y_{0t}$ variables, constraints (3.15) can be eliminated as follows. For $y_{0t}$ variables taking value '1', corresponding constraints of (3.15) become redundant since left-hand side of (3.15) is contained by left-hand side of (3.21). For $y_{0t}$ variables taking value '0', *W* variables in (3.15) take value '0' due to (3.15) and are removed from the formulation together with the corresponding constraints (3.15). Then, the remaining formulation with constraints (3.21) and (3.22) decomposes for each retailer, and each of the decomposed problems defines the convex hull of feasible solutions of an ULS problem. Hence, continuous $y_{it}$ variables ($1 \leq i \leq N, 1 \leq t \leq T$) naturally take integral values. □

The number of binary variables in *TP* and *TP-c* is $O(T)$ while the number of continuous variables and constraints are $O(NT^3)$ and $O(NT^2)$, respectively.

### 3.1.3 A new combined transportation and shortest path based formulation

In this section, we propose a new stronger formulation, referred to as the combined transportation and shortest path based formulation (*SP*) where we represent lot sizing problem of retailers using shortest path. Lot sizing problem of the warehouse is represented in the same manner as *TP*. Since timing and magnitude of demands realized at the warehouse (due to the replenishment of retailers) cannot be known in advance, which is necessary to compute the cost figures at the warehouse in a shortest path representation, using the shortest path type lot sizing representation for the warehouse (as it is) is not possible.

Additional parameters and variables used in the *SP* formulation are as follows. Let $G_{itk}$ be the total variable cost of satisfying total demand of retailer *i* from period *t* through $k$, i.e. $G_{itk} = p_{it} D_{itk} + \sum_{l=t}^{k-1} h_{il} D_{i,l+1,k}$. Let $Z_{itk}$ be the fraction of the total demand of retailer $i$ from period $t$ through $k$ satisfied in $t$, and $U_{iqtk}$ be the fraction of the quantity ordered by the warehouse in period $q$ and sent to the retailer $i$ in period $t$ to satisfy the total demand of $i$ from period $t$ through $k$ ($1 \le q \le t \le k \le T$).

$$SP\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{0qt} D_{itk} U_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} G_{itk} Z_{itk} \quad (3.23)$$

*s.t.* (3.5) and

$$\sum_{q=1}^{t} U_{iqtk} = Z_{itk} \qquad 1 \le i \le N, 1 \le t \le k \le T \quad (3.24)$$

$$\sum_{k=q}^{t}\sum_{r=t}^{T} a_{ikr} U_{iqkr} \le y_{0q} \qquad 1 \le i \le N, 1 \le q \le t \le T \quad (3.25)$$

$$\sum_{t=1}^{T} Z_{i1t} = 1 \qquad 1 \le i \le N \quad (3.26)$$

$$-\sum_{k=1}^{t-1} Z_{ik,t-1} + \sum_{k=t}^{T} Z_{itk} = 0 \qquad 1 \le i \le N,\ 2 \le t \le T \quad (3.27)$$

$$\sum_{k=t}^{T} a_{itk} Z_{itk} \le y_{it} \qquad 1 \le i \le N, 1 \le t \le T \quad (3.28)$$

$$Z_{itk} \ge 0 \qquad 1 \le i \le N, 1 \le t \le k \le T \quad (3.29)$$

$$U_{iqtk} \ge 0 \qquad 1 \le i \le N, 1 \le q \le t \le k \le T \quad (3.30)$$

where $a_{itk} = \begin{cases} 1 \text{ if } D_{itk} > 0 \\ 0 \text{ otherwise.} \end{cases}$

The objective function (3.23) is the sum of fixed costs, variable order costs and inventory holding costs at the warehouse and retailers. Constraints (3.24) ensure that if retailer $i$ places an order in period $t$, then it is satisfied by placing an order for the warehouse prior to or at $t$. Constraints (3.25) guarantee that a fixed order cost is incurred at the warehouse if an order is placed by the warehouse in a period. Constraints (3.26) and (3.27) are the shortest path representation constraints for the retailers' replenishment problems. Constraints (3.28) stipulate that a fixed order cost is incurred at retailer *i* if *i* places an order in a period. Constraints (3.29) and (3.30) are for nonnegativity of variables. For having a more compact formulation, referred to as *SP-c*, $Z_{itk}$ variables in *SP* can be eliminated using (3.24).

$$SP\text{-}c\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T}(H'_{0qt}D_{itk} + G_{itk})U_{iqtk} \qquad (3.31)$$

*s.t.* (3.5), (3.25), (3.30) and

$$\sum_{t=1}^{T} U_{i11t} = 1 \qquad 1 \le i \le N \qquad (3.32)$$

$$-\sum_{k=1}^{t-1}\sum_{q=1}^{k} U_{iqk,t-1} + \sum_{k=t}^{T}\sum_{q=1}^{t} U_{iqtk} = 0 \qquad 1 \le i \le N,\ 2 \le t \le T \qquad (3.33)$$

$$\sum_{k=t}^{T}\sum_{q=1}^{t} a_{itk}U_{iqtk} \le y_{it} \qquad 1 \le i \le N, 1 \le t \le T \qquad (3.34)$$

where $a_{itk} = \begin{cases} 1 \text{ if } D_{itk} > 0 \\ 0 \text{ otherwise.} \end{cases}$

Similar to the case in *TP-c*, continuous $y_{it}$ variables ($1 \le i \le N, 1 \le t \le T$) in *SP-c* automatically take integral values provided that $y_{0t}$ variables are integral in any solution.

**Proposition 3.2.** Given the integral values of $y_{0t}$ variables, the optimal solution of the *SP-c* formulation yields integral values of continuous $y_{it}$ variables ($1 \le i \le N, 1 \le t \le T$).

**Proof.** Since $f_{it} \ge 0$, $y_{it}$ variables will take the smallest possible value which means constraints (3.34) will be satisfied as equality in the optimal solution for that *i* and *t*. Thus, (3.34) can be eliminated by replacing $y_{it}$ with the left-hand side of (3.34) in (3.31) for all *i* and *t*. Given the integral values of $y_{0t}$ variables, constraints (3.25) can be eliminated as follows. For $y_{0t}$ variables taking value '1', corresponding constraints of (3.25) become redundant since left-hand side of (3.25) cannot take a value greater than ‘1’ due to (3.32) and (3.33). For $y_{0t}$ variables taking value ‘0’, *U* variables take value ‘0’ due to (3.25) and are removed from the formulation together with the corresponding constraints (3.25). The remaining constraints (3.32) and (3.33) define a shortest path problem, which is known to have a totally unimodular coefficient matrix. Hence, *U*-variables take integral values, which imply integral $y_{it}$ values ($1 \le i \le N, 1 \le t \le T$). □

As a result, the number of binary variables in *SP* and *SP-c* is $O(T)$ while the number of continuous variables and constraints are $O(NT^3)$ and $O(NT^2)$, respectively. Table 3.1 summarizes the number of constraints, integer and continuous variables of formulations.

**Table 3.1** Number of constraints, integer and continuous variables in formulations

| Formulation | Constraints | Integer variables | Continuous variables |
|---|---|---|---|
| *SES* | $O(NT^2)$ | $O(NT)$ | $O(NT^2)$ |
| *TP-c* | $O(NT^2)$ | $O(T)$ | $O(NT^3)$ |
| *SP-c* | $O(NT^2)$ | $O(T)$ | $O(NT^3)$ |

### 3.1.4 Analysis of LP relaxations of formulations

In this section, we study the strength of formulations in terms of their LP relaxation solution values. In the LP relaxations of *SES*, *TP* and *SP*, we replace their constraints (3.5) with the following constraints.

$$0 \le y_{it} \le 1 \qquad 0 \le i \le N,\ 1 \le t \le T \tag{3.35}$$

Define *v*(.) as the optimal LP relaxation solution value and *F*(.) as the feasible solution space of LP relaxation of formulation (.). Then, we have

$(X, y) \equiv \{(X_{itk}, y_{it}) | 0 \le i \le N, 1 \le t \le k \le T\} \in F(SES)$,

$(W,X,y) \equiv \{(W_{iqtk}, X_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 1 \le q \le t \le k \le T\} \in F(TP)$, and

$(U,Z,y) \equiv \{(U_{iqtk}, Z_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 1 \le q \le t \le k \le T\} \in F(SP)$.

Before showing the relations among LP relaxations of formulation, we present an example problem instance, which will help us in making the proofs.

**Example:** Consider an instance of OWMR problem with $T$=4, $N$=1, $d_{1t} = 1$ for $1 \le t \le T$; $p_{0t} = p_{1t} = 0$ for $1 \le t \le T$; $h_{01} = 2, h_{02} = 1$, $h_{03} = 1, h_{04} = 0$, $h_{11} = 4, h_{12} = 3$, $h_{13} = 2, h_{14} = 0$, $f_{01} = 0, f_{02} = 4$, $f_{03} = 6, f_{04} = 2$ and $f_{11} = 0, f_{12} = 4$, $f_{13} = 4, f_{14} = 2$.
For this instance, the LP optimal solutions are as follows:

- *SES*: $v(SES) = 14.33$ with $y_{01} = y_{11} = 1$, $y_{02} = y_{13} = 0.67$, $y_{04} = y_{12} = y_{14} = 0.33$, $y_{03} = 0$.
- *TP*: $v(TP) = 14.5$, with $y_{01} = y_{11} = 1$, $y_{02} = y_{04} = y_{12} = y_{13} = y_{14} = 0.5$, $y_{03} = 0$.
- *SP*: $v(SP) = 15$ with $y_{01} = y_{04} = y_{11} = y_{13} = y_{14} = 1$.

The *SP* gives the integer optimal.

**Theorem 3.1.** $v(SES) \le v(TP)$

**Proof.** Let $j_{SES}(X,y)$ and $j_{TP}(W,X,y)$ be LP relaxation solution values of $(X,y) \in F(SES)$ and $(W,X,y) \in F(TP)$, respectively. To make the proof, it suffices to show $F(TP) \subseteq F(SES)$ and give an instance for which $v(SES) < v(TP)$.

(i) To show $F(TP) \subseteq F(SES)$, we take a feasible solution $(W,X,y) \in F(TP)$ and construct a feasible solution $(X,y) \in F(SES)$ with the same objective function value as follows. By convention, we have

$$X_{0qk} = \sum_{i=1}^{N} \sum_{t=q}^{k} W_{iqtk} \qquad \text{for} \qquad 1 \le q \le k \le T \qquad \text{(i.1)}$$

Note that since $X_{itk}$ for $1 \le i \le N, 1 \le t \le k \le T$, and $y_{it}$ for $0 \le i \le N, 1 \le t \le T$ are the same for both *SES* and *TP*, we directly map them. Now, we show that $(X,y)$ constructed using (i.1) is feasible to $F(SES)$.

(a) Constraints (3.9): For $1 \le i \le N, 1 \le k \le T,$ constraints (3.16) are equivalent to (3.9). For *i* = 0, $1 \le k \le T,$ we sum (3.14) over all *t* and *i* which gives

$$\sum_{q=1}^{t}\sum_{i=1}^{N}\sum_{t=1}^{k} W_{iqtk} = \sum_{i=1}^{N}\sum_{t=1}^{k} X_{itk} \qquad 1 \le k \le T.$$

Since $q \le t,$ we can modify the summation bounds above, which gives

$$\sum_{q=1}^{k}\sum_{i=1}^{N}\sum_{t=q}^{k} W_{iqtk} = \sum_{i=1}^{N}\sum_{t=1}^{k} X_{itk} \qquad 1 \le k \le T.$$

The term in right-hand side above is equal to $d_{0k}$ $(= \sum_{i=1}^{N} d_{ik}$ ) due to (3.16). Substituting $X_{0qk}$ in place of $\sum_{i=1}^{N}\sum_{t=q}^{k} W_{iqtk}$ above due to (i.1) gives

$$\sum_{q=1}^{k} X_{0qk} = d_{0k} \qquad 1 \le k \le T,$$

which is equivalent to (3.9) for *i* = 0, $1 \le k \le T$ . Thus, constraints (3.9) hold.

(b) Constraints (3.10): For $1 \le i \le N, 1 \le t \le k \le T,$ constraints (3.17) are equivalent to (3.10). For *i* = 0, $1 \le t \le k \le T,$ we sum (3.15) over all *i*, which gives

$$\sum_{i=1}^{N}\sum_{t=q}^{k} W_{iqtk} \le (\sum_{i=1}^{N} d_{ik}) y_{0q} = d_{0k} y_{0q} \qquad 1 \le q \le k \le T.$$

Substituting $X_{0qk}$ in place of $\sum_{i=1}^{N}\sum_{t=q}^{k} W_{iqtk}$ above due to (i.1) gives

$$X_{0qk} \le d_{0k} y_{0q} \qquad 1 \le q \le k \le T,$$

which is equivalent to (3.10) for *i* = 0, $1 \le q \le k \le T$ . Thus, constraints (3.10) hold.

(c) Constraints (3.11): Summing constraints (3.14) over *t* (from *r* = 1 to *t*) and all *i*, *k*, we obtain

$$\sum_{i=1}^{N}\sum_{q=1}^{r}\sum_{r=1}^{t}\sum_{k=r}^{T} W_{iqrk} = \sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T} X_{irk} \qquad 1 \le t \le T.$$

Since $q \le r$ , we can rewrite the above equation as

$$\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{k=r}^{T} W_{iqrk} = \sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T} X_{irk} \qquad 1 \le t \le T.$$

Adding the same term to both sides, we have

$$\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=q}^{t}\sum_{k=r}^{T}W_{iqrk}+\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=t+1}^{T}\sum_{k=r}^{T}W_{iqrk}=\sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T}X_{irk}+\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=t+1}^{T}\sum_{k=r}^{T}W_{iqrk}.$$

The left-hand side of the above equation reduces to $\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=q}^{T}\sum_{k=r}^{T}W_{iqrk}$ which is indeed equal to $\sum_{r=1}^{t}\sum_{k=r}^{T}X_{0rk}$ due to (i.1). Thus, it becomes

$$\sum_{r=1}^{t}\sum_{k=r}^{T}X_{0rk}=\sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T}X_{irk}+\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{r=t+1}^{T}\sum_{k=r}^{T}W_{iqrk}\qquad 1\le t\le T,$$

which ensures that

$$\sum_{r=1}^{t}\sum_{k=r}^{T}X_{0rk}\ge\sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T}X_{irk}\qquad 1\le t\le T.$$

Thus, constraints (3.11) hold.

(d) To show that $j_{SES}(X,y)$ = $j_{TP}(W,X,y)$ for $(X,y)\in F(SES)$ and $(W,X,y)\in F(TP)$, we start with $j_{TP}(W,X,y)$, which is equal to

$$\sum_{i=0}^{N}\sum_{t=1}^{T}f_{it}y_{it}+\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T}(p_{0q}+\sum_{l=q}^{t-1}h_{0l})W_{iqtk}+\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(p_{it}+\sum_{r=t}^{k-1}h_{ir})X_{itk}.$$

Let the second term of the right-hand side of above equation be equal to

$$\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T}(p_{0q}+\sum_{l=q}^{k-1}h_{0l})W_{iqtk}-\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T}(\sum_{l=t}^{k-1}h_{0l})W_{iqtk}.$$

Since $q\le t\le k$ we can modify the bounds in the summation signs and obtain

$$\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{k}\sum_{k=q}^{T}(p_{0q}+\sum_{l=q}^{k-1}h_{0l})W_{iqtk}-\sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{l=t}^{k-1}h_{0l})W_{iqtk}$$

$$=\sum_{q=1}^{T}\sum_{k=q}^{T}(p_{0q}+\sum_{l=q}^{k-1}h_{0l})\sum_{i=1}^{N}\sum_{t=q}^{k}W_{iqtk}-\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{l=t}^{k-1}h_{0l})\sum_{q=1}^{t}W_{iqtk}.$$

Using (i.1) for the first term and (3.14) for the second term above, we obtain

$$\sum_{q=1}^{T}\sum_{k=q}^{T}H_{0qk}X_{0qk}-\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{l=t}^{k-1}h_{0l})X_{itk}.$$

Then, replacing the above with the second term of $j_{TP}(W,X,y)$, $j_{TP}(W,X,y)$ becomes

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{q=1}^{T}\sum_{k=q}^{T} H_{0qk}X_{0qk} - \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{l=t}^{k-1} h_{0l})X_{itk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(p_{it} + \sum_{r=t}^{k-1} h_{ir})X_{itk},$$

which is equal to $j_{SES}(X, y)$.

(ii) As presented in the example instance, $v(SES) < v(TP)$. □

**Theorem 3.2.** $v(TP) \le v(SP)$

**Proof.** Let $j_{TP}(W, X, y)$ and $j_{SP}(U, Z, y)$ be LP relaxation solution values of $(W, X, y) \in F(TP)$ and $(U, Z, y) \in F(SP)$, respectively. To make the proof, it suffices to show $F(SP) \subseteq F(TP)$ and give an instance for which $v(TP) < v(SP)$.

(i) To show $F(SP) \subseteq F(TP)$, we take a feasible solution $(U, Z, y) \in F(SP)$ and construct a feasible solution $(W, X, y) \in F(TP)$ with the same objective function value as follows. By definition, we have

$$X_{itk} = d_{ik}\sum_{j=k}^{T} Z_{itj} \qquad \text{for} \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad \text{(i.1)}$$

$$W_{iqtk} = d_{ik}\sum_{j=k}^{T} U_{iqtj} \qquad \text{for} \qquad 1 \le i \le N, 1 \le q \le t \le k \le T \qquad \text{(i.2)}$$

Note that since $y_{it}$ ($0 \le i \le N, 1 \le t \le T$) are the same for both *TP* and *SP*, we directly map them. Now, we show that $(W, X, y)$ constructed using (i.1) and (i.2) is feasible to $F(TP)$.

(a) Constraints (3.14): Summing constraints (3.24) over *k* (from *j*=*k* to *j*=*T*) and multiplying both sides by $d_{ik}$, we obtain

$$\sum_{q=1}^{t} d_{ik}\sum_{j=k}^{T} U_{iqtj} = d_{ik}\sum_{j=k}^{T} Z_{itj} \qquad 1 \le i \le N, 1 \le t \le k \le T.$$

Substituting (i.1) and (i.2) into the above equation gives

$$\sum_{q=1}^{t} W_{iqtk} = X_{itk} \qquad 1 \le i \le N, 1 \le t \le k \le T,$$

which is equivalent to (3.14). Thus, constraints (3.14) hold.

(b) Constraints (3.15): Note that (i.2) can be rewritten as $W_{iqtk} = d_{ik} \sum_{j=k}^{T} a_{itj} U_{iqtj}$ since if $d_{ik} > 0$ then $a_{itj} = 1$ for $t \le k \le j \le T$, else $W_{iqtk}$ becomes zero. Then, we substitute $W_{iqkt} / d_{it}$ in place of $\sum_{r=t}^{T} a_{ikr} U_{iqkr}$ in (3.25), which gives

$$\sum_{k=q}^{t} (W_{iqkt} / d_{it}) \le y_{0q} \qquad 1 \le i \le N, 1 \le q \le t \le T.$$

Thus, constraints (3.15) hold.

(c) Constraints (3.16): Summing constraints (3.26) and (3.27) from $t$=2 to $t$=$k$, we obtain

$$\sum_{r=1}^{k} \sum_{j=k}^{T} Z_{irj} = 1 \qquad 1 \le i \le N, 1 \le k \le T.$$

Substituting $X_{irk} / d_{ik}$ in place of $\sum_{j=k}^{T} Z_{irj}$ due to (i.1) gives

$$\sum_{r=1}^{k} (X_{irk} / d_{ik}) = 1 \quad 1 \le i \le N, 1 \le k \le T,$$

which is equivalent to (3.16). Thus, constraints (3.16) hold.

(d) Constraints (3.17): Note that (3.28) can be rewritten as

$$\sum_{j=k}^{T} a_{itj} Z_{itj} \le y_{it} \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad (3.28')$$

which actually encompasses (3.28). (i.1) can be rewritten as $X_{itk} = d_{ik} \sum_{j=k}^{T} a_{itj} Z_{itj}$, as done for $W$ variables in part (b). Substituting $X_{itk} / d_{ik}$ in place of $\sum_{j=k}^{T} a_{itj} Z_{itj}$ in (3.28') gives

$$(X_{itk} / d_{it}) \le y_{it} \qquad 1 \le i \le N, 1 \le t \le k \le T,$$

which is equivalent to (3.17). Thus, constraints (3.17) hold.

(e) To show that $j_{TP}(W, X, y) = j_{SP}(U, Z, y)$ for $(W, X, y) \in F(TP)$ and $(U, Z, y) \in F(SP)$, we start with

$$j_{TP}(W,X,y) = \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} H'_{0qt}W_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk}X_{itk}.$$

Substituting (i.1) and (i.2) into $j_{TP}(W,X,y)$, we obtain

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} H'_{0qt}\left(d_{ik}\sum_{j=k}^{T}U_{iqtj}\right) + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk}\left(d_{ik}\sum_{j=k}^{T}Z_{itj}\right) \quad \text{(e.1)}$$

We can rewrite the second term in (e.1) as

$$\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T} H'_{0qt}\sum_{k=t}^{T}\left(d_{ik}\sum_{j=k}^{T}U_{iqtj}\right) = \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T} H'_{0qt}\{d_{it}U_{iqtt} + d_{it}U_{iqt,t+1} + \ldots + d_{it}U_{iqtT}$$
$$+ d_{i,t+1}U_{iqt,t+1} + \ldots + d_{i,t+1}U_{iqtT}$$
$$\ldots$$
$$+ d_{iT}U_{iqtT}\}.$$

Thus, the second term in (e.1) can be rewritten as

$$\sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} H'_{0qt}D_{itk}U_{iqtk} \quad \text{(e.2)}$$

We can rewrite the third term in (e.1) as

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}\left((p_{it}d_{ik} + \sum_{l=t}^{k-1}h_{il}d_{ik})\sum_{j=k}^{T}Z_{itj}\right),$$ which is equal to

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\{p_{it}d_{it}Z_{itt} + p_{it}d_{it}Z_{it.t+1} + \ldots + p_{it}d_{it}Z_{itT}$$
$$+ (p_{it}d_{i,t+1} + h_{it}d_{i,t+1})Z_{it.t+1} + \ldots + (p_{it}d_{i,t+1} + h_{it}d_{i,t+1})Z_{itT}$$
$$\ldots$$
$$+ (p_{it}d_{iT} + h_{it}d_{iT} + h_{i,t+1}d_{iT} + \ldots + h_{i,T-1}d_{iT})Z_{itT}\}.$$

Thus, the third term in (e.1) can be rewritten as

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(p_{it}D_{itk} + \sum_{l=t}^{k-1}h_{il}D_{i,l+1,k})Z_{itk} \quad \text{(e.3)}$$

Thus, summing the first term in (e.1), (e.2) and (e.3), we obtain

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{0qt}D_{itk}U_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} G_{itk}Z_{itk},$$

which is equal to $j_{SP}(U,Z,y)$.

(ii) As presented in the example instance, $v(TP) < v(SP)$. □

## 3.2 Single warehouse-single retailer (SWSR) problem

Except the echelon stock formulation, the two other formulations are based on the same principle: The warehouse is considered to be divided into $N$ departments, each of which is responsible for replenishing a specific retailer. Thus, the OWMR problem is set as the assemblage of $N$ many SWSR problems. All the SWSR problems are linked so that a fixed order cost at the warehouse is incurred whenever any department places an order. Therefore, the SWSR problem deserves a detailed analysis. The SWSR problem is also important since it is the two-level case of the multi-level problem in Zangwill (1969) and uncapacitated case of the two-level problem in van Hoesel et al. (2005). In the following we adapt the *SP-c* formulation to the SWSR problem, referred to as the *SSP-c* formulation, by dropping subscript $i$ from the *SP-c*.

$$\textit{SSP-c}: \ \textit{Min} \ \sum_{t=1}^{T}(f_{0t}y_{0t}+f_{1t}y_{1t})+\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T}(H'_{0qt}D_{tk}+G_{tk})U_{qtk} \tag{3.36}$$

*s.t.*

$$\sum_{t=1}^{T}U_{11t}=1 \tag{3.37}$$

$$-\sum_{k=1}^{t-1}\sum_{q=1}^{k}U_{qk,t-1}+\sum_{k=t}^{T}\sum_{q=1}^{t}U_{qtk}=0 \qquad 2\le t\le T \tag{3.38}$$

$$\sum_{k=t}^{T}\sum_{q=1}^{t}a_{tk}U_{qtk}\le y_{1t} \qquad 1\le t\le T \tag{3.39}$$

$$\sum_{k=q}^{t}\sum_{r=t}^{T}a_{kr}U_{qkr}\le y_{0q} \qquad 1\le q\le T \tag{3.40}$$

$$U_{qtk}\ge 0 \qquad 1\le q\le t\le k\le T \tag{3.41}$$

$$y_{0t},y_{1t}\in\{0,1\} \qquad 1\le t\le T \tag{3.42}$$

Below we slightly modify *SSP-c* before showing that it defines the convex hull of the feasible solutions of SWSR problem.

**Lemma 3.1.** The inequalities

$$\sum_{r=q}^{T} a_{qr} U_{qqr} \le y_{0q} \qquad 1 \le q \le T \qquad (3.43)$$

$$\sum_{k=q}^{t-1} a_{k,t-1} U_{qk,t-1} - \sum_{k=t}^{T} a_{tk} U_{qtk} \ge 0 \qquad 1 \le q < t \le T \qquad (3.44)$$

where $a_{k,t-1} \ne 0$ for at least one $k$ ($q \le k \le t-1$) are valid for *SSP-c*.

**Proof.** Inequalities (3.43) are valid for *SSP-c* since they correspond to constraints (3.40) when $t = q$. Inequalities (3.44) are actually the simplified version of the following constraints (some terms appear on both sides of (3.45) and cancel each other):

$$\sum_{k=q}^{t} \sum_{r=t}^{T} a_{kr} U_{qkr} \le \sum_{k=q}^{t-1} \sum_{r=t-1}^{T} a_{kr} U_{qkr} \qquad 1 \le q < t \le T \qquad (3.45)$$

where $a_{k,t-1} \ne 0$ for at least one $k$ ($q \le k \le t-1$) to account for the zero demand case. As the optimal policy at the warehouse has the well-known Wagner-Whitin property (Federgruen and Tzur, 1999), if a quantity is ordered by the warehouse in period $q$ to satisfy the demand of retailer $i$ from period $t$ through $k$ ($t \le k \le T$) then the demand of retailer $i$ from period $j$ ($q \le j < t$) through $t$-1 must also be met by an order in period $q$ by the warehouse. Since this is ensured by constraints (3.44), they are valid for *SSP-c*. □

Since constraints (3.43) and (3.44) do not change the feasible region of *SSP-c* with regard to $y$ variables, we use them in place of (3.39). Thus, in the sequel by *SSP-c* we mean the formulation: *Min* (3.36) *s.t.* (3.37)–(3.39) and (3.41)–(3.44).

**Theorem 3.3.** *SSP-c* defines the convex hull of feasible solutions of the SWSR problem.

**Proof.** We show that the associated constraint matrix of *SSP-c* is totally unimodular (TU) using the following well known rule:

$$-1 \le \sum_{i \in C_1} c_{ij} - \sum_{i \in C_2} c_{ij} \le 1 \qquad \text{for all } j,$$

where $j$ denotes the columns, and $c_{ij}$ is the technological coefficient of $j^{th}$ column in $i^{th}$ row. For any subset $C$ of constraints of *SSP-c*, $C$ would be partitioned into two disjoint sets, $C_1$ and $C_2$ where $C = C_1 \cup C_2$, such that the difference between the total of coefficients in $C_1$ and $C_2$ for each column equals to 0, 1 or -1. First note that constraints (3.39) and (3.43) can be eliminated in a similar manner of the proof of Proposition 3.2. Then, the constraint matrix of *SSP-c* is composed of constraints (3.37), (3.38) and (3.44). Note that each variable appears at most twice with coefficients -1 and 1 in (3.37) and (3.38). The same argument is valid for (3.44) as well. Thus, the TU rule is satisfied if $C$ involves only rows from (3.37) and (3.38) or only from (3.44). On the other hand, if $C$ involves rows from (3.37), (3.38) and (3.44) at the same time, we propose the following partitioning scheme:

- Assign all the rows in $C$ from (3.37) and (3.38) into $C_1$.
- Assign rows from (3.44) for a given period $t$ in $C$ into $C_1$ if row from (3.38) for that $t$ is in $C$, and into $C_2$ otherwise. Note that row from (3.38) for a given period $t$ contains all variables in constraints (3.44) for that $t$ with just the opposite sign.

Due to the TU property, $U$ variables take integral values which in turn imply integral $y$ variables. □

To show that *SES* and *TP* formulations do not represent the convex hull of feasible solutions of the SWSR problem, we use the example already presented before Theorem 3.1. Indeed, this example has been used by Pochet and Wolsey (1994) to disprove the conjecture that the LP relaxation of multi-commodity formulation solves the uncapacitated multi-level lot sizing problem in series. To the best of our knowledge, *SSP-c* is the first formulation that defines the convex-hull of feasible solutions of the SWSR problem (see Solyalı and Süral, 2008a and Chapter 4 for a similar argument on a variant of the SWSR problem where retailers apply order-up-to level inventory policy).

### 3.3 Joint replenishment problem

In this subsection, using Theorem 3.2 we show that transportation and shortest path based formulations for the JRP yield the same objective value for the LP relaxation.

We start with the *TP* for the OWMR problem and obtain the formulation for the JRP, referred to as *TP-JRP*. Note that *W* variables are not needed any more since whenever a retailer places an order, the warehouse also places an order in the same period. Thus, there is no need for constraints (3.14), and constraints (3.15) can be rewritten as $X_{itk} \le d_{ik} y_{0t}$. Constraints (3.17) and $X_{itk} \le d_{ik} y_{0t}$ imply the following constraints.

$$y_{it} \le y_{0t} \qquad 1 \le i \le N, 1 \le t \le T \tag{3.46}$$

As a result, *TP-JRP* is as follows.

$$TP\text{-}JRP\text{: } Min \ \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk} X_{itk} \tag{3.47}$$

*s.t.* (3.5), (3.16)–(3.18), and (3.46)

Next, we derive the *SP* formulation for JRP, referred to as *SP-JRP*. *U* variables in *SP* are not needed any more due to the same reason stated above as for the *TP*. There is no need for constraints (3.24), and constraints (3.25) can be rewritten as $\sum_{r=t}^{T} a_{itr} Z_{itr} \le y_{0t}$, which in turn imply constraints (3.46) due to (3.28). Thus, *SP-JRP* is as follows.

$$SP\text{-}JRP\text{: } Min \ \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} G_{itk} Z_{itk} \tag{3.48}$$

*s.t.* (3.5), (3.26)–(3.29), and (3.46)

In the following, we present the LP equivalence result of *TP-JRP* and *SP-JRP*. Note that we replace (3.5) in both formulations with (3.35). The solutions in *F*(*TP-JRP*) and *F*(*SP-JRP*) can be identified respectively as

$(X, y) \equiv \{(X_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 1 \le t \le k \le T\} \in F(TP-JRP)$ and

$(Z, y) \equiv \{(Z_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 1 \le t \le k \le T\} \in F(SP-JRP).$

**Theorem 3.4.** $v(TP-JRP) = v(SP-JRP)$

**Proof.** Let $j_{TP-JRP}(X, y)$ and $j_{SP-JRP}(Z, y)$ be LP relaxation solution values of $(X, y) \in F(TP-JRP)$ and $(Z, y) \in F(SP-JRP)$. To make the proof, it suffices to show (i) *F*(*SP-JRP*) $\subseteq$ *F*(*TP-JRP*), (ii) *F*(*TP-JRP*) $\subseteq$ *F*(*SP-JRP*), and (iii) $j_{TP-JRP}(X, y)$ = $j_{SP-JRP}(Z, y)$ for any $(X, y) \in F(TP-JRP)$ and its corresponding $(Z, y) \in F(SP-JRP)$, or vice versa.

(i) As the JRP is a special case of the OWMR problem, it is immediate from Theorem 3.2 that the relation *F*(*SP-JRP*) $\subseteq$ *F*(*TP-JRP*) holds.

(ii) To show *F*(*TP-JRP*) $\subseteq$ *F*(*SP-JRP*), we take a feasible solution $(X, y) \in F(TP-JRP)$ and construct an associated feasible solution $(Z, y) \in F(SP-JRP)$ as follows. By definition, we have

$$Z_{itT} = V_{itT} \quad \text{for} \quad 1 \le i \le N, 1 \le t \le T \tag{ii.1}$$

$$Z_{itk} = V_{itk} - V_{it,k+1} \quad \text{for} \quad 1 \le i \le N, 1 \le t \le k < T \tag{ii.2}$$

where $V_{itk} = \begin{cases} X_{itk}/d_{ik} \text{ if } d_{ik} > 0 \\ 1 \text{ if } d_{ik} = 0 \text{ and } t = k \\ 0 \text{ otherwise.} \end{cases}$

Since $y_{it}$ variables ($0 \le i \le N, 1 \le t \le T$) are the same for both *TP-JRP* and *SP-JRP*, we directly map them. The construction above is defined by Denizel et al. (2008) in the context of the capacitated multi-item lot sizing problem with setup times. Now, we should show that $(Z, y)$ constructed using (ii.1) and (ii.2) is feasible to *F*(*SP-JRP*). Since constraints (3.26)–(3.29) have been shown to be feasible by Denizel et al. (2008) and constraints (3.46) are common to both formulations, feasibility of $(Z, y)$ is proved.

(iii) To show $j_{TP-JRP}(X,y) = j_{SP-JRP}(Z,y)$ for any $(X,y)\in F(TP-JRP)$ and the associated $(Z,y)\in F(SP-JRP)$ or vice versa, we start with

$$j_{TP-JRP}(X,y) = \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk}X_{itk},$$

which is equivalent to

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\left[ p_{it}d_{it}(X_{itt}/d_{it}) + \sum_{k=t+1}^{T}(p_{it} + \sum_{l=t}^{k-1} h_{il})d_{ik}(X_{itk}/d_{ik})\right].$$

We can safely insert $V_{itk}$ in place of $(X_{itk}/d_{ik})$ above regardless of the value of $d_{ik}$ since by definition $V_{itk} = X_{itk}/d_{ik}$ if $d_{ik} > 0$, and $p_{it}d_{it}V_{itt} = 0$ or $(p_{it} + \sum_{l=t}^{k-1} h_{il})d_{ik}V_{itk} = 0$ if $d_{ik} = 0$. So, we have

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\left[ p_{it}d_{it}(V_{itt}) + \sum_{k=t+1}^{T}(p_{it} + \sum_{l=t}^{k-1} h_{il})d_{ik}(V_{itk})\right].$$

Since $d_{ik} = D_{itk} - D_{it,k-1}$ for $t \le k-1$, $\sum_{k=t+1}^{T}(p_{it} + \sum_{l=t}^{k-1} h_{il})d_{ik}(V_{itk})$ equals to

$$= \sum_{k=t+1}^{T}\left[ p_{it}(D_{itk} - D_{it,k-1}) + \sum_{l=t}^{k-2} h_{il}(D_{i,l+1,k} - D_{i,l+1,k-1}) + h_{i,k-1}D_{ikk}\right]V_{itk}$$

$$= \sum_{k=t+1}^{T}\left[ p_{it}D_{itk} + \sum_{l=t}^{k-1} h_{il}D_{i,l+1,k} - p_{it}D_{it,k-1} - \sum_{l=t}^{k-2} h_{il}D_{i,l+1,k-1}\right]V_{itk}$$

$$= \sum_{k=t+1}^{T}\left[ G_{itk} - G_{it,k-1}\right]V_{itk}$$

Thus, $j_{TP-JRP}(X,y) = \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\left[ G_{itt}V_{itt} + \sum_{k=t+1}^{T}(G_{itk} - G_{it,k-1})V_{itk}\right].$

Rewriting the second term of the above relation, we obtain

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\left[ \sum_{k=t}^{T-1} G_{itk}(V_{itk} - V_{it,k+1}) + G_{itT}V_{itT}\right].$$

Inserting $Z$ variables using (ii.1) and (ii.2) to the above relation gives

$$\sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} G_{itk}Z_{itk},$$

which is $j_{SP-JRP}(Z,y)$. □

### 3.4 Extending formulations to initial inventory case

Our theoretical results in Section 3.1 are extended to the case of nonzero initial inventory at the warehouse (i.e. $I_{00} > 0$). We explicitly address nonzero initial inventories in contrast to the most of the studies in the multi-level lot sizing literature where initial inventories are usually ignored (see Zangwill, 1969; Federgruen and Tzur, 1999; Levi et al., 2008; Pochet and Wolsey, 2006). To the best of our knowledge, only van Hoesel et al. (2005) explicitly consider initial inventory issue. Although the models presented in Section 3.2 yield zero ending inventories at all facilities in optimality, such planning models are mostly used within a rolling horizon framework, which necessarily implies presence of initial inventories. Thus, initial inventory at the warehouse is an important issue and must be explicitly considered in the formulations. Note that it is not possible to simply deduce demands of retailers from the initial inventory of warehouse, $I_{00}$, until $I_{00}$ becomes zero since the replenishment of retailers is not known in advance. An exception to this is the echelon stock case where the warehouse is supposed to face with the total system-wide demand $d_{0t}$ and one can deduce $d_{0t}$ values from $I_{00}$ until $I_{00}$ equal to zero.

We should note that Stadtler (1996; 1997) consider nonzero initial inventories for a multi-level lot sizing problem with a general product structure, but use echelon stock type formulation, which can reduce initial inventories to zero.

#### 3.4.1 The *SES* formulation with nonzero initial inventory

Now, we extend the *SES* formulation to the nonzero initial inventory case, referred to as *SES-I*. As lot sizing decisions at the warehouse are separated from lot sizing decisions of retailers in *SES* and the warehouse is faced with total system-wide demand $d_{0t}$, one can deduce $d_{0t}$ values from $I_{00}$ by finding a period $j$ such that $I_{00} - D_{01j} \geq 0$, and $I_{00} - D_{01,j+1} < 0$. Then, the total cost (*IC*) of satisfying demand by the initial inventory at the warehouse is equal to $\sum_{r=1}^{j} h_{0r}(I_{00} - D_{01r})$. Next, we should modify the total system-wide demand values as $d_{0t} = 0$ for $1 \le t \le j$ and $d_{0,j+1} = D_{01,j+1} - I_{00}$. Then,

$$SES\text{-}I\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=0}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H_{itk} X_{itk} + IC \qquad (3.49)$$

*s.t.* (3.5), (3.9), (3.10), (3.12) and

$$\sum_{r=1}^{t}\sum_{k=r}^{T} X_{0rk} + I_{00} \ge \sum_{i=1}^{N}\sum_{r=1}^{t}\sum_{k=r}^{T} X_{irk} \qquad 1 \le t \le T \qquad (3.50)$$

where $H_{0tk} = p_{0t} + \sum_{l=t}^{k-1} h_{0l}$, $H_{itk} = p_{it} + \sum_{l=t}^{k-1} (h_{il} - h_{0l})$ for $1 \le i \le N, 1 \le t \le k \le T$, $d_{0t} = 0$ for $1 \le t \le j$ and $d_{0,j+1} = D_{01,j+1} - I_{00}$.

Objective function (3.49) involves a constant term *IC* that is the inventory holding cost due to the initial inventory at the warehouse in addition to the original objective function (3.8). Constraints (3.50) stipulate that the total amount ordered for the warehouse up to and including period *t* plus initial inventory at the warehouse must be greater than or equal to the total amount ordered for all of the retailers up to and including period *t*. Note that $I_{00}$ appears only in constraints (3.50) and is used to modify original $d_{0t}$ for $1 \le t \le j+1$.

Although it could be conjectured that the demand quantities $d_{0t}$ for $1 \le t \le j$ and portion of $d_{0,j+1}$ (i.e. $I_{00} - D_{01j}$) should be satisfied by the initial inventory available at the warehouse in an optimal solution of *SES-I*, this may not be the case. For example, we have $I_{00} = 2600$, $d_{01} = 2464$ and $d_{02} = 2683$ in one of the instances we addressed in Section 3.5. Note that *j*=1 in this instance and it could be expected that demand quantities 2464 ($= d_{01}$) for $t = 1$ and 136 ($= I_{00} - D_{011}$) for $t = 2$ would be satisfied by $I_{00}$ in the optimal solution. However, this is not true. At optimality, the retailers prefer to have an amount of 2598 units in period *t*=1, which is supplied by $I_{00}$. The remaining 2 units of $I_{00}$ and an amount ordered in *t*=2 at the warehouse are used to satisfy the order of retailers in $t \geq 2$. Thanks to the trick in $H_{itk}$ term for retailers and constraints (3.50), *SES-I* is able to find the true optimum solution with the true optimum solution value.

### 3.4.2 The *TP* formulation with nonzero initial inventory

The *TP* formulation presented in Levi et al. (2008) does not consider initial inventories. Let $W_{i0tk}$ be fraction of the demand of retailer *i* in period *k* satisfied from the initial inventory of the warehouse in period *t*. We refer to *TP* with initial inventories at the warehouse as *TP-I*, which is given below.

$$TP\text{-}I\text{: } Min \ \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} H'_{0qt}W_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{itk}X_{itk}$$

$$+\sum_{r=1}^{T} h_{0r}I_{00} - \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{r=t}^{T} h_{0r})W_{i0tk} \qquad (3.51)$$

*s.t.* (3.5), (3.15)–(3.19),

$$\sum_{q=1}^{t} W_{iqtk} + W_{i0tk} = X_{itk} \qquad 1 \leq i \leq N, 1 \leq t \leq k \leq T \qquad (3.52)$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} W_{i0tk} \leq I_{00} \qquad (3.53)$$

$$W_{i0tk} \geq 0 \qquad 1 \leq i \leq N, 1 \leq t \leq k \leq T \qquad (3.54)$$

where $H'_{itk} = p_{it} + \sum_{r=t}^{k-1} h_{ir}$ for $0 \leq i \leq N, 1 \leq t \leq k \leq T$.

The objective function (3.51) consists of inventory holding cost due to the initial inventory at the warehouse besides the cost terms in (3.13). Constraints (3.52) ensure that if retailer *i* places an order in period *t*, then this order is either satisfied by the initial inventory available at the warehouse or by placing an order for the warehouse prior to or at period *t*. Constraint (3.53) assures that the total amount of demand supplied by the initial inventory of the warehouse cannot exceed the available amount. Constraints (3.54) are for nonnegativity. As in Section 3.1, one can substitute the left-hand side of (3.52) in place of $X_{itk}$ variables for $1 \le i \le N, 1 \le t \le k \le T$ and obtain a more compact formulation, referred to as *TP-I-c*.

### 3.4.3 The *SP* formulation with nonzero initial inventory

Let $U_{i0tk}$ be the fraction of the quantity supplied by the initial inventory of warehouse to satisfy the total demand of retailer *i* from period *t* through *k*. Then, the *SP* formulation with initial inventories at the warehouse explicitly modeled, referred to as *SP-I*, is as follows.

$$SP\text{-}I\text{: } Min \ \sum_{i=0}^{N}\sum_{t=1}^{T} f_{it}y_{it} + \sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{t=1}^{T}\sum_{k=t}^{T} H'_{0qt}D_{itk}U_{iqtk} + \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} G_{itk}Z_{itk}$$

$$+\sum_{r=1}^{T} h_{0r}I_{00} - \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(\sum_{r=t}^{T} h_{0r})D_{itk}U_{i0tk} \qquad (3.55)$$

*s.t.* (3.5), (3.25)–(3.30),

$$\sum_{q=1}^{t} U_{iqtk} + U_{i0tk} = Z_{itk} \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad (3.56)$$

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T} D_{itk}U_{i0tk} \le I_{00} \qquad (3.57)$$

$$U_{i0tk} \ge 0 \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad (3.58)$$

The objective function (3.55) is the total of inventory holding costs due to the initial inventory at the warehouse and cost terms in (3.23). Constraints (3.56) ensure that if retailer *i* places a positive order in period *t*, then this order is either satisfied by the initial inventory available at the warehouse or by placing an order for the warehouse prior to or at period *t*. Constraint (3.57) assures that the total amount of demand supplied by the initial inventory of the warehouse cannot exceed the available amount. Constraints (3.58) are for nonnegativity of variables. Like in Section 3.1, one can substitute the left-hand side of (3.56) in place of $Z_{itk}$ variables for $1 \le i \le N, 1 \le t \le k \le T$ and obtain a more compact formulation, referred to as *SP-I-c*.

Initial inventory at the warehouse actually acts as a capacitated source of alternative supply as opposed to the replenishment of warehouse using an uncapacitated source of supply. As a result, there exist both capacitated and uncapacitated sources of supplies in the presence of initial inventory at the warehouse, which decreases the strength of *TP-I* (and *TP-I-c*) as well as *SP-I* (and *SP-I-c*) formulations.

### 3.4.4 Analysis of LP relaxations of formulations with $I_{00} > 0$

In this subsection, we extend the analysis in Section 3.1.4 to the case with $I_{00} > 0$. Since the LP relaxations of formulations are concerned, constraints (3.5) are replaced with (3.35) in *SES-I*, *TP-I*, and *SP-I*. The solutions in *F*(*SES-I*), *F*(*TP-I*) and *F*(*SP-I*) can be identified respectively as

$(X, y) \equiv \{(X_{itk}, y_{it}) | 0 \le i \le N, 1 \le t \le k \le T\} \in F(SES-I),$

$(W, X, y) \equiv \{(W_{iqtk}, X_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 0 \le q \le t \le k \le T\} \in F(TP-I),$ and

$(U, Z, y) \equiv \{(U_{iqtk}, Z_{itk}, y_{jt}) | 1 \le i \le N, 0 \le j \le N, 0 \le q \le t \le k \le T\} \in F(SP-I).$

**Theorem 3.5.** $v(TP-I) \le v(SP-I)$

**Proof.** Let $j_{TP-I}(W, X, y)$ and $j_{SP-I}(U, Z, y)$ be LP relaxation solution values of $(W, X, y) \in F(TP-I)$ and $(U, Z, y) \in F(SP-I)$, respectively. To make the proof, it suffices to show *F*(*SP-I*) $\subseteq$ *F*(*TP-I*) and give an instance for which $v(TP-I) < v(SP-I)$.

(i) To show $F$(*SP-I*) $\subseteq$ $F$(*TP-I*), we take a feasible solution $(U,Z,y)\in F(SP-I)$ and construct a feasible solution $(W,X,y)\in F(TP-I)$ with the same objective function value as follows. By definition, we have

$$X_{itk}=d_{ik}\sum_{j=k}^{T}Z_{itj} \qquad \text{for} \qquad 1\le i\le N,\, 1\le t\le k\le T \qquad \text{(i.1)}$$

$$W_{iqtk}=d_{ik}\sum_{j=k}^{T}U_{iqtj} \qquad \text{for} \qquad 1\le i\le N,\, 0\le q\le t\le k\le T \quad \text{(i.2)}$$

Note that since $y_{it}$ $(0\le i\le N, 1\le t\le T)$ are the same for both *TP-I* and *SP-I*, we directly map them. Now, we should show that $(W,X,y)$ constructed using (i.1) and (i.2) is feasible to $F$(*TP-I*).

(a) Constraints (3.15)–(3.17): These constraints have already been shown to be feasible in Theorem 3.2.

(b) Constraints (3.52): Summing constraints (3.56) over *k* (from *j*=*k* to *j*=*T*) and multiplying both sides of (3.56) by $d_{ik}$, we obtain

$$\sum_{q=1}^{t}d_{ik}\sum_{j=k}^{T}U_{iqtj}+d_{ik}\sum_{j=k}^{T}U_{i0tj}=d_{ik}\sum_{j=k}^{T}Z_{itj} \qquad 1\le i\le N,\, 1\le t\le k\le T.$$

Substituting (i.1) and (i.2) into the above equation gives

$$\sum_{q=1}^{t}W_{iqtk}+W_{i0tk}=X_{itk} \quad 1\le i\le N,\, 1\le t\le k\le T,$$

which is (3.52). Thus, constraints (3.52) hold.

(c) Constraints (3.53): Constraint (3.57) is

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}D_{itk}U_{i0tk}=\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{k=t}^{T}(d_{it}+d_{i,t+1}+\ldots+d_{ik})U_{i0tk}\le I_{00}.$$

The above inequality can be rewritten as

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\left[d_{it}U_{i0tt}+(d_{it}+d_{i,t+1})U_{i0t,t+1}+\ldots+(d_{it}+d_{i,t+1}+\ldots+d_{iT})U_{i0tT}\right]$$

$$=\sum_{i=1}^{N}\sum_{t=1}^{T}\left[d_{it}\sum_{k=t}^{T}U_{i0tk}+d_{i,t+1}\sum_{k=t+1}^{T}U_{i0tk}+\ldots+d_{iT}\sum_{k=T}^{T}U_{i0tk}\right]$$

$$= \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} \left[ d_{ik} \sum_{j=k}^{T} U_{i0tj} \right] \leq I_{00}.$$

Substituting (i.2) for $q$=0 into above expression gives

$$\sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} W_{i0tk} \leq I_{00}.$$

Thus, constraints (3.53) hold.

(d) To show that $j_{TP-I}(W, X, y) = j_{SP-I}(U, Z, y)$ for $(W, X, y) \in F(TP-I)$ and $(U, Z, y) \in F(SP-I)$, we start with $j_{TP-I}(W, X, y),$ which is equal to

$$j_{TP}(W, X, y) + \sum_{r=1}^{T} h_{0r} I_{00} - \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} (\sum_{r=t}^{T} h_{0r}) W_{i0tk}.$$

We have already shown in Theorem 3.2 that $j_{TP}(W, X, y) = j_{SP}(U, Z, y)$, i.e.

$$\sum_{i=0}^{N} \sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=1}^{N} \sum_{q=1}^{T} \sum_{t=q}^{T} \sum_{k=t}^{T} H'_{0qt} W_{iqtk} + \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} H'_{itk} X_{itk}$$

$$= \sum_{i=0}^{N} \sum_{t=1}^{T} f_{it} y_{it} + \sum_{i=1}^{N} \sum_{q=1}^{t} \sum_{t=1}^{T} \sum_{k=t}^{T} H'_{0qt} D_{itk} U_{iqtk} + \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} G_{itk} Z_{itk}.$$

Since $D_{itk} U_{i0tk} = W_{i0tk}$ as shown in part (c) above, $j_{TP-I}(W, X, y)$ can be rewritten as

$$j_{TP}(W, X, y) + \sum_{r=1}^{T} h_{0r} I_{00} - \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{k=t}^{T} (\sum_{r=t}^{T} h_{0r}) D_{itk} U_{i0tk},$$

which is equal to $j_{SP-I}(U, Z, y).$

(ii) Consider an instance with $T$=4, $N$=2, $I_{00} = 65$, $p_{0t} = p_{1t} = 0$, $h_{0t} = 0.5$, $h_{1t} = 0.68$, $h_{2t} = 0.58$ for $1 \leq t \leq T$; $I_{01} = I_{02} = 0$, $f_{01} = 70$, $f_{02} = 60$, $f_{03} = 70$, $f_{04} = 50$, $f_{11} = 40$, $f_{12} = 60$, $f_{13} = 20$, $f_{14} = 40$, $f_{21} = 20$, $f_{22} = 40$, $f_{23} = 50$, $f_{24} = 40$, $d_{11} = 100$, $d_{12} = 77$, $d_{13} = 28$, $d_{14} = 66$, $d_{21} = 65$, $d_{22} = 89$, $d_{23} = 11$, $d_{24} = 35$. For this instance, $v(SP-I) = 393.49$ while $v(TP-I) = 392.718$. □

On the other hand, there is no dominance relation between $v(SES-I)$ and $v(SP-I)$ (or $v(TP-I)$). In the following, we present the two instances that have $v(SES-I) > v(SP-I)$ and $v(SES-I) < v(SP-I)$, respectively. Our first instance is as follows: Consider an instance with $T$=4, $N$=2, $I_{00} = 34$; $p_{0t} = p_{1t} = 0$, $h_{0t} = 0.5$, $h_{1t} = 0.7$, $h_{2t} = 0.5$, $f_{0t} = 60$, $f_{1t} = 40$, $f_{2t} = 30$ for $1 \le t \le T$; $I_{01} = I_{02} = 0$, $d_{11} = 100$, $d_{12} = 77$, $d_{13} = 28$, $d_{14} = 66$, $d_{21} = 65$, $d_{22} = 89$, $d_{23} = 11$, $d_{24} = 35$. For this instance, $v(SP-I) = v(TP-I) = 397.04$ whereas $v(SES-I) = 397.311$. Our second instance is the same as the first instance except that $I_{00} = 33$. For this instance, $v(SP-I) = v(TP-I) = 397.38$ whereas $v(SES-I) = 397.311$.

### 3.5 Computational experiments

We perform a set of computational experiments on randomly generated instances to assess the empirical performance of the formulations when the solution tool is a standard general purpose MIP solver. We generate our instances as follows. Number of retailers is set equal to 50. Two different horizon lengths are considered: $T$ = 15 or 30. External demand at retailers $d_{it}$ is generated as an integer for static demand case (i.e. $d_{it} = d_i$ for $1 \le t \le T$) and dynamic demand case from U[5,100]. Fixed cost at the warehouse $f_{0t}$ is either static (i.e. $f_{0t} = f_0$ for $1 \le t \le T$) or dynamic over time and generated from U[1500,4500] as an integer. Fixed cost at the retailers $f_{it}$ is generated as an integer from U[5,100]. Inventory holding cost at the warehouse is set equal to 0.5, while inventory holding cost at the retailers is static over time and generated from U[0.5,1]. Variable order cost at facility $i$ ($0 \le i \le N$) $p_{it}$ is equal to 0. For each combination of parameters, we generate 10 random instances; thus, we obtain 80 instances in total. Besides, two levels for initial inventory level $I_{00}$ are considered: zero and nonzero, where $I_{00} = \sum_{i=1}^{N} d_i$ for static demand case and $I_{00} = \lfloor (5+100)/2 \rfloor * N$ for dynamic demand case. Thus, we solve 160 instances in total.

We use *SES*, *TP-c*, *SP-c* and their variants with $I_{00}$ (i.e. *SES-I*, *TP-I-c* and *SP-I-c*) in the computational experiments. All these formulations are solved using callable library of CPLEX 10.1 under a time limit of 7200 seconds on a Pentium IV 3.2 GHz PC with 1GB RAM running under Windows XP. We have tested some of the default features of CPLEX in our preliminary experiments, and decided not to allow CPLEX MIP cuts in our experiments and to set presolver and aggregator off in solving all formulations, except *SES* and *SES-I*.

We present the average computational results over 10 instances with $I_{00}$=0 in Table 3.2. In the table, columns 1–3 show horizon length, type of demand (static or dynamic) and type of fixed cost at the warehouse (static or dynamic), respectively. Columns 4–6 indicate the integrality gap (%Gap) between the optimal objective value ($z^*$) and the objective value in the LP relaxation for formulation (.), i.e. %Gap = 100*($z^*$ - $v$(.))/$v$(.). Columns 7–9 and 10–12 show the elapsed time in seconds and the number of nodes explored in solving the corresponding formulation by CPLEX. The numbers in parenthesis in Column 7 indicate the number of instances that could not be solved within the time limit of two hours.

Empirical results given in Table 3.2 are in accordance with the theoretical results in Section 3.1.4, as expected. Results reveal that *SP-c* and *TP-c* formulations perform significantly better than *SES* formulation. *TP-c* and *SP-c* achieve integrality gaps which are very close to zero whereas *SES* gives around 7.3% integrality gap on average. This success of *TP-c* and *SP-c* can be attributed to disaggregation of replenishment decisions at the warehouse into separate retailers, which is contrary to the *SES*. The integrality gap and elapsed time figures for *SP-c* and *TP-c* are quite close to each other. Regarding the integrality gaps, the largest integrality gap difference between *SP-c* and *TP-c* is 0.1% among the 80 instances generated. Note that this difference was around 3.4% for the example instance of the single retailer case in Section 3.1.4. It seems that instances with dynamic fixed cost at the warehouse are relatively easier to solve for all formulations than those with static rates. Besides, instances with static demand and static fixed cost rate at the warehouse are the most challenging ones for all formulations.

**Table 3.2** Average results when $I_{00}$=0

| | | | *%Gap* | | | *Seconds* | | | *Nodes* | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ***T*** | $d_{it}$ | $f_{0t}$ | ***SES*** | ***TP-c*** | ***SP-c*** | ***SES*** | ***TP-c*** | ***SP-c*** | ***SES*** | ***TP-c*** | ***SP-c*** |
| **15** | **S** | **S** | 8.310 | 0.013 | 0.013 | 13.6 | 1.2 | 0.7 | 326.6 | 0.0 | 0.0 |
| | | **D** | 5.731 | 0.000 | 0.000 | 5.7 | 0.5 | 0.3 | 88.5 | 0.0 | 0.0 |
| | **D** | **S** | 7.485 | 0.003 | 0.003 | 10.3 | 1.1 | 0.6 | 221.5 | 0.0 | 0.0 |
| | | **D** | 5.864 | 0.000 | 0.000 | 5.5 | 0.5 | 0.4 | 103.7 | 0.0 | 0.0 |
| **30** | **S** | **S** | 9.511 | 0.044 | 0.030 | 5464.7 (4) | 14.8 | 13.9 | 49253.6 | 1.3 | 0.9 |
| | | **D** | 6.313 | 0.001 | 0.000 | 479.5 | 4.4 | 4.6 | 3536.4 | 0.0 | 0.0 |
| | **D** | **S** | 9.170 | 0.031 | 0.030 | 4765.4 (3) | 11.0 | 9.5 | 43914.7 | 0.6 | 0.6 |
| | | **D** | 6.389 | 0.000 | 0.000 | 520.3 | 4.2 | 4.2 | 4594.7 | 0.0 | 0.0 |
| **Average** | | | **7.347** | **0.012** | **0.010** | **1408.1** | **4.7** | **4.3** | **12755.0** | **0.2** | **0.2** |

S: Static, D: Dynamic

Average computational results over 10 instances with $I_{00} > 0$ are provided in Table 3.3, which has the same format with Table 3.2. Results in Table 3.3 indicate that *SP-I-c* and *TP-I-c* still perform far better than the *SES-I*. However, *SP-I-c* and *TP-I-c* yield results not as good as those results that are given for the case with $I_{00} = 0$ while the impact on *SES-I* is not significant compared to the results for *SES*. Nevertheless, both *SP-I-c* and *TP-I-c* are quite successful in that they find the optimal solution within one minute on average and within three minutes in general. Note that *SES-I* needs around 25 minutes on average to solve the instances to optimality and it could not find the optimal solution in 10 out of 80 instances.

**Table 3.3** Average results when $I_{00}>0$

| | | | *%Gap* | | | *Seconds* | | | *Nodes* | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ***T*** | $d_{it}$ | $f_{0t}$ | ***SES-I*** | ***TP-I-c*** | ***SP-I-c*** | ***SES-I*** | ***TP-I-c*** | ***SP-I-c*** | ***SES-I*** | ***TP-I-c*** | ***SP-I-c*** |
| **15** | **S** | **S** | 8.935 | 0.910 | 0.894 | 22.0 | 13.5 | 11.2 | 304.1 | 0.7 | 1.4 |
| | | **D** | 6.760 | 1.558 | 1.558 | 6.8 | 4.0 | 4.4 | 113.2 | 0.3 | 1.3 |
| | **D** | **S** | 7.445 | 1.667 | 1.653 | 10.3 | 17.7 | 15.4 | 218.9 | 27.0 | 21.1 |
| | | **D** | 6.259 | 2.529 | 2.528 | 6.3 | 5.8 | 6.9 | 137.1 | 24.6 | 26.9 |
| **30** | **S** | **S** | 10.029 | 0.635 | 0.589 | 6098.8 (6) | 149.4 | 127.4 | 50693.5 | 16.4 | 3.4 |
| | | **D** | 6.657 | 0.713 | 0.711 | 606.6 | 34.7 | 28.0 | 4344.1 | 1.0 | 1.4 |
| | **D** | **S** | 9.338 | 1.030 | 1.029 | 5268.1 (4) | 92.9 | 87.2 | 48142.1 | 39.6 | 36.7 |
| | | **D** | 6.674 | 1.216 | 1.216 | 702.6 | 25.8 | 35.4 | 6687.8 | 20.7 | 23.2 |
| **Average** | | | **7.762** | **1.282** | **1.272** | **1590.2** | **43.0** | **39.5** | **13830.1** | **16.3** | **14.4** |

S: Static, D: Dynamic

# CHAPTER 4

# ONE-WAREHOUSE MULTI-RETAILER PROBLEM WITH ORDER-UP-TO LEVEL POLICY

In this chapter, we address the one-warehouse multi-retailer problem with order-up-to level inventory control policy (OWMR-O) where the warehouse orders from a higher echelon (supplier) to be able to serve the retailers (endogenously defined inventory control policy). When a retailer is replenished by the warehouse, its inventory level has to be brought up to a predetermined maximum level (exogenously defined inventory control policy). This inventory policy is called deterministic order-up-to level inventory control policy and introduced in Bertazzi et al. (2002) for an inventory routing problem. The same policy is studied in Bertazzi et al. (2005), Pınar and Süral (2006), Archetti et al. (2007a), and Solyalı and Süral (2008a; 2008b). In practice, order-up-to level policy is frequently observed in distribution of industrial gases (Dror and Ball, 1987), in replenishment of vending machines and shelf-spaces of groceries where replenishment raises inventory up to the maximum level. Unlike Bertazzi et al. (2002; 2005), Pınar and Süral (2006), Solyalı and Süral (2008b), and Archetti et al. (2007a), we consider direct shipments in delivery to the retailers. Our problem, therefore, is closely related to the one-warehouse multi-retailer (OWMR) problem in that when we relax the order-up-to level control policy in OWMR-O, we obtain the OWMR problem (studied in Chapter 3) where the inventory control policies at both levels are endogenously defined. To the best of our knowledge, this is the first study considering the OWMR-O problem.

As the echelon stock and transportation based formulations of the OWMR problem given in Chapter 3 cannot be adapted for the OWMR-O problem (reasons are given in Section 4.2) and the standard formulation of the OWMR-O problem cannot solve large size instances to optimality (see Section 4.4), one of our aims in this chapter is to propose a strong formulation for the OWMR-O which can solve large size instances to optimality by means of an off-the-shelf MIP solver. The OWMR-O problem is not only important in its own right, but also arises as a subproblem in its variants with capacities over replenishment quantities and/or multi-stop routing. Therefore, introduction of a strong formulation for the OWMR-O problem is important since it creates an opportunity in solving such complex variants. As discussed in Chapter 1, nonzero initial inventories cannot be treated as zero and they may increase the complexity of the problem. In the current study, for instance, only $O(T)$ binary variables are needed in the formulation if no initial inventory exists at the warehouse in the OWMR-O problem whereas the presence of initial inventories increases the number to $O(NT^2)$. Thus, analyzing the effect of nonzero initial inventories to the OWMR-O problem is another aim in this chapter.

In this chapter, we show that the OMWR-O problem is *NP*-hard. We formulate the problem as a mixed integer program and the formulation is rather unique due to three reasons in comparison to its weak representations in Pınar and Süral (2006), Solyalı and Süral (2008b), and Archetti et al. (2007a). First, we provide a stronger formulation for the retailers' replenishment problem using a shortest path network representation. Second, we decompose the warehouse's replenishment problem into independent retailers and represent each with a novel set of constraints. Third, we show that the resulting formulation leads to the convex hull of the feasible region in the single retailer case of OMWR-O. Computational experiments reveal that our strong formulation is able to solve large-scale instances in reasonable time whereas the standard formulation does not. Through computational experiments, we also show that the vendor-managed approach (i.e. solving the MIPs) provides considerable savings compared to the traditional retailer-managed approach.

The remainder of this chapter is organized as follows. We present the problem definition and the computational complexity of the problem in Section 4.1. In Section 4.2, we present strong mixed integer program formulations for the problem and the convex hull proof for the single retailer case. We examine a greedy policy for controlling retailers' inventories in Section 4.3. Its implementation reduces the OWMR-O problem into a set of single-level lot sizing problems. Section 4.4 is devoted to a computational study on randomly generated problem instances in order to find out the integrality gaps created by strong formulations. Note that the notation and abbreviations defined in this chapter is only valid in this chapter.

### 4.1 Problem definition

We consider a two-level vendor-managed system where a warehouse (vendor) replenishes multiple retailers with direct shipments over a finite time horizon. Retailer $i$ ($1 \leq i \leq N$) faces external deterministic dynamic demand $d_{it}$ in period $t$ ($1 \leq t \leq T$) and may keep inventory, $I_{it}$, at the end of period $t$ to satisfy demands of future periods $k$, where $t+1 \leq k \leq T$. Retailer $i$ employs an order-up-to level inventory control policy such that its inventory level is brought up to a maximum level $S_i$ whenever it is replenished by an amount of $Q_{it}$ by the warehouse. The warehouse ($i$=0) manages the entire inventories in the system and has to order an amount of $Q_{0t}$ from its supplier to replenish the retailers. The warehouse like retailers may keep inventory, $I_{0t}$, to satisfy future demands. It uses a direct shipment transportation policy to replenish each retailer and ships an amount equal to the maximum inventory level of the retailer less its inventory level at the end of the previous period ($S_i$-$I_{i,t-1}$) whenever a replenishment is made. Figure 4.1 clarifies the order of events occurring in retailers.

We assume, without loss of generality, that there is no lead time for the shipments between the warehouse and the retailers and between the supplier and the warehouse. The shipments to the warehouse incur a fixed order cost, $f_{0t}$, independent of the size of shipment and a variable purchasing cost $p_t$, which is charged for each unit purchased in $t$. A fixed order cost, $f_{it}$, is also incurred whenever retailer $i$ receives a shipment in $t$. Both the warehouse and the retailers incur a linear holding cost for each item carried at the end of a period, $h_{it}$. We assume that all parameters are nonnegative. In addition, we assume that initial inventory level at retailer $i$, $I_{i0}$, is less than total demand of retailer $i$ over the planning horizon so that at least one replenishment is required for retailer $i$; otherwise, retailer $i$ can trivially be eliminated from the problem. The problem is to simultaneously determine lot sizing decisions of the warehouse and replenishment decisions of the retailers such that the total of inventory holding costs and fixed order costs at the warehouse and at the retailers, and the purchasing costs are minimized. We provide a standard MIP formulation of the problem in the following.

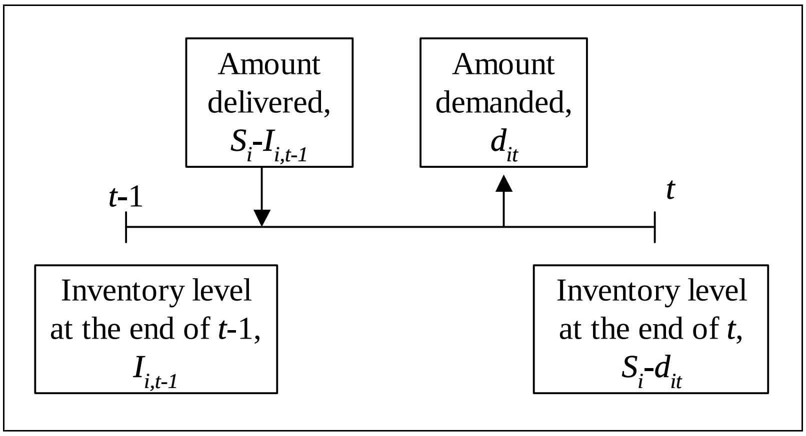


**Figure 4.1** Order of events at the retailers

$$P: \ Min \ \sum_{i=0}^{N}\sum_{t=1}^{T}(f_{it}y_{it} + h_{it}I_{it}) + \sum_{t=1}^{T} p_t Q_{0t} \tag{4.1}$$

*s.t.*

$$I_{0,t-1} + Q_{0t} = \sum_{i=1}^{N} Q_{it} + I_{0t} \qquad 1 \le t \le T \tag{4.2}$$

$$I_{i,t-1} + Q_{it} = d_{it} + I_{it} \qquad 1 \le i \le N,\ 1 \le t \le T \tag{4.3}$$

$$Q_{0t} \le My_{0t} \qquad 1 \le t \le T \tag{4.4}$$

$$Q_{it} \le S_i y_{it} \qquad 1 \le i \le N,\ 1 \le t \le T \tag{4.5}$$

$$Q_{it} \le S_i - I_{i,t-1} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad (4.6)$$

$$Q_{it} \ge S_i y_{it} - I_{i,t-1} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad (4.7)$$

$$Q_{it}, I_{it} \ge 0 \qquad 0 \le i \le N,\ 1 \le t \le T \qquad (4.8)$$

$$y_{it} \in \{0,1\} \qquad 0 \le i \le N,\ 1 \le t \le T \qquad (4.9)$$

where $M$ is a large number, and $y_{it}$ is equal to 1 if an order for facility $i$ ( $0 \le i \le N$ ) is placed in period $t$ and 0 otherwise.

Objective function (4.1) is the total of fixed order and inventory holding costs at the warehouse and retailers as well as per-unit order costs at the warehouse. Constraints (4.2) and (4.3) are the inventory balance equations for the warehouse and retailers respectively. Constraints (4.4) ensure that a fixed order cost is incurred if warehouse places an order in a period. Constraints (4.5)-(4.7) are the either-or type constraints ensuring order-up-to level policy at the retailers. Constraints (4.8) are for nonnegativity of variables while (4.9) are for integrality of variables.

**Theorem 4.1.** The OWMR-O problem is *NP*-hard.

**Proof.** We prove by reducing the *NP*-hard uncapacitated facility location problem (UFLP) (Cornuejols et al., 1990) to the OWMR-O problem. Consider a simple instance of the OWMR-O problem: for every retailer $i$, $d_{it} = 0$ for $1 \le t \le T-1$ and $d_{iT} = S_i$; $h_{it} = 0$ for all $i$ and $t$; and $h_{0t} = M$ and $p_t$ =0 for $1 \le t \le T$ where $M$ is a very large number. All shipments incur fixed order costs $f_{0t}$ ($f_{it}$) for the warehouse (the retailers) in $t$. There are no initial inventories in the system. This instance suggests the following optimal policy.

- Each retailer $i$ makes a single replenishment $S_i$ throughout the entire horizon.
- Warehouse does not keep inventory at all, and in any period it orders from its supplier an amount just enough to ship to the retailers.

This optimal policy is equivalent to solving an instance of the *NP*-hard UFLP where there are $T$ alternative sites to locate facilities with a fixed establishment cost $f_{0t}$ for each $1 \leq t \leq T$, a service cost from facility $t$ to retailer $i$ being equal to $f_{it}$, and the problem is to decide the number of facilities to establish (total number of orders/shipments), their sites (their periods) and their service regions (the retailers to be served in these periods). Thus, solving the described instance of the OWMR-O problem will also solve the above instance of UFLP. □

## 4.2 Strong formulations for the OWMR-O problem

Due to the fixed-charge cost structure at the warehouse and order-up-to level policies of the retailers, formulation $P$ provides a weak LP relaxation solution value (see Section 4.4), which makes solving even small-size problems to optimality difficult. In this section, we propose a strong formulation for the OWMR-O problem, which enables us to solve reasonable sized problems to optimality using an off-the-shelf solver.

The MIP formulated below includes integration of two components: replenishment problem of retailers and lot sizing problem of the warehouse. All of the previous studies in the literature (Archetti et al., 2007a; Pınar and Süral, 2006; Solyalı and Süral, 2008b) have modeled the retailer's replenishment problem using either-or type (weak) constraints (see constraints (4.5)–(4.7) in $P$ formulation) as a retailer $i$ receives either nothing or a quantity raising its inventory to $S_i$ at any period. We use the shortest path network representation of the retailer's replenishment problem that gives the convex hull of a single retailer's replenishment problem (see Bertazzi et al., 2002; Solyalı and Süral, 2008a for usages of network representation in different contexts). The lot sizing problem of the warehouse, on the other hand, is different from the single-level problem because its decisions regarding how much to ship to the retailers are *endogenously* specified here (note that, unlike the OWMR problem, the total amount ordered from the warehouse in the OWMR-O problem may be greater than or equal to the total external demand). We therefore redefine the variables related with the warehouse in $P$ formulation and model the entire problem as a set of combined $N$ single warehouse-single retailer problems. Additional parameters and decision variables used in the formulation are as follows.

*Parameters*

$H_{ikt}$ : Cost of serving retailer $i$ in period $t$ when the last replenishment has occurred in period $k$.

$b_{ikt}$ : Quantity shipped to retailer $i$ in period $t$ when the last replenishment has occurred in period $k$.

$D_{ikt}$ : Demand of retailer $i$ from period $k$ to period $t$, i.e. $D_{ikt} = \sum_{r=k}^{t} d_{ir}$ .

$I_{i0}$ : Level of initial inventory at facility $i$ ( $0 \le i \le N$ ).

$p(i,t)$ : The earliest period starting from which retailer $i$ does not stock out until replenished in period $t$, that is $p(i,1) = 0$ for $1 \le i \le N$, $p(i,t) = \min\{(0 \mid I_{i0} - D_{i1,t-1} \ge 0), (r \mid S_i - D_{ir,t-1} \ge 0)\}$ where $1 \le r \le t-1$ for all $1 \le i \le N$, $2 \le t \le T+1$.

$m(i,t)$ : The latest period that retailer $i$ can be replenished before being stocked out when the previous replenishment has occurred in period $t$, i.e., $m(i,0) = \max\{1, (r \mid I_{i0} - D_{i1,r-1} \ge 0)\}$ where $2 \le r \le T$ for all $1 \le i \le N$, $m(i,t) = \max(r \mid S_i - D_{it,r-1} \ge 0)$ where $t+1 \le r \le T+1$ for all $1 \le i \le N$, $1 \le t \le T$.

*Variables*

$X_{ikt}$ : 1 if retailer $i$ is replenished in period $t$ ( $1 \le t \le T$ ) when the last replenishment has occurred in period $k$ ( $p(i,t) \le k < t$ ), 0 otherwise.

$X_{ik,T+1}$ : 1 if the last replenishment to the retailer has occurred in period $k$ ( $p(i,T+1) \le k < T+1$ ) and no replenishments occur until the end of the horizon, 0 otherwise.

$y_t$ : 1 if an order for warehouse is placed in period $t$ and 0 otherwise.

$U_{iqkt}$ : Quantity the warehouse orders in period $q$ to serve retailer $i$ in period $t$ when the last replenishment to $i$ has occurred in period $k$.

$V_{ikt}$ : Quantity supplied from the initial inventory at the warehouse to serve retailer $i$ in period $t$ when the last replenishment to $i$ has occurred in period $k$.

$$F(I)\text{:}\ Min \sum_{t=1}^{T} f_{0t} y_t + \sum_{i=1}^{N} \sum_{q=1}^{t} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T} g_{qt} U_{iqkt} + \sum_{r=1}^{T} h_{0r} (I_{00} - \sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{r} V_{ikt})$$

$$+ \sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T+1} H_{ikt} X_{ikt} \qquad (4.10)$$

*s.t.*

$$\sum_{q=1}^{t} U_{iqkt} + V_{ikt} = b_{ikt} X_{ikt} \qquad 1 \le i \le N,\ p(i,t) \le k < t, 1 \le t \le T \qquad (4.11)$$

$$U_{iqkt} \le b_{ikt} y_q \qquad 1 \le i \le N,\ p(i,t) \le k < t,\ 1 \le q \le t \le T \qquad (4.12)$$

$$\sum_{t=1}^{m(i,0)} X_{i0t} = 1 \qquad 1 \le i \le N \qquad (4.13)$$

$$\sum_{k=t+1}^{m(i,t)} X_{itk} - \sum_{k=p(i,t)}^{t-1} X_{ikt} = 0 \qquad 1 \le i \le N,\ 1 \le t \le T \qquad (4.14)$$

$$- \sum_{k=p(i,T+1)}^{T} X_{ik,T+1} = -1 \qquad 1 \le i \le N \qquad (4.15)$$

$$\sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T} V_{ikt} \le I_{00} \qquad (4.16)$$

$$X_{ikt} \in \{0,1\} \qquad 1 \le i \le N,\ p(i,t) \le k < t, 1 \le t \le T+1 \qquad (4.17)$$

$$y_t \in \{0,1\} \qquad 1 \le t \le T \qquad (4.18)$$

$$U_{iqkt} \ge 0 \qquad 1 \le i \le N,\ p(i,t) \le k < t,\ 1 \le q \le t \le T \qquad (4.19)$$

$$V_{ikt} \ge 0 \qquad 1 \le i \le N,\ p(i,t) \le k < t, 1 \le t \le T \qquad (4.20)$$

where $g_{qt} = p_q + \sum_{r=q}^{t-1} h_{0r}$ for $1 \le q \le t \le T$,

$$
b_{ikt} = \begin{cases} S_i - I_{i0} + D_{i1,t-1} & \text{if } k = 0, 1 \le t \le m(i,0) \\ D_{ik,t-1} & \text{if } 1 \le k \le T, k < t \le m(i,k), t \ne T+1 \\ 0 & \text{otherwise,} \end{cases}
$$

$$
H_{ikt} = \begin{cases} f_{it} + \sum_{l=1}^{t-1} h_{il}(I_{i0} - D_{i1l}) + h_{it}(S_i - d_{it}) & \text{if } k = 0, 1 \le t \le m(i,0) \\ f_{it} + \sum_{l=k+1}^{t-1} h_{il}(S_i - D_{ikl}) + h_{it}(S_i - d_{it}) & \text{if } p(i,t) \le k < t, 1 \le t \le T, k \ne 0 \\ \sum_{l=k+1}^{T} h_{il}(S_i - D_{ikl}) & \text{if } p(i,T+1) \le k \le T, t = T+1 \\ 0 & \text{if } k = T. \end{cases}
$$

The objective function (4.10) of the model consists of fixed order, purchasing and inventory holding costs at the warehouse, and fixed order as well as inventory holding costs at the retailers. Constraints (4.11) ensure that the sum of the quantity supplied from initial inventory at the warehouse and the quantity ordered from the supplier from period 1 through period *t* for retailer *i* is equal to the quantity shipped to *i* in *t* when the last replenishment has occurred in period *k*. Constraints (4.12) guarantee that a fixed order cost is incurred if the warehouse places an order in a period. Constraints (4.13)–(4.15) are flow conservation constraints on the shortest path network problem accounting for the replenishment decisions of retailers over the horizon. They ensure that the inventory level at a retailer is brought up to the maximum level if a delivery is made. Note that $p(i,t)$ and $m(i,t)$ are used to define feasible replenishment periods. That is, by the use of $p(i,t)$ and $m(i,t)$, infeasible arcs (variables) representing stock out cases are never generated. For instance, if a retailer *i* is replenished in period *t* then it has to be replenished no later than period *k* ($t < k \le T$) when $S_i - D_{it,k-1} \ge 0$ and $S_i - D_{itk} < 0$ in order not to stock out. Then, in this specific example, feasibility is ensured by setting $m(i,t)$ equal to *k*. Similarly, for example, if a retailer *i* is replenished in period $t > 2$ then it can satisfy external demands from period *k*+1 through *t*-1 from its inventory at the end of period *k* ($2 \le k \le t-1$) without being stocked out when $S_i - D_{ik,t-1} \ge 0$ and $S_i - D_{i,k-1,t-1} < 0$. Then, in this specific example, feasibility is ensured by setting $p(i,t)$ equal to *k*. An example shortest path network for the replenishment problem of a single retailer for a three-period planning horizon where $m(0)=3$, $m(1)=m(2)=m(3)=4$, $p(1)=p(2)=p(3)=0$ and $p(4)=1$ is depicted in Figure 4.2.

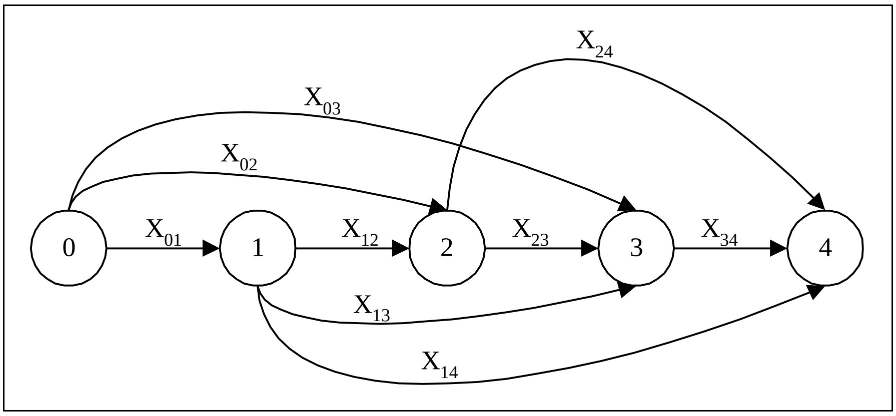


**Figure 4.2** Shortest path network representation of the single retailer replenishment problem for *T*=3

In the network, nodes except first and last represent time periods while arcs represent the replenishment decisions. Constraint (4.16) assures that the amount shipped from initial inventory is not more than the available amount. Constraints (4.17) and (4.18) assure the integrality of variables while constraints (4.19) and (4.20) are for the nonnegativity of variables.

*A stronger representation of OWMR-O*

Formulation *F(I)* can be further strengthened by the following variable redefinitions. Let $W_{iqkt}$ $(=U_{iqkt}/b_{ikt})$ be the fraction of the quantity ordered at the warehouse in period *q* to serve retailer *i* in period *t* when the last replenishment to *i* has occurred in period *k*, and $Z_{ikt}$ $(=V_{ikt}/b_{ikt})$ be the fraction of the quantity supplied by the initial inventory of warehouse to serve retailer *i* in period *t* when the last replenishment to *i* has occurred in period *k*. Then, constraints (4.11), (4.12) and (4.16) can be rewritten respectively as

$$\sum_{q=1}^{t} W_{iqkt} + Z_{ikt} = X_{ikt} \qquad 1 \le i \le N,\ p(i,t) \le k < t, 1 \le t \le T \qquad (4.21)$$

$$W_{iqkt} \le y_q \qquad 1 \le i \le N,\ p(i,t) \le k < t,\ 1 \le q \le t \le T \qquad (4.22)$$

$$\sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T} b_{ikt} Z_{ikt} \le I_{00} \qquad (4.23)$$

Constraints (4.21)–(4.23) are the results of straightforward conversion of variables and affect neither the total number of constraints nor the strength of the formulation. However, variable redefinition enables us to derive a stronger and reduced-size formulation as we show below. Theorem 4.2 is based on the fact that the warehouse must incur a fixed cost in period $q$ if an order is placed in period $q$ to replenish retailer $i$ in period $t$ when the last replenishment to the retailer might have occurred in *any* period $k$ ($p(i,t) \le k < t$). Therefore, left hand side of (4.22) can be summed over $k$.

**Theorem 4.2.** The inequalities

$$\sum_{k=p(i,t)}^{t-1} W_{iqkt} \le y_q \qquad 1 \le i \le N,\ 1 \le q \le t \le T \qquad (4.24)$$

are valid for the OWMR-O problem and they are tighter than constraints (4.12) and (4.22).

**Proof.** Because of the flow conservation constraints (4.13)–(4.15), the relation $\sum_{k=p(i,t)}^{t-1} X_{ikt} \le 1$ holds. Due to (4.21) and $\sum_{k=p(i,t)}^{t-1} X_{ikt} \le 1$, $\sum_{k=p(i,t)}^{t-1} W_{iqkt}$ cannot be greater than 1. Thus, constraints (4.24) are valid inequalities for the OWMR-O problem. Since (4.12) are equivalent to (4.22) and left-hand side of (4.24) is greater than or equal to that of (4.22), (4.24) are tighter than both (4.12) and (4.22). □

Note that contrary to (4.22), it is not possible to sum left-hand side of (4.12) over $k$. Constraints (4.24) not only reduce the total number of constraints from $O(NT^3)$ to $O(NT^2)$ but also tighten the formulation. However, there is still room for improvement as shown in the following theorem. Theorem 4.3 is based on the fact that the warehouse must incur a fixed cost in period $q$ if an order is placed in period $q$ to replenish retailer $i$ for any period $t$ or later when the last replenishment to the retailer might have occurred in *any* period $k$ ($0 \le k < t$). Thus, left hand side of (4.24) can be summed over $r \ge t$.

**Theorem 4.3.** The inequalities

$$\sum_{k=0}^{t-1} \sum_{r=t}^{\min\{m(i,k),T\}} W_{iqkr} \le y_q \qquad 1 \le i \le N, 1 \le q \le t \le T \qquad (4.25)$$

are valid for the OWMR-O problem and they are tighter than constraints (4.24).

**Proof.** Consider the shortest path network representation of the replenishment problem for each retailer by replacing $X$ variables with $W$ variables as suggested by (4.21). Now consider the partial networks involving only those arcs defined by (4.25). Each such partial network contains nodes only with either outgoing or incoming arcs with $W$ variables but not both. Because there is unit flow over the network and no node has both incoming and outgoing arcs with positive $W$ variables, the left-hand side of (4.25) cannot be greater than one (see Figure 4.3 as an illustrative example for (4.25) with $q$ = 1 and $t$ = 1, 2 and 3 on a partial network assuming $m(0) = m(1) = m(2) = 3$). Thus, (4.25) constitute valid inequalities for the OWMR-O problem.

We rewrite left-hand side of (4.25) separately for $r = t$ and $r \ge t+1$ as

$$\sum_{k=p(i,t)}^{t-1} W_{iqkt} + \sum_{k=0}^{t-1} \sum_{r=t+1}^{\min\{m(i,k),T\}} W_{iqkr} \qquad (4.25')$$

Recall that feasible variables (arcs) are assured by either $p$ or $m$. While $m$ is used to define feasible variables in the left-hand side of (4.25), $k$ is set to $p(i,t)$ instead of 0 in the first term of (4.25'), since $r$ is set equal to $t$ and only feasible arcs incoming to period $t$ should be defined. Constraints (4.25) are tighter than (4.24) since $\sum_{k=0}^{t-1} \sum_{r=t}^{\min\{m(i,k),T\}} W_{iqkr} = \sum_{k=p(i,t)}^{t-1} W_{iqkt} + \sum_{k=0}^{t-1} \sum_{r=t+1}^{\min\{m(i,k),T\}} W_{iqkr} \ge \sum_{k=p(i,t)}^{t-1} W_{iqkt}$. □

The stronger formulation is given as follows.

$$SF(I)\text{: } Min \sum_{t=1}^{T} f_{0t} y_t + \sum_{i=1}^{N} \sum_{q=1}^{t} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T} g_{qt} b_{ikt} W_{iqkt} + \sum_{r=1}^{T} h_{0r} (I_{00} - \sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{r} b_{ikt} Z_{ikt})$$

$$+ \sum_{i=1}^{N} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T+1} H_{ikt} X_{ikt} \quad (4.26)$$

*s.t.* (4.13)–(4.15), (4.17), (4.18), (4.21), (4.23), (4.25)

$$W_{iqkt} \geq 0 \qquad 1 \leq i \leq N,\ p(i,t) \leq k < t,\ 1 \leq q \leq t \leq T \quad (4.27)$$

$$Z_{ikt} \geq 0 \qquad 1 \leq i \leq N,\ p(i,t) \leq k < t,\ 1 \leq t \leq T \quad (4.28)$$

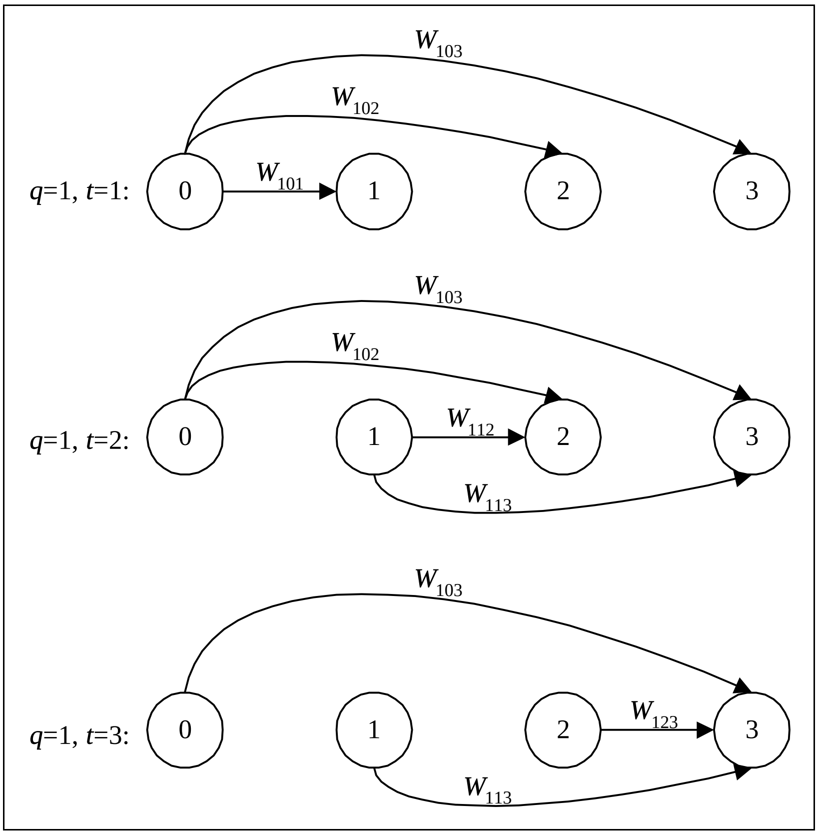


**Figure 4.3** An illustrative example for constraints (4.25) for *q*=1 when *T*=3

Note that *SF(I)* has $O(NT^2)$ constraints and $O(NT^3)$ variables, $O(NT^2)$ of which are binary. We should note that in the OWMR problem, the optimal replenishment policy at retailers is of Wagner-Whitin type, and the warehouse orders a quantity equal to the total external demand of retailers over the entire horizon. The echelon stock formulation for the OWMR problem cannot simply be extended to model the OWMR-O problem since the observation of demand at the warehouse in a period being equal to total external demand of retailers in that period (Federgruen and Tzur, 1999) is not valid for the OWMR-O problem. Similarly, the formulation proposed in Levi et al. (2008) cannot be directly used since it is based on replenishing the warehouse according to the retailers' given external demands in a period, which is non-germane to the OWMR-O problem.

*A special case: OWMR-O with* $I_{00} = 0$

In the absence of initial inventory at the warehouse, all $Z_{ikt}$ variables and constraints (4.23) would be removed from the formulation *SF(I)*. For $1 \le i \le N,\, p(i,t) \le k < t, 1 \le t \le T$, letting $X_{ikt} = \sum_{q=1}^{t} W_{iqkt}$ due to (4.21), the formulation for the *OWMR-O* problem with $I_{00} = 0$, referred to as the formulation *SF*, can be written as

$$SF\text{: } Min \sum_{t=1}^{T} f_{0t} y_t + \sum_{i=1}^{N} \sum_{q=1}^{t} \sum_{k=p(i,t)}^{t-1} \sum_{t=1}^{T} (H_{ikt} + g_{qt} b_{ikt}) W_{iqkt} + \sum_{i=1}^{N} \sum_{k=p(i,T+1)}^{T} H_{ik,T+1} X_{ik,T+1} \quad (4.29)$$

*s.t.* (4.18), (4.25), (4.27) and

$$\sum_{q=1}^{t} \sum_{t=1}^{m(i,0)} W_{iq0t} = 1 \qquad 1 \le i \le N \qquad (4.30)$$

$$a_{it} X_{it,T+1} + \sum_{k=t+1}^{\min\{m(i,t),T\}} \sum_{q=1}^{k} W_{iqtk} - \sum_{k=p(i,t)}^{t-1} \sum_{q=1}^{t} W_{iqkt} = 0 \qquad 1 \le i \le N,\, 1 \le t \le T \qquad (4.31)$$

$$-\sum_{t=1}^{T} a_{it} X_{it,T+1} = -1 \qquad 1 \le i \le N \qquad (4.32)$$

$$X_{it,T+1} \ge 0 \qquad 1 \le i \le N,\, p(i,T+1) \le t \le T \qquad (4.33)$$

where $a_{it} = \begin{cases} 1 \text{ if } p(i,T+1) \le t \\ 0 \text{ otherwise.} \end{cases}$

Note that $X$'s are set as nonnegative continuous variables and its legitimacy is proven below. Hence, *SF* has only $O(T)$ binary variables.

**Theorem 4.4.** The formulation *SF* has an optimal solution with integral values for $X$.

**Proof.** Consider a partial solution for *SF* in which all $y$-variables are known. To reach a complete solution, set $W_{iqkt} = 0$, $1 \le i \le N, p(i,t) \le k < t, 1 \le t \le T$, for those $q$'s with $y_q = 0$, in (4.25). For those $q$'s with $y_q$=1, eliminate the associated (4.25), since, due to constraints (4.30)–(4.32), $\sum_{k=0}^{t-1} \sum_{r=t}^{\min\{m(i,k),T\}} W_{iqkr} \le 1$ becomes redundant. Simplify the objective function (4.29) by letting the first term be constant. This reduces *SF* to a formulation of a collection of $N$ replenishment problems, one for each retailer, whose constraint matrix (defined by constraints (4.30)–(4.32)) is totally unimodular since all coefficients are elements of {0, -1, +1}, each $W$ and $X$ variable appear twice with coefficients -1 and +1, and there exists a partition $(R_1 = R, R_2 = \varnothing)$ of the set $R$ of rows such that the difference between summation of coefficients in $R_1$ and summation of coefficients in $R_2$ is zero for each variable. Thus, $W$-variables and $X$-variables take integral values in an optimal solution to the formulation *SF*. □

Below we show that *SF* describes the convex-hull of feasible solutions of the OWMR-O problem with $I_{00} = 0$ when there is a single retailer. For this purpose, we remove $i$ subscript representing retailer from *SF* since $N$ = 1, and refer to the resulting single retailer *SF* formulation as *SF-SR*.

**Lemma 4.1.** Let *FeasibleSR* be the set of feasible solutions to the single retailer variant of the OWMR-O problem with $I_{00} = 0$. The following inequalities

$$\sum_{k=0}^{T-1}\sum_{r=T}^{\min\{m(k),T\}} W_{qkr} \le \sum_{k=0}^{T-2}\sum_{r=T-1}^{\min\{m(k),T\}} W_{qkr} \le \ldots \le \sum_{k=0}^{q}\sum_{r=q+1}^{\min\{m(k),T\}} W_{qkr} \le \sum_{k=0}^{q-1}\sum_{r=q}^{\min\{m(k),T\}} W_{qkr} \le y_q$$

$$\text{for } 1 \le q \le T \qquad (4.34)$$

are valid for ***FeasibleSR***.

**Proof.** Inequalities (4.34) can equivalently be represented as

$$y_q - \sum_{k=0}^{q-1}\sum_{r=q}^{\min\{m(k),T\}} W_{qkr} \ge 0 \qquad 1 \le q \le T \qquad (4.35)$$

$$\sum_{k=0}^{t-1}\sum_{r=t}^{\min\{m(k),T\}} W_{qkr} - \sum_{k=0}^{t}\sum_{r=t+1}^{\min\{m(k),T\}} W_{qkr} \ge 0 \qquad 1 \le q \le t \le T-1 \qquad (4.36)$$

Validity of constraints (4.35) has already been shown in Theorem 4.3. Inequalities (4.36) can be simplified and rewritten as

$$\sum_{k=p(t)}^{t-1} W_{qkt} - \sum_{k=t+1}^{\min\{m(t),T\}} W_{qtk} \ge 0 \qquad 1 \le q \le t \le T-1 \qquad (4.37)$$

Note that first term of (4.37) denotes the replenishment of retailer in period *t* realized by the quantity ordered at the warehouse in period *q* while second term of (4.37) denotes the replenishment of retailer in any period *s* ($s > t$) where the previous replenishment has occurred in period *t* by the quantity ordered at the warehouse in period *q*. Since the optimal replenishment policy at the warehouse has the well-known Wagner-Whitin property (Solyalı and Süral, 2008a), if the quantity shipped to retailer in period *s* is ordered to the warehouse in period *q*, then the quantity shipped to retailer in period *t* ($q \le t < s$) must also be ordered to the warehouse in period *q*. As this is ensured by (4.37), (4.37) are valid. □

Thus, the modified but equivalent *SF-SR* is as follows.

$$\textit{SF-SR: Min } \sum_{t=1}^{T} f_{0t} y_t + \sum_{q=1}^{t}\sum_{k=p(t)}^{t-1}\sum_{t=1}^{T}(H_{kt} + g_{qt} b_{kt}) W_{qkt} + \sum_{k=p(T+1)}^{T} H_{k,T+1} X_{k,T+1} \qquad (4.38)$$

*s.t.* (4.18), (4.35), (4.37) and

$$\sum_{q=1}^{t}\sum_{t=1}^{m(0)} W_{q0t} = 1 \qquad (4.39)$$

$$a_t X_{t,T+1} + \sum_{k=t+1}^{\min\{m(t),T\}} \sum_{q=1}^{k} W_{qtk} - \sum_{k=p(t)}^{t-1} \sum_{q=1}^{t} W_{qkt} = 0 \quad 1 \le t \le T \tag{4.40}$$

$$-\sum_{t=1}^{T} a_t X_{t,T+1} = -1 \tag{4.41}$$

$$W_{qkt} \ge 0 \qquad p(t) \le k < t, 1 \le q \le t \le T \tag{4.42}$$

$$X_{t,T+1} \ge 0 \qquad p(T+1) \le t \le T \tag{4.43}$$

**Theorem 4.5.** The LP relaxation of the *SF-SR* formulation has an optimal solution with integral *y*, *W* and *X*.

**Proof.** As $f_{0q} \ge 0$, (4.35) will be satisfied as equality in the optimal LP relaxation solution of *SF-SR* and $y_q$ can be eliminated by substituting $\sum_{k=0}^{q-1} \sum_{r=q}^{\min\{m(k),T\}} W_{qkr}$ in place of $y_q$ in (4.38). Let *R* denote the constraint matrix composed of the remaining constraints (4.37) and (4.39)–(4.41), and $R_s$ denote the subset of rows of *R*. We will show that *R* is totally unimodular by using the following sufficient condition: For any $R_s$, there exists a partition of $R_s$ into $R_1$ and $R_2$ such that

$$\left| \sum_{i \in R_1} r_{ij} - \sum_{i \in R_2} r_{ij} \right| \le 1 \qquad \text{for all columns (variables)} \tag{4.44}$$

where $r_{ij}$ denotes the technological coefficient of $j^{\text{th}}$ variable in $i^{\text{th}}$ row. Note that $r_{ij} \in \{0, -1, +1\}$ for all *i*, *j*. Our partitioning scheme is as follows: We assign all rows of $R_s$ corresponding to constraints (4.39)–(4.41) into $R_1$. We assign the rows of $R_s$ corresponding to constraint (4.37) for *q*, *t* $(1 \le q \le t \le T-1)$ into $R_1$ if the row corresponding to constraint (4.40) for *t* $(1 \le t \le T)$ exists in $R_s$, otherwise into $R_2$.

With this partitioning scheme it is obvious that condition (4.44) holds for the columns corresponding to each $X_{kT}$ variable for $p(T) \le k < T$ and each $W_{qkT}$ variable for $p(T) \le k < T$, $1 \le q \le T$ where $k < q$ since those variables appear twice in each column with coefficients -1 and +1 due to constraints (4.39)–(4.41) (i.e. $R_1 = R_s, R_2 = \varnothing$). Also, if $R_s$ involves only the rows corresponding to constraints (4.39)–(4.41), then (4.44) holds due to the same reason. Similarly, if $R_s$ involves only the rows corresponding to (4.37), (4.44) holds since each variable appears either two times with coefficients -1 and +1 or only once with coefficient -1 or +1 due to (4.37) (i.e. $R_2 = R_s, R_1 = \varnothing$). If $R_s$ involves the rows corresponding to both constraints (4.39)–(4.41) and (4.37), then there are two cases to consider.

**Case 1.** For any $t$ $(1 \le t \le T)$, $R_s$ involves (4.37) for $q$, $t$ $(1 \le q \le t \le T-1)$ and (4.40) for $t$: Note that variables in (4.37) for $q$, $t$ $(1 \le q \le t \le T-1)$ are all involved in (4.40) for $t$ $(1 \le t \le T)$ with just the opposite coefficient signs. Thus, those opposite signs cancel each other since all those rows are assigned to $R_1$ according to the proposed partitioning scheme.

**Case 2.** For any $t$ $(1 \le t \le T)$, $R_s$ involves (4.37) for $q$, $t$ $(1 \le q \le t \le T-1)$ but not (4.40) for $t$: Note that variables in (4.37) for $q$, $t$ $(1 \le q \le t \le T-1)$ also exist in (4.40) for some $k \ne t$ $(1 \le k \le T)$ with the same coefficient sign. Thus, those same signs cancel each other since rows corresponding to (4.40) are assigned to $R_1$ whereas rows corresponding to (4.37) are assigned to $R_2$ according to the proposed partitioning scheme.

Due to cancellations of nonzero coefficients as depicted by case 1 and case 2, left-hand side of (4.44) takes values 0 or +1 for all columns which means (4.44) holds for $R_s$ involving rows corresponding to both constraints (4.39)–(4.41) and (4.37). Since the constraint matrix is totally unimodular, the proof is done. □

### 4.3 Vendor-managed approach versus retailer-managed approach

In contrast to the above vendor-managed inventory system, suppose that the supply chain operates under a retailer-managed inventory system so that each retailer wishes to replenish its own stock just prior to the period in which it stocks out and dictates this to the warehouse. This replenishment policy is a deterministic variant of the classical (*s*, *S*) policy, which we call *latest ordering* up-to level policy, determined in $O(T)$ time for a single retailer. Given all retailers' replenishment schedules, the optimal replenishment decisions at the warehouse are found in $O(T \log T)$ time by solving a single-item uncapacitated lot sizing problem in which demands to the warehouse are specified as the sum of individual retailer's replenishment quantities in every period of the schedule. Thus, such a retailer-managed approach requires $O(NT + T \log T)$ time, which is quite efficient compared to the vendor-managed approach in Sections 4.1 and 4.2. Below we analyze the structural properties of the latest ordering up-to level policy. In the following theorem, we first show that the latest ordering up-to level policy is optimal for the single retailer replenishment problem when three major problem parameters are constant.

**Theorem 4.6.** The latest ordering up-to level policy is optimal for the single retailer problem with order-up-to policy if order cost, holding cost, and external demand are all constant over the planning horizon.

**Proof.** Under the latest ordering up-to level policy, we can define at most three successive stages over the planning horizon regarding the replenishment process, assuming at least one replenishment is needed during the horizon. Stage 1 starts with the very first period where the demand is satisfied from initial inventory and lasts until the first replenishment is inevitably done. Stage 2 is the one in between two consecutive replenishment periods *k* and *t*, *k*<*t* such that $S-(t-k)d \geq 0$ and $S-(t-k+1)d < 0$. Stage 3 starts with the last replenishment period and ends with *T*. Note that stages 1 and 3 can be equivalent to stage 2 when $I_0 = S - d$ and when the difference between the last replenishment period and *T* is equal to $t-k$, respectively. It is easy to see that any other replenishment cannot reduce the inventory holding cost components associated with stage 1 and stage 2(s). The number of replenishments specified by the latest ordering up-to level policy cannot be decreased either. Only, the cost associated with stage 3 may lead to a higher cost because of the possibility of a leftover stock after *T*. Although a replenishment before the last possible moment (period in which stock-out occurs) may eliminate stage 3, the resulting cost of such a policy is at least equal to the cost incurred by the latest ordering up-to level policy. □

We next illustrate by counter-examples how optimality of the latest ordering up-to level policy vanishes even if only one of these three parameters changes dynamically. Below we assume that $T = 4$, $S = 75$ and $I_0 = 50$.

i. *Demand is dynamic*: Let $h_t = h$ and $f_t = f$ for $1 \le t \le 4$, and $d_1 = 40$, $d_2 = 15$, $d_3 = 50$, $d_4 = 11$. Optimal policy sends 25 and 55 units in periods 1 and 3, respectively, yielding a total cost of 2*f* + 94*h*, whereas the latest ordering up-to level policy sends 65 units in periods 2 and 4, and its associated cost is 2*f* +144*h*.

ii. *Unit holding cost changes over time*: Suppose that $h_t = h$ for $1 \le t \le 3$, $h_4 = h^*$ where $h^* >> h$, and $f_t = f$ and $d_t = 15$ for $1 \le t \le 4$. The optimal solution is to make a replenishment of 25 units in period 1 with a total cost of $f + 135h + 15h^*$ whereas the cost of the latest ordering up-to level policy is $f + 60h + 60h^*$.

iii. *Fixed order cost changes over time*: Consider an instance with the following data: $f_t = f$ for $1 \le t \le 3$, $f_4 = f^*$ where $f^* >> f$, and $h_t = h$ and $d_t = 15$ for $1 \le t \le 4$. The optimal solution is to make a replenishment of 25 units in period 1 with a total cost of $f + 150h$ whereas the cost of the latest ordering up-to level policy is $f^* + 120h$.

### 4.4 Computational experiments

We performed computational experiments on randomly generated instances to test the computational performance of our MIP formulations. In this section we present our results. We compare the results obtained under the vendor-managed inventory approach using MIPs with those of the retailer-managed approach implementing latest ordering policy. MIPs were solved using callable library of CPLEX 10.1 and latest ordering up-to level policy is coded in C within MS Visual C++ 6.0. All the computational experiments were done on an Intel Pentium IV 3.2 GHz PC with 1 GB RAM. Instances are generated with the following settings.

**Table 4.1** Results for $N$ = 50, $T$ = 15 and $I_{00} = 0$

| | | | | | | | Optimal | | | Latest | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_{it}$ | $p_t$ | $f_{0t}$ | $h_{0t}$ | %Gap | CPU | %LD | WC | TC | RC | WC | TC | RC |
| **SD** | **> 0** | **D** | **0.3** | 0.00 | 0.90 | 15.26 | 24.16 | 12.30 | 63.54 | 32.76 | 11.30 | 55.94 |
| | | | **0.8** | 0.01 | 1.18 | 17.93 | 27.16 | 12.82 | 60.02 | 41.22 | 10.29 | 48.49 |
| | | **L** | **0.3** | 0.02 | 1.53 | 14.95 | 17.19 | 13.55 | 69.26 | 23.36 | 13.33 | 63.31 |
| | | | **0.8** | 0.06 | 2.47 | 15.73 | 19.54 | 14.94 | 65.52 | 29.80 | 13.28 | 56.92 |
| | | **H** | **0.3** | 0.00 | 0.80 | 15.75 | 34.01 | 12.66 | 53.33 | 44.20 | 11.13 | 44.67 |
| | | | **0.8** | 0.02 | 1.72 | 17.57 | 36.19 | 11.38 | 52.44 | 51.66 | 8.29 | 40.05 |
| **Average** | | | | 0.02 | 1.43 | 16.20 | 26.37 | 12.94 | 60.69 | 37.17 | 11.27 | 51.56 |
| **SD** | **0** | **D** | **0.3** | 0.08 | 0.85 | 13.77 | 22.45 | 12.26 | 65.30 | 32.76 | 11.30 | 55.94 |
| | | | **0.8** | 0.34 | 1.82 | 23.07 | 24.30 | 12.91 | 62.78 | 41.22 | 10.29 | 48.49 |
| | | **L** | **0.3** | 0.63 | 3.72 | 9.99 | 16.49 | 13.37 | 70.14 | 23.36 | 13.33 | 63.31 |
| | | | **0.8** | 2.36 | 4.84 | 14.05 | 18.78 | 14.18 | 67.04 | 29.80 | 13.28 | 56.92 |
| | | **H** | **0.3** | 0.10 | 1.57 | 16.70 | 33.84 | 12.29 | 53.87 | 44.20 | 11.13 | 44.67 |
| | | | **0.8** | 0.89 | 4.18 | 28.12 | 34.60 | 11.22 | 54.18 | 51.66 | 8.29 | 40.05 |
| **Average** | | | | 0.73 | 2.83 | 17.62 | 25.08 | 12.71 | 62.22 | 37.17 | 11.27 | 51.56 |
| **DD** | **> 0** | **D** | **0.3** | 0.00 | 0.64 | 11.13 | 26.00 | 14.58 | 59.42 | 32.96 | 13.89 | 53.16 |
| | | | **0.8** | 0.01 | 1.03 | 11.16 | 28.85 | 13.84 | 57.32 | 41.10 | 11.82 | 47.09 |
| | | **L** | **0.3** | 0.05 | 1.79 | 10.59 | 19.32 | 16.40 | 64.28 | 23.39 | 16.73 | 59.88 |
| | | | **0.8** | 0.04 | 1.76 | 12.40 | 22.49 | 15.71 | 61.80 | 28.45 | 15.33 | 56.21 |
| | | **H** | **0.3** | 0.01 | 1.18 | 10.41 | 34.78 | 13.27 | 51.95 | 41.24 | 12.57 | 46.19 |
| | | | **0.8** | 0.06 | 2.66 | 12.84 | 38.77 | 11.91 | 49.32 | 50.60 | 9.95 | 39.45 |
| **Average** | | | | 0.03 | 1.51 | 11.42 | 28.37 | 14.28 | 57.35 | 36.29 | 13.38 | 50.33 |
| **DD** | **0** | **D** | **0.3** | 0.04 | 0.86 | 10.76 | 25.96 | 14.59 | 59.45 | 32.96 | 13.89 | 53.16 |
| | | | **0.8** | 0.22 | 1.63 | 15.37 | 27.70 | 14.03 | 58.26 | 41.10 | 11.82 | 47.09 |
| | | **L** | **0.3** | 0.53 | 3.05 | 7.60 | 19.33 | 16.51 | 64.16 | 23.39 | 16.73 | 59.88 |
| | | | **0.8** | 1.64 | 2.82 | 9.88 | 21.28 | 16.08 | 62.64 | 28.45 | 15.33 | 56.21 |
| | | **H** | **0.3** | 0.16 | 2.12 | 10.20 | 34.18 | 13.29 | 52.52 | 41.24 | 12.57 | 46.19 |
| | | | **0.8** | 0.91 | 4.45 | 18.53 | 36.68 | 12.46 | 50.86 | 50.60 | 9.95 | 39.45 |
| **Average** | | | | 0.58 | 2.49 | 12.06 | 27.52 | 14.50 | 57.98 | 36.29 | 13.38 | 50.33 |
| **Overall average** | | | | 0.34 | 2.06 | 14.32 | 26.84 | 13.61 | 59.56 | 36.73 | 12.33 | 50.95 |

The number of retailers, *N*, is set to 50 whereas the number of time periods, *T*, is set equal to 15 and 30. External demands are considered either constant (*SD*) or dynamic (*DD*), which are randomly generated as integers from U[5,100]. Maximum inventory level at retailer *i*, $S_i$, is set equal to its mean_demand**g*, where mean_demand = $d_i$ (mean_demand $= \lfloor (5+100)/2 \rfloor$) if demands are constant (dynamic), and $g$ is randomly generated as an integer from U[2,8] if $T$ = 15 and from U[2,15] if $T$ = 30. Retailer inventory holding costs are generated for each $i$ and $t$ from U[0.05,1.00]. Retailers' fixed cost for each $t$ is randomly generated as an integer from U[5,100]. We test the warehouse's fixed costs under three settings: two different levels, 1500 ($L$) and 6000 ($H$), are tried for constant fixed cost, whereas dynamic fixed costs ($D$) are randomly generated as integers from U[1500,6000].

**Table 4.2** Results for $N$ = 50, $T$ = 30 and $I_{00} = 0$

| | | | | | | | Optimal | | | Latest | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_{it}$ | $p_t$ | $f_{0t}$ | $h_{0t}$ | %Gap | CPU | %LD | WC | TC | RC | WC | TC | RC |
| SD | > 0 | D | 0.3 | 0.02 | 12.57 | 15.05 | 13.79 | 5.90 | 80.31 | 22.55 | 5.48 | 71.97 |
| | | | 0.8 | 0.02 | 23.73 | 15.41 | 16.57 | 6.39 | 77.04 | 29.20 | 5.27 | 65.54 |
| | | L | 0.3 | 0.09 | 66.59 | 13.91 | 10.29 | 6.34 | 83.37 | 15.15 | 6.39 | 78.46 |
| | | | 0.8 | 0.08 | 70.76 | 13.72 | 12.69 | 6.80 | 80.52 | 19.97 | 6.43 | 73.61 |
| | | H | 0.3 | 0.12 | 80.14 | 13.05 | 22.44 | 6.18 | 71.38 | 32.27 | 5.63 | 62.11 |
| | | | 0.8 | 0.14 | 106.89 | 14.62 | 23.29 | 5.72 | 70.99 | 38.57 | 4.23 | 57.20 |
| Average | | | | 0.08 | 60.11 | 14.29 | 16.51 | 6.22 | 77.27 | 26.28 | 5.57 | 68.15 |
| SD | 0 | D | 0.3 | 0.15 | 30.68 | 10.60 | 12.78 | 5.83 | 81.39 | 22.55 | 5.48 | 71.97 |
| | | | 0.8 | 0.34 | 62.40 | 16.78 | 14.48 | 6.49 | 79.03 | 29.20 | 5.27 | 65.54 |
| | | L | 0.3 | 0.89 | 173.16 | 6.77 | 9.64 | 6.21 | 84.14 | 15.15 | 6.39 | 78.46 |
| | | | 0.8 | 1.76 | 305.16 | 9.35 | 11.52 | 6.84 | 81.65 | 19.97 | 6.43 | 73.61 |
| | | H | 0.3 | 1.39 | 188.80 | 12.87 | 21.52 | 6.09 | 72.38 | 32.27 | 5.63 | 62.11 |
| | | | 0.8 | 2.35 | 268.59 | 21.28 | 21.53 | 5.72 | 72.75 | 38.57 | 4.23 | 57.20 |
| Average | | | | 1.15 | 171.46 | 12.94 | 15.25 | 6.20 | 78.56 | 26.28 | 5.57 | 68.15 |
| DD | > 0 | D | 0.3 | 0.01 | 14.89 | 10.93 | 16.37 | 7.23 | 76.39 | 22.70 | 6.93 | 70.37 |
| | | | 0.8 | 0.02 | 24.42 | 12.61 | 18.47 | 6.97 | 74.56 | 29.93 | 5.96 | 64.12 |
| | | L | 0.3 | 0.09 | 57.01 | 11.02 | 12.31 | 7.75 | 79.94 | 15.95 | 7.71 | 76.34 |
| | | | 0.8 | 0.06 | 36.05 | 12.43 | 13.62 | 7.67 | 78.70 | 19.46 | 7.15 | 73.40 |
| | | H | 0.3 | 0.12 | 73.01 | 11.05 | 23.41 | 6.92 | 69.67 | 30.50 | 6.54 | 62.97 |
| | | | 0.8 | 0.12 | 91.70 | 12.48 | 26.34 | 6.85 | 66.81 | 40.23 | 5.38 | 54.40 |
| Average | | | | 0.07 | 49.51 | 11.75 | 18.42 | 7.23 | 74.35 | 26.46 | 6.61 | 66.93 |
| DD | 0 | D | 0.3 | 0.09 | 22.97 | 9.16 | 16.07 | 7.31 | 76.62 | 22.70 | 6.93 | 70.37 |
| | | | 0.8 | 0.59 | 63.73 | 14.86 | 17.05 | 7.05 | 75.90 | 29.93 | 5.96 | 64.12 |
| | | L | 0.3 | 0.77 | 128.03 | 6.78 | 12.16 | 7.76 | 80.08 | 15.95 | 7.71 | 76.34 |
| | | | 0.8 | 1.25 | 110.06 | 8.84 | 13.20 | 7.71 | 79.10 | 19.46 | 7.15 | 73.40 |
| | | H | 0.3 | 0.94 | 148.70 | 9.80 | 22.82 | 7.05 | 70.12 | 30.50 | 6.54 | 62.97 |
| | | | 0.8 | 2.04 | 225.25 | 18.13 | 25.84 | 6.76 | 67.41 | 40.23 | 5.38 | 54.40 |
| Average | | | | 0.95 | 116.45 | 11.26 | 17.86 | 7.27 | 74.87 | 26.46 | 6.61 | 66.93 |
| Overall average | | | | 0.56 | 99.39 | 12.56 | 17.01 | 6.73 | 76.26 | 26.37 | 6.09 | 67.54 |

Warehouse's holding costs, $h_{0t}$, are set at two levels, 0.3 and 0.8. Purchasing costs, $p_t$, are set at two levels: either to 0 or 10 when $h_{0t}$=0.3 and 30 when $h_{0t}$=0.8. If the warehouse has nonzero initial inventory, then its level is taken as $\sum_{i=1}^{N} d_i$ if demands are constant and $N * \lfloor (5+100)/2 \rfloor$ if demands are dynamic. Retailers' initial inventory is taken as $I_{i0} = \lfloor r * S_i \rfloor$ where $r$ is randomly generated from U[0.01,0.99]. Both for the zero and nonzero initial inventory cases, we generate 10 random instances for each combination of the parameters, thus we have 960 instances in total.

Computational results on instances with $T$ = 15 and 30 in the absence of initial inventory at the warehouse are presented in Tables 4.1 and 4.2, respectively. Results for which the initial inventory at the warehouse is nonzero are given in Tables 4.3 and 4.4 for $T$ = 15 and 30, respectively. In Tables 4.1–4.4, columns 1–4 show the type of demand pattern, unit purchasing cost, fixed order cost at the warehouse and unit inventory holding cost at the warehouse, respectively. Column 5 shows the percentage gap (%Gap) between the optimal solution value (Opt) of a MIP formulation (*SF* in Tables 4.1–4.2 and *SF*(*I*) in Tables 4.3–4.4) and the LP relaxation solution value (LP) of the MIP formulation, i.e. %Gap=100*(Opt – LP)/LP. Column 6 lists the time elapsed in seconds to obtain the optimal solution (CPU). Column 7 gives the percentage deviation (%LD) of the solution value found by latest ordering up-to level policy (Latest) from the optimal solution value, i.e. %LD=100*(Latest – Opt)/Opt. WC, TC and RC in columns 8–10 (for MIPs) and in columns 11–13 (for latest ordering up-to level policy) denote percentages of the warehouse's cost (fixed order plus inventory holding costs at warehouse), transportation cost (fixed cost of shipments from warehouse to retailers) and retailers' cost (inventory holding cost at retailers), respectively, over the total cost less purchasing cost. Purchasing cost is disregarded because it is assumed to be a system cost shared by all parties. Each entry of Tables 4.1–4.4 is the average results of 10 instances.

In the absence of initial inventories at the warehouse, all instances with $T$ = 15 are solved less than five seconds and integrality gaps are quite small, about 0.3% on average. When $T$ increases to 30, integrality gaps are still about 0.6% on average and CPU times increase from seconds to 1.7 minutes. This can be explained by the fact that as the number of time periods increases, the number of continuous variables and constraints increase in cubic and quadratic terms, respectively.

**Table 4.3** Results for $N$ = 50, $T$ = 15 and $I_{00} > 0$

| | | | | | | | Optimal | | | Latest | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d_{it}$ | $p_t$ | $f_{0t}$ | $h_{0t}$ | %Gap | CPU | %LD | WC | TC | RC | WC | TC | RC |
| SD | > 0 | D | 0.3 | 0.24 | 14.77 | 16.37 | 23.14 | 12.46 | 64.40 | 31.87 | 11.43 | 56.70 |
| | | | 0.8 | 0.14 | 23.99 | 19.35 | 26.67 | 12.81 | 60.51 | 39.96 | 10.50 | 49.54 |
| | | L | 0.3 | 0.14 | 31.92 | 16.01 | 16.48 | 13.59 | 69.93 | 22.38 | 13.51 | 64.11 |
| | | | 0.8 | 0.10 | 42.58 | 17.08 | 19.32 | 14.58 | 66.10 | 28.51 | 13.53 | 57.97 |
| | | H | 0.3 | 0.26 | 32.51 | 16.80 | 33.13 | 12.66 | 54.21 | 42.80 | 11.41 | 45.79 |
| | | | 0.8 | 0.23 | 60.10 | 18.88 | 35.49 | 11.46 | 53.05 | 50.59 | 8.47 | 40.94 |
| Average | | | | 0.18 | 34.31 | 17.42 | 25.71 | 12.93 | 61.37 | 36.02 | 11.48 | 52.51 |
| SD | 0 | D | 0.3 | 1.36 | 17.28 | 13.63 | 21.56 | 12.30 | 66.15 | 31.87 | 11.43 | 56.70 |
| | | | 0.8 | 2.23 | 35.62 | 22.54 | 23.78 | 12.90 | 63.32 | 39.96 | 10.50 | 49.54 |
| | | L | 0.3 | 1.40 | 71.50 | 9.98 | 15.41 | 13.63 | 70.96 | 22.38 | 13.51 | 64.11 |
| | | | 0.8 | 2.88 | 138.89 | 15.11 | 17.06 | 14.51 | 68.42 | 28.51 | 13.53 | 57.97 |
| | | H | 0.3 | 1.79 | 61.95 | 16.88 | 32.39 | 12.57 | 55.04 | 42.80 | 11.41 | 45.79 |
| | | | 0.8 | 3.58 | 122.31 | 27.67 | 33.69 | 11.29 | 55.02 | 50.59 | 8.47 | 40.94 |
| Average | | | | 2.21 | 74.59 | 17.63 | 23.98 | 12.87 | 63.15 | 36.02 | 11.48 | 52.51 |
| DD | > 0 | D | 0.3 | 0.18 | 9.02 | 11.91 | 24.95 | 14.70 | 60.36 | 31.77 | 14.14 | 54.09 |
| | | | 0.8 | 0.09 | 19.21 | 11.99 | 27.91 | 14.06 | 58.03 | 39.64 | 12.11 | 48.25 |
| | | L | 0.3 | 0.14 | 26.41 | 11.37 | 18.70 | 16.47 | 64.83 | 22.63 | 16.90 | 60.48 |
| | | | 0.8 | 0.09 | 22.11 | 13.34 | 21.81 | 15.88 | 62.31 | 27.78 | 15.48 | 56.73 |
| | | H | 0.3 | 0.25 | 27.49 | 11.11 | 33.84 | 13.52 | 52.64 | 40.24 | 12.79 | 46.98 |
| | | | 0.8 | 0.15 | 43.84 | 13.88 | 37.33 | 12.34 | 50.33 | 49.35 | 10.20 | 40.45 |
| Average | | | | 0.15 | 24.68 | 12.27 | 27.42 | 14.49 | 58.08 | 35.23 | 13.60 | 51.16 |
| DD | 0 | D | 0.3 | 0.98 | 18.79 | 11.01 | 24.66 | 14.80 | 60.54 | 31.77 | 14.14 | 54.09 |
| | | | 0.8 | 1.38 | 27.75 | 15.29 | 27.18 | 14.18 | 58.64 | 39.64 | 12.11 | 48.25 |
| | | L | 0.3 | 1.17 | 49.34 | 7.84 | 18.84 | 16.37 | 64.79 | 22.63 | 16.90 | 60.48 |
| | | | 0.8 | 2.40 | 36.75 | 9.98 | 20.70 | 16.27 | 63.02 | 27.78 | 15.48 | 56.73 |
| | | H | 0.3 | 1.46 | 65.66 | 10.48 | 33.26 | 13.51 | 53.23 | 40.24 | 12.79 | 46.98 |
| | | | 0.8 | 2.30 | 87.80 | 19.12 | 35.58 | 12.76 | 51.66 | 49.35 | 10.20 | 40.45 |
| Average | | | | 1.61 | 47.68 | 12.29 | 26.70 | 14.65 | 58.65 | 35.23 | 13.60 | 51.16 |
| Overall average | | | | 1.04 | 45.32 | 14.90 | 25.95 | 13.73 | 60.31 | 35.63 | 12.54 | 51.83 |

Nevertheless, all of the 480 instances are solved to optimality in less than five minutes when $I_{00} = 0$. When the initial inventory at the warehouse is nonzero ($I_{00} > 0$), all of the 240 instances are solved to optimality within three minutes for *T*=15. For *T*=30, 47 out of 240 instances could not be solved to optimality within a 2 hour time limit and the number of those instances is indicated in CPU column within parenthesis in Table 4.4. Although average integrality gaps in the experiment are about 1%, it is evident that presence of initial inventories at the warehouse significantly increases the computational requirements for solving the problem.

Tables 4.1–4.4 indicate that the instances with zero purchasing cost are “difficult” compared to those with nonzero purchasing cost, independently from initial inventory status. Especially, those with static demand at retailers and static fixed cost at the warehouse require more computational effort. Besides, the amount shipped to the retailers tends to deviate from the exact required amount at the retailers (i.e., $\sum_{i=1}^{N}(D_{i1T} - I_{i0})$), as expected.

Either $I_{00} = 0$ or $I_{00} > 0$, the total cost values by the latest ordering up-to level policy are 14% worse than the optimal total cost values on average, ranging from 4% to 32%. This policy is most successful when the setting involves no purchasing cost, static low fixed cost at the warehouse, and low inventory holding cost rates at the warehouse. This is because myopic decisions can be penalized only up to a certain degree when cost parameters at the warehouse are low. Composition of total costs for the two approaches is considerably different: retailers’ costs are lower but warehouse’s costs are higher for the latest ordering up-to level policy, it is just the opposite as for the optimal policy. Transportation cost figures are similar for both policies. As expected vendor-managed inventory approach coordinates replenishments better than the retailer-managed inventory approach leading to significant reductions in cost.

**Table 4.4** Results for $N = 50$, $T = 30$ and $I_{00} > 0$

| $d_{it}$ | $p_t$ | $f_{0t}$ | $h_{0t}$ | %Gap | CPU | %LD | Optimal* WC | Optimal* TC | Optimal* RC | Latest WC | Latest TC | Latest RC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **SD** | **> 0** | **D** | **0.3** | 0.08 | 161.19 | 15.51 | 13.35 | 5.90 | 80.75 | 22.01 | 5.51 | 72.48 |
| | | | **0.8** | 0.08 | 253.61 | 15.98 | 16.25 | 6.41 | 77.34 | 28.89 | 5.29 | 65.83 |
| | | **L** | **0.3** | 0.17 | 1067.31 | 14.33 | 10.05 | 6.33 | 83.62 | 14.90 | 6.41 | 78.69 |
| | | | **0.8** | 0.10 | 1109.17 | 14.25 | 12.43 | 6.80 | 80.77 | 19.56 | 6.46 | 73.98 |
| | | **H** | **0.3** | 0.30 | 2645.34 (1) | 13.42 | 21.77 | 6.27 | 71.96 | 31.67 | 5.68 | 62.65 |
| | | | **0.8** | 0.19 | 3206.09 (2) | 15.14 | 22.85 | 5.69 | 71.46 | 37.79 | 4.29 | 57.92 |
| **Average** | | | | 0.15 | 1407.12 | 14.77 | 16.12 | 6.23 | 77.65 | 25.80 | 5.61 | 68.59 |
| **SD** | **0** | **D** | **0.3** | 0.50 | 423.97 | 10.56 | 12.25 | 5.87 | 81.88 | 22.01 | 5.51 | 72.48 |
| | | | **0.8** | 1.00 | 868.90 | 16.87 | 14.04 | 6.52 | 79.44 | 28.89 | 5.29 | 65.83 |
| | | **L** | **0.3** | 1.29 | 4340.24 (3) | 6.67 | 9.51 | 6.22 | 84.27 | 14.90 | 6.41 | 78.69 |
| | | | **0.8** | 2.16 | 4787.76 (4) | 9.46 | 11.22 | 6.88 | 81.90 | 19.56 | 6.46 | 73.98 |
| | | **H** | **0.3** | 2.54 | 7114.62 (9) | 12.42 | 20.86 | 6.22 | 72.92 | 31.67 | 5.68 | 62.65 |
| | | | **0.8** | 3.44 | 6521.43 (9) | 20.72 | 21.37 | 5.66 | 72.98 | 37.79 | 4.29 | 57.92 |
| **Average** | | | | 1.82 | 4009.49 | 12.78 | 14.87 | 6.23 | 78.90 | 25.80 | 5.61 | 68.59 |
| **DD** | **> 0** | **D** | **0.3** | 0.12 | 130.50 | 11.25 | 16.03 | 7.29 | 76.68 | 22.29 | 6.97 | 70.74 |
| | | | **0.8** | 0.07 | 355.63 | 13.07 | 17.87 | 7.01 | 75.12 | 29.25 | 6.02 | 64.73 |
| | | **L** | **0.3** | 0.14 | 607.98 | 11.39 | 11.92 | 7.78 | 80.30 | 15.60 | 7.74 | 76.66 |
| | | | **0.8** | 0.09 | 589.10 | 12.87 | 13.50 | 7.64 | 78.86 | 19.08 | 7.18 | 73.74 |
| | | **H** | **0.3** | 0.25 | 1969.71 | 11.41 | 22.55 | 7.00 | 70.44 | 29.85 | 6.60 | 63.55 |
| | | | **0.8** | 0.17 | 3062.10 (2) | 12.91 | 25.84 | 6.85 | 67.30 | 39.34 | 5.46 | 55.21 |
| **Average** | | | | 0.14 | 1119.17 | 12.15 | 17.95 | 7.26 | 74.79 | 25.90 | 6.66 | 67.44 |
| **DD** | **0** | **D** | **0.3** | 0.52 | 174.92 | 9.14 | 15.73 | 7.37 | 76.90 | 22.29 | 6.97 | 70.74 |
| | | | **0.8** | 0.96 | 1596.40 (1) | 15.04 | 16.64 | 7.07 | 76.29 | 29.25 | 6.02 | 64.73 |
| | | **L** | **0.3** | 1.03 | 2889.54 | 6.87 | 11.89 | 7.77 | 80.34 | 15.60 | 7.74 | 76.66 |
| | | | **0.8** | 1.58 | 2769.73 (1) | 8.88 | 13.00 | 7.64 | 79.36 | 19.08 | 7.18 | 73.74 |
| | | **H** | **0.3** | 1.55 | 6524.60 (6) | 9.80 | 22.36 | 7.08 | 70.55 | 29.85 | 6.60 | 63.55 |
| | | | **0.8** | 2.94 | 6975.46 (9) | 17.74 | 24.91 | 6.92 | 68.17 | 39.34 | 5.46 | 55.21 |
| **Average** | | | | 1.43 | 3488.44 | 11.24 | 17.42 | 7.31 | 75.27 | 25.90 | 6.66 | 67.44 |
| **Overall average** | | | | 0.89 | 2506.05 | 12.74 | 16.59 | 6.76 | 76.65 | 25.85 | 6.13 | 68.01 |

*Some instances could not be solved to optimality due to a 2 hour time limit. %GAP for such instances is computed by using the best integer feasible solution value found within the limit instead of Opt.

We also attempt to solve the instances using the standard (weak) MIP formulation ($P$). We set the $M$ value in constraints (4.4) of $P$ equal to $\sum_{i=1}^{N}\sum_{k=p(i,t)}^{T-1} d_{ik}$ for each $t$ $(1 \le t \le T)$ where $d_{i0} = S_i - I_{i0}$ so that constraints (4.4) are as tight as possible using information regarding the total requirements of retailers in period $t$ and afterwards. We use $P$ to solve the first ten instances generated (first row of Table

4.1) which corresponds to instances with $N$=50, $T$=15, static demand at retailers, nonzero production cost, low inventory holding cost rate for warehouse, dynamic fixed cost and zero initial inventory at the warehouse. Computational results with a 2 hour time limit are given in Table 4.5. In Table 4.5, column 1 lists the instances. Column 2 is the same as column 5 in Tables 4.1–4.4. Column 3 shows the percentage gap (%BO) between the best solution value found by $P$ and the optimal solution value, i.e. %BO = 100 * (Best – Opt) / Opt. Column 4 shows the remaining percentage gap (%Rgap) between the best solution value found by $P$ (Best) and the minimum of the objective function values of unexplored nodes (BestNode), i.e. %Rgap = 100 * (Best – BestNode) / BestNode.

Table 4.5 indicates that none of the instances are solved to optimality by using $P$: Seven of them due to lack of memory and the rest due to time limit. On average, there is still 2% remaining gap (%Rgap) between the best solution and best lower bound when CPLEX terminated. Integrality gaps (%Gap) are on average 15.9% for $P$ whereas it is almost zero for *SF* for the same group of instances. Although the best solution values found by $P$ are quite close to the optimal solution values, note that *SF* finds those values and proves that they are optimal within a few seconds.

**Table 4.5** Results using $P$ with a 2-hour time limit

| Instance | %Gap | %BO | %Rgap |
|---|---|---|---|
| **1** | 15.33 | 0.00 | 1.87 |
| **2** | 15.63 | 0.81 | 2.61 |
| **3** | 12.37 | 0.13 | 1.11 |
| **4** | 14.27 | 0.43 | 1.94 |
| **5** | 16.88 | 0.36 | 2.98 |
| **6** | 19.68 | 0.68 | 3.00 |
| **7** | 16.49 | 0.44 | 1.93 |
| **8** | 15.54 | 0.35 | 2.13 |
| **9** | 16.06 | 0.00 | 1.78 |
| **10** | 16.27 | 0.00 | 1.62 |
| **Average** | 15.85 | 0.32 | 2.10 |

# CHAPTER 5

# THE INVENTORY ROUTING PROBLEM WITH ORDER-UP-TO LEVEL POLICY

In this chapter, we consider the inventory routing problem with order-up-to level policy, the same problem as in Bertazzi et al. (2002), Pınar and Süral (2006), and Archetti et al. (2007a). A supplier (vendor) receives a given amount of a single product each period and distributes to multiple retailers controlled by order-up-to level policy in a vendor-managed inventory (VMI) setting over a finite time horizon using a capacitated vehicle. It is called deterministic VMI routing with order-up-to level (VMIR-OU) problem in Archetti et al. (2007a). This policy is also considered in production-distribution routing problems (Bertazzi et al., 2005; Solyalı and Süral, 2008b) and production-distribution problems with direct shipments (Chapter 4; Solyalı and Süral, 2008a).

The inventory routing problem (IRP) has been widely studied in the literature under various settings, as discussed in Chapter 2. Recently, Archetti et al. (2007a), Abdelmaguid et al. (2008), Yugang et al. (2008), and Savelsbergh and Song (2008) try to find the optimal solution or a lower bound for their multi-period (finite) IRPs using mathematical programming formulations. The common feature of all these studies is that they use weak representations for the inventory replenishment problem of retailers. Since strong formulations lend themselves to an exact solution, their use in IRPs seems promising. To the best of our knowledge, this study is the first to consider strong formulations for inventory routing problems in developing solution algorithms.

We view the VMIR-OU problem as an integration of vehicle routing problem and inventory replenishment problems, and model the problem by using a strong formulation for its replenishment decisions and a computationally attractive formulation for routing decisions. We develop a branch-and-cut algorithm and an a priori tour heuristic, both based on the strong formulation proposed. Computational results reveal that our algorithms perform better than their competitors in the literature. We also discuss how to implement our approach to the two related problems in which order-up-to level policy is relaxed.

The rest of the chapter is organized as follows. We give a formal problem definition in Section 5.1. The notation and description of the problem draw on Archetti et al. (2007a). In Section 5.2, we present the strong formulation for the problem. Section 5.3 describes the branch-and-cut and the heuristic algorithms in detail. In Section 5.4, we provide a computational study on randomly generated instances to test the performance of algorithms and compare them with those available in the literature. Also, we discuss how to extend our approach to the two related VMI routing problems. Note that the notation and abbreviations defined in this chapter is only valid in this chapter and Appendices B, C, and D.

### 5.1 Problem definition

We consider a distribution system in which a supplier distributes a single product to $n$ retailers over a finite time horizon $H$ with a vehicle of capacity $C$. Retailer $i \in M = \{1, 2, ..., n\}$ faces external demand $r_{it}$ in each discrete time period $t \in t = \{1, 2, ..., H\}$ and keeps inventory $I_{it}$ to meet the demand without backlogging. Besides, retailer $i \in M$ is controlled by an order-up-to level inventory, and in any period $t \in t$ it receives either no replenishment or a quantity $U_i - I_{it}$ raising its inventory level $I_{it}$ to its maximum level $U_i$ whenever replenished by the supplier. The supplier, denoted by $i = 0$, manages the inventories at the retailers by deciding on when and how much to ship to each retailer $i \in M$, and guarantees that no retailers will stock-out (i.e. $I_{it} \geq 0$) in any period $t \in t$. The supplier receives a quantity $r_{0t}$ in every period $t \in t$ and may ship to the retailers immediately or keep inventory $I_{0t}$ for replenishing retailers in later periods. $I_{0t}$ and $I_{it}$ respectively denote inventory levels of the supplier and retailers at the beginning of period $t \in t$. $H$+1 accounts for the impact of decisions given in the last period $H$. Each unit kept at inventory in $t \in t'$, $t' = t \cup \{H+1\}$, incurs a holding cost $h_i$ at facility $i \in M'$ where $M' = M \cup \{0\}$. The vehicle can visit several retailers in a multi-stop route, departing from and returning back to the supplier's depot, without exceeding the vehicle capacity. A visit from facility $i \in M'$ to facility $j \in M'$ incurs a transportation cost $c_{ij}$. We assume that the vehicle performs at most a single tour in every period. The VMIR-OU problem is to decide on when and in what sequence to visit retailers and how much to ship to each retailer in a trip such that the sum of transportation costs and inventory carrying costs at the supplier and retailers is minimized. The VMIR-OU problem is known to be a strongly *NP*-hard problem (Bertazzi et al., 2002). In the following theorem, we show that even the feasibility problem of the VMIR-OU problem is *NP*-complete in the strong sense.

**Theorem 5.1.** The feasibility problem of the VMIR-OU problem is strongly *NP*-complete.

**Proof.** Let *Feas* denote the associated feasibility problem of the (optimization) VMIR-OU problem. *Feas* is obviously in NP. We prove by reducing the strongly NP-complete 3-partition problem (Garey and Johnson, 1979) to the *Feas*. The 3-partition problem can be described as follows. Given a finite set $A$ of 3$q$ elements, a bound $B \in Z^+$ and a "size" $d_k \geq 0$ satisfying $\frac{1}{4}B < d_k < \frac{1}{2}B$ for each $k \in A$ such that $\sum_{k \in A} d_k = qB$, can $A$ be partitioned into $q$ disjoint sets, i.e. $A_1, A_2, ..., A_s, ..., A_q$ such that each set $s$ ($1 \leq s \leq q$) satisfy $\sum_{k \in A_s} d_k = B$?

Consider the following instance of the *Feas*: Let $n = 3q$, $H = q$; for each $i \in M$, $U_i = d_i$, $r_{it} = 0$ for $t \in t \setminus \{H\}$ and $r_{iH} = d_i$. Let $r_{01} = \sum_{i=1}^{n} d_i$, $r_{0t} = 0$ for

$t \in t \setminus \{1\}$, $C = B$, $\sum_{i=1}^{3q}\sum_{t=1}^{q} r_{it} = qB$ and $\frac{1}{4}B < d_i < \frac{1}{2}B$ for each $i \in M$. Note that solving the above instance of *Feas* will also solve the 3-partition problem. □

The VMIR-OU problem is formulated as a mixed integer program (MIP) by Pınar and Süral (2006) and Archetti et al. (2007a). The main difference between two MIPs is their way of modeling routing part of the problem. The former uses a Miller-Tucker-Zemlin based formulation to model routing decision while the latter uses a two-index vehicle flow based formulation. We present the latter formulation, referred to as formulation *F*, in Appendix B.

### 5.2 Strong formulation for the VMIR-OU problem

A two-index vehicle flow representation of routing problem in *F* is one of few effective representations for the symmetric vehicle routing problem in the literature (Laporte, 2007). Inventory replenishment problem of retailers in *F*, on the other hand, is formulated using either-or type constraints (see constraints (B.5)–(B.7) in Appendix B), later strengthened with some valid inequalities (see constraints (B.15)–(B.17) in Appendix B). However, such a representation is not tight as it is shown by an example below. Apparently, the replenishment problem of a single retailer can be represented as a shortest path problem, which can be solved in $O(H^2)$ time (see Bertazzi et al., 2002, and Solyalı and Süral, 2008a for alternative shortest path representations). We reformulate the VMIR-OU problem so that the routing uses the two-index vehicle flow representation and the convex-hull of inventory replenishment problem is represented using the shortest path problem. Additional parameters and variables needed in the strong formulation are as follows.

Define $b_{ikt}$ as the quantity shipped to retailer $i$ in period $t$ when the last replenishment has occurred in period $k$ and $R_{ikt}$ as the sum of demand between periods $k$ and $t$ to retailer $i$, i.e. $R_{ikt} = \sum_{j=k}^{t} r_{ij}$. $p(i,t)$ denotes the earliest period starting from which retailer $i$ does not stock out until replenished in period $t$, where $p(i,1) = 0$ for $i \in M$, $p(i,t) = \min\{(0 \mid I_{i0} - R_{i1,t-1} \geq 0), (k \mid U_i - R_{ik,t-1} \geq 0)\}$ for all $i \in M$, $1 \leq k \leq t-1$, $t \in t' \setminus \{1\}$. $m(i,t)$ indicates the latest period that retailer $i$ can be replenished before being stock out when the previous replenishment has occurred in period $t$, where $m(i,0) = \max\{1, (k \mid I_{i0} - R_{i1,k-1} \geq 0)\}$ for all $i \in M$, $2 \leq k \leq H$ and $m(i,t) = \max\{(k \mid U_i - R_{it,k-1} \geq 0)\}$ for all $i \in M$, $t+1 \leq k \leq H+1$, $t \in t$. Let $w_{ikt}$ be 1 if retailer $i$ is replenished in period $t$ ($1 \leq t \leq H$) when the last replenishment has occurred in period $k$ ($p(i,t) \leq k < t$) and 0 otherwise; $w_{ik,H+1}$ be 1 if the final replenishment to the retailer has occurred in period $k$ ($p(i,H+1) \leq k < H+1$) and no replenishment occurs any more, and 0 otherwise; $z_{it}$ be 1 if retailer $i \in M$ is replenished in period $t \in t$ and 0 otherwise; $z_{0t}$ be 1 if vehicle departs from the supplier in period $t \in t$ and 0 otherwise; and $y_{ji}^{t}$ be 1 if vehicle visits facility $i \in M'$ immediately after facility $j \in M'$ in period $t \in t$ and 0 otherwise. Then, the strong formulation we propose is as follows.

$$SF\text{: } Min \sum_{i \in M'} \sum_{t \in t'} h_i I_{it} + \sum_{i \in M'} \sum_{j \in M', j<i} \sum_{t \in t} c_{ij} y_{ij}^{t} \tag{5.1}$$

*s.t.*

$$I_{0t} = I_{0,t-1} + r_{0,t-1} - \sum_{i \in M} \sum_{k=p(i,t-1)}^{t-2} b_{ik,t-1} w_{ik,t-1} \qquad t \in t' \tag{5.2}$$

$$I_{0t} \geq \sum_{i \in M} \sum_{k=p(i,t)}^{t-1} b_{ikt} w_{ikt} \qquad t \in t \tag{5.3}$$

$$I_{it} = I_{i,t-1} + \sum_{k=p(i,t-1)}^{t-2} b_{ik,t-1} w_{ik,t-1} - r_{i,t-1} \qquad i \in M, t \in t' \tag{5.4}$$

$$\sum_{k=1}^{m(i,0)} w_{i0k} = 1 \qquad i \in M \tag{5.5}$$

$$\sum_{k=t+1}^{m(i,t)} w_{itk} - \sum_{k=p(i,t)}^{t-1} w_{ikt} = 0 \qquad i \in M, t \in t \tag{5.6}$$

$$-\sum_{k=p(i,H+1)}^{H} w_{ik,H+1} = -1 \qquad i \in M \qquad (5.7)$$

$$\sum_{i \in M} \sum_{k=p(i,t)}^{t-1} b_{ikt} w_{ikt} \le C z_{0t} \qquad t \in t \qquad (5.8)$$

$$\sum_{k=p(i,t)}^{t-1} w_{ikt} = z_{it} \qquad i \in M, t \in t \qquad (5.9)$$

$$\sum_{j \in M', j<i} y_{ij}^{t} + \sum_{j \in M', j>i} y_{ji}^{t} = 2 z_{it} \qquad i \in M', t \in t \qquad (5.10)$$

$$\sum_{i \in S} \sum_{j \in S, j<i} y_{ij}^{t} \le \sum_{i \in S} z_{it} - z_{kt} \qquad S \subseteq M, t \in t, \text{some } k \in S \qquad (5.11)$$

$$z_{it} \le z_{0t} \qquad i \in M, t \in t \qquad (5.12)$$

$$y_{ij}^{t} \le z_{it} \qquad i \in M, j \in M, t \in t \qquad (5.13)$$

$$y_{ij}^{t} \in \{0,1\} \qquad i \in M, j \in M, j < i, t \in t \qquad (5.14)$$

$$y_{i0}^{t} \in \{0,1,2\} \qquad i \in M, t \in t \qquad (5.15)$$

$$z_{it} \in \{0,1\} \qquad i \in M', t \in t \qquad (5.16)$$

$$I_{0t} \ge 0 \qquad t \in t' \qquad (5.17)$$

$$w_{ikt} \ge 0 \qquad i \in M, p(i,t) \le k < t, t \in t' \qquad (5.18)$$

where $w_{ik0} = r_{i0} = 0$ for $i \in M$ and

$$b_{ikt} = \begin{cases} U_i - I_{i0} + R_{i1,t-1} & \text{if } k = 0, 1 \le t \le m(i,0) \\ R_{ik,t-1} & \text{if } 1 \le k \le H, k < t \le m(i,k), t \ne H+1 \\ 0 & \text{otherwise.} \end{cases}$$

Objective function (5.1) is the sum of inventory holding costs at the supplier and retailers as well as transportation costs, respectively. Constraints (5.2) are inventory balance equations for the supplier. Constraints (5.3) ensure that the total amount shipped to the retailers in a period cannot exceed the available amount at the supplier in the beginning of that period. Constraints (5.4) are inventory balance equations for retailers. Constraints (5.5)–(5.7) define the shortest path network representation of order-up-to level policy at each retailer $i \in M$. Its network consists of nodes for each time period $t \in t'$ and a dummy node 0. An arc from node $k$ to $t$

($t \neq H+1$) represents that a quantity $b_{ikt}$ is shipped to retailer $i$ in period $t$ where the last replenishment has occurred in $k$. An arc from any $k$ to $H$+1 represents that the final replenishment to retailer $i$ has occurred in $k$ and no replenishments occur any more. It is well known that the associated matrix of the shortest path problem formulation is totally unimodular and such a formulation describes the convex hull of the single retailer replenishment problem with order-up-to level policy (see Chapter 4). Note that $p(i,t)$ and $m(i,t)$ define arcs corresponding to feasible replenishment policies on the network (i.e. arcs representing occurrences of stock-out are cancelled out). An example network for $H$ = 4 where all possible arcs are assumed to be feasible is shown in Figure 5.1. Constraints (5.8) stipulate that the total amount shipped to the retailers in a period cannot exceed the capacity of the vehicle. Constraints (5.9) assure that if retailer $i \in M$ is replenished in $t \in t$ then that retailer must be replenished prior to $t$. Constraints (5.10) are degree constraints ensuring that two edges are incident to node (retailer) $i$ in a period if $i$ is visited in that period. Constraints (5.11) are generalized subtour elimination constraints. Constraints (5.12) and (5.13) are actually not needed in formulating the VMIR-OU problem but added a priori as being in Archetti et al. (2007a) to strengthen *SF*. Constraints (5.14)–(5.16) are for integrality. Note that $y_{i0}^{t}$ can take 2 to account for a single stop (at retailer $i$) tour from supplier. Constraints (5.17) and (5.18) are for nonnegativity. Note that $w$ must take a binary value for which the order-up-to level policy restriction holds at the retailers. However, due to (5.9), imposing integrality on $z$ variables is sufficient and the optimal solution of *SF* always gives integral $w$ variables. Actually, constraints (5.2) and (5.4) are not necessary in *SF* and they are just needed to keep track of cost accounts. Instead, one can represent those costs in terms of $w$ variables in (5.1).

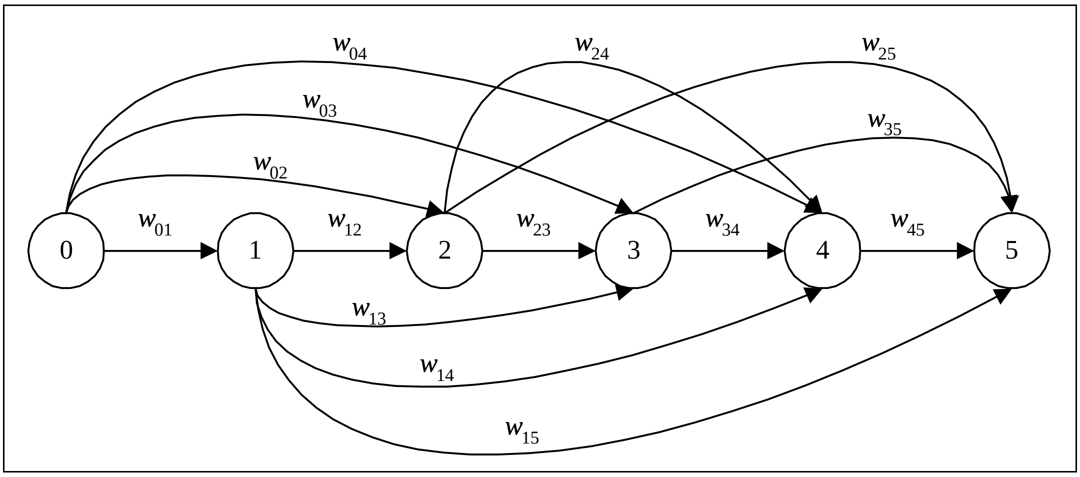


**Figure 5.1** An example network with $H$=4 for the single retailer replenishment problem

We have now a few remarks about the formulation $F$ given in Appendix B. We start with an example showing that the representation of the single retailer inventory replenishment problems in $F$ is not tight. For instance, consider a single retailer $i$ (i.e. $|M|=1$) with $h_{it}=1,\ U_i=5,\ I_{i0}=3, r_{it}=2$ for all $t \in t$ where $H$ = 4. The corresponding formulation $F$ for retailer $i$ is

$$Min\ \ I_{i1}+I_{i2}+I_{i3}+I_{i4}+I_{i5}$$

*s.t.* (B.4)–(B.7), (B.15)–(B.17), and (B.20)–(B.22)

The optimal solution has an objective function value of 11 whereas its linear programming (LP) relaxation yields a value of 5.92 with a fractional solution composed of $z_{i1}$ = 0, $z_{i2}$ = 0.5, $z_{i3}$ = 0.64 and $z_{i4}$ = 0.4. Note that the duality gap is about 86%. However, the LP relaxation of the formulation $SF$ (i.e. $Min$ $I_{i1}+I_{i2}+I_{i3}+I_{i4}+I_{i5}$ *s.t.* (5.4)–(5.7), (5.9), (5.16) and (5.18)) would have an integral optimal solution, as clarified before.

The second remark is about an extension of $F$. Archetti et al. (2007a) consider two related problems in which order-up-to level policy is relaxed. First is vendor-managed inventory routing with maximum level (VMIR-ML) problem where the amount shipped to a retailer in a period plus inventory carried from the previous period cannot exceed retailer's maximum level, and the second is vendor-managed inventory routing (VMIR) problem where any amount can be shipped to a retailer in a period. Archetti et al. (2007a) claim that by eliminating constraints (B.5) and (B.7) from *F* one can model the VMIR-ML problem, and by eliminating (B.5)–(B.7) from *F* one can model the VMIR problem. However, the resulting reduced formulations cannot have the true link between *x* and *z* variables, where the first one denotes the amount shipped to retailer $i \in M$ in $t \in t$ and the second indicates whether retailer $i \in M$ has received a shipment or not in $t \in t$. Therefore, the given formulations in their article need to be revised with adding the following constraints.

$$x_{it} \le Cz_{it} \qquad i \in M, t \in t \tag{5.19}$$

As a final remark, we will show that constraints (B.8), (B.14) and (B.15) of *F* are indeed dominated by other constraints in *F*. It is clear that $Cz_{0t} \le C$ so constraints (B.9) dominate (B.8). Constraints (B.10) can be rewritten as $2z_{it} = \sum_{j \in M', j<i} y_{ij}^t + \sum_{j \in M', j>i} y_{ji}^t = y_{i0}^t + \sum_{j \in M, j<i} y_{ij}^t + \sum_{j \in M', j>i} y_{ji}^t \ge y_{i0}^t$. Thus, constraints (B.14) are dominated by (B.10). When *k*=0, constraints (B.16) become $I_{it} \ge r_{it}(1 - z_{it})$ for $i \in M, t \in t$ which are equivalent to (B.15). Thus, constraints (B.15) are encompassed by (B.16).

### 5.3 Solution algorithms for the VMIR-OU problem

#### 5.3.1 Branch-and-cut algorithm

In the branch-and-cut algorithm, we consider all constraints in *SF* except (5.11) and all integralities, so that (5.11) would be added dynamically in a cutting plane fashion if it is violated. Whenever a violated constraint has been found, it would be added to the LP relaxation on hand and we reoptimize it until no constraint of (5.11) is violated. We use the separation algorithm of Padberg and Rinaldi (1991) to find violated constraints of (5.11). This separation algorithm is for classical traveling salesman problem (TSP) subtour elimination constraints whose right-hand side is |*S*|-1. Right-hand side of (5.11) is tighter than |*S*|-1 since it depends on whether the retailers in *S* are visited or not. In constraints (5.11), we select $z_{kt}$ variable with largest value in that iteration, i.e. $k = \arg\max_i \{z_{it}\}$, to subtract from summation of *z* variables in *S*. This cutting plane generation procedure is repeated in each node of the branch-and-bound tree. For branching variable selection, we first branch on $z_{it}$ and then on $y_{ij}^t$ variables. We use a best-node-first strategy (i.e. the node with the best objective function value is selected) as a node selection rule. An initial upper bound is found by a heuristic which is described in detail in the next section. For a detailed explanation of the branch-and-cut algorithm proposed, we refer the reader to Appendix C. Note that our branch-and-cut principle is almost the same as that of Archetti et al. (2007a) except that we use *SF* formulation within our branch-and-cut algorithm and a new heuristic to find an initial upper bound. Archetti et al. (2007a) use *F* formulation within their branch-and-cut algorithm and a heuristic proposed in Bertazzi et al. (2002), referred to as *BPS*, to find an initial upper bound. In the rest of the chapter, we will refer to our branch-and-cut algorithm as BC(*SF*) whereas that of Archetti et al. (2007a) will be referred to as BC(*F*).

### 5.3.2 A priori tour based heuristic

The main idea of our heuristic is to replace routing decision problem with a simple sequencing decision problem so that the vehicle following a priori route always skips the retailers that would not be visited on the route, but is being loyal to the predetermined visiting order of retailers that would be visited. In other words, given a TSP tour (optimal or not) involving the supplier and all retailers (i.e. all facilities), the precedence order of the facilities on the tour is fixed to determine facilities $j \in M'$ that can be visited before visiting facility *i* (denoted by set $b_i$) and facilities $j \in M'$ that can be visited after visiting facility *i* (denoted by set $a_i$), for each facility $i \in M'$. Note that sets $b_i$ and $a_i$ for $i \in M$ always involve supplier, and $b_0$ and $a_0$ cover all the retailers. Imposing a predetermined tour into *SF* eliminates the need to solve inherent $H$ many TSPs and significantly simplifies the routing decision. This idea is firstly introduced by Pınar and Süral (2006) for the VMIR-OU problem. The strong formulation we propose for the VMIR-OU problem with a priori tour, called *APF*, is as follows.

$$APF\text{: } Min \sum_{i \in M'} \sum_{t \in t'} h_i I_{it} + \sum_{i \in M'} \sum_{j \in M', i \neq j} \sum_{t \in t} c_{ij} y_{ij}^t \tag{5.20}$$

*s.t.* (5.2)–(5.9), (5.12), (5.16)–(5.18),

$$\sum_{j \in a_i} y_{ij}^t = z_{it} \qquad i \in M', t \in t \tag{5.21}$$

$$\sum_{j \in b_i} y_{ji}^t = z_{it} \qquad i \in M', t \in t \tag{5.22}$$

$$y_{ij}^t \in \{0,1\} \qquad i \in M', j \in M', i \neq j, t \in t \tag{5.23}$$

Objective function (5.20) is the same as (5.1). Constraints (5.21) and (5.22) are the assignment constraints which ensure that if facility $i \in M'$ is visited in any period $t \in t$ then it will be visited in the order imposed by a priori tour. Note that non-visited retailers will be skipped. Since a priori tour imposed is indeed a directed tour, it necessitates the definition of $y$ variables in (5.23) to be based on arcs rather than on edges in (5.14) and (5.15).

A complete presentation of the relaxation heuristic, referred to as a priori tour heuristic, is as follows.

*A priori tour heuristic*:

**S1:** Solve a TSP instance with all $i \in M'$ and store the (optimal) solution denoted by $s(TSP)$

**S2:** Use $s(TSP)$ to find $b_i$ and $a_i$ for all $i \in M'$:

$b_i = \{ j : j$ is visited before $i$ in $s(TSP)$, $j \in M' \}$,

$a_i = \{ j : j$ is visited after $i$ in $s(TSP), j \in M' \}$

**S3:** Construct *APF* using $b_i$ and $a_i$ for all $i \in M'$, solve it to optimality and store the optimal solution $s(APF)$ and the optimal objective value $z(APF)$. $z(APF)$ is a valid upper bound for the VMIR-OU problem

**S4: (Improvement step)** $z(APF)$ may be further improved:

Set $z'(APF) := \sum_{i \in M'} \sum_{t \in t'} h_i I_{it}$ where $I_{it} \in s(APF)$

**for** $t$:=1 to $H$ **do**

Solve a TSP over all $i \in M' \ni z_{it} = 1$ in $s(APF)$ and store the optimal objective value $z(TSP^t)$

Set $z'(APF) := z'(APF) + z(TSP^t)$

**end**

$z'(APF)$ is a valid upper bound for the VMIR-OU problem

We refer to a priori tour heuristic without the improvement step S4 as *APT* whereas we refer to the complete procedure from S1 to S4 as $APT^+$. $APT^+$ requires to solve $H$ many TSPs in addition to computational requirements of *APT*. Computational effectiveness of the heuristic depends on the solvers used for solving *APF* and TSPs. Note that any feasible solution to the *APF* formulation yields an upper bound to the VMIR-OU problem. In Appendix D, a small example is given so as to explain how a priori tour heuristic works.

## 5.4 Computational experiments

We perform computational experiments on instances generated by Archetti et al. (2007a) as well as a set of new instances introduced by us for assessment of the performance of the algorithms. The computational platform used is a Pentium IV 3.2GHz PC with 1GB RAM running under Windows XP. We code all the algorithms in C++ on MS Visual Studio.NET 2005 using Concert Technology 2.2

and CPLEX 10.1. We use CPLEX 10.1 to solve *APF*, and CONCORDE (Applegate et al., 2007) to solve TSPs to optimality. Below we present the properties of instances in Archetti et al. (2007a).

Two different horizon lengths ($H$ = 3 or 6) are considered. When $H$ = 3 ($H$ = 6), instances with up to 50 (30) retailers are generated. External demands $r_{it}$ are considered constant over time (i.e. $r_{it} = r_i$) and generated as integers from U[10,100]. The quantity received by the supplier $r_{0t}$ is set equal to $\sum_{i \in M} r_i$. Maximum inventory level at retailers $U_i$ is set equal to $g_i r_i$ where $g_i$ is randomly selected from the set {2, 3} and denotes the number of periods needed to consume inventory at retailers. Initial inventory level at the supplier $I_{00}$ is set equal to $\sum_{i \in M} U_i$ while initial inventory level at retailers $I_{i0}$ is set equal to $U_i - r_i$. Inventory carrying cost rate at retailers $h_i$ is generated from U[0.01,0.05] and U[0.1,0.5] while inventory carrying cost rate at the supplier $h_0$ is set equal to 0.03 if $h_i$ is generated from U[0.01,0.05] and 0.3 if $h_i$ is generated from U[0.1,0.5]. Vehicle's capacity $C$ is set equal to $\frac{3}{2}\sum_{i \in M} r_i$. Transportation cost $c_{ij}$ is set equal to $\left\lfloor \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2} \right\rfloor$ where $X_i, X_j, Y_i$ and $Y_j$ are generated as integers from U[0,500]. Five random instances are generated for each combination of the parameters.

Using the above generation scheme, we generate new larger instances for $H$ = 3 and $H$ = 6 with up to 65 and 45 retailers, respectively. We also generate new larger instances with up to 35 retailers for $H$ = 9 and up to 25 retailers for $H$ = 12. $g_i$ is selected from the set {2, 3, 4} for $H$ = 9 and from the set {2, 3, 4, 5} for $H$ = 12, respectively. In the sequel, *abls* refers to the instances generated by Archetti et al. (2007a) while *ss* refers to the new instances.

**Table 5.1** Average results on *abls* instances with and without CPLEX cuts†

| | | | | With Cplex Cuts | | | | Without Cplex Cuts | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | BC(*F*)⁻ | | BC(*SF*)⁻ | | BC(*F*)⁻ | | BC(*SF*)⁻ | |
| *n* | *H* | $h_i$ & $h_0$ | *%LP* | Seconds | Nodes | Seconds | Nodes | Seconds | Nodes | Seconds | Nodes |
| 5 | 3 | low | 52.1 | 0.1 | 2.8 | 0.1 | 4.0 | 0.1 | 2.4 | 0.1 | 9.4 |
| 10 | 3 | low | 42.9 | 0.6 | 44.2 | 0.5 | 37.8 | 0.6 | 43.4 | 0.5 | 35.4 |
| 15 | 3 | low | 35.0 | 2.0 | 63.0 | 2.9 | 95.2 | 2.4 | 75.0 | 2.2 | 70.0 |
| 20 | 3 | low | 26.7 | 8.3 | 103.4 | 11.1 | 156.4 | 12.3 | 181.8 | 10.1 | 143.4 |
| 25 | 3 | low | 35.7 | 30.4 | 185.8 | 27.9 | 190.0 | 43.1 | 238.0 | 32.6 | 202.8 |
| 30 | 3 | low | 36.0 | 74.1 | 254.6 | 74.3 | 250.4 | 117.0 | 444.0 | 93.8 | 336.6 |
| 35 | 3 | low | 33.6 | 170.0 | 338.6 | 195.3 | 400.6 | 338.2 | 576.4 | 206.8 | 444.8 |
| 40 | 3 | low | 32.9 | 565.3 | 701.8 | 642.7 | 796.8 | 678.2 | 879.4 | 565.0 | 696.8 |
| 45 | 3 | low | 30.8 | 1267.4 | 838.8 | 1099.9 | 940.2 | 1683.7 | 1077.2 | 1132.4 | 776.4 |
| 50 | 3 | low | 28.4 | 3424.5 (1) | 1347.8 | 2937.7 | 1505.0 | 4812.6 (1) | 1932.4 | 2105.0 | 1215.0 |
| Average | | | 35.4 | 554.3 | 388.1 | 499.2 | 437.6 | 768.8 | 545.0 | 414.8 | 393.1 |
| 5 | 3 | high | 21.6 | 0.1 | 3.2 | 0.1 | 4.4 | 0.1 | 6.4 | 0.1 | 7.4 |
| 10 | 3 | high | 14.2 | 0.5 | 38.8 | 0.6 | 42.0 | 0.5 | 43.4 | 0.5 | 41.0 |
| 15 | 3 | high | 10.3 | 2.1 | 72.4 | 2.4 | 87.2 | 2.6 | 88.0 | 1.9 | 68.8 |
| 20 | 3 | high | 7.4 | 8.3 | 120.0 | 8.4 | 115.2 | 13.9 | 184.4 | 7.7 | 113.2 |
| 25 | 3 | high | 8.8 | 22.5 | 125.2 | 24.4 | 132.8 | 37.4 | 242.2 | 39.7 | 247.6 |
| 30 | 3 | high | 8.5 | 94.3 | 351.6 | 83.9 | 283.6 | 122.2 | 459.2 | 83.1 | 336.0 |
| 35 | 3 | high | 7.9 | 234.5 | 509.2 | 197.1 | 454.8 | 286.5 | 546.4 | 213.0 | 455.8 |
| 40 | 3 | high | 7.6 | 431.5 | 449.4 | 551.1 | 600.0 | 899.5 | 937.6 | 516.2 | 575.0 |
| 45 | 3 | high | 7.2 | 738.3 | 496.8 | 965.6 | 737.2 | 1601.3 | 1025.0 | 1146.0 | 878.6 |
| 50 | 3 | high | 6.8 | 3783.6 (1) | 1801.0 | 2712.7 | 1283.2 | 3693.5 (1) | 1853.4 | 3012.5 | 1701.8 |
| Average | | | 10.0 | 531.6 | 396.8 | 454.6 | 374.0 | 665.8 | 538.6 | 502.1 | 442.5 |
| 5 | 6 | low | 23.5 | 0.5 | 51.6 | 0.4 | 43.6 | 0.5 | 150.6 | 0.3 | 64.6 |
| 10 | 6 | low | 21.9 | 5.1 | 157.2 | 3.8 | 127.0 | 11.9 | 731.8 | 3.6 | 127.8 |
| 15 | 6 | low | 20.9 | 23.6 | 214.8 | 19.3 | 223.8 | 87.1 | 1677.2 | 15.4 | 149.8 |
| 20 | 6 | low | 19.5 | 266.0 | 1572.2 | 198.2 | 996.6 | 2396.3 | 20278.8 | 207.7 | 1096.4 |
| 25 | 6 | low | 21.4 | 395.3 | 602.2 | 322.1 | 458.8 | 4420.4 (2) | 10805.0 | 470.4 | 653.6 |
| 30 | 6 | low | 21.3 | 2076.1 | 2088.8 | 1687.7 | 1614.0 | 7200.1 (5) | 6658.0 | 1606.3 | 1410.8 |
| Average | | | 21.4 | 461.1 | 781.1 | 371.9 | 577.3 | 2352.7 | 6716.9 | 383.9 | 583.8 |
| 5 | 6 | high | 11.7 | 0.4 | 44.4 | 0.3 | 47.0 | 0.5 | 137.0 | 0.3 | 59.2 |
| 10 | 6 | high | 10.0 | 5.7 | 186.0 | 3.9 | 133.0 | 12.9 | 869.6 | 3.9 | 133.2 |
| 15 | 6 | high | 8.3 | 20.2 | 187.4 | 14.6 | 125.8 | 91.9 | 1987.4 | 16.5 | 153.2 |
| 20 | 6 | high | 7.4 | 319.3 | 2084.2 | 200.8 | 1071.4 | 3326.6 (1) | 25166.8 | 222.3 | 1155.2 |
| 25 | 6 | high | 7.7 | 482.3 | 841.8 | 307.4 | 467.0 | 4685.2 (3) | 11089.2 | 316.7 | 430.6 |
| 30 | 6 | high | 7.3 | 2096.6 | 2095.6 | 1503.3 | 1455.8 | 7200.2 (5) | 6735.6 | 1785.2 | 1757.4 |
| Average | | | 8.7 | 487.4 | 906.6 | 338.4 | 550.0 | 2552.9 | 7664.3 | 390.8 | 614.8 |
| Overall average | | | 19.8 | 517.2 | 561.7 | 431.3 | 465.0 | 1368.1 | 3035.1 | 431.8 | 485.9 |

†low: $h_i$ = [0.01, 0.05] and $h_0$ = 0.03; high: $h_i$ = [0.1, 0.5] and $h_0$ = 0.3. The numbers in parentheses on columns 5 and 9 represent the number of instances that could not be solved to optimality within 2 hour-time limit.

We compare the LP relaxation solution values of *F* and *SF* without constraints (5.11) on ***abls*** instances to see the strength of formulations relative to each other. Also, in order to fairly compare branch-and-cut algorithms BC(*F*) and BC(*SF*), we test them on ***abls*** instances without using any initial upper bound. We refer to those branch-and-cut algorithms without their initial upper bounding heuristics as BC(*SF*)$^-$ and BC(*F*)$^-$, respectively. Furthermore, we conduct computational experiments to see the impact of CPLEX's cuts such as clique inequalities, cover inequalities, mixed integer rounding cuts, etc. (for a detailed information on available cuts see the User Manual of CPLEX 10.1) on the performance of BC(*F*)$^-$ and BC(*SF*)$^-$. By default, CPLEX 10.1 automatically decides on whether generating a class of cuts or not. Average computational results (over five instances) using branch-and-cut algorithms with and without CPLEX's cuts on ***abls*** instances are given in Table 5.1. In the table, columns 1–3 show the number of retailers, horizon length and inventory carrying cost rates for retailers and supplier, respectively. Column 4 shows the percentage gap between the LP relaxation solution values of the formulations, computed as the difference between the LP relaxation solution values of the formulations divided by the LP relaxation solution values of *F*. Columns titled as "Seconds" and "Nodes" show elapsed time in seconds and the number of nodes explored in the branch-and-bound tree for the algorithm with and without CPLEX's cuts, respectively.

In Table 5.1, %LP column ranging from 7 to 52 indicates that the LP relaxation solution value of *SF* is better than that of *F*, as expected. In particular, the lower the inventory carrying cost rate is, the larger the gap is. Results reveal that the performance of BC(*F*)$^-$ depends on CPLEX's cuts whereas BC(*SF*)$^-$ performs well even in the absence of those cuts. Without cuts, BC(*SF*)$^-$ outperforms BC(*F*)$^-$ in that the former is more than 3 times faster than BC(*F*)$^-$, and the latter explores 6 times more nodes than the former. Besides, BC(*SF*)$^-$ solved all instances to optimality while BC(*F*)$^-$ could not solve 16 out of 60 instances to optimality when $H$ = 6. When CPLEX cuts are allowed, BC(*SF*)$^-$ is on average 17% faster than BC(*F*)$^-$, and

BC(*SF*)$^-$ is able to solve all instances well under 2 hour-time limit whereas BC(*F*)$^-$ could not solve the two larger instances within the time limit.

We also test our heuristics (*APT* and *APT*$^+$) on ***abls*** instances and compare them with the *BPS* heuristic. Average computational results are given in Table 5.2. Column 4 shows the elapsed time in seconds for *APT*$^+$ to run. Columns 5–7 designate the percent deviation (%Dev) of solution values found by the heuristics *APT*$^+$, *APT* and *BPS* respectively from the optimal solution value, i.e. %Dev = 100 * (*Heur – Opt*) / *Opt* where *Heur* represents the solution value found by the corresponding heuristic and *Opt* denotes the optimal solution value.

All heuristics perform well. Especially, *APT*$^+$ heuristic yields higher quality solutions within a few seconds (on average 0.6% deviation from the optimal solution value) and has found the optimal solution in 73 out of 160 instances. We should also note that time required to solve TSPs for *APT*$^+$ heuristic at the outset is negligible (not greater than 3 seconds even for the largest *ss* instances solved) compared to the time required to solve *APF* formulation. It seems that instances with low inventory carrying cost rates are more difficult to solve for all than those with high rates. More success on instances with high inventory carrying cost rates can be related with the fact that transportation cost constitutes a smaller percentage of the total cost in these instances compared to the instances with low rates.

To see whether using *APT*$^+$ as an initial upper bound has a significant effect on the performance of BC(*SF*), we test BC(*SF*) on ***abls*** instances. In these experiments, we use the solution value obtained by *APT*$^+$ as an initial upper bound within BC(*SF*) and allow CPLEX to add its cuts. Average computational results (over five instances) on ***abls*** instances are given in Table 5.3. Columns 6 and 7 list the percentage reduction achieved in elapsed time of the algorithm (%Rsec) and the number of nodes explored by the algorithm (%Rnode), where %Rsec (%Rnode) is found as the percentage of the difference between values in column 7 (8) of Table 5.1 and column 4 (5) of Table 5.3 divided by values of column 7 (8) of Table 5.1.

**Table 5.2** Average results for heuristics on *abls* instances†

| | | | | %Dev | | |
|---|---|---|---|---|---|---|
| ***n*** | ***H*** | ***$h_i$ & $h_0$*** | **Seconds** | ***APT*** $^+$ | ***APT*** | ***BPS*** |
| 5 | 3 | low | 0.61 | 0.00 | 0.09 | 2.88 |
| 10 | 3 | low | 0.81 | 0.95 | 0.95 | 0.78 |
| 15 | 3 | low | 1.01 | 0.21 | 0.35 | 2.56 |
| 20 | 3 | low | 1.28 | 0.43 | 1.00 | 3.83 |
| 25 | 3 | low | 1.46 | 1.03 | 1.98 | 2.99 |
| 30 | 3 | low | 1.97 | 2.20 | 3.26 | 3.60 |
| 35 | 3 | low | 1.85 | 0.55 | 1.10 | 4.46 |
| 40 | 3 | low | 2.72 | 1.15 | 1.94 | 6.46 |
| 45 | 3 | low | 3.16 | 2.60 | 3.51 | 7.60 |
| 50 | 3 | low | 3.72 | 0.88 | 2.22 | 5.81 |
| Average | | | 1.86 | 1.00 | 1.64 | 4.10 |
| 5 | 3 | high | 0.58 | 0.00 | 0.06 | 1.31 |
| 10 | 3 | high | 0.80 | 0.36 | 0.36 | 1.74 |
| 15 | 3 | high | 1.00 | 0.07 | 0.13 | 2.18 |
| 20 | 3 | high | 1.19 | 0.12 | 0.36 | 3.30 |
| 25 | 3 | high | 1.56 | 0.43 | 0.78 | 1.06 |
| 30 | 3 | high | 1.88 | 0.65 | 0.95 | 1.21 |
| 35 | 3 | high | 1.98 | 0.22 | 0.39 | 2.25 |
| 40 | 3 | high | 2.80 | 0.23 | 0.63 | 2.26 |
| 45 | 3 | high | 3.41 | 0.52 | 1.08 | 2.49 |
| 50 | 3 | high | 3.57 | 0.16 | 0.66 | 1.57 |
| Average | | | 1.88 | 0.28 | 0.54 | 1.94 |
| 5 | 6 | low | 1.24 | 0.03 | 0.13 | 1.64 |
| 10 | 6 | low | 1.62 | 0.35 | 0.55 | 1.36 |
| 15 | 6 | low | 2.32 | 0.89 | 1.19 | 4.27 |
| 20 | 6 | low | 3.30 | 0.31 | 0.89 | 2.95 |
| 25 | 6 | low | 3.65 | 0.59 | 1.22 | 6.19 |
| 30 | 6 | low | 5.43 | 1.94 | 4.23 | 4.64 |
| Average | | | 2.93 | 0.68 | 1.37 | 3.51 |
| 5 | 6 | high | 1.07 | 0.04 | 0.11 | 0.34 |
| 10 | 6 | high | 1.63 | 0.17 | 0.31 | 1.87 |
| 15 | 6 | high | 2.29 | 0.42 | 0.58 | 1.20 |
| 20 | 6 | high | 3.53 | 0.16 | 0.42 | 2.09 |
| 25 | 6 | high | 4.00 | 0.20 | 0.70 | 2.12 |
| 30 | 6 | high | 5.16 | 0.87 | 1.56 | 2.56 |
| Average | | | 2.95 | 0.31 | 0.61 | 1.70 |
| Overall average | | | 2.27 | 0.59 | 1.05 | 2.86 |

†low: $h_i$ = [0.01, 0.05] and $h_0$ = 0.03; high: $h_i$ = [0.1, 0.5] and $h_0$ = 0.3.

**Table 5.3** Average results on *abls* instances for BC(*SF*)[†]

| ***n*** | ***H*** | **$h_i$ & $h_0$** | **Seconds** | **Nodes** | **%Rsec** | **%Rnode** |
|---|---|---|---|---|---|---|
| 5 | 3 | low | 0.1 | 1.0 | 0.0 | 75.0 |
| 10 | 3 | low | 0.4 | 25.8 | 18.9 | 31.7 |
| 15 | 3 | low | 1.6 | 59.8 | 45.2 | 37.2 |
| 20 | 3 | low | 6.0 | 82.0 | 46.5 | 47.6 |
| 25 | 3 | low | 22.6 | 135.2 | 19.1 | 28.8 |
| 30 | 3 | low | 62.5 | 227.2 | 15.9 | 9.3 |
| 35 | 3 | low | 163.0 | 370.8 | 16.6 | 7.4 |
| 40 | 3 | low | 409.3 | 581.6 | 36.3 | 27.0 |
| 45 | 3 | low | 618.3 | 453.0 | 43.8 | 51.8 |
| 50 | 3 | low | 1593.6 | 901.6 | 45.8 | 40.1 |
| Average | | | 287.7 | 283.8 | 28.8 | 35.6 |
| 5 | 3 | high | 0.1 | 2.0 | 8.5 | 54.5 |
| 10 | 3 | high | 0.4 | 35.6 | 32.2 | 15.2 |
| 15 | 3 | high | 1.6 | 50.0 | 33.9 | 42.7 |
| 20 | 3 | high | 5.9 | 79.6 | 29.7 | 30.9 |
| 25 | 3 | high | 19.1 | 131.4 | 22.0 | 1.1 |
| 30 | 3 | high | 70.3 | 272.8 | 16.3 | 3.8 |
| 35 | 3 | high | 140.1 | 275.6 | 28.9 | 39.4 |
| 40 | 3 | high | 323.1 | 385.0 | 41.4 | 35.8 |
| 45 | 3 | high | 596.2 | 471.8 | 38.3 | 36.0 |
| 50 | 3 | high | 1883.2 | 1179.6 | 30.6 | 8.1 |
| Average | | | 304.0 | 288.3 | 28.2 | 26.8 |
| 5 | 6 | low | 0.3 | 30.4 | 19.1 | 30.3 |
| 10 | 6 | low | 2.7 | 88.4 | 27.7 | 30.4 |
| 15 | 6 | low | 11.3 | 99.4 | 41.5 | 55.6 |
| 20 | 6 | low | 120.4 | 626.6 | 39.3 | 37.1 |
| 25 | 6 | low | 218.8 | 384.6 | 32.1 | 16.2 |
| 30 | 6 | low | 1065.3 | 1129.6 | 36.9 | 30.0 |
| Average | | | 236.5 | 393.2 | 32.8 | 33.3 |
| 5 | 6 | high | 0.3 | 33.8 | 16.0 | 28.1 |
| 10 | 6 | high | 2.8 | 91.8 | 27.3 | 31.0 |
| 15 | 6 | high | 10.4 | 93.8 | 28.9 | 25.4 |
| 20 | 6 | high | 124.4 | 643.0 | 38.0 | 40.0 |
| 25 | 6 | high | 220.9 | 329.0 | 28.1 | 29.6 |
| 30 | 6 | high | 1090.4 | 1037.4 | 27.5 | 28.7 |
| Average | | | 241.5 | 371.5 | 27.6 | 30.5 |
| Overall average | | | 274.5 | 322.2 | 29.1 | 31.4 |

[†]low: $h_i$ = [0.01, 0.05] and $h_0$ = 0.03; high: $h_i$ = [0.1, 0.5] and $h_0$ = 0.3.

Results indicate that using $APT^{+}$ as an initial upper bound enhances the performance of BC(*SF*). On average, there exist 30% reduction in both the elapsed time and number of nodes explored regardless of the horizon length and level of inventory carrying cost rates.

In all experiments on larger *ss* instances, we allow $APT^{+}$ to find an initial bound and CPLEX cuts within BC(*SF*). Average computational results (over five instances) are reported in Table 5.4 for the first group of *ss* instances. Those largest *ss* instances (i.e. the second group of *ss* instances) that could not be solved to optimality in reasonable times are solved with a time limit of 4 hours and results for each instance are reported in Tables 5.5 and 5.6. In Table 5.4, column 6 shows the elapsed time in seconds for $APT^{+}$ to run, and column 7 indicates the percentage deviation (%Dev) of the upper bound found by $APT^{+}$ from the optimal solution value. In Tables 5.5 and 5.6, columns 1 refers to test instance number, and column 6 shows the remaining percentage gap (%Gap) between the best upper bound ($UB^{*}$) and lower bound ($LB^{*}$) found (i.e. %Gap = 100 * ($UB^{*}$ – $LB^{*}$) / $LB^{*}$) by BC(*SF*). A "–" sign in %Gap column means that the optimal solution of the corresponding instance is proved. Columns 8 is the same as column 7 in Table 5.4 but note that column 8 indicates the percentage deviation of the upper bound found by $APT^{+}$ from the optimal or the best available lower bound solution value.

As seen in Table 5.4, BC(*SF*) is consistently able to solve to optimality instances containing up to 60, 35, 25 and 15 retailers with horizon length of 3, 6, 9 and 12, respectively under both low and high inventory carrying cost rates within reasonable times (less than 2.5 hours). Also, $APT^{+}$ finds high quality solutions within a few minutes. Similar to the results on *abls* instances, $APT^{+}$ is more successful (with regard to the deviation from optimal solution value) on *ss* instances with high inventory carrying cost rates. Results on Tables 5.5 and 5.6 indicate that BC(*SF*) has found the optimal solution in 18 (15) out of 35 (35) instances with low (high) inventory carrying cost rates for the largest *ss* instances. Furthermore, BC(*SF*) has succeeded in finding a small gap between the best upper bound and lower bound for those instances that could not be solved to optimality within limited time. Except six out of 70 instances, all %Gaps are less than 2.7%. Results with $APT^+$ are in accordance with previous comments.

**Table 5.4** Average results on "optimally solvable" *ss* instances†

| | | | BC(*SF*) | | $APT^+$ | |
|---|---|---|---|---|---|---|
| ***n*** | ***H*** | $h_i$ **&** $h_0$ | **Seconds** | **Nodes** | **Seconds** | **%Dev** |
| 55 | 3 | low | 6669.4 | 3683.4 | 5.1 | 0.93 |
| 60 | 3 | low | 7080.5 | 2762.8 | 5.7 | 1.34 |
| 55 | 3 | high | 5034.7 | 2350.0 | 4.6 | 0.28 |
| 60 | 3 | high | 8621.0 | 3370.4 | 6.9 | 0.45 |
| 35 | 6 | low | 5207.5 | 3949.6 | 8.6 | 1.03 |
| 35 | 6 | high | 3494.9 | 2129.2 | 7.3 | 0.49 |
| 5 | 9 | low | 1.7 | 264.0 | 2.2 | 0.00 |
| 10 | 9 | low | 16.4 | 402.4 | 3.9 | 0.15 |
| 15 | 9 | low | 197.4 | 1424.0 | 12.5 | 0.23 |
| 20 | 9 | low | 450.4 | 958.8 | 11.1 | 0.95 |
| 25 | 9 | low | 3903.0 | 4105.0 | 29.1 | 0.40 |
| 5 | 9 | high | 1.4 | 213.8 | 1.9 | 0.00 |
| 10 | 9 | high | 13.4 | 327.4 | 3.0 | 0.24 |
| 15 | 9 | high | 135.0 | 971.4 | 8.7 | 0.05 |
| 20 | 9 | high | 995.9 | 2689.2 | 19.1 | 0.07 |
| 25 | 9 | high | 6240.6 | 6403.4 | 25.6 | 0.20 |
| 5 | 12 | low | 8.8 | 1214.2 | 5.2 | 0.00 |
| 10 | 12 | low | 181.4 | 3486.2 | 23.2 | 0.25 |
| 15 | 12 | low | 2253.2 | 9033.4 | 55.8 | 1.04 |
| 5 | 12 | high | 24.0 | 3150.2 | 6.7 | 0.00 |
| 10 | 12 | high | 119.4 | 1723.2 | 24.2 | 0.13 |
| 15 | 12 | high | 962.5 | 4550.2 | 33.2 | 0.11 |

†low: $h_i$ = [0.01, 0.05] and $h_0$ = 0.03; high: $h_i$ = [0.1, 0.5] and $h_0$ = 0.3.

**Table 5.5** Results on low-cost *ss* instances

| | | | BC(*SF*) | | | *APT*$^+$ | |
|---|---|---|---|---|---|---|---|
| **#** | ***n*** | ***H*** | **Seconds** | **Nodes** | **%Gap** | **Seconds** | **%Dev** |
| 1 | 65 | 3 | 12172.3 | 3542 | – | 5.9 | 0.55 |
| 2 | 65 | 3 | 14541.3 | 2324 | 4.41 | 8.1 | 4.41 |
| 3 | 65 | 3 | 10544.7 | 3340 | – | 5.8 | 1.56 |
| 4 | 65 | 3 | 3197.7 | 921 | – | 5.0 | 0.85 |
| 5 | 65 | 3 | 14435.9 | 3345 | 1.35 | 7.4 | 1.85 |
| 1 | 40 | 6 | 5231.8 | 2472 | – | 6.8 | 0.66 |
| 2 | 40 | 6 | 6387.7 | 2021 | – | 8.6 | 1.02 |
| 3 | 40 | 6 | 14400.9 | 4428 | 2.54 | 15.3 | 2.54 |
| 4 | 40 | 6 | 4903.2 | 1930 | – | 9.5 | 1.37 |
| 5 | 40 | 6 | 8519.9 | 3413 | – | 11.8 | 1.90 |
| 1 | 45 | 6 | 14402.1 | 2389 | 2.68 | 10.3 | 3.30 |
| 2 | 45 | 6 | 14401.6 | 2831 | 0.85 | 13.7 | 1.49 |
| 3 | 45 | 6 | 14400.1 | 2771 | 1.26 | 17.4 | 2.29 |
| 4 | 45 | 6 | 14400.1 | 1683 | 0.89 | 18.4 | 4.32 |
| 5 | 45 | 6 | 10221.0 | 2574 | – | 9.5 | 1.46 |
| 1 | 30 | 9 | 5634.3 | 2966 | – | 60.0 | 1.28 |
| 2 | 30 | 9 | 10523.8 | 5837 | – | 25.1 | 0.81 |
| 3 | 30 | 9 | 5174.6 | 1681 | – | 79.4 | 4.06 |
| 4 | 30 | 9 | 10808.9 | 5700 | – | 53.9 | 2.37 |
| 5 | 30 | 9 | 11718.0 | 6959 | – | 42.0 | 1.69 |
| 1 | 35 | 9 | 14400.4 | 2585 | 2.03 | 30.9 | 4.12 |
| 2 | 35 | 9 | 14400.1 | 2279 | 3.33 | 106.6 | 3.33 |
| 3 | 35 | 9 | 14400.0 | 1678 | 3.90 | 45.1 | 4.44 |
| 4 | 35 | 9 | 14400.3 | 2198 | 8.04 | 47.3 | 8.04 |
| 5 | 35 | 9 | 4365.1 | 962 | – | 44.0 | 0.00 |
| 1 | 20 | 12 | 9301.3 | 13947 | – | 96.3 | 0.21 |
| 2 | 20 | 12 | 2840.3 | 3143 | – | 94.9 | 0.80 |
| 3 | 20 | 12 | 14400.8 | 13209 | 4.15 | 1046.5 | 4.15 |
| 4 | 20 | 12 | 2747.8 | 4185 | – | 69.3 | 0.00 |
| 5 | 20 | 12 | 3867.5 | 6324 | – | 154.5 | 0.03 |
| 1 | 25 | 12 | 14400.7 | 4765 | 2.05 | 191.3 | 2.05 |
| 2 | 25 | 12 | 14400.2 | 5927 | 1.16 | 260.2 | 1.16 |
| 3 | 25 | 12 | 14400.4 | 3893 | 1.95 | 181.7 | 2.26 |
| 4 | 25 | 12 | 14401.3 | 4101 | 1.69 | 180.3 | 1.69 |
| 5 | 25 | 12 | 14400.3 | 4062 | 7.23 | 1309.0 | 7.23 |

**Table 5.6** Results on high-cost *ss* instances

| | | | BC(*SF*) | | | *APT*$^+$ | |
|---|---|---|---|---|---|---|---|
| **#** | ***n*** | ***H*** | **Seconds** | **Nodes** | **%Gap** | **Seconds** | **%Dev** |
| 1 | 65 | 3 | 7786.2 | 1322 | – | 7.4 | 1.00 |
| 2 | 65 | 3 | 3785.6 | 1131 | – | 5.6 | 0.07 |
| 3 | 65 | 3 | 11543.0 | 3181 | – | 7.7 | 0.11 |
| 4 | 65 | 3 | 7504.2 | 2115 | – | 7.7 | 0.40 |
| 5 | 65 | 3 | 5810.8 | 1527 | – | 6.1 | 0.05 |
| 1 | 40 | 6 | 5281.3 | 1763 | – | 9.3 | 0.35 |
| 2 | 40 | 6 | 14400.7 | 4818 | 0.07 | 18.7 | 1.23 |
| 3 | 40 | 6 | 2236.2 | 626 | – | 7.9 | 0.38 |
| 4 | 40 | 6 | 4359.2 | 1470 | – | 8.9 | 0.40 |
| 5 | 40 | 6 | 5850.8 | 3361 | – | 8.3 | 0.18 |
| 1 | 45 | 6 | 5835.4 | 1123 | – | 9.4 | 0.51 |
| 2 | 45 | 6 | 14401.0 | 3229 | 0.07 | 13.2 | 0.16 |
| 3 | 45 | 6 | 14400.8 | 1986 | 0.74 | 10.3 | 1.00 |
| 4 | 45 | 6 | 14410.9 | 1756 | 1.71 | 7.4 | 2.12 |
| 5 | 45 | 6 | 7906.4 | 1463 | – | 8.2 | 0.62 |
| 1 | 30 | 9 | 14401.2 | 5741 | 0.21 | 59.6 | 0.90 |
| 2 | 30 | 9 | 14400.7 | 5430 | 0.26 | 39.7 | 1.19 |
| 3 | 30 | 9 | 14400.1 | 5580 | 0.74 | 14.8 | 0.74 |
| 4 | 30 | 9 | 14404.3 | 4851 | 0.62 | 21.6 | 0.82 |
| 5 | 30 | 9 | 6404.2 | 3699 | – | 20.7 | 0.20 |
| 1 | 35 | 9 | 14400.9 | 3216 | 0.21 | 18.8 | 0.21 |
| 2 | 35 | 9 | 14401.6 | 2721 | 2.43 | 132.9 | 2.43 |
| 3 | 35 | 9 | 14400.0 | 2300 | 1.91 | 72.1 | 1.91 |
| 4 | 35 | 9 | 14400.5 | 2998 | 0.28 | 65.8 | 0.28 |
| 5 | 35 | 9 | 14400.1 | 2641 | 0.50 | 32.5 | 1.96 |
| 1 | 20 | 12 | 4850.8 | 4420 | – | 19.5 | 0.06 |
| 2 | 20 | 12 | 5312.4 | 7267 | – | 71.7 | 0.13 |
| 3 | 20 | 12 | 14400.2 | 11126 | 0.24 | 106.2 | 0.35 |
| 4 | 20 | 12 | 5767.5 | 8693 | – | 243.7 | 0.03 |
| 5 | 20 | 12 | 14400.2 | 20939 | 0.11 | 52.7 | 0.32 |
| 1 | 25 | 12 | 14401.3 | 3861 | 0.73 | 277.1 | 0.98 |
| 2 | 25 | 12 | 14400.4 | 5794 | 0.66 | 248.2 | 0.94 |
| 3 | 25 | 12 | 14400.9 | 4074 | 1.58 | 279.4 | 1.58 |
| 4 | 25 | 12 | 14400.1 | 7282 | 0.25 | 57.1 | 0.25 |
| 5 | 25 | 12 | 14400.1 | 4759 | 0.32 | 25.0 | 1.23 |

Our approach in strong formulations can also be extended to the two related inventory routing problems, namely VMIR-ML and VMIR problems. Retailers' inventory replenishment problem in VMIR-ML problem is a lot-sizing problem with bounds on inventory (Love, 1973), which can be solved polynomially in $O(H^2)$ by a dynamic programming (*DP*) algorithm (see Atamtürk and Küçükyavuz, 2008). Similarly, the retailers' inventory replenishment problem in VMIR problem is an uncapacitated lot-sizing problem (Wagner and Whitin, 1958) which can be solved polynomially in $O(H \log H)$ by a *DP* algorithm (Federgruen and Tzur, 1991). Using DP recursions, one can construct shortest path formulations for the replenishment problems. Another direction might be to study inventory routing problems with multiple vehicles. However, exact solution of such problems will be quite challenging since the known valid inequalities for the vehicle routing problem (VRP) cannot be trivially adapted here due to not knowing how much a retailer will receive in each period. Let alone the additional valid inequalities, even the so called rounded capacity inequalities in VRP cannot be used because of the same reason and one has to resort to fractional capacity inequalities which are not strong enough.

# CHAPTER 6

# A PRODUCTION-DISTRIBUTION-ROUTING PROBLEM WITH ENDOGENOUS POLICY

In this chapter, we consider a production-distribution-routing problem with endogenous inventory control policy (PDR) where a supplier (vendor) decides when and how much to order/produce a single product and distributes to multiple retailers in a VMI setting over a finite time horizon using a capacitated vehicle.

Although production-distribution problems arise in many settings, there are very few studies in the literature that consider production-distribution-routing problem, as presented in Chapter 2, which may be due to the problem being very complex. Most of the studies propose heuristic solution approaches to their problems without having a lower bounding procedure to gauge the effectiveness of their heuristics. Only a few studies to the best of our knowledge, such as Fumero and Vercellis (1999), Archetti et al. (2007b), Bard and Nananukul (2008), consider obtaining lower bounds. All these studies use weak representations for the replenishment decisions at the suppliers and retailers which limit their approaches in obtaining exact solutions. In particular, the only study that tries to obtain exact solutions is the study of Archetti et al. (2007b).

In this study, we consider strong representations of replenishment decisions and a computationally attractive formulation for the routing of vehicles to develop a branch-and-cut algorithm. To the best of our knowledge, this is the first exact algorithm that is based on a strong formulation in the context of production-distribution-routing problems. We also use the proposed strong formulation to develop a mathematical programming based heuristic. Computational experiments show that our branch-and-cut and heuristic algorithms perform better than their competitors in the literature.

The rest of the paper is organized as follows. In Section 6.1, we describe the PDR problem in detail. We present the formulation we propose for the problem in Section 6.2. In Section 6.3, we provide the details of our branch-and-cut algorithm and mathematical programming based heuristic. Section 6.4 is devoted to the computational experiments over the test instances to assess the performance of the proposed algorithms. Note that the notation and abbreviations defined in this chapter is only valid in this chapter and in Appendix E.

## 6.1 Problem definition

We consider a production-distribution system in which a supplier orders (or produces) a single product and distributes to $N$ retailers over a finite time horizon $T$ with a capacitated vehicle. Retailer $i$ ($1 \le i \le N$) faces external customer demand $d_{it}$ in each discrete time period $t$ ($1 \le t \le T$) and may keep inventory $I_{it}$ to meet the demand without backordering. The supplier, denoted by $i$=0, manages the inventories at the retailers by deciding on when and how much to ship to each retailer $i$, and guarantees that neither retailers nor itself will stock-out (i.e. $I_{it} \ge 0$ for $0 \le i \le N$) in any period $t$. The supplier decides on how much to order in each period $t$, and may ship to the retailers immediately or keep inventory $I_{0t}$ for replenishing retailers in later periods. We assume the beginning inventory level $I_{00}$ is zero. When an order is placed at the supplier in a period $t$, a fixed order cost $f_t$ independent of the size of order and a variable order cost $p_t$ per unit ordered are incurred. For each unit kept in inventory at facility $i$ ($0 \le i \le N$) in a period $t$, a holding cost $h_{it}$ is incurred. The vehicle based at the supplier can visit multiple retailers in a multi-stop route without exceeding its capacity $C$. A vehicle traveling from facility $i$ ($0 \le i \le N$) to facility $j$ ($0 \le j \le N$) incurs a transportation cost $c_{ij}$, where $c_{ij} = c_{ji}$. We assume that the vehicle can only perform a single tour in every period. The PDR problem is to decide on when and how much to order at the supplier, when and how much to ship to each retailer, and the routing of vehicles such that the sum of fixed and variable order costs, transportation cost as well as inventory carrying costs at the supplier and retailers is minimized. It is an extension of the OWMR problem, studied in Chapter 3, so that a multi-stop routing as a transportation policy is imposed to the OWMR problem instead of the direct shipment. The PDR problem, a strongly ***NP***-hard problem, is formulated as a mixed integer program (MIP) in the following. We also present a standard MIP formulation, referred to as formulation ***F-ML***, due to Archetti et al. (2007b) in the Appendix E to inform the reader of the type of formulations usually proposed in the literature for the PDR problems.

### 6.2 Strong formulation for the PDR problem

We view the PDR problem as an integration of a two-level lot sizing problem (very similar to the OWMR problem in Chapter 3) and routing problem of vehicles. The formulation that we will propose is a strong formulation since we use effective mathematical representations for these two problems. For the lot sizing part of the problem, we use the shortest path based representation we propose in Chapter 3 for the OWMR problem as a basis to develop an effective representation for the replenishment decisions of the PDR problem since it has proved to be the strongest one in Chapter 3. Besides, we use a two-index vehicle flow based formulation, which proved to be one of the effective formulations for vehicle routing problems (see Laporte, 2007 for a general discussion and Chapter 5 for a usage in inventory routing), in formulating the routing problem of vehicles.

Define $D_{itk}$ as the total demand of retailer $i$ from period $t$ through $k$, i.e. $D_{itk} = \sum_{r=t}^{k} d_{ir}$, and $a_{itk}$ as being equal to 1 if $D_{itk} > 0$, 0 otherwise. Letting $x_{ijt}$ be 1 if the vehicle visits facility $j$ immediately after facility $i$ ($j < i$) in period $t$ and 0 otherwise; $z_{0t}$ be 1 if the vehicle departs from the supplier in period $t$ and 0 otherwise; $z_{it}$ be 1 if retailer $i$ is visited in period $t$ and 0 otherwise; $y_t$ be 1 if an order is placed at the supplier in period $t$ and 0 otherwise; $W_{itk}$ be the fraction of the total demand of retailer $i$ from period $t$ through $k$ satisfied in period $t$; and $U_{iqtk}$ be the fraction of the amount ordered at the supplier in period $q$ to meet the demand of retailer $i$ from period $t$ through $k$, the strong formulation *SF-PR* we propose is as follows.

$$\textit{SF-PR}:\ Min\ \sum_{t=1}^{T} f_t y_t + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} p_t D_{itk} U_{iqtk} + \sum_{i=0}^{N}\sum_{t=1}^{T} h_{it} I_{it} + \sum_{i=1}^{N}\sum_{j=0}^{i-1}\sum_{t=1}^{T} c_{ij} x_{ijt} \qquad (6.1)$$

*s.t.*

$$I_{0t} = I_{0,t-1} + \sum_{i=1}^{N}\sum_{r=t}^{T}\sum_{k=r}^{T} D_{irk} U_{itrk} - \sum_{i=1}^{N}\sum_{q=1}^{t}\sum_{k=t}^{T} D_{itk} U_{iqtk} \qquad 1 \le t \le T \qquad (6.2)$$

$$I_{it} = I_{i,t-1} + \sum_{k=t}^{T} D_{itk} W_{itk} - d_{it} \qquad 1 \le i \le N, 1 \le t \le T \qquad (6.3)$$

$$\sum_{q=1}^{t} U_{iqtk} = W_{itk} \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad (6.4)$$

$$\sum_{k=q}^{t}\sum_{r=t}^{T} a_{ikr} U_{iqkr} \le y_q \qquad 1 \le i \le N, 1 \le q \le t \le T \qquad (6.5)$$

$$\sum_{k=1}^{T} W_{i1k} = 1 \qquad 1 \le i \le N \qquad (6.6)$$

$$\sum_{k=t}^{T} W_{itk} - \sum_{k=1}^{t-1} W_{ik,t-1} = 0 \qquad 1 \le i \le N,\ 2 \le t \le T \qquad (6.7)$$

$$\sum_{i=1}^{N}\sum_{k=t}^{T} D_{itk} W_{itk} \le C z_{0t} \qquad 1 \le t \le T \qquad (6.8)$$

$$\sum_{k=t}^{T} a_{itk} W_{itk} \le z_{it} \qquad 1 \le i \le N, 1 \le t \le T \qquad (6.9)$$

$$\sum_{j=0}^{i-1} x_{ijt} + \sum_{j=i+1}^{N} x_{jit} = 2z_{it} \qquad 0 \le i \le N, 1 \le t \le T \qquad (6.10)$$

$$\sum_{i \in S} \sum_{j \in S, j<i} x_{ijt} \le \sum_{i \in S} z_{it} - z_{kt} \qquad S \subseteq \{1,2,...,N\}, 1 \le t \le T, \text{some } k \in S \qquad (6.11)$$

$$z_{it} \le z_{0t} \qquad 1 \le i \le N, 1 \le t \le T \qquad (6.12)$$

$$x_{ijt} \le z_{it} \qquad 1 \le j < i \le N, 1 \le t \le T \qquad (6.13)$$

$$x_{ijt} \le z_{jt} \qquad 1 \le j < i \le N, 1 \le t \le T \qquad (6.14)$$

$$x_{ijt} \in \{0,1\} \qquad 0 \le j < i \le N, 1 \le t \le T \qquad (6.15)$$

$$x_{i0t} \in \{0,1,2\} \qquad 1 \le i \le N, 1 \le t \le T \qquad (6.16)$$

$$z_{it} \in \{0,1\} \qquad 0 \le i \le N, 1 \le t \le T \qquad (6.17)$$

$$y_t \in \{0,1\} \qquad 1 \le t \le T \qquad (6.18)$$

$$W_{itk} \ge 0 \qquad 1 \le i \le N, 1 \le t \le k \le T \qquad (6.19)$$

$$U_{iqtk} \ge 0 \qquad 1 \le i \le N, 1 \le q \le t \le k \le T \qquad (6.20)$$

where $I_{i0} = 0$ for $0 \le i \le N$.

Objective function (6.1) is the sum of fixed and variable order costs, inventory holding costs at the supplier and retailers as well as transportation costs, respectively. Constraints (6.2) and (6.3) are inventory balance equations for the supplier and retailers, respectively. They are actually not needed but used to compute inventory holding costs. Constraints (6.4) ensure that if retailer $i$ is shipped a quantity in period $t$ then it is satisfied by placing an order to the supplier in period $q$ ($1 \le q \le t$). Constraints (6.5) stipulate that a fixed cost is incurred when an order is placed to the supplier. Constraints (6.6) and (6.7) are the flow conservation equations of the shortest path representation of each retailer $i$. Constraints (6.8) ensure that the total amount shipped to the retailers in period $t$ cannot exceed the capacity of the vehicle. Constraints (6.9) guarantee that if any replenishment occurs to a retailer $i$ in period $t$ then $i$ must be visited in $t$. Constraints (6.10) are degree constraints for ensuring that two edges are incident to retailer $i$ in a period $t$ if $i$ is visited in $t$. Constraints (6.11) are the generalized subtour elimination constraints.

Constraints (6.12)–(6.14) are indeed not needed here but they are added to strengthen the formulation. Constraints (6.15)–(6.18) are for integrality while constraints (6.19) and (6.20) are for nonnegativity.

As the nonzero initial inventories at the retailers (i.e. $I_{i0} > 0$ for $1 \le i \le N$) can be treated as zero by deducing external demands at the retailers from $I_{i0}$ until it becomes zero and adding its cost to the objective function value, the *SF-PR* formulation given above is assuming zero initial inventories. Although *SF-PR* is also assuming zero initial inventory at the supplier, it is easy to incorporate nonzero initial inventories into *SF-PR* as done in Chapter 3.

The *SF-PR* formulation we propose is a quite flexible formulation in that it can handle a variety of issues considered in the literature. In the following, we briefly discuss these issues and show how *SF-PR* can cope with them.

§ *Inventory bound constraints*: Some researchers consider problems in which the amount of inventory carried in any period at the retailers and/or at the supplier cannot exceed a maximum level $UP_i$ ($0 \le i \le N$). For example, Lei et al. (2006) consider inventory bound constraints both at the supplier and at the retailers while Archetti et al. (2007b) consider these constraints at the retailers (they call maximum level policy). We can incorporate such a constraint into *SF-PR* as follows.

$$I_{it} \le UP_i \qquad 1 \le i \le N, 1 \le t \le T \tag{6.21}$$

§ *Capacity over replenishment quantities to supplier*: Lei et al. (2006) and Boudia et al. (2007) consider a capacity constraint $C_s$ on the amount that can be produced or ordered at the supplier in a period. This can also be easily handled in *SF-PR* by adding the following constraints.

$$\sum_{i=1}^{N}\sum_{t=q}^{T}\sum_{k=t}^{T} D_{itk} U_{iqtk} \le C_s y_q \qquad 1 \le q \le T \tag{6.22}$$

- *Vehicle fleet*: Although we consider a single vehicle, it is possible to adapt *SF-PR* to the case of homogeneous/heterogeneous vehicles by adding an index *v* into transportation cost and vehicle capacity parameters, i.e. $c_{ijv}$ and $C_v$ as well as into the variables *x*, *z*, *W*, and modifying related constraints in *SF-PR* accordingly. Also, it is possible to adapt *SF-PR* for an unlimited fleet size as in Chandra and Fisher (1994) and Archetti et al. (2007b).

- *Multi-product case*: It is also easy to address multi-product case by incorporating an additional index for each product into the variables $I, U, W, y$ and writing all constraints in *SF-PR* for each product except (6.8) and (6.10)–(6.14).

Although our focus is on the PDR problem, we also consider the PDR problem with bounded inventory (i.e. the PDR problem with constraints (6.21)) for benchmarking of our formulations with that of Archetti et al. (2007b). For the PDR problem with bounded inventory, we add (6.21) to *SF-PR*, which we refer to as *SF-ML* formulation. Note that *SF-ML* reduces to *SF-PR* if $UP_i$ is sufficiently large (e.g. $UP_i = D_{i1T}$) for all *i* ($1 \le i \le N$).

Since *SF-PR* (*SF-ML*) has an exponential number of constraints due to (6.11), one cannot directly attempt to solve the complete *SF-PR* (*SF-ML*) formulation even for a few retailers. Instead, we add constraints (6.11) dynamically to *SF-PR* (*SF-ML*), which leads to a branch-and-cut algorithm, as described next.

## 6.3 Solution algorithms for the PDR problems

### 6.3.1 Branch-and-cut algorithm

The branch-and-cut algorithm proposed for the PDR problem is based on dynamic addition of constraints (6.11). We first start with *SF-PR* without constraints (6.11) and integrality requirements on variables. At each node of the branch-and-bound tree, first, the current solution at this node is checked to see whether there are inequalities of (6.11) that are violated by the current solution. If there are such inequalities then they are added to the formulation and it is reoptimized. This procedure repeats until no such violated inequalities are found. We use the separation algorithm of Padberg and Rinaldi (1991) to detect violated inequalities of (6.11) (see Appendix C for further details), where $z_{kt}$ variable with largest value in any iteration, i.e. $k = \arg\max_i\{z_{it}\}$, is selected to subtract from summation of *z* variables in the retailer subset *S*. Whenever there is no violated inequality of (6.11), branching occurs. Regarding the branching variable selection, we first branch on *y* variables, then on *z* variables and lastly on *x* variables. As a node selection rule we use best-bound first rule (i.e. the node with the best objective function value is selected). We also use an initial upper bound found by a heuristic described in the next subsection. Our branch-and-cut algorithm is referred to as BC(*SF-PR*). Similarly, using *SF-ML* in place of *SF-PR*, we obtain a branch-and-cut algorithm for the PDR problem with bounded inventory, which is referred to as BC(*SF-ML*). Note that our branch-and-cut algorithms are almost the same as that of Archetti et al. (2007b), which is referred to as BC(*F-ML*). There are only two differences between our and their branch-and-cut algorithms in that we use *SF-PR* or *SF-ML* formulation and a mathematical programming based heuristic for initial upper bounding whereas they use *F-ML* formulation and an improvement heuristic, referred to as *ABPS*, for initial upper bounding.

### 6.3.2 A priori tour based heuristic

In this subsection, we present a mathematical programming based heuristic, a priori tour heuristic, for the PDR problem. The idea of replacing combinatorial routing decision problem with a simpler ordering decision problem has already been explained in Chapter 5. Adaptation of the algorithm given in Section 5.3.2 to the PDR problem is undemanding. We replace the formulation given in Section 5.3.2 for the VMIR-OU problem with the following formulation for the PDR problem.

$$\textit{A-PR}: \ Min \sum_{t=1}^{T} f_t y_t + \sum_{i=1}^{N}\sum_{q=1}^{T}\sum_{t=q}^{T}\sum_{k=t}^{T} p_t D_{itk} U_{iqtk} + \sum_{i=0}^{N}\sum_{t=1}^{T} h_{it} I_{it} + \sum_{i=0}^{N}\sum_{j=0, i\neq j}^{N}\sum_{t=1}^{T} c_{ij} x_{ijt} \qquad (6.23)$$

*s.t.* (6.2)–(6.9), (6.12) and (6.17)–(6.20)

$$\sum_{j\in a_i} x_{ijt} = z_{it} \qquad 0 \le i \le N, 1 \le t \le T \qquad (6.24)$$

$$\sum_{j\in b_i} x_{jit} = z_{it} \qquad 0 \le i \le N, 1 \le t \le T \qquad (6.25)$$

$$x_{ijt} \in \{0,1\} \qquad 0 \le i \le N, 0 \le j \le N, j \neq i, 1 \le t \le T \qquad (6.26)$$

where the set $b_i$ ($0 \le i \le N$) contains facilities *j* ($0 \le j \le N$) that can be visited before visiting facility *i*, the set $a_i$ contains facilities *j* ($0 \le j \le N$) that can be visited after visiting facility *i*, and $x_{ijt}$ takes value 1 if facility *j* is visited immediately after facility *i* in period *t* and 0 otherwise.

As in Section 5.3.2, we denote the a priori tour heuristic without the improvement step (i.e. solving *T* many traveling salesman problems to improve the tours in each period) for the PDR problem as *APT-PR* while the complete procedure including the improvement step is denoted as $APT^{+}$*-PR*. To obtain a priori tour heuristic for the PDR problem with bounded inventory, we add (6.21) to *A-PR*, which is referred to as *A-ML* formulation. Using *A-ML* in place of *A-PR* in *APT-PR* and $APT^{+}$*-PR*, we obtain *APT-ML* and $APT^{+}$*-ML* heuristics, respectively.

### 6.4 Computational experiments

We conduct computational experiments on instances generated by Archetti et al. (2007b) to evaluate the performance of the algorithms. The computational platform used is a Pentium IV Core 2 Duo 2.33GHz PC with 1GB RAM running under Windows XP. We code all the algorithms in C++ on MS Visual Studio.NET 2005 using Concert Technology 2.2 and CPLEX 10.1. We use CPLEX 10.1 to solve *A-PR* and *A-ML*, and CONCORDE (Applegate et al., 2007) to solve TSPs to optimality. In the following, we present the properties of instances in Archetti et al. (2007b).

The number of retailers $N$ and horizon length $T$ are set equal to 19 and 6, respectively. External demands $d_{it}$ are constant over time (i.e. $d_{it} = d_i$) and generated from U[5,25] as an integer. Maximum inventory level at retailers $UP_i$ is set equal to $g_i d_i$ where $g_i$ is randomly selected from the set {2,3,6} and denotes the number of periods needed to consume inventory at retailers. There is a single vehicle with capacity $C$, which is set equal to $UP', 3UP'/2$ and $2UP'$ where $UP' = \max_{1 \le i \le N}\{UP_i + d_i\}$. Initial inventory level at the supplier is set equal to 0 while the initial inventory level at retailers $I_{i0}$ is set equal to $UP_i - d_i$. It is provided that all the problem instances are feasible. All the cost parameters are generated as constant over time. Inventory carrying cost rate at retailers $h_i$ is generated from U[1,5] and U[6,10] while inventory carrying cost rate at the supplier $h_0$ is set equal to 3 and 8. Variable order (production) cost $p$ is set equal to $10h_0$ and fixed order cost $f$ is set equal to $100p$. Transportation cost $c_{ij}$ between two facilities $i$ and $j$ is set equal to $\left\lfloor \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2} + 0.5 \right\rfloor$ where $X_i, X_j, Y_i$ and $Y_j$ are the coordinates of facilities $i$ and $j$ and are generated from U[0,500] and U[0,1000] as integers. Thus, 24 test instances are obtained in the first class of instances. In the second class of instances, only the variable order cost parameter $p$ is changed from $10h_0$ to $100h_0$ (note that $p$ affects $f$). The third class of instances is created by multiplying the coordinates of the supplier and retailers with five and the rest remains the same. In the fourth class of instances, instances 1–6 and 13–18 of the first class, and instances 7–12 and 19–24 of the second class are selected and their $h_i$ is set equal to 0. Thus, there are 96 instances in total. We give the properties of instances of the first class corresponding to the combination of parameters $h_0$, $h_i$, coordinates ($X_i$, $X_j$, $Y_i$, $Y_j$) and $C$ in the Table 6.1.

**Table 6.1** Properties of instances

| **Instance** | $h_0$ | $h_i$ | $X_i, X_j, Y_i, Y_j$ | $C$ |
|---|---|---|---|---|
| 1 | 3 | [6,10] | [0,500] | $2UP'$ |
| 2 | 3 | [6,10] | [0,500] | $3UP'/2$ |
| 3 | 3 | [6,10] | [0,500] | $UP'$ |
| 4 | 8 | [6,10] | [0,500] | $2UP'$ |
| 5 | 8 | [6,10] | [0,500] | $3UP'/2$ |
| 6 | 8 | [6,10] | [0,500] | $UP'$ |
| 7 | 3 | [1,5] | [0,500] | $2UP'$ |
| 8 | 3 | [1,5] | [0,500] | $3UP'/2$ |
| 9 | 3 | [1,5] | [0,500] | $UP'$ |
| 10 | 8 | [1,5] | [0,500] | $2UP'$ |
| 11 | 8 | [1,5] | [0,500] | $3UP'/2$ |
| 12 | 8 | [1,5] | [0,500] | $UP'$ |
| 13 | 3 | [6,10] | [0,1000] | $2UP'$ |
| 14 | 3 | [6,10] | [0,1000] | $3UP'/2$ |
| 15 | 3 | [6,10] | [0,1000] | $UP'$ |
| 16 | 8 | [6,10] | [0,1000] | $2UP'$ |
| 17 | 8 | [6,10] | [0,1000] | $3UP'/2$ |
| 18 | 8 | [6,10] | [0,1000] | $UP'$ |
| 19 | 3 | [1,5] | [0,1000] | $2UP'$ |
| 20 | 3 | [1,5] | [0,1000] | $3UP'/2$ |
| 21 | 3 | [1,5] | [0,1000] | $UP'$ |
| 22 | 8 | [1,5] | [0,1000] | $2UP'$ |
| 23 | 8 | [1,5] | [0,1000] | $3UP'/2$ |
| 24 | 8 | [1,5] | [0,1000] | $UP'$ |

We start with basic experiments in order to assess the impact of CPLEX cuts and initial upper bounding on our algorithm using the test instances of Archetti et al. (2007b) with inventory bound constraints. First, we run BC(*SF-ML*) without an initial upper bound, referred to as BC(*SF-ML*)$^-$, under three different settings of CPLEX cuts: (i) All cuts are allowed, (ii) Only Gomory and implied bound cuts are allowed, and (iii) No cuts are allowed. Results are given in Table 6.2 where column 1 indicates the class of instances, column 2 indicates both elapsed time in seconds (averaged over 24 instances in each class) and number of instances that are solved to optimality, and columns 3–5 show BC(*SF-ML*)$^-$ with all cuts (All), with only Gomory and implied bound cuts (G&IB) and no cuts (None), respectively. Table 6.2 indicates that the worst performance of the algorithm is the case where no cuts are allowed. BC(*SF-ML*)$^-$ with G&IB is better than BC(*SF-ML*)$^-$ with All for the first two classes while BC(*SF-ML*)$^-$ with All is better than BC(*SF-ML*)$^-$ with G&IB for the last class. They are even for the third class. Second, we try BC(*SF-ML*), that is the branch-and-cut with an initial upper bound, with All and with G&IB. Results are given in columns 6 and 7, respectively in Table 6.2. We have decided to use BC(*SF-ML*) with G&IB in subsequent experiments. Although providing an initial upper bound to the branch-and-cut algorithm may not yield better results (see the results for second class), the overall results justify feeding an initial upper bound to the algorithm.

Unfortunately we do not have the algorithm of Archetti et al (2007b). Because of this, we were not able to directly compare BC(*F-ML*) and BC(*SF-ML*) (or BC(*SF-ML*)$^-$). Average results by BC(*F-ML*), BC(*SF-ML*)$^-$ and BC(*SF-ML*) over 24 instances in each class are given in Table 6.3. In this table, column 1–5 show the algorithm used, the elapsed time in seconds, the number of nodes explored, the remaining percentage gap (%Gap) between the best upper (UB) and lower bounds (LB) (i.e. %Gap = 100 * (UB – LB) / LB), and the number of instances solved to optimality, respectively.

BC(*F-ML*) could not solve 52 out of 96 instances to optimality within 2-hour time limit and the results did not change when it was allowed to run for two hours more as noted in Archetti et al. (2007b). BC(*SF-ML*)$^-$ and BC(*SF-ML*) yield quite satisfactory results proving optimality in all instances in reasonable times (except one instance for BC(*SF-ML*)$^-$). In spite of the differences in the computational platform and solver in the experiments of Archetti et al. (2007b), it can be said that our branch-and-cut algorithms outperform their competitor.

**Table 6.2** Average results for the impact of CPLEX cuts and initial upper bounding

| | | **BC(*SF-ML*)$^-$** | | | **BC(*SF-ML*)** | |
|---|---|---|---|---|---|---|
| **Class** | **Info** | **All** | **G&IB** | **None** | **All** | **G&IB** |
| **1st** | **Seconds** | 404.4 | 377.4 | 426.9 | 364.6 | 352.2 |
| | **# solved** | 24 | 24 | 24 | 24 | 24 |
| **2nd** | **Seconds** | 975.8 | 842.8 | 1404.7 | 1323.5 | 1189.3 |
| | **# solved** | 24 | 24 | 24 | 24 | 24 |
| **3rd** | **Seconds** | 1463.8 | 1464.7 | 1697.9 | 1118.8 | 939.9 |
| | **# solved** | 23 | 23 | 22 | 22 | 24 |
| **4th** | **Seconds** | 1004.1 | 1384.8 | 1754.0 | 912.6 | 1057.1 |
| | **# solved** | 24 | 24 | 23 | 24 | 24 |
| **Average** | **Seconds** | 962.0 | 1017.4 | 1320.9 | 929.9 | 884.6 |
| **Total** | **# solved** | 95 | 95 | 93 | 94 | 96 |

**Table 6.3** Average results for BC(*F-ML*)[†], BC(*SF-ML*)$^-$ and BC(*SF-ML*)

| **Class** | **Algorithm** | **Seconds** | **Nodes** | **%Gap** | **# solved** |
|---|---|---|---|---|---|
| **1st** | **BC(*F-LM*)** | 6131.0 | 28375.0 | 1.24 | 4 |
| | **BC(*SF-ML*)$^-$** | 377.4 | 2740.8 | 0.00 | 24 |
| | **BC(*SF-ML*)** | 352.2 | 3115.0 | 0.00 | 24 |
| **2nd** | **BC(*F-ML*)** | 2013.0 | 8389.8 | 0.01 | 18 |
| | **BC(*SF-ML*)$^-$** | 842.8 | 6953.0 | 0.00 | 24 |
| | **BC(*SF-ML*)** | 1189.3 | 11020.4 | 0.00 | 24 |
| **3rd** | **BC(*F-ML*)** | 4252.5 | 19203.7 | 1.34 | 11 |
| | **BC(*SF-ML*)$^-$** | 1464.7 | 10768.6 | 0.04 | 23 |
| | **BC(*SF-ML*)** | 939.9 | 8213.5 | 0.00 | 24 |
| **4th** | **BC(*F-ML*)** | 4318.1 | 18716.0 | 1.03 | 11 |
| | **BC(*SF-ML*)$^-$** | 1384.8 | 10530.2 | 0.00 | 24 |
| | **BC(*SF-ML*)** | 1057.1 | 10149.1 | 0.00 | 24 |

[†]Results of BC(*F-ML*) are found using CPLEX 9.0 on a Pentium IV 2.8GHz PC with 1GB RAM running under Windows XP.

We also compare our heuristics, *APT-ML* and *APT*$^+$*-ML*, with that of Archetti et al. (2007b), called *ABPS*. Results are provided in Table 6.4, where columns 2–3 show the elapsed time in seconds for *APT-ML* and *APT*$^+$*-ML* respectively, and columns 4–6 indicate the percentage deviation (%Dev) of the heuristic value (Heur) from the optimal solution value (Opt) (i.e. %Dev = 100 * (Heur – Opt) / Opt). According to the results, all heuristics perform well but our heuristics are slightly better. In particular, *APT*$^+$*-ML* yields solutions with an overall average percentage deviation of 0.07% and achieves to find the optimal solution in 29 out of 96 instances whereas *APT-ML* and *ABPS* could not find the optimal solution in any of the 96 instances. Besides, the time required to solve *T* many TSPs to improve the solutions of *APT-ML* is negligible compared to the time required to solve the *A-ML* formulation.

**Table 6.4** Average results for *APT-ML*, *APT*$^+$*-ML* and *ABPS*

| | **Seconds** | | **%Dev** | | |
|---|---|---|---|---|---|
| **Class** | ***APT-ML*** | ***APT*$^+$*-ML*** | ***APT-ML*** | ***APT*$^+$*-ML*** | ***ABPS*** |
| **1st** | 13.0 | 14.3 | 0.20 | 0.07 | 1.46 |
| **2nd** | 22.9 | 24.2 | 0.03 | 0.02 | 0.19 |
| **3rd** | 48.9 | 50.1 | 0.68 | 0.19 | 2.05 |
| **4th** | 37.2 | 38.4 | 0.13 | 0.02 | 0.47 |
| **Average** | 30.5 | 31.8 | 0.26 | 0.07 | 1.05 |

Next, we perform experiments without setting the inventory bound constraints, i.e. testing the PDR problem. We again implement BC(*SF-PR*)$^-$ and BC(*SF-PR*) with G&IB. Average results are presented in Table 6.5, where it can be seen that both BC(*SF-PR*)$^-$ and BC(*SF-PR*) perform well. Note that for those instances BC(*SF-PR*) could not solve two instances in the third class to optimality whereas BC(*SF-PR*)$^-$ is able to find the optimal solution in all instances. These results imply that the impact of initial upper bounding diminishes when inventory bounds on stocking are relaxed.

**Table 6.5** Average results for BC(*SF-PR*)$^{-}$ and BC(*SF-PR*)

| Class | Algorithm | Seconds | Nodes | %Gap |
|---|---|---|---|---|
| **1st** | **BC(*SF-PR*)$^{-}$** | 384.6 | 3037.1 | 0.00 |
| | **BC(*SF-PR*)** | 309.6 | 2947.3 | 0.00 |
| **2nd** | **BC(*SF-PR*)$^{-}$** | 812.8 | 7192.6 | 0.00 |
| | **BC(*SF-PR*)** | 644.1 | 6541.9 | 0.00 |
| **3rd** | **BC(*SF-PR*)$^{-}$** | 1606.4 | 12773.4 | 0.00 |
| | **BC(*SF-PR*)** | 1624.0 | 15033.5 | 0.08 |
| **4th** | **BC(*SF-PR*)$^{-}$** | 1306.5 | 11627.1 | 0.00 |
| | **BC(*SF-PR*)** | 1301.2 | 14042.1 | 0.00 |
| **Average** | **BC(*SF-PR*)$^{-}$** | 1027.6 | 8657.5 | 0.00 |
| | **BC(*SF-PR*)** | 969.7 | 9641.2 | 0.02 |

**Table 6.6** Average results for *APT-PR* and *APT$^{+}$-PR*

| | **Seconds** | | **%Dev** | |
|---|---|---|---|---|
| **Class** | ***APT-PR*** | ***APT$^{+}$-PR*** | ***APT-PR*** | ***APT$^{+}$-PR*** |
| **1st** | 9.0 | 10.4 | 0.15 | 0.07 |
| **2nd** | 17.9 | 19.1 | 0.03 | 0.01 |
| **3rd** | 27.1 | 28.3 | 0.49 | 0.10 |
| **4th** | 20.3 | 21.6 | 0.11 | 0.03 |
| **Average** | 18.6 | 19.8 | 0.19 | 0.05 |

As shown in Table 6.6, our heuristics *APT-PR* and *APT$^{+}$-PR* yield superior results in short times even if inventory bounds on stocking are relaxed. Specifically, *APT$^{+}$-PR* finds the optimal solution in 44 out of 96 instances.

Archetti et al. (2007b) have analyzed the consequences of using maximum level policy and order-up-to level policy (discussed in Chapters 4 and 5) both theoretically and empirically and shown that a solution found using the order-up-to level policy can be significantly worse than the maximum level policy in terms of costs. As a by-product of our computational experiments, we have an opportunity to compare the consequences of using maximum level policy (ML) and the endogenous policy (E). We compute the percentage difference (%Dif) between the optimal solution values of two policies (i.e. %Dif = 100 * (ML – E) / ML) in two different ways: Considering the total cost (TC) term and the total cost less constant terms (TC-Const), which does not have any affect on the optimal solution. These constant terms involve variable order costs and costs due to initial inventories at the retailers. Since per unit order cost is constant over time and the total amount to be ordered to the supplier is known, one can compute the total variable order cost in advance. Also, costs due to initial inventories at the retailers can be computed in advance. Average and Maximum %Dev figures over 24 instances in each class are provided in Table 6.7 through columns 2–3 and 4–5, respectively. Columns 2 and 4 (3 and 5) indicate results obtained with the total cost (the total cost less constant terms). Apparently, there is not much difference between the two policies so that the optimal solution values obtained under the maximum level policy can be used as an upper bound on the objective function of the PDR problem with endogenous policy.

**Table 6.7** Differences between the maximum level and endogenous policy

| | **%AveDif** | | **%MaxDif** | |
|---|---|---|---|---|
| **Class** | **TC** | **TC-Const** | **TC** | **TC-Const** |
| **1st** | 0.38 | 1.18 | 1.58 | 4.49 |
| **2nd** | 0.07 | 0.49 | 0.23 | 1.65 |
| **3rd** | 2.72 | 5.45 | 6.56 | 10.89 |
| **4th** | 1.04 | 4.06 | 4.42 | 13.59 |

# CHAPTER 7

# CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

In this thesis, we have addressed one-warehouse multi-retailer problems under two different inventory control (endogenous and order-up-to level) and two different shipment policies (direct shipping and multi-stop routing), namely,

- § One-warehouse multi-retailer problem with endogenous policy (OWMR)
- § One-warehouse multi-retailer problem with order-up-to level policy (OWMR-O)
- § Inventory routing problem with order-up-to level policy (VMIR-OU)
- § Production-distribution-routing problem with endogenous policy (PDR)

We have addressed the OWMR problem by proposing a new shortest path based formulation and showing that it is stronger than the previously proposed transportation based formulation, which in turn is stronger than the strengthened version of another previously proposed one, echelon stock formulation. The new formulation is a strong formulation since it defines the convex hull of feasible solutions of the single-warehouse single-retailer (SWSR) problem. We have also revealed that the shortest path and transportation based formulations are equivalent in strength for the joint replenishment problem (JRP), which is an important special case of OWMR problem. Moreover, we have explicitly considered the case of nonzero initial inventory at the warehouse which is truly a neglected issue in the multi-level lot sizing literature although it is natural for only some specific representations. We have shown both theoretical and empirical implications of nonzero initial inventories to the problem complexity. Our computational experiments over the test instances have revealed that the shortest path and transportation based formulations perform significantly better than the echelon stock formulation both under zero and nonzero initial inventories at the warehouse. In particular, our shortest path based formulation being the best yields integrality gaps of 0.01% and 1.3% on average in the absence and presence of initial inventories, respectively. This formulation also achieves to find the optimal solution for all instances within less than three minutes for large instances involving 50 retailers and 30 time periods.

We have proposed strong formulations for the OWMR-O problem by explicitly considering the case of nonzero initial inventory at the warehouse besides the case of zero initial inventory. We have shown that in the case of single retailer and zero initial inventory at the warehouse, our strong formulation defines the convex hull of feasible solutions of the problem. We have shown that problem becomes more difficult in the presence of initial inventories at the warehouse in contrast to the single-level lot sizing problems where initial inventories can be easily handled. Computational experiments performed on a set of randomly generated instances have provided that our strong formulations are very successful in solving the problem to optimality. They are very effective in closing the gap between the integer and the continuous solutions with 1% gap on average. Moreover, we have performed a limited number of experiments with the standard (weak) formulation. The results have shown that our strong formulation significantly outperforms the standard formulation such that the standard formulation could not solve any of the instances to optimality within a two-hour time limit whereas our strong formulation solves them to optimality only within a few seconds. We have also shown that significant cost savings (14% on average) can be obtained by using the vendor managed inventory system (by solving MIPs) over the retailer managed inventory system (by implementing latest ordering up-to level policy).

We have addressed the VMIR-OU problem by proposing a branch-and-cut algorithm and a heuristic, based on strong formulations. To the best of our knowledge, this study is the first to consider strong formulations for inventory routing problems. Computational results indicate that our exact and heuristic algorithms have outperformed their competitors in the literature. Our branch-and-cut algorithm is able to find the optimal solution values of the larger problem instances with size up to $nxH = \{65x3, 45x6, 35x9, 20x12\}$ where $n$ and $H$ denote the number of retailers and horizon length, respectively. Our heuristic achieves to find high quality solutions deviating 1% from the optimal solution on average within a few minutes.

We have considered the PDR problem (and also PDR problem with bounded inventory) and proposed branch-and-cut and heuristic algorithms based on strong formulations. Computational experiments have shown that the proposed algorithms have outperformed their competitors in the literature. Our branch-and-cut algorithm solves all instances to optimality for both the PDR problem and the PDR problem with bounded inventory whereas an existing exact algorithm for the PDR problem with bounded inventory could not solve more than half of all instances. Within a few minutes, our heuristic achieves to find solutions with 0.05% and 0.07% deviation on average from the optimal solution values for PDR problem and PDR problem with bounded inventory, respectively. We have also empirically compared the optimal solution values of PDR problem and PDR problem with bounded inventory, and we have shown that the difference is indeed small, which is contrary to the result of huge difference between optimal solution values of PDR problem with order-up-to level policy and PDR with bounded inventory (Archetti et al., 2007b).

One of our overall conclusions is as follows. Since the strong formulations we have proposed for four related but different one-warehouse multi-retailer problems are proved to be effective and they are flexible with regard to handling additional side constraints, they can be used within the decision support systems for planning purposes in different levels in VMI settings discussed in Chapter 1. For instance, direct shipment formulations can be used for tactical level planning such as constructing master production schedules, while multi-stop routing formulations can be used for operational level planning such as constructing detailed production and distribution schedules. In case of multi-stop routing policy, the proposed heuristics can also be used in place of the exact algorithms as they are effective and flexible as well.

As we formulate the retailers' replenishment problem using strong shortest path representation and effectively extend this representation to the warehouse operations for different inventory control policies (like endogenous and order-up-to level policies), we conclude that our approach can be applicable to different replenishment problems (at retailers) provided that a strong shortest path representation of the replenishment problem can be developed. For example, the existence of strong shortest path representations for problems with stationary capacity on replenishment quantities and problems with backorders (see Pochet and Wolsey, 2006) lends these types of lot sizing problems to our approach.

Another overall conclusion we have reached is that the presence of initial inventory at the warehouse adds a significant complexity to the problem at hand.

*Further research issues*

A research direction is to use the approximate strong formulation idea of Van Vyve and Wolsey (2006), instead of incorporating the complete strong formulation. The authors add only some part of the complete formulation as cuts and obtain the best results in the context of the multi-item capacitated lot sizing problem with setup times. Since our strong formulations involve a large number of variables and constraints, decreasing the formulation size using the approximate strong formulation idea seems promising for solving larger problem instances. Another research direction is to use our strong formulations within decomposition based customized algorithms that might help in solving larger problem instances.

Since we assume predetermined order-up-to level ($S_i$) for each retailer $i$ in the OWMR-O problem, one interesting research question that might immediately arise is whether it is possible to optimize order-up-to levels or not. Indeed, this question is partially answered in Solyalı and Süral (2008a) where the authors propose a pseudo-polynomial dynamic programming algorithm for the single retailer case. As discussed in Solyalı and Süral (2008a), each $S_i$ should take a value in between retailer $i$'s maximum and total demand over the horizon. Thus, by incorporating an additional index accounting for the possible values of $S_i$ into the shortest path representation variables, one could obtain a promising strong formulation for this problem. The disadvantage of the resulting formulation might be the large number of variables arising due to accounting for each possible $S_i$ value. However, such a handicap might be addressed by generating a variable only when it improves the solution (i.e. using column generation).

As a further research issue, one can easily extend the formulations considered in this thesis to the multi-item case (as in Federgruen and Tzur, 1999). A promising research avenue is to adapt the shortest path based formulation for the multi-level lot sizing problems with serial, assembly or general structure. In particular, although there exists a polynomial time algorithm for the problem with serial structure, there is no known explicit convex hull defining formulation for it. As the shortest path based formulation is tight for the two-level problem with serial structure (SWSR problem), its extension to more than two levels would be nice contributions to the literature. Extension of the shortest path based formulation for the problem with assembly or general structure seems also promising anyway as the mathematical programming studies almost are based on the echelon stock idea (e.g. Stadtler, 1997).

Another research direction might be to study multiple vehicles for problems with multi-stop routing policy. However, exact solution of such problems will be quite challenging since the known valid inequalities for the vehicle routing problem cannot be trivially adapted here due to not knowing how much a retailer will receive in each period. Let alone the additional valid inequalities, even the so called rounded capacity inequalities in VRP cannot be used because of the same reason and one has to resort to fractional capacity inequalities which are not strong enough.

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## APPENDIX A

## A STANDARD FORMULATION FOR THE OWMR PROBLEM

In this appendix, we provide a standard formulation of the OWMR problem, referred to as *P*. Define *N* as the number of retailers, *T* as the number of periods in the time horizon, $d_{it}$ as the external demand faced by retailer *i* ($1 \le i \le N$), $i = 0$ as the warehouse, $f_{it}$ as the fixed order cost incurred when an order for facility *i* ($0 \le i \le N$) is placed in period *t* ($1 \le t \le T$), $p_{it}$ as the variable order cost incurred per unit ordered by facility *i* in period *t*, and $h_{it}$ as the inventory holding cost at facility *i* incurred for each unit kept at the end of period *t*. Let $I_{0t}$ and $I_{it}$ respectively be inventory levels of the supplier and retailers at the end of period $t$, $Q_{it}$ be the quantity ordered to facility *i* in period *t*, and $y_{it}$ be 1 if an order for facility *i* is placed in period *t* and 0 otherwise.

$$P\text{: } Min \sum_{i=0}^{N}\sum_{t=1}^{T}\left( f_{it}y_{it} + p_{it}Q_{it} + h_{it}I_{it}\right) \qquad \text{(A.1)}$$

*s.t.*

$$I_{0,t-1} + Q_{0t} = \sum_{i=1}^{N} Q_{it} + I_{0t} \qquad 1 \le t \le T \qquad \text{(A.2)}$$

$$I_{i,t-1} + Q_{it} = d_{it} + I_{it} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(A.3)}$$

$$Q_{it} \le D_{itT}\, y_{it} \qquad 0 \le i \le N, 1 \le t \le T \qquad \text{(A.4)}$$

$$y_{it} \in \{0,1\} \qquad 0 \le i \le N,\ 1 \le t \le T \qquad \text{(A.5)}$$

$$Q_{it} \ge 0 \qquad 0 \le i \le N,\ 1 \le t \le T \qquad \text{(A.6)}$$

$$I_{it} \ge 0 \qquad 0 \le i \le N,\ 1 \le t \le T \qquad \text{(A.7)}$$

where $I_{i0}$ is the known initial inventory level at facility $i$ ($0 \le i \le N$), $d_{0t} = \sum_{i=1}^{N} d_{it}$ for $1 \le i \le N$ and $D_{itk} = \sum_{r=t}^{k} d_{ir}$ for $0 \le i \le N, 1 \le t \le k \le T$.

The objective function (A.1) is the sum of fixed and variable order costs and inventory holding costs at the warehouse and retailers. Constraints (A.2) and (A.3) are the inventory balance constraints for the warehouse and retailers, respectively. Constraints (A.4) ensure that a fixed order cost is incurred at facility $i$ ($0 \le i \le N$) if an order is placed any time for $i$. Constraints (A.5) are for integrality while (A.6) and (A.7) are for nonnegativity.

## APPENDIX B

## A STANDARD FORMULATION FOR THE VMIR-OU PROBLEM

In this appendix, we provide the standard formulation of the VMIR-OU problem due to Archetti et al. (2007a). Define *n* as the number of retailers, *H* as the number of periods in the time horizon, $r_{it}$ as the external demand faced by retailer $i \in M = \{1, 2, ..., n\}$, $r_{0t}$ as the amount received by the supplier, denoted by $i = 0$, in period $t \in t = \{1, 2, ..., H\}$, $C$ as the capacity of the vehicle, $U_i$ as the maximum inventory level at retailer $i \in M$, $h_i$ as the inventory holding cost at facility $i \in M' = M \text{ U } \{0\}$ incurred for each unit kept at inventory in $t \in t'$ where $t' = t \text{ U } \{H+1\}$, and $c_{ij}$ as the transportation cost of traveling from facility $i \in M'$ to facility $j \in M'$. Let $I_{0t}$ and $I_{it}$ respectively be inventory levels of the supplier and retailers at the beginning of period $t \in t$ ; $x_{it}$ be the amount shipped to retailer $i \in M$ in period $t \in t$ ; $z_{it}$ be 1 if retailer $i \in M$ is replenished in period $t \in t$ and 0 otherwise; $z_{0t}$ be 1 if vehicle departs from the supplier in period $t \in t$ and 0 otherwise; and $y_{ji}^t$ be 1 if vehicle visits facility $i \in M'$ immediately after facility $j \in M'$ in period $t \in t$ and 0 otherwise. Then, the formulation of the VMIR-OU problem due to Archetti et al. (2007a), referred to as *F*, is as follows.

$$F\text{: } Min \sum_{i \in M'} \sum_{t \in t'} h_i I_{it} + \sum_{i \in M'} \sum_{j \in M', j<i} \sum_{t \in t} c_{ij} y_{ij}^t \qquad \text{(B.1)}$$

*s.t.*

$$I_{0t} = I_{0,t-1} + r_{0,t-1} - \sum_{i \in M} x_{i,t-1} \qquad t \in t' \qquad \text{(B.2)}$$

$$I_{0t} \geq \sum_{i \in M} x_{it} \qquad t \in t \qquad \text{(B.3)}$$

$$I_{it} = I_{i,t-1} + x_{i,t-1} - r_{i,t-1} \qquad i \in M, t \in t' \qquad \text{(B.4)}$$

$$x_{it} \geq U_i z_{it} - I_{it} \qquad i \in M, t \in t \qquad \text{(B.5)}$$

$$x_{it} \leq U_i - I_{it} \qquad i \in M, t \in t \qquad \text{(B.6)}$$

$$x_{it} \leq U_i z_{it} \qquad i \in M, t \in t \qquad \text{(B.7)}$$

$$\sum_{i \in M} x_{it} \leq C \qquad t \in t \qquad \text{(B.8)}$$

$$\sum_{i \in M} x_{it} \leq C z_{0t} \qquad t \in t \qquad \text{(B.9)}$$

$$\sum_{j \in M', j<i} y_{ij}^t + \sum_{j \in M', j>i} y_{ji}^t = 2 z_{it} \qquad i \in M', t \in t \qquad \text{(B.10)}$$

$$\sum_{i \in S} \sum_{j \in S, j<i} y_{ij}^t \leq \sum_{i \in S} z_{it} - z_{kt} \qquad S \subseteq M, t \in t, \text{some } k \in S \qquad \text{(B.11)}$$

$$z_{it} \leq z_{0t} \qquad i \in M, t \in t \qquad \text{(B.12)}$$

$$y_{ij}^t \leq z_{it} \qquad i \in M, j \in M, t \in t \qquad \text{(B.13)}$$

$$y_{i0}^t \leq 2 z_{it} \qquad i \in M, t \in t \qquad \text{(B.14)}$$

$$I_{it} \geq (1 - z_{it}) r_{it} \qquad i \in M, t \in t \qquad \text{(B.15)}$$

$$I_{i,t-k} \geq \left( \sum_{j=0}^{k} r_{i,t-j} \right) \left( 1 - \sum_{j=0}^{k} z_{i,t-j} \right) \qquad i \in M, t \in t, k = 0,1,\ldots,t-1 \qquad \text{(B.16)}$$

$$I_{it} \geq U_i z_{i,t-k} - \sum_{j=t-k}^{t-1} r_{ij} \qquad i \in M, t \in t, k = 0,1,\ldots,t-1 \qquad \text{(B.17)}$$

$$y_{ij}^t \in \{0,1\} \qquad i \in M, j \in M, j < i, t \in t \qquad \text{(B.18)}$$

$$y_{i0}^t \in \{0,1,2\} \qquad i \in M, t \in t \qquad \text{(B.19)}$$

$$z_{it} \in \{0,1\} \qquad i \in M', t \in t \qquad \text{(B.20)}$$

$$I_{it} \geq 0 \qquad i \in M', t \in t' \qquad \text{(B.21)}$$

$$x_{it} \geq 0 \qquad i \in M, t \in t \qquad \text{(B.22)}$$

where $x_{i0} = r_{i0} = 0$ for $i \in M$.

Objective function (B.1) is exactly the same as (5.1). Constraints (B.2)–(B.4) are respectively the same as (5.2)–(5.4) except that $w$ variables are replaced with $x$ variables. Constraints (B.5)–(B.7) guarantee that the order-up-to level policy is satisfied at the retailers. Constraints (B.8) and (B.9) stipulate that the total amount shipped to the retailers in a period cannot exceed the capacity of the vehicle. Indeed, only constraints (B.8) or (B.9) are sufficient to ensure that capacity of the vehicle is not exceeded. Constraints (B.10) and (B.11) are exactly the same as (5.10) and (5.11), respectively. Constraints (B.12)–(B.17) are actually not needed in formulating the VMIR-OU problem but they are added to $F$ a priori by Archetti et al. (2007a) to strengthen the formulation. Constraints (B.12)–(B.14) are for strengthening retailers' replenishment part of the problem while constraints (B.15)–(B.17) are for strengthening routing part of the problem. Constraints (B.18)–(B.20) are for integrality while (B.21) and (B.22) are for nonnegativity of variables.

## APPENDIX C

## AN OVERVIEW OF THE BRANCH-AND-CUT ALGORITHM

In this appendix, we present the branch-and-cut algorithm proposed in Chapter 5 in detail.

As cutting planes, we use constraints (5.11) in Chapter 5 and those of CPLEX 10.1 mentioned in Section 2.1 with the default version. We use the exact separation algorithm of Padberg and Rinaldi (1991) to find violated inequalities of (5.11). The separation algorithm of Padberg and Rinaldi (1991), a polynomial time algorithm, is based on determination of the minimum weighted cut. At each node $j$ of the branch-and-bound tree, we add constraints (5.11) as long as the separation algorithm detects any violated inequality. As for branching variable selection, we only give priority to $z$ variables over $y$ variables and the decision of which $z$ or $y$ variable to select among eligible ones is left to the MIP solver of CPLEX. In Figure C.1, we present our branch-and-cut algorithm with a flow chart, which is an adaptation of the generic flow chart given in Wolsey (1998). In the flow chart, $UB^*$ denotes the best objective value, $x^*$ denotes the best solution obtained, $z'(APF)$ denotes the solution value obtained by $APT^+$, $s'(APF)$ denotes the solution obtained by $APT^+$, $SF^j$ denotes the $SF$ formulation at node $j$, and $SF^{j,k}$ denotes the $SF$ formulation at node $j$ in iteration $k$.

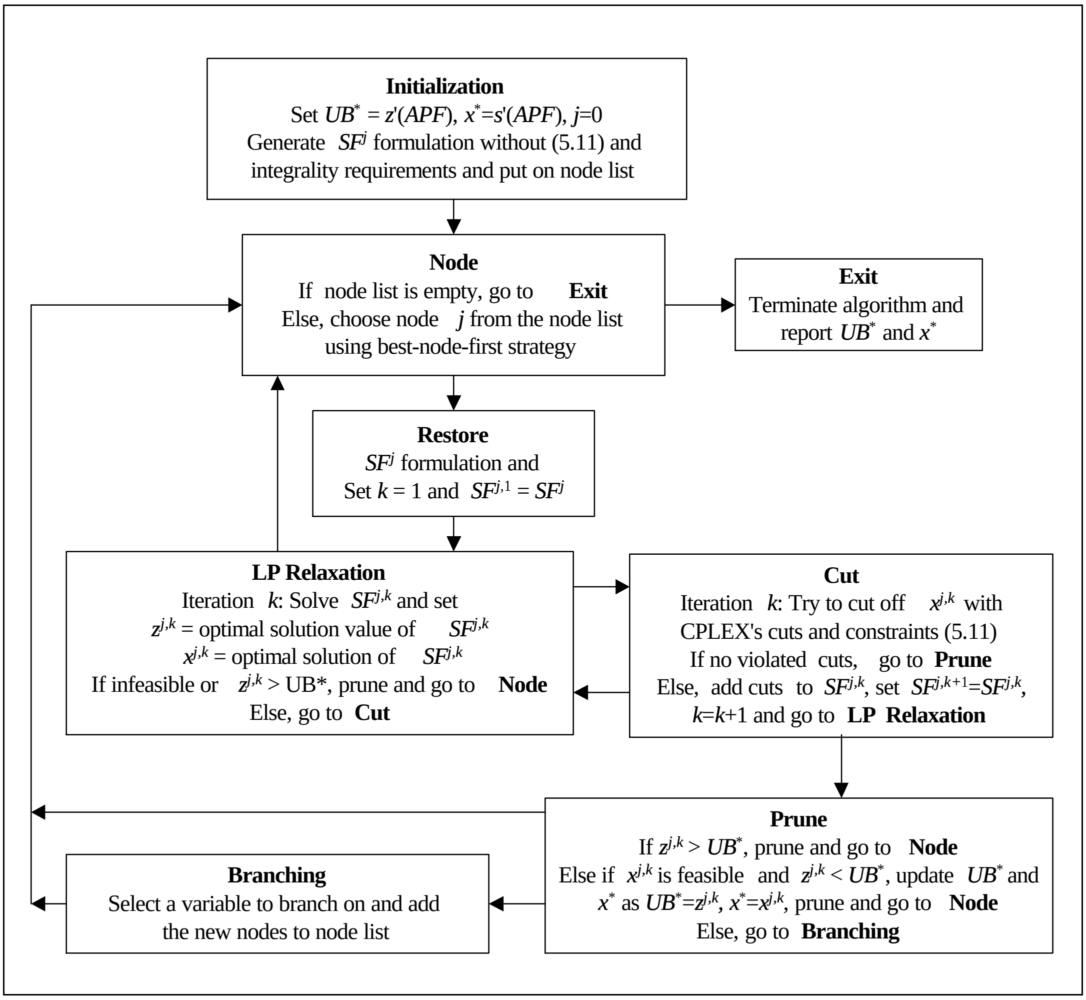


**Figure C.1** Flowchart of our branch-and-cut algorithm

# APPENDIX D

## AN OVERVIEW OF A PRIORI TOUR HEURISTIC

Here, we describe how our a priori tour heuristic works on a small example. Consider a VMIR-OU problem instance with four retailers, i.e. $M' = \{0,1,2,3,4\}$. Using $c_{ij}$ values of the instance, we construct a TSP instance and solve it to optimality. Suppose that the optimal tour is depicted by 0 – 2 – 3 – 1 – 4 – 0 as given in Figure D.1. Then, we derive the sets $b_i$ and $a_i$ for $i \in M'$ using the optimal tour as follows:

$$
\begin{array}{ll}
b_0 = \{1,2,3,4\} & a_0 = \{1,2,3,4\} \\
b_1 = \{0,2,3\} & a_1 = \{0,4\} \\
b_2 = \{0\} & a_2 = \{0,1,3,4\} \\
b_3 = \{0,2\} & a_3 = \{0,1,4\} \\
b_4 = \{0,1,2,3\} & a_4 = \{0\}
\end{array}
$$

Then, we construct *APF* using $b_i$ and $a_i$ for $i \in M'$ and solve it to optimality. In any feasible solution of *APF*, one obtains a tour that follows the ***a priori*** tour by skipping the unvisited retailers in each period the vehicle departs from the supplier. For example, suppose that only retailers 1 and 2 are visited in a period. Then, the tour that the vehicle follows in this case is shown with dashed lines in the following figure.

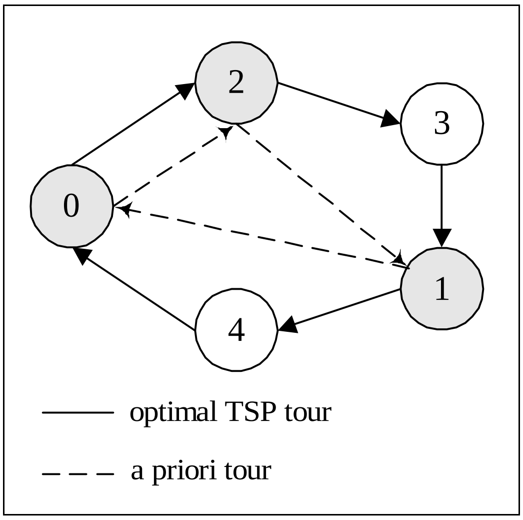


**Figure D.1** Example for a priori tour

# APPENDIX E

## A STANDARD FORMULATION FOR THE PDR PROBLEM

In this appendix, we provide a formulation due to Archetti et al. (2007b). Define $N$ as the number of retailers, $T$ as the number of periods in the time horizon, $d_{it}$ as the external demand faced by retailer $i$ ($1 \le i \le N$), $i$=0 as the supplier, $C$ as the capacity of the vehicle, $UP_i$ as the maximum inventory level at retailer $i$ ($1 \le i \le N$), $f_t$ as the fixed order cost incurred independent of the size of order, $p_t$ as the variable order cost incurred per unit ordered to the supplier, $h_i$ as the inventory holding cost at facility $i$ ($0 \le i \le N$) incurred for each unit stocked in the end of $t$ ($1 \le t \le T$), and $c_{ij}$ as the transportation cost of traveling from facility $i$ ($0 \le i \le N$) to facility $j$ ($0 \le j \le N$).

Let $I_{0t}$ and $I_{it}$, respectively, be inventory levels of the supplier and retailers at the end of period $t$ ($1 \le t \le T$); $U_t$ be the amount ordered to the supplier in period $t$; $W_{it}$ be the amount shipped to retailer $i$ ($1 \le i \le N$) in period $t$; $x_{ijt}$ be 1 if the vehicle visits facility $j$ ($0 \le j \le N$) immediately after facility $i$ ($0 \le i \le N$) in period $t$ and 0 otherwise; $z_{0t}$ be 1 if the vehicle departs from the supplier in period $t$ and 0 otherwise; $z_{it}$ be 1 if retailer $i$ is visited in period $t$ and 0 otherwise; and $y_t$ be 1 if an order is placed at the supplier in period $t$ and 0 otherwise. Then, the formulation due to Archetti et al. (2007b), referred to as *F-ML*, is as follows.

$$\text{F-ML: } Min \sum_{t=1}^{T}(f_t y_t + p_t U_t) + \sum_{i=0}^{N}\sum_{t=1}^{T} h_i I_{it} + \sum_{i=1}^{N}\sum_{j=0}^{i-1}\sum_{t=1}^{T} c_{ij} x_{ijt} \qquad \text{(E.1)}$$

*s.t.*

$$I_{0t} = I_{0,t-1} + U_t - \sum_{i=1}^{N} W_{it} \qquad 1 \le t \le T \qquad \text{(E.2)}$$

$$I_{it} = I_{i,t-1} + W_{it} - d_{it} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(E.3)}$$

$$U_t \le \sum_{i=1}^{N} \sum_{r=t}^{T} d_{ir} y_t \qquad 1 \le t \le T \qquad \text{(E.4)}$$

$$I_{it} \le UP_i \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(E.5)}$$

$$W_{it} \le \min\{UP_i + d_{it}, C, \sum\nolimits_{r=t}^{T} d_{ir}\} z_{it} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(E.6)}$$

$$\sum_{i \in M} W_{it} \le C \qquad 1 \le t \le T \qquad \text{(E.7)}$$

$$\sum_{j=0}^{i-1} x_{ijt} + \sum_{j=i+1}^{N} x_{jit} = 2 z_{it} \qquad 0 \le i \le N,\ 1 \le t \le T \qquad \text{(E.8)}$$

$$\sum_{i \in S} \sum_{j \in S, j < i} x_{ijt} \le \sum_{i \in S} z_{it} - z_{kt} \qquad S \subseteq \{1, 2, ..., N\}, 1 \le t \le T, \text{some } k \in S \qquad \text{(E.9)}$$

$$z_{it} \le z_{0t} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(E.10)}$$

$$x_{ijt} \le z_{it} \qquad 1 \le j < i \le N, 1 \le t \le T \qquad \text{(E.11)}$$

$$x_{ijt} \le z_{jt} \qquad 1 \le j < i \le N, 1 \le t \le T \qquad \text{(E.12)}$$

$$I_{i,t-k} \ge \left( \sum_{j=0}^{k} r_{i,t-j} \right) \left( 1 - \sum_{j=0}^{k} z_{i,t-j} \right) \qquad 1 \le i \le N,\ 1 \le t \le T\ \ 0 \le k \le t-1 \qquad \text{(E.13)}$$

$$I_{0,t-1} \le \sum_{i=1}^{N} \sum_{r=t}^{T} d_{ir} (1 - y_t) \qquad 1 \le t \le T \qquad \text{(E.14)}$$

$$U_t \ge \frac{K}{h_0 j} (y_{t-j} + y_t - 1) \qquad 2 \le t \le T, 1 \le j \le t-1 \qquad \text{(E.15)}$$

$$\sum_{t=1}^{t'} y_t \ge 1 \qquad \text{(E.16)}$$

$$\sum_{t=1}^{t'} U_t \ge \sum_{i=1}^{N} \max\{0, \sum_{t=1}^{t'} d_{it} - I_{i0}\} \qquad \text{(E.17)}$$

$$\sum_{t=1}^{t'} \sum_{i=1}^{N} x_{i0t} \ge \left\lceil \sum_{i=1}^{N} \max\{0, \sum_{t=1}^{t'} d_{it} - I_{i0}\} / C \right\rceil \qquad \text{(E.18)}$$

$$x_{ijt} \in \{0,1\} \qquad 0 \le j < i \le N, 1 \le t \le T \qquad \text{(E.19)}$$

$$x_{i0t} \in \{0,1,2\} \qquad 1 \le i \le N,\ 1 \le t \le T \qquad \text{(E.20)}$$

$$z_{it} \in \{0,1\} \qquad 0 \le i \le N, 1 \le t \le T \qquad \text{(E.21)}$$

$$y_t \in \{0,1\} \qquad 1 \le t \le T \qquad \text{(E.22)}$$

$$I_{it} \ge 0 \qquad 0 \le i \le N, 1 \le t \le T \qquad \text{(E.23)}$$

$$W_{it} \ge 0 \qquad 1 \le i \le N, 1 \le t \le T \qquad \text{(E.24)}$$

$$U_t \ge 0 \qquad 1 \le t \le T \qquad \text{(E.25)}$$

where $t' = \min_{1 \le i \le N} \{t'_i\}$, $t'_i = \min_{1 \le j \le T} \{ j \mid I_{i0} - \sum_{t=1}^{j} d_{it} < 0\}$ and $I_{00} = 0$.

Objective function (E.1) is the total of fixed and variable order costs at the supplier, inventory holding costs at the supplier and retailers as well as transportation costs. Constraints (E.2) and (E.3) are inventory balance equations for the supplier and retailers, respectively. Constraints (E.4) stipulate that a fixed order cost is incurred whenever the supplier places an order. Constraints (E.5) ensure that the amount of inventory carried at a retailer cannot exceed its maximum level. These constraints are defined by Archetti et al. (2007b) since they consider a variant of PDR in which there are constraints on the amount of inventory that can be carried in any period. Thus, constraints (E.5) must be removed from ***F-ML*** to obtain a valid formulation for the PDR problem. Constraints (E.6) guarantee that a retailer is visited in a period in which it is replenished. Constraints (E.7) ensure that the total amount shipped to the retailers in a period cannot exceed the capacity of the vehicle. Constraints (E.8)–(E.12) are exactly the same as (6.10)–(6.14), respectively. Indeed, constraints (E.11)–(E.18) are not needed in formulating the problem but they are shown to be valid inequalities and added to ***F-ML*** a priori by Archetti et al. (2007b) to strengthen the formulation. Specifically, constraints (E.11), (E.12) and (E.18) are for strengthening routing part of the problem while constraints (E.13)–(E.17) are for strengthening inventory replenishment part of the problem. Constraints (E.19)–(E.22) are for integrality while (E.23)–(E.25) are for nonnegativity of variables.

## CURRICULUM VITAE

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### EDUCATION

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### PROFESSIONAL EXPERIENCE

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### AWARDS AND HONORS

**Awarded post-doctoral fellowship,** Canada Research Chair in Distribution Management & Canada Research Chair in Logistics and Transportation, HEC Montreal, September 2009-September 2010

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**Awarded research & teaching assistantship**, Middle East Technical University, Ankara, Turkey, February 2005-June 2006

**Ranked 2nd among 2003 graduates**, Department of Environmental Engineering, Middle East Technical University, Ankara, Turkey

**Member of the 1st team**, TRNC 4th Mathematics Team Competition among High Schools, 1998

**Awarded Prof. Dr. Cahit Arf Encouragement Award**, TRNC 4th Mathematics Competition among High School Students, 1998

## RESEARCH INTERESTS

Optimization

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Facility Location

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Supply Chain Management

## JOURNAL PUBLICATIONS

**O. Solyalı** and H. Süral, "A single supplier-single retailer system with order-up-to level inventory policy", *Operations Research Letters* 36, 543-546, 2008

**O. Solyalı** and Ö. Özpeynirci, "Operational fixed job scheduling problem under spread time constraints: a branch-and-price algorithm", *International Journal of Production Research*, doi:10.1080/00207540701666204, 2007

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**O. Solyalı** and B. Kat, "A bicriteria single allocation hub location problem", Proceedings of the 27th National Conference on Operations Research and Industrial Engineering, in CD format, 2007

**O. Solyalı** and Ö. Özpeynirci, "Operational fixed job scheduling problem under spread time constraints: a branch-and-price algorithm", Proceedings of the 27th National Conference on Operations Research and Industrial Engineering, in CD format, 2007

**O. Solyalı**, "Solving the single allocation p-hub center problem using genetic algorithms", Proceedings of the 35th International Conference on Computers and Industrial Engineering, Durmuşoğlu M.B. and Kahraman C. (Eds.): Vol 2, 1777-1782, 2005

## CONFERENCE/WORKSHOP PRESENTATIONS

**O. Solyalı**, H. Süral and M. Denizel, "Strong formulations for the one-warehouse multi-retailer problem", International Symposium on Combinatorial Optimization (CO 2008), University of Warwick, Coventry, UK, March 16-19, 2008

**O. Solyalı**, H. Süral and M. Denizel, "One-warehouse multi-retailer inventory management with order-up-to level policy and direct shipments: Optimization by strong formulations", IE/OR PhD Students Colloquium, Koç University, İstanbul, Turkey, November 23-24, 2007 (in Turkish)

H. Süral, **O. Solyalı** and M. Denizel, "Inventory management with direct shipments", INFORMS 2007, Seattle, USA, November 4-7, 2007

**O. Solyalı** and Ö. Özpeynirci, "Operational fixed job scheduling problem under spread time constraints: a branch-and-price algorithm", METU-IE and TU/e-OPAC 2nd joint workshop, Technische Universiteit

Eindhoven, The Netherlands, November 20-21, 2006

H. Süral and **O. Solyalı**, “Coordinated inventory control and vehicle routing: Lagrangian relaxation based solution approach”, 19th International Symposium on Mathematical Programming, Rio de Janeiro, Brazil, July 30-August 4, 2006

**O. Solyalı**, “A genetic algorithm for the single allocation p-hub center problem”, 25th National Conference on Operations Research and Industrial Engineering, İstanbul, Turkey, July 4-6, 2005 (in Turkish)

**O. Solyalı** and H. Süral, “An inventory routing problem with order-up-to level inventory policy”, 25th National Conference on Operations Research and Industrial Engineering, İstanbul, Turkey, July 4-6, 2005 (in Turkish)

## LANGUAGES

English (Fluent)

## COMPUTER SKILLS

General: MS Windows, MS Office, Latex

Programming Language: C, C++

Mathematical Software: Matlab

Optimization Software: Cplex, Lindo, Lingo

Simulation Software: Arena, Siman

Statistics Software: Minitab

## PROFESSIONAL MEMBERSHIPS

INFORMS (2008 – present)

Operational Research Society, Turkey (ORST) (2004 – present)

## PROFESSIONAL SERVICES

Consultant for :
Solid Waste Collection System Design for the Turkish Municipality of Nicosia, October – December 2006