Nonmonotonic Reasoning and Multiple Belief Revision

Dongmo Zhang[1,2] Shifu Chen[1] Wujia Zhu[1,2] Hongbing Li[1]
[1] State Key Lab. for Novel Software Technology
Department of Computer Science and Technology
Nanjing University, Nanjing, 210093, China
[2] Department of Computer Science, Nanjing University
of Aeronautics and Astronautics, Nanjing, 210016, China
e-mail:aics@nuaa.edu.cn

Abstract

The aim of the present paper is to reveal the interrelation between general patterns of non-monotonic reasoning and multiple belief revision. For this purpose we define a nonmonotonic inference frame in which individual inference rules have been proposed in the literature but their combination as a system has not been investigated. It is shown that such a system is so strong that almost all the rules (including the supracompactness) suggested for nonmonotonic inference relations in the literature hold in it. We prove that this nonmonotonic inference frame is strictly correspendent with multiple belief revision operation. On the basis of this result we analyse a specific paradigm of defult theory which satisfies all the rules under consideration and discuss limitations of methods based on consequence relations for the study of nonmonotonic reasoning.

1 Introduction

In recent years much work has been done on the relationship between nonmonotonic reasoning and belief revision [Makinson and Gardenfors 1991] [Brewka 1991] [Nebel 1992] [Cravoand Martins 1993] [Li 1993][Gardenfors and Makinson 1994] [Boutilier 1994] [Gardenfors and Rott 1995] [Zhang 1996]. A very close correspondence between them has been found based on the following formal translation:

$$A \mid\!\sim_K C \text{ iff } C \in K * A$$

The main idea is to identify revision of a belief set *K* by a proposition *A* with nonmonotonic inference from *A* under the guidance of the background knowledge A'. With this connection, it has been shown in [Makinson and Gardenfors 1991] [Gardenfors and Rott 1995] that each postulate for the belief revision function * can be translated into a plausible conditions on the nonmonotonic inference relation |~; conversely, almost all the plausible conditions on the nonmonotonic inference relation in the literature can also be translated into conditions on * that are consequences of the postulates for the revision function. In fact, it is not difficult to verify that the revision function * satisfies all eight postulates in [Gärdenfors 1988] if and only if $\mid\!\sim$ satisfies the following five inference rules:

1. If $A \vdash B$, then $A \mid\!\sim B$ (Supraclassicality).
2. If $A \mid\!\sim \bot$, then $A \vdash \bot$ (Consistency Preservation).
3. If $A \mid\!\sim B_i$ for all $B_i \in \Gamma$, $\Gamma \vdash C$, then $A \mid\!\sim C$ (Closure).
4. If $A \wedge B \mid\!\sim C$, then $A \mid\!\sim B \rightarrow C$ (Conditionalization).
5. If $A \not\mid\!\sim \neg B$ and $A \mid\!\sim C$, then $A \wedge B \mid\!\sim C$ (Rational Monotony).

This translation may be extended to the finite case. If Γ is a finite set of propositions, written by $\{A_1, \cdots, A_n\}$, then:

$$\Gamma \mid\!\sim_K A \text{ iff } A \in K * (A_1 \wedge \cdots \wedge A_n)$$

As mentioned in [Makinson 1993], however, this extension muddies the 'neat' distinction between $A_1, \cdots, A_n \mid\!\sim A$ and $A_1 \wedge \cdots \wedge A_n \mid\!\sim A$. A possible improvement is to replace the revision operation with some sort of multiple revision function. Suppose we have had a multiple revision function $\otimes$ such that $K \otimes F$ represents the result of revising a belief set K with a set F of propositions. The translation given below would be more natural:

$$\Gamma \mid\!\sim_K A \text{ iff } A \in K \otimes \Gamma$$

This extension is also essential because it enables a treatment of inference relation in which premises are arbitrary sets of propositions, including infinite sets.

The questions arises naturally now that:

- how the nonmonotonic inference rules on |~ are extended to the infinite level so that they are still plausible for nonmonotonic reasoners;
- how an infinite revision framework is constructed so that it is a natural generalization of the original one;
- whether the strict correspondence between belief revision and nonmonotonic reasoning can be preserved in the setting of the extended frameworks.

Fortunately, the first question has been widely investigated in the literature [Makinson 1989] [Freund 1990]

[Makinson 1993] [Herre 1994], only the presentation of the extended rules is mostly in the Tarski-style's inference operation *C*.

As far as the generalization of belief revision are concerned, [Zhang 1996] presented a kind of multiple revision framework, called *general revision,* which enables a treatment of revisions of belief set by arbitrary set of sentences. [Zhang *et al.* 1997] further developed the framework by providing two presentation theorems and suggesting an additional postulate to characterize the infinite properties of revision operations.

This paper is devoted to the last question. In the next section, we combine some of the nonmonotonic inference rules which have been suggested in the literature into a system of nonmonotonic reasoning, called RN, and discuss its properties. Section 3 outlines the general belief revision, and then, section 4 investigates the relationship between the system RN and the general belief revision. Section 5 presents a specific system of default reasoning which satisfies all the inference rules of RN. The last section discusses the inference power of RN and concludes the paper.

2 Rational Nonmonotonic Frame

This section will define a nonmonotonic frame of inference through combining generalized rules of the five nonmonotonic relations of inference mentioned above into a system, named RN. Although each of the generalized rules has been suggested in the literature, their properties as a whole have not been investigated. We start with the syntax of RN and then discuss its properties and derived rules.

We shall restrict the language of the indented system within any propositional language $\mathcal{L}$ with the standard logical connectives $\neg$, $\vee$, $\wedge$ and $\rightarrow$. Elements of $\mathcal{L}$ are called formulas which are denoted by A,B,C. Sets of formulas are denoted by Γ, Δ, F and etc. There are two relations of inference between premises on the left and conclusions on the right: $\vdash$, denoting the classical propositional derivability, and $\mid\!\sim$, used for a nonmonotonic relation of inference. An associated Tarski-style's consequence operation may be defined by each of the relations of inference in such manner:

$Cn(\Gamma) = \{A : \Gamma \vdash A\}$
$C(\Gamma) = \{A : \Gamma \mid\!\sim A\}$.

It is presupposed that the inference relation $\vdash$ satisfies all the inference rules of the classical propositional logic so it is compact:

$\Gamma \vdash A$ iff there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash A$.

A set Γ of formulas is said to be *closed* if $\Gamma = Cn(\Gamma)$. $\Gamma \mid\!\sim (\vdash) \Delta$ indicates that $\Gamma \mid\!\sim (\vdash) A$ for all $A \in \Delta$ (Δ may be empty); $\Gamma \not\mid\!\sim A$ indicates that $\Gamma \mid\!\sim A$ does not hold.

Definition 2.1 *A system* $\mathbf{RN} = (\mathcal{L}, \mid\!\sim)$ *is said to be a rational nonmonotonic frame if* $\mathcal{L}$ *is a language of classical propositional logic at least including propositional connectives* $(\neg, \wedge, \vee$ *and* $\rightarrow)$ *and* $\mid\!\sim$ *is a relation from* $2^{\mathcal{L}}$ *to* $\mathcal{L}$, *called the rational nonmonotonic inference relation, if it satisfies:*

(RN1) *If* $\Gamma \vdash A$, *then* $\Gamma \mid\!\sim A$ *(Supraclassicality).*

(RN2) *If* $\Gamma \mid\!\sim \bot$, *then* $\Gamma \vdash \bot$ *(Consistency Preservation).*

(RN3) *If* $\Gamma \mid\!\sim \Delta \vdash A$, *then* $\Gamma \mid\!\sim A$ *(Closure or Weak Transitivity).*

(RN4) *If* $\Gamma \cup \Delta \mid\!\sim A$ *and* $\Delta \neq \phi$, *then there are* $A_1, \cdots, A_n \in \Delta$ *such that* $\Gamma \mid\!\sim (A_1 \wedge \cdots \wedge A_n) \rightarrow A$ *(Infinite Conditionalization).*

(RN5) *If* $\Gamma \not\mid\!\sim \neg(A_1 \wedge \cdots \wedge A_n)$ *for all* $A_1, \cdots, A_n \in \Delta$, *then* $\Gamma \mid\!\sim A$ *implies* $\Gamma \cup \Delta \mid\!\sim A$ *(Infinite Rational Monotonicity).*

Furthermore, a rational nonmonotonic frame is said to be finite supracompact if it satisfies:

(RN6) $\Gamma \mid\!\sim A$ *iff there exists a finite subset* Γ_0 *of* Γ *such that* $\Gamma_0 \cup \Gamma' \mid\!\sim A$ *for every finite subset* Γ' *of* $Cn(\Gamma)$ *(Finite Supracompactness).*

The name 'rational' follows from [Lehmann and Magidor 1992] [Herre 1994] but the rational inference relation here is stronger because the consistency preservation is added.

For those who are familiar with Tarski-style's nonmonotonic consequence operations, the following equivalent presentation of the conditions **(RN1)** – **(RN6)** would be preferential.

1. $Cn(\Gamma) \subseteq C(\Gamma)$(Supraclassicality).
2. If $Cn(\Gamma) \neq \mathcal{L}$, then $C(\Gamma) \neq \mathcal{L}$(Consistency Preservation).
3. $Cn(C(\Gamma)) \subseteq C(\Gamma)$(Closure).
4. $C(\Gamma \cup \Delta) \subseteq Cn(\Gamma \cup C(\Delta))$(Infinite Conditionalization).
5. IF $\Delta \cup C(\Gamma) \neq \mathcal{L}$, then $C(\Gamma) \subseteq C(\Gamma \cup \Delta)$(Rational Monotony).
6. $\Gamma \mid\!\sim A$ iff there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \cup \Delta \mid\!\sim A$ for every finite subset Δ of $Cn(\Gamma)$(Finite Supracompactness).

It should be noted that none of the above conditions is the authors' invention. They all have been suggested for nonmonotonic reasonings in the literature. In fact, the conditions 1-4 were presented in [Makinson 1993] and the last two conditions are found in [Herre 1994][1]

[1]In [Herre 1994] the finite supracompactness refers to that $\Gamma \mid\!\sim A$ *if* there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \cup \Delta \mid\!\sim A$ for every finite subset Δ of $Cn(\Gamma)$. However the complete Δ_2-compactness is just the meaning of the finite supracompatness in this paper.

In order to reveal the power of **RN**, we shall show that most of the inference rules for nonmonotonic reasoning suggested in the literature are derived rules of **RN**.

Lemma 2.2 *The following rules are derived rules of* **RN**:

(1). $\Gamma \mid\!\sim \Gamma$ *(Reflexivity)*

(2). *If* $\Gamma, A \mid\!\sim B, \neg B$, *then* $\Gamma \mid\!\sim \neg A$ *(Reductio ad Absurdum).*

(3). *If* $\Gamma, A \mid\!\sim B$, *then* $\Gamma \mid\!\sim A \rightarrow B$ *(Deduction Theorem).*

(4). *If* $\Gamma \mid\!\sim A \rightarrow B$ *and* $\Gamma \mid\!\sim A$, *then* $\Gamma \mid\!\sim B$ *(Modus Ponens).*

Proof: (1) follows **(RN1)**. (2) follows **(RN2)** and **(RN3)**. (3) is the special case of **(RN4)** where $\Delta = \{A\}$. For (4), since $\Gamma \mid\!\sim A \rightarrow B$ and $\Gamma \mid\!\sim A$, so $\Gamma \mid\!\sim \{A \rightarrow B, A\} \vdash B$. By **(RN3)** we get $\Gamma \mid\!\sim B$. □

The above theorem shows that $\mid\!\sim$ satisfies all the formal inference rules of classical propositional logic except for the following deductive transitivity:

If $\Gamma \mid\!\sim \Delta \mid\!\sim A (\Delta \neq \phi)$, then $\Gamma \mid\!\sim A$.

Lemma 2.3 *The following rules are derived rules of* **RN**:

(1). *If* $\Gamma \mid\!\sim \Delta$ *and* $\Gamma \cup \Delta \mid\!\sim A$, *then* $\Gamma \mid\!\sim A$ *(Cumulative Transitivity)*

(2). *If* $\Gamma \mid\!\sim \Delta$ *and* $\Gamma \mid\!\sim A$, *then* $\Gamma \cup \Delta \mid\!\sim A$ *(Cautious Monotony)*

(3). *If* $\Gamma \mid\!\sim \Delta$ *and* $\Delta \mid\!\sim \Gamma$, *then* $\Gamma \mid\!\sim A$ *if and only if* $\Delta \mid\!\sim A$ *(Reciprocity)*

(4). *If* $\Gamma \dashv\vdash \Delta$, *then* $\Gamma \mid\!\sim A$ *if and only if* $\Delta \mid\!\sim A$. *(Left Logical Equivalence)*

(5). *If* $A \dashv\vdash B$, *then* $\Gamma \mid\!\sim A$ *if and only if* $\Gamma \mid\!\sim B$. *(Right Logical Equivalence)*

Proof: For (1), suppose that $\Gamma \mid\!\sim \Delta (\Delta \neq \phi)$ and $\Gamma \cup \Delta \mid\!\sim A$. Then, by **(RN4)**, there exists $A_1, \cdots, A_n \in \Delta$ such that $\Gamma \mid\!\sim (A_1 \wedge \cdots \wedge A_n) \rightarrow A$. Since $A_1, \cdots, A_n \in \Delta$ implies $\Delta \vdash A_1 \wedge \cdots \wedge A_n$, hence we obtain $\Gamma \mid\!\sim A$ by **(RN3)** and Theorem 2.2 (4).

For (2), suppose that $\Gamma \mid\!\sim \Delta$ and $\Gamma \mid\!\sim A$. If there are $A_1, \cdots, A_n \in \Delta$ such that $\Gamma \mid\!\sim \neg (A_1 \wedge \cdots \wedge A_n)$, since $\Gamma \mid\!\sim \Delta$ implies $\Gamma \mid\!\sim A_1 \wedge \cdots \wedge A_n$ by **(RN3)**, then we have $\Gamma \mid\!\sim \bot$ again by **(RN3)**. It follows from **(RN2)** that $\Gamma \vdash \bot$. By the compactness of the classical propositional logic, $\Gamma \cup \Delta \vdash \bot$, so $\Gamma \cup \Delta \vdash A$. By the Supraclassicality, we have $\Gamma \cup \Delta \mid\!\sim A$. If $\Gamma \not\mid\!\sim \neg (A_1 \wedge \cdots A_n)$ for any $A_1, \cdots, A_n \in \Delta$, then by **(RN5)** and $\Gamma \mid\!\sim A$, we have $\Gamma \cup \Delta \mid\!\sim A$ as desired.

(3) follows from (1) and (2). (4) follows from **(RN1)** and (3). (5) follows from **(RN3)**. □

As shown by [Makinson 1993], the Infinite Conditionalization along with other rules implies the following Distributivity.

Lemma 2.4 *If* $\Gamma \cup \Delta_1 \mid\!\sim A$, $\Gamma \cup \Delta_2 \mid\!\sim A$, *then* $\Gamma \cup (\Delta_1 \bigvee \Delta_2) \mid\!\sim A$.
Specially, if $\Delta_1 \mid\!\sim A$ *and* $\Delta_2 \mid\!\sim A$, *then* $\Delta_1 \bigvee \Delta_2 \mid\!\sim A$ *(Distribution).*
where $\Delta_1 \bigvee \Delta_2 = \{A \vee B : A \in \Delta_1 \text{ and } \Delta_2\}$.

It is well-known that compactness is a very important property of the classical logic which provides a bridge between inferences of finite and infinite premises: $\Gamma \vdash A$ iff $\Gamma_0 \vdash A$ for some finite subset Γ_0 of Γ. But such equivalence implies monotony, so this kind of compactness must fail in any nonmonotonic logic. This does not mean that there are no properties of compactness for the nonmonotonic logic. In fact there are a number of alternative versions of compactness for nonmonotonic reasoning proposed([Freund 1990] [Makinson 1993] [Herre 1994]).

[Freund 1990] suggested the following *Supracompactness* for nonmonotonic inference:
$\Gamma \mid\!\sim A$ iff there exists a finite subset Γ_0 of Γ such that for any set of formulas Δ, $\Gamma \mid\!\sim \Delta$ implies $\Gamma_0 \cup \Delta \mid\!\sim A$.

The following theorem shows that such supracompactness follows from the finite supracompactness. This was also noted by [Freund 1990] and [Makinson 1993] with a little different setting.

Theorem 2.5 *Any finite supracompact rational inference relation satisfies the Supracompactness.*

Proof: It is enough to show that if $\Gamma \mid\!\sim A$ then there exists a finite subset Γ_0 of Γ such that $\Gamma \mid\!\sim \Delta$ implies $\Gamma_0 \cup \Delta \mid\!\sim A$. For this, let $\Gamma \mid\!\sim A$. By Finite Supracompactness, there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \cup \Gamma' \mid\!\sim A$ for every finite subset Γ' of $Cn(\Gamma)$. Suppose that $\Gamma \mid\!\sim \Delta$. Since $Cn((\Gamma_0 \cup \Delta) \bigvee \Gamma) \subseteq Cn(\Gamma)$, the finite supracompactness implies that $\Gamma_0 \cup \Gamma' \mid\!\sim A$ for every finite subset Γ' of $Cn((\Gamma_0 \cup \Delta) \bigvee \Gamma)$. Again by the finite supracompactness, we have $Cn((\Gamma_0 \cup \Delta) \bigvee \Gamma) \mid\!\sim A$ (noting that $\Gamma_0 \subseteq Cn((\Gamma_0 \cup \Delta) \bigvee \Gamma)$). It follows by the left logical equivalence that $(\Gamma_0 \cup \Delta) \bigvee \Gamma \mid\!\sim A$. On the other hand, by $\Gamma \mid\!\sim \Delta$ and the supraclassicality as well as Lemma 2.4, it is not difficult to verify that $(\Gamma_0 \cup \Delta) \bigvee \Gamma \mid\!\sim \Delta$. Thus by the cautious monotony we have $\Delta \cup ((\Gamma_0 \cup \Delta) \bigvee \Gamma) \mid\!\sim A$. Noting that $\Delta \cup ((\Gamma_0 \cup \Delta) \bigvee \Gamma) \dashv\vdash \Gamma_0 \cup \Delta$, we conclude from the left logical equivalence that $\Gamma_0 \cup \Delta \mid\!\sim A$ as desired. □

On the basis of Makinson's work on general patterns in nonmonotonic reasoning, it is not difficult to see that any rational nonmonotonic relation of inference also satisfies conditions such as Absorption, Cut, Cumulativity, Loop, Negation Rationality(see [Makinson 1989] and [Makinson 1993]).

3 Multiple Belief Revision

This section recalls definitions and results on the multiple belief revision. [Zhang 1995][Zhang 1996] introduced and further developed by [Zhang *et al.* 1997] a framework for multiple belief changes through extending the AGM theory([Gärdenfors 1988]). The extended revision function was called the general revision. Formally, a function $K\otimes^{2}: 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ with respect to a given belief set K is said to be a *general revision function over* K if it satisfies the following nine postulates:

$(K\otimes 1)$ $K\otimes F = Cn(K\otimes F)$.

$(K\otimes 2)$ $F \subseteq K\otimes F$.

$(K\otimes 3)$ $K\otimes F \subseteq K + F$.

$(K\otimes 4)$ If $K \cup F$ is consistent, then $K + F \subseteq K\otimes F$.

$(K\otimes 5)$ $K\otimes F$ is inconsistent iff F is inconsistent.

$(K\otimes 6)$ If $Cn(F_1) = Cn(F_2)$, then $K\otimes F_1 = K\otimes F_2$.

$(K\otimes 7)$ $K\otimes (F_1 \cup F_2) \subseteq K\otimes F_1 + F_2$.

$(K\otimes 8)$ If $F_2 \cup (K\otimes F_1)$ is consistent, then $(K\otimes F_1) + F_2 \subseteq K\otimes (F_1 \cup F_2)$.

$(K\otimes LP)$ $$K\otimes F = \bigcup_{\bar{F}\in \mathcal{C}_F} \bigcap_{\substack{\bar{F}\subseteq \bar{F}' \\ \bar{F}'\in \mathcal{C}_F}} K\otimes \bar{F}'$$

where $\mathcal{C}_F = \{\bar{F} : \bar{F} \subseteq Cn(F)$ and $\bar{F}$ is finite $\}$.

The postulates $(K\otimes 1)$-$(K\otimes 8)$ were presented in [Zhang 1996] and the last one, called *the Limit Postulate*, was introduced by [Zhang *et al.* 1997]. The representation theorem for all nine postulates was given in [Zhang *et al.* 1997] based on the following notions:

For any set Γ of formulas, let $\mathcal{P}$ be a partition [3] of Γ and $<$ a total-ordering(well-ordering) relation on $\mathcal{P}$. For any $p \in \mathcal{P}$, if $A \in p$, p is called the *rank* of A, denoted by $b(A)$.

The triple $\Sigma = (\Gamma, \mathcal{P}, <)$ is called a *nice-ordered partition*(NOP) (*perfect-ordered partition*(POP)) of Γ if it satisfies the following *Logical Constraint*:

If $A_1, \cdots, A_n \vdash B$, *then* $\sup\{b(A_1), \cdots, b(A_n)\} \geq b(B)$.

Now let K be a closed set of formulas and $\Sigma = (K, \mathcal{P}, <)$ a nice-ordered partition . A function $\otimes : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is said to be *the revision function generated by* Σ if for any $F \subseteq \mathcal{L}$,

i). if $F \cup K$ is consistent, then $K\otimes F = K + F$; otherwise,

ii). $B \in K\otimes F$ if and only if $B \in K + F$ and there exists $A \in K$ such that $F \vdash \neg A$ and

$$\forall C \in K(A \vdash C \wedge F \vdash \neg C \to (b(C \vee B) < b(C) \vee \vdash C \vee B))$$

The original presentation of the representation theorem is based on the contraction function(see [Zhang *et al.* 1997]). The following theorem is obtained by using the interrelation of revision and contraction.

Theorem 3.1 *For any closed set* K *of formulas, a revision function* $\otimes$ *satisfies* $(K\otimes 1) - (K\otimes 8)$ *as well as* $(K\otimes LP)$ *if and only if there exists a nice-ordered partition* $\Sigma = (K, \mathcal{P}, <)$ *such that* $\otimes$ *is the revision function generated by* Σ.

[2] In the present paper, $K\otimes$ is also written as $\otimes_K$ or $\otimes$ if without confusion.

[3] A partition of a set Γ is a disjoint family $\mathcal{P}$ of subsets of Γ such that $\Gamma = \bigcup\{p : p \in \mathcal{P}\}$.

4 Representation Theorem

In order to reveal the interrelation between **RN** and the multiple belief revision, we shall take revision operations as the semantic of **RN** rather than follow the traditional approach of Shoham's preferential models.

Theorem 4.1 *(Soundness)Let* $\mathcal{L}$ *be a language of propositional logic and* K *a consistent closed set in* $\mathcal{L}$. *Let* $\otimes$ *be a general revision function over* K. *Define a relation* $\mid\!\sim\, \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ *as follows: for any set* $\Gamma \subseteq \mathcal{L}$ *and any formula* $A \in \mathcal{L}$,

$$\Gamma \mid\!\sim A \text{ iff } A \in K\otimes \Gamma$$

then $(\mathcal{L}, \mid\!\sim)$ *is a finite supracompact rational nonmonotonic frame.*

Proof: We need to show $\mid\!\sim$ satisfies the rules **(RN1)**-**(RN6)**. For **(RN1)**, assume that $\Gamma \mid\!\sim A$. Since $\Gamma \subseteq K\otimes\Gamma$ and $K\otimes\Gamma$ is closed, thus $A \in K\otimes\Gamma$, that is, $\Gamma \mid\!\sim A$.

For **(RN2)**, assume that $\Gamma \mid\!\sim \bot$, i.e., $\bot \in K\otimes\Gamma$, which means that $K\otimes\Gamma$ is inconsistent. It follows by $(K\otimes 5)$ that Γ is inconsistent. Thus $\Gamma \vdash \bot$.

For **(RN3)**, assume that $\Gamma \mid\!\sim \Delta \vdash A$ which means that $\Delta \subseteq K\otimes\Gamma$ and $A \in Cn(\Delta)$. By $(K\otimes 1)$, we have $A \in K\otimes\Gamma$, i.e., $\Gamma \mid\!\sim A$.

For **(RN4)**, assume that $\Gamma \cup \Delta \mid\!\sim A$, that is, $A \in K\otimes(\Gamma\cup\Delta)$. By $(K\otimes 7)$, $K\otimes(\Gamma\cup\Delta) \subseteq K\otimes\Gamma + \Delta$, so we have $A \in K\otimes\Gamma + \Delta$. There exist then $A_1, \cdots, A_n \in \Delta$ such that $(A_1 \wedge \cdots \wedge A_n) \to A \in K\otimes\Gamma$, that is $\Gamma \mid\!\sim (A_1 \wedge \cdots \wedge A_n) \to A$.

For **(RN5)**, if for any $A_1 \wedge \cdots \wedge A_n \in \Delta$, $\Gamma \not\mid\!\sim \neg(A_1 \wedge \cdots \wedge A_n)$, or $\neg(A_1 \wedge \cdots \wedge A_n) \notin K\otimes\Gamma$, then $\Delta \cup (K\otimes\Gamma)$ is consistent. Therefore, when $\Gamma \mid\!\sim A$, or $A \in K\otimes\Gamma$, we conclude by $(K\otimes 8)$ that $A \in K\otimes(\Gamma\cup\Delta)$, so $\Gamma\cup\Delta \mid\!\sim A$ as desired.

For the finite supracompactness, suppose that $\otimes$ satisfies $(K\otimes LP)$, that is, $A \in K\otimes\Gamma$ iff there exists a finite subset Γ_1 of $Cn(\Gamma)$ such that for any finite subset $\Gamma_2 \subseteq Cn(\Gamma)$, $\Gamma_1 \subseteq \Gamma_2$ implies $A \in K\otimes\Gamma_2$. It is easy to see that we only need to show that $\Gamma \mid\!\sim A$ implies that there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \cup \Delta \mid\!\sim A$ for every finite subset Δ of $Cn(\Gamma)$. To this end, assume that $\Gamma \mid\!\sim A$, that is $A \in K\otimes\Gamma$. By $(K\otimes LP)$, there exists a finite subset Γ_1 of $Cn(\Gamma)$ such that for any finite subset Γ_2 of $Cn(\Gamma)$, if $\Gamma_1 \subseteq \Gamma_2$, then $A \in K\otimes\Gamma_2$. Let Γ_0 be a finite subset of Γ such that $\Gamma_1 \in Cn(\Gamma_0)$. For any finite subset Δ of $Cn(\Gamma)$, since $\Gamma_0 \cup \Gamma_1 \cup \Delta$ is finite and also a subset of $Cn(\Gamma)$, we have $A \in K\otimes(\Gamma_0 \cup \Gamma_1 \cup \Delta)$. It follows from $(K\otimes 6)$ that $A \in K\otimes Cn(\Gamma_0 \cup \Gamma_1 \cup \Delta)$. On the other hand, $\Gamma_1 \subseteq Cn(\Gamma_0)$ implies $Cn(\Gamma_0 \cup \Delta) = Cn(\Gamma_0 \cup \Gamma_1 \cup \Delta)$. Thus we obtain that $A \in K\otimes Cn(\Gamma_0\cup\Delta)$. It follows from $(K\otimes 6)$ again that $A \in K\otimes(\Gamma_0\cup\Delta)$, that is $\Gamma_0\cup\Delta \mid\!\sim A$. □

Theorem 4.2 *(Completeness) Let* $(\mathcal{L}, \mid\!\sim)$ *be a finite supracompact rational nonmonotonic frame. Let* $K = \{A \in \mathcal{L} : \phi \mid\!\sim A\}$. *Define a function* $\otimes_K : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ *as follows: for any* $F \subseteq \mathcal{L}$,

$$\otimes_K(F) = \{A \in \mathcal{L} : F \mid\!\sim A\}$$

Then $\otimes_K$ is a general belief revision function over K.

Proof: We first prove that K is closed and consistent. The consistency of K follows easily from **(RN2)**. To show that K is closed, let us assume that $K \vdash A$. There are then $A_1, \cdots, A_n \in K$ such that $A_1, \cdots, A_n \vdash A$. Hence $\phi \mid\!\sim A_1, \cdots, \phi \mid\!\sim A_n$, that is $\phi \mid\!\sim \{A_1, \cdots, A_n\}$. By **(RN3)**, we see $\phi \mid\!\sim A$ and then $A \in K$.

We now turn to show that $\otimes_K$ satisfies all nine postulates for the general belief revision.

Proof of $(K \otimes 1)$ is similar to that of closeness of K. $(K \otimes 2)$ follows immediately from the Reflexivity. $(K \otimes 3)$ and $(K \otimes 4)$ are special cases of $(K \otimes 7)$ and $(K \otimes 8)$, respectively. $(K \otimes 5)$ follows directly from **(RN2)**. $(K \otimes 6)$ follows from the Reciprocity.

For $(K \otimes 7)$, assume that $A \in \otimes_K(F_1 \cup F_2)$, or $F_1 \cup F_2 \mid\!\sim A$. Then by **(RN4)**, there are $A_1, \cdots, A_n \in F_2$ such that $F_1 \mid\!\sim (A_1 \wedge \cdots \wedge A_n) \rightarrow A$, that is, $(A_1 \wedge \cdots \wedge A_n) \rightarrow A \in \otimes_K(F_1)$. Consequently we have $A \in \otimes_K(F_1) + F_2$. Therefore, $\otimes_K(F_1 \cup F_2) \subseteq \otimes_K(F_1) + F_2$.

For $(K \otimes 8)$, assume that $F_2 \cup \otimes_K(F_1)$ is consistent, which means that for any $A_1, \cdots, A_n \in F, \neg(A_1 \wedge \cdots \wedge A_n) \notin \otimes_K(F_1)$, or $F_1 \not\mid\!\sim \neg(A_1 \wedge \cdots \wedge A_n)$. Now suppose $A \in \otimes_K(F_1) + F_2$, then there exist $B_1, \cdots, B_m \in F_2$ such that $(B_1 \wedge \cdots \wedge B_m) \rightarrow A \in \otimes_K(F_1)$, or $F_1 \mid\!\sim (B_1 \wedge \cdots B_m) \rightarrow A$. It follows from **(RN5)** that $F_1 \cup F_2 \mid\!\sim (B_1 \wedge \cdots \wedge B_m) \rightarrow A$. Since $B_1, \cdots, B_m \in F_2$ implies $F_1 \cup F_2 \mid\!\sim B_1 \wedge \cdots \wedge B_m$, we conclude, by Theorem 2.2(4), that $F_1 \cup F_2 \mid\!\sim A$, that is, $A \in \otimes_K(F_1 \cup F_2)$. Therefore we have proven that $\otimes_K(F_1) + F_2 \subseteq \otimes_K(F_1 \cup F_2)$ as desired.

The proof of the limit postulate is similar to that of the soundness. □

5 A Paradigm of Default Reasoning

Following the general considerations of the previous sections, we now look at a specific approach to nonmonotonic reasoning. We aim to seek a 'natural' system of nonmonotonic logic which satisfies all the inference rules for the rational nonmonotonic frame. On the basis of Makinson's 'satisfaction table' in [Makinson 1993], only Poole's system without constraints based on finite set of defaults in the systems of nonmonotonic logic considered in that paper satisfies all the inference rules of RN except the rational monotony. There is a disadvantage of Poole's approach, however, that it does not allow to represent priorities between defaults, which causes that the inference relations generated by Poole's system happen to collapse into the classical one when the default set is closed. [Nebel 1992] developed a system of default logic, called *ranked default theory* (RDT), which efficiently overcame this shortage. We here reformulate Nebel's system in a more general fashion.

Let (F, *D)* be a default theory, where *F* and *D* are both sets of propositions, interpreted as 'facts' and 'defaults', respectively. (F, *D)* is said to be a *perfect-ordered partitioned default theory* (POP DT) w.r.t. E if $\Sigma = (D, \mathcal{P}, <)$ is a perfect-ordered partition(see section 3). The order-type η of $\mathcal{P}$ is called the type of (F, D), denoted by η_D. The partition $\mathcal{P}$ is denoted as $\{D_\alpha : \alpha < \eta_D\}$.

A set E of propositions is a *syntax-based extension of* (F, D) if $E = Cn((\bigcup_{\alpha<\eta_D} R_\alpha) \cup F)$ such that for all $\alpha < \eta_D$,

$R_\alpha \subseteq D_\alpha$ and R_α is maximal (with respect to set-inclusion) among the subsets of D_α such that $(\bigcup_{\gamma\leq\alpha} R_\gamma) \cup F$ is consistent.

A proposition A is *strongly provable* in (F, D), denoted by $F \mid\!\sim_D A$, iff for every extension E of (F, D), $A \in E$.

It is easy to see that Poole' system without constraints is a limiting case of POP DT when $\mathcal{P} = \{D\}$ and Nebel' RDT is the special case when η_D is finite. Unfortunately, as pointed out by [Nebel 1992], the inference relation $\mid\!\sim_D$ generated by syntax-based extensions still fails to satisfy the rational monotony. [Zhang 1996] modified the definition of extensions into the following form:

a set E is a *syntax-independent extension of* (F, D) if $E = Cn((\bigcup_{\alpha<\eta_D} R_\alpha) \cup F)$ such that for all $\alpha < \eta_D$,

$R_\alpha \subseteq Cn(\bigcup_{\gamma\leq\alpha} D_\gamma)$ and R_α is maximal among the subsets of $Cn(\bigcup_{\gamma\leq\alpha} D_\gamma)$ such that $(\bigcup_{\gamma\leq\alpha} R_\gamma) \cup F$ is consistent.

This approach, though slightly complicated, can yet be regarded as 'natural'. The only difference between two types of extensions is that the former does not satisfies the principle of irrelevance of syntax but the latter does.

On the basis of the notion of syntax-independent extensions, we have the following result:

Theorem 5.1 *Let D be a set of formulas in a language $\mathcal{L}$. For any perfect-ordered partition Σ of D, $(\mathcal{L}, \mid\!\sim_D)$ is a finite supracompact rational nonmonotonic frame.*

The limited space does not afford a direct proof of the theorem. An indirect one may be done by using the result in [Zhang 1996] that $F \mid\!\sim_D A$ iff $A \in Cn(D) \otimes F$.

6 Discussions and Conclusions

We have established a very close connection between the general patterns of nonmonotonic reasoning and the multiple belief revision. This enables us to take the strategy to use methods from belief revision, set-theoretical, to contribute to a better understanding of nonmonotonic reasoning. We have seen that RN is such a strong system that almost all the rules suggested for nonmonotonic inference in the literature are the derived rules of RN. One may think that much more consequences would be derived in RN than in the classical logic from the same premises. This is clearly false when none of the pieces of background knowledge is available. Precisely specking, we have

Proposition 6.1 *Let $(\mathcal{L}, \mid\!\sim)$ be a rational nonmonotonic inference frame. If $K = \{B : \phi \mid\!\sim B\} = Cn(\phi)$, then*

$$\Gamma \mid\!\sim A \quad \text{iff} \quad \Gamma \vdash A.$$

Furthermore, even though we equip with the whole background knowledge, the upshot is still less optimistic.

Proposition 6.2 *For any propositional language $\mathcal{L}$, there is a rational nonmonotonic frame $(\mathcal{L}, \mid\!\sim)$ such that for any $\Gamma \subseteq \mathcal{L}$ and any formula $A \in \mathcal{L}$,*
i). if $K \cup \Gamma$ is consistent, then $\Gamma \mid\!\sim A$ iff $K \cup \Gamma \vdash A$;
ii). if $K \cup \Gamma$ is inconsistent, then $\Gamma \mid\!\sim A$ iff $\Gamma \vdash A$.
where $K = \{B : \phi \mid\!\sim B\}$.

This means that we can not always count on entailing more information from nonmonotonic inference rules alone than from classical ones. For example, even if we are told that $\phi \mid\!\sim p \rightarrow q$ and $\phi \mid\!\sim \neg p$, we still can not conduct the inference $p \mid\!\sim q$. There are two ways to surmount this obstacle. One is to construct some sort of ordering for the background knowledge such as nice-(perfect-)ordered partition, epistemic entrenchment or expectation ordering. The other is to transform the background knowledge into a conditional knowledge base as [Kraus *et al.* 1990] and [Lehmann and Magidor 1992] have already done. After all, the less we know, the less we can do.

References

[Boutilier 1994] Craig Boutilier, Unifying default reasoning and belief revision in a modal framework, *Artificial Intelligence,* 68(1994), 33-85.

[Brewka 1989] Gerhard Brewka, Preferred subtheories: an extended logical framework for default reasoning, in: *Proceedings IJCAI-89,* (Detroit, Mich., 1989) 1034-1048.

[Brewka 1991] Gerhard Brewka, Belief revision in a framework for default reasoning, in: A. Fuhrmann and M. Morreau eds. *The Logic of Theory Change,* (LNCS 465, Springer-Verlag, Berlin, Germany, 1991) 185-205.

[Cravo and Martins 1993] Maria R. Cravo and Joao P. Martins, A unified approach to default reasoning and belief revision, in: M. Filgueiras and L. Damas eds., *Progress in Artificial Intelligence,* (LNA1 727, Springer-Verlag, 1993), 226-241.

[Freund 1990] Michael Freund, Supracompact inference operations, in: J. Dix, K. P. Jantke and P. H. Schmitt, eds. *Non-monotonic and Inductive Logic,* (LNAI 543, Springer-Verlag, 1990), 59-73.

[Gardenfors 1988] P. Gardenfors, *Knowledge in Flux: Modeling the Dynamics of Epistemic States* (The MIT Press, 1988).

[Gardenfors and Makinson 1994] Peter Gardenfors and David Makinson, Nonmonotonic inference based on expectations, *Artificial Intelligence* 65(1994), 197-245.

[Gardenfors and Rott 1995] P. Gardenfors and H. Rott, Belief revision, in: D. M. Gabbay C. J. Hogger and J. A. Robinson eds., *Handbook of Logic in Artificial Intelligence and Logic Programming,* Clarendon Press, Oxford, 1995, 35-132.

[Herre 1994] Heinrich Herre, Compactness properties of nonmonotonic inference operations, in: C. MacNish, D. Pearce and L. M. Pereira eds., *Logics in Artificial Intelligence,* (LNAI 838, Springer-Verlag, 1994), 19-33.

[Kraus *et al.* 1990] Sarit Kraus, Daniel Lehmann and Menachem Magidor, Nonmonotonic reasoning, preferential models and cumulative logics, *Artificial Intelligence,* 44(1990), 167-207.

[Lehmann and Magidor 1992] Daniel Lehmann and Menachem Magidor, What does a conditional knowledge base entail?, *Artificial Intelligence,* 55(1992), 1-60.

[Li 1993] Wei Li, An open logic system, *Scientia Sinica* (Series A), March ,1993.

[Makinson 1989] David Makinson, General theory of cumulative inference, in: M. Reinfrank, J. de Kleer, M. L. Ginsberg and E. Sandewall, eds., *Non-monotonic Reasoning,* (LNAI 346, Springer-Verlag, 1989), 1-18.

[Makinson 1993] David Makinson, General patterns in nonmonotonic reasoning, in: D. Gabbay, ed., *Handbook of Logic in Artificial Intelligence and Logic Programming,* (Oxford University Press, 1993), 35-110.

[Makinson and Gardenfors 1991] David Makinson, Peter Gardenfors, Relations between the logic of theory change and nonmonotonic logic, in: A. Fuhrmann and M. Morreau eds. *The Logic of Theory Change,* (LNCS 465, Springer-Verlag, Berlin, Germany, 1991) 185-205.

[Nebel 1992] Bernhard Nebel, Syntax based approaches to belief revision, in: P. Gardenfors ed., *Belief Revision* (Cambridge University Press, Cambridge, 1992) 52-88.

[Poole 1988] David Poole, A logical framework for default reasoning, *Artificial Intelligence,* 36(1988), 27-47.

[Ryan 1991] Mark Ryan, Defaults and revision in structured theories, in: *1991 IEEE 6th Annual Symposium on Logic in Computer Science,* IEEE Computer Society Press, Los Alarnitos, California, 1991.

[Zhang 1995] Dongmo Zhang, A general framework for belief revision, in:Proc. *4th Int. Confi for Young Computer Scientists* (Peking University Press, 1995) 574-581.

[Zhang 1996] Zhang Dongmo, Belief revision by sets of sentences, *Journal of Computer Science and Technology,* 1996, 11(2), 1-19.

[Zhang *et al* 1997] Dongmo Zhang, Shifu Chen, Wujia Zhu, and Zhaoqian Chen, Representation theorems for multiple belief change, *this volume.*