Dyck paths with a first return decomposition constrained by height

Jean-Luc Baril, Sergey Kirgizov and Armen Petrossian
LE2I UMR-CNRS 6306, Université de Bourgogne
B.P. 47 870, 21078 DIJON-Cedex France
e-mail:{barjl,sergey.kirgizov,armen.petrossian}@u-bourgogne.fr

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Abstract

We study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. We exhibit new restricted sets of Dyck paths counted by the Motzkin numbers, and we give a constructive bijection between these objects and Motzkin paths. As a byproduct, we provide a generating function for the number of Motzkin paths of height k with a flat (resp. with no flats) at the maximal height.

Keywords: Enumeration, Dyck and Motzkin paths, first return decomposition, statistics, height, peak.

1 Introduction and notations

A *Dyck path* of semilength $n \geq 0$ is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{D}_n$, $n \geq 0$, be the set of all Dyck paths of semilength n, and let $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is given by the nth Catalan number, which is the general term $\frac{1}{n+1}\binom{2n}{n}$ of the sequence A000108 in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [17]. A large number of various classes of combinatorial objects are enumerated by the Catalan numbers such as planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on. A list of over 60 types of such combinatorial classes has been compiled by Stanley [19].

In combinatorics, many papers deal with Dyck paths. Most of them consist in enumerating Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x-axis (see for instance [2,8–16,18]). Other studies investigate restricted classes of Dyck paths avoiding some patterns or having a specific structure. For instance, Barcucci et al. [1] consider non-decreasing Dyck paths which are those having a non-decreasing sequence of heights of valleys (see also [6,7]), and it is well known [5] that Dyck paths avoiding the triple rise UUU are enumerated by the Motzkin numbers (see A001006 in [17]).

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [8] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in $\mathcal{D}$. See Figure 1 for an illustration of this decomposition.

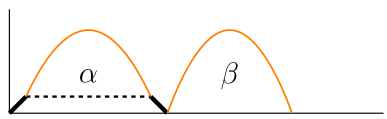

Figure 1: First return decomposition $U\alpha D\beta$ of a Dyck path $P \in \mathcal{D}$.

Based on this decomposition, we construct a new collection of subsets of $\mathcal{D}$ as follows. Given a function $s : \mathcal{D} \to \mathbb{N}$, called *statistic*, and a comparison operator $\diamond$ on $\mathbb{N}$ (for instance $\geq$ or $>$), the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases} \alpha, \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta). \end{cases} \tag{1}$$

For $n \geq 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$. Thus, we have $\mathcal{D}^{s,\diamond} = \bigcup_{n\geq 0} \mathcal{D}_n^{s,\diamond}$.

For instance, if the operator $\diamond$ is $=$ and s is a constant statistic (*i.e.*, $s(P) = 0$ for any $P \in \mathcal{D}$), then we obviously have $\mathcal{D}_n^{s,\diamond} = \mathcal{D}_n$ for $n \geq 0$.

If s is the number of returns (*i.e.*, $s(P)$ is the number of down steps D that return the path P to the x-axis) and $s(U\alpha D) \diamond s(\beta)$ is $s(U\alpha D) \geq s(\beta)$, then it is straightforward to see that $\mathcal{D}_n^{s,\geq}$ consists of Dyck paths built over the grammar $S \to \epsilon \mid USD \mid USDUSD$. So, the generating function $S(x)$ for the cardinalities of $\mathcal{D}_n^{s,\geq}$, $n \geq 0$, satisfies the functional equation $S(x) = 1 + xS(x) + x^2S(x)^2$. The solution of this equation is the well-known generating function for the Motzkin numbers (A001006 in [17]).

In this paper, we focus on the sets $\mathcal{D}^{h,\diamond}$ where the statistic h is the maximal height of a Dyck path, *i.e.*, $h(P)$ is the maximal ordinate reached by the path P.

In Section 2, we deal with the case in which operator $\diamond$ is a strict inequality $>$. We prove that the cardinalities of the sets $\mathcal{D}_n^{h,>}$, $n \geq 0$, are given by the sequence A045761 in [17]. This sequence corresponds to the coefficients of the series $\lim_{k\to\infty} P_k(x)$ where $P_k(x)$ is a polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_k(x) = P_{k-1}(x) + P_{k-2}(x)$ if k is even, and $P_k(x) = P_{k-1}(x) \cdot P_{k-2}(x)$ if k is odd.

In Section 3, we focus on the set $\mathcal{D}^{h,\geq}$ where $h(U\alpha D) \geq h(\beta)$ (the operator $\diamond$ is $\geq$). Using generating functions, we prove that the cardinalities of the $\mathcal{D}_n^{h,\geq}$, $n \geq 0$, are given by the Motzkin numbers (A001006 in [17]). Moreover, we give a constructive one-to-one correspondence ϕ between Dyck paths in $\mathcal{D}_n^{h,\geq}$ and Motzkin paths of length n. Also, we show how ϕ transforms peaks UD into peaks UD and flats F in Motzkin paths. Finally, we deduce generating function for the total number of peaks in $\mathcal{D}_n^{h,\geq}$.

Table 1 presents the two main enumerative results of $\mathcal{D}_n^{h,\diamond}$ obtained in Sections 2 and 3.

$\diamond$-constraint	Sequence	OEIS	$a_n, 1 \leq n \leq 9$
$h(U\alpha D) > h(\beta)$		A045761	$1, 1, 2, 3, 6, 12, 24, 50, 107$
$h(U\alpha D) \geq h(\beta)$	Motzkin	A001006	$1, 2, 4, 9, 21, 51, 127, 323, 835$

Table 1: Cardinalities of $\mathcal{D}_n^{h,\diamond}$ according to the $\diamond$-constraint.

2 Enumeration of $\mathcal{D}_n^{h,>}$

In this section, we enumerate the set $\mathcal{D}_n^{h,>}$ of Dyck paths of semilength $n \geq 0$ with a first return decomposition satisfying $h(U\alpha D) > h(\beta)$ where h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,>} = \{UD\}$, $\mathcal{D}_2^{h,>} = \{UUDD\}$, and $\mathcal{D}_3^{h,>} = \{UUDDUD, UUUDDD\}$ (see Figure 2).

Let $A_k(x) = \sum_{n\geq 0} a_{n,k} x^n$ (resp. $B_k(x) = \sum_{n\geq 0} b_{n,k} x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,>}$ with a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^{k} A_i(x)$ and the generating function $B(x)$ for the set $\mathcal{D}^{h,>}$

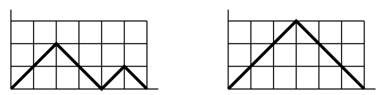

Figure 2: The two Dyck paths in $\mathcal{D}_3^{h,>}$.

is given by $B(x) = \lim_{k \to +\infty} B_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{h,>}$ is either empty, or of the form $U\alpha D\beta$ where α is a Dyck path in $\mathcal{D}^{h,>}$ of height $k-1$ and $\beta \in \mathcal{D}^{h,>}$ such that $h(\beta) \leq k-1$. So, we deduce easily the following functional equations:

$$\begin{cases} A_0(x) & = B_0(x) = 1, \\ A_k(x) & = xA_{k-1}(x) \cdot B_{k-1}(x) \text{ for } k \geq 1. \end{cases} \tag{2}$$

Theorem 1 *We have for $k \geq 0$,*

$$B_k(x) = \frac{P_{2k}(x)}{x}$$

where P_k is the polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_{2k}(x) = P_{2k-1}(x) + P_{2k-2}(x)$ and $P_{2k+1}(x) = P_{2k}(x) \cdot P_{2k-1}(x)$. As consequence, we have for $k \geq 1$

$$A_k(x) = \frac{P_{2k}(x) - P_{2k-2}(x)}{x},$$

and $B(x)$ is generating function of the sequence A045761 in [17].

Proof. We proceed by induction on k. For $k = 0$, the property holds since $P_0(x) = x = xB_0(x)$. Assuming the property for $0 \leq i \leq k-1$, we prove it for k. Taking into account that $A_k(x) = B_k(x) - B_{k-1}(x)$ in equation (2), we obtain

$$xB_k(x) \quad = xB_{k-1}(x) + x^2 B_{k-1}(x)\big(B_{k-1}(x) - B_{k-2}(x)\big).$$

With the recurrence hypothesis, we have

$$\begin{aligned} xB_k(x) &= P_{2k-2}(x) + P_{2k-2}(x)\big(P_{2k-2}(x) - P_{2k-4}(x)\big) \\ &= P_{2k-2}(x) + P_{2k-2}(x)P_{2k-3}(x) \\ &= P_{2k-2}(x) + P_{2k-1}(x) = P_{2k}(x). \end{aligned}$$

An induction on k completes the proof and the expression of $A_k(x)$ is deduced from $A_k(x) = B_k(x) - B_{k-1}(x)$. $\square$

For instance, we have $B_2(x) = 1 + x + x^2 + x^3$, $B_3(x) = 1 + x + x^2 + 2\,x^3 + 2\,x^4 + 2\,x^5 + 2\,x^6 + x^7$, and the first ten terms of $B(x)$ are $1 + x + x^2 + 2\,x^3 + 3\,x^4 + 6\,x^5 + 12\,x^6 + 24\,x^7 + 50\,x^8 + 107\,x^9$. We refer to Table 2 for an overview of the coefficients $a_{n,k}$ for $1 \le n \le 10$ and $1 \le k \le 9$.

Notice that the family of sets $\mathcal{D}^{h,>}$, $n \ge 1$, seems to be the first example of combinatorial objects enumerated by the sequence A045761 in [17].

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1	1									
2		1	1							
3			1	2	2	2	1			
4				1	3	5	8	11	13	15
5					1	4	9	18	33	56
6						1	5	14	33	71
7							1	6	20	54
8								1	7	27
9									1	$\dots$
$\sum$	1	1	2	3	6	12	24	50	107	$\dots$

Table 2: Number $a_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,>}$, $1 \le n \le 10$ and $1 \le k \le 9$.

3 Enumeration of $\mathcal{D}_n^{h,\ge}$

3.1 Enumeration using generating functions

In this section, we enumerate the set $\mathcal{D}_n^{h,\ge}$ of Dyck paths of semilength $n \ge 0$ with a first return decomposition satisfying $h(U\alpha D) \ge h(\beta)$ where $\alpha, \beta \in \mathcal{D}^{h,\ge}$ and h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,\ge} = \{UD\}$, $\mathcal{D}_2^{h,\ge} = \{UDUD, UUDD\}$, and $\mathcal{D}_3^{h,\ge}$ consists of the four Dyck paths $UDUDUD, UUDDUD, UUDUDD$ and $UUUDDD$ (see Figure 3).

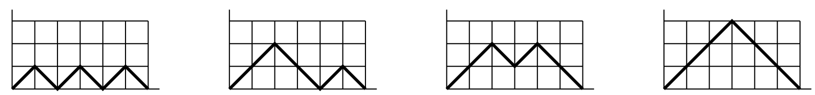

Figure 3: The four Dyck paths in $\mathcal{D}_3^{h,\geq}$.

Let $C_k(x) = \sum_{n\geq 0} c_{n,k}x^n$ (resp. $D_k(x) = \sum_{n\geq 0} d_{n,k}x^n$) be the generating function where the coefficient $c_{n,k}$ (resp. $d_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,\geq}$ with a maximal height equal to k (resp. of at most k). So, we have $D_k(x) = \sum\limits_{i=0}^{k} C_i(x)$ and the generating function $D(x)$ for the set $\mathcal{D}_n^{h,\geq}$ is given by $D(x) = \lim\limits_{k\to+\infty} D_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{h,\geq}$ is either empty, or of the form $U\alpha D\beta$ where α (resp. β) is a Dyck path in $\mathcal{D}^{h,\geq}$ of height $k-1$ (resp. of height at most k). So, we deduce the following functional equations:

$$\begin{cases} C_0(x) & = D_0(x) = 1, \\ C_k(x) & = xC_{k-1}(x) \cdot D_k(x) \text{ for } k \geq 1. \end{cases} \tag{3}$$

Theorem 2 provides recursive expressions for the two generating functions $D_k(x)$ and $C_k(x)$. As a consequence, $D(x)$ can be expressed as an infinite product of terms $\frac{1}{1-xC_k(x)}$.

Theorem 2 *We have $D_0(x) = C_0(x) = 1$, $C_1(x) = \frac{x}{1-x}$ and*

$$D_k(x) = \prod_{i=0}^{k-1} \big(1 - xC_i(x)\big)^{-1} \text{ for } k \geq 1,$$

$$C_k(x) = \frac{C_1(x)^k}{\prod_{i=1}^{k-1} \big(1 - xC_i(x)\big)^{k-i}} \text{ for } k \geq 2,$$

and

$$D(x) = \prod_{i=0}^{\infty} \big(1 - xC_i(x)\big)^{-1}.$$

Proof. Since we have $C_k(x) = D_k(x) - D_{k-1}(x)$, equation (3) implies

$$D_k(x) = \frac{D_{k-1}(x)}{1 - xC_{k-1}(x)},$$

and starting from $D_0(x) = 1$, a straightforward induction on k provides $D_k(x) = \prod_{i=0}^{k-1} \big(1 - xC_i(x)\big)^{-1}$. Moreover, from equation (3) and $C_1(x) = \frac{x}{1-x}$, we deduce

$$\frac{C_k(x)}{C_{k-1}(x)} = xD_k(x) = C_1(x) \prod_{i=1}^{k-1} \big(1 - xC_i(x)\big)^{-1}.$$

An induction on k completes the proof. $\square$

Lemma 1 *For $k \geq 1$, we have $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$.*

Proof. We proceed by induction on k. Since $D_0(x) = 1$ and $D_1(x) = \frac{1}{1-x}$, it is easy to check that $D_0(x) = \frac{D_1(x)-1}{x(xD_1(x)+1)}$.

Assuming $D_{i-1}(x) = \frac{D_i(x)-1}{x(xD_i(x)+1)}$ for $1 \leq i \leq k-2$, we prove the result for $i = k-1$. From equation (3) and the recurrence hypothesis on $D_{k-2}(x)$ we obtain

$$D_k(x) = \frac{D_{k-1}(x)}{1 - x\left(D_{k-1}(x) - \frac{D_{k-1}(x)-1}{x\left(xD_{k-1}(x)+1\right)}\right)}.$$

Isolating $D_{k-1}(x)$, we obtain $D_{k-1}(x) = \frac{D_k(x)-1}{x(xD_k(x)+1)}$ which completes the induction. $\square$

Theorem 3 *The sets $\mathcal{D}_n^{h,\geq}$, $n \geq 0$, are enumerated by the Motzkin numbers.*

Proof. Lemma 1 induces $x^2D_k(x)D_{k-1}(x) + xD_{k-1}(x) = D_k(x) - 1$ for $k \geq 1$. By taking the limits when k converges to infinity, one gets $x^2D(x)^2 + (x-1)D(x) + 1 = 0$, which is the functional equation of the generating function of Motzkin numbers. $\square$

Now, we will show how $D_k(x)$, $k \geq 0$, can be expressed as a closed form. For this, let us define the function

$$f : u \mapsto \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - x^2u}}}.$$

For $n \geq 1$, we denote by f^n the function recursively defined by $f^n(u) = f\big(f^{n-1}(u)\big)$ anchored with $f^0(u) = u$. A simple calculation (using Maple for instance) proves that the map f satisfies Remark 1.

Remark 1 *If* $X = \frac{Y-1}{x(xY+1)}$ *then the map* f *satisfies*

$$f\left(\frac{Y}{1-x(Y-X)}\right) = \frac{f(Y)}{1-x\big(f(Y)-f(X)\big)}.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		1	2	4	7	12	20	33	54	88
3			1	3	8	19	43	94	201	423
4				1	4	13	37	99	254	634
5					1	5	19	62	187	536
6						1	6	26	95	316
7							1	7	34	137
8								1	8	43
9									1	...
$\sum$	1	2	4	9	21	51	127	323	835	...

Table 3: Number $c_{n,k}$ of Dyck paths of height k in $\mathcal{D}_n^{h,\geq}$, $1 \leq n \leq 10$ and $1 \leq k \leq 9$.

Theorem 4 *For* $k \geq 0$*, we have*

$$D_k(x) = f^{\lfloor \frac{k}{4} \rfloor}\big(D_{k \bmod 4}(x)\big)$$

with the initial cases $D_0(x) = 1$*,* $D_1(x) = \frac{1}{1-x}$*,* $D_2(x) = \frac{1}{1-x-x^2}$ *and* $D_3(x) = \frac{1}{1-\frac{x}{1-x}}$*.* □

Proof. We proceed by induction on k. Since we have $D_0(x) = f^0\big(D_0(x)\big)$, $D_1(x) = f^0\big(D_1(x)\big)$, $D_2(x) = f^0\big(D_2(x)\big)$ and $D_3(x) = f^0\big(D_3(x)\big)$, the basic case holds.

Assuming the result for $0 \leq i \leq k$, we prove it for $k+1$.

From equation (3) we have $D_{k+1}(x) = \frac{D_k(x)}{1-x\big(D_k(x)-D_{k-1}(x)\big)}$. Using the recurrence hypothesis for $D_k(x)$ and $D_{k-1}(x)$, we obtain for $k \geq 4$:

$$
\begin{aligned}
D_{k+1}(x) &= \frac{f^{\lfloor \frac{k}{4} \rfloor}\left(D_{k \bmod 4}(x)\right)}{1-x\left(f^{\lfloor \frac{k}{4} \rfloor}\left(D_{k \bmod 4}(x)\right)-f^{\lfloor \frac{k-1}{4} \rfloor}\left(D_{k-1 \bmod 4}(x)\right)\right)} \\
&= \frac{f\left(f^{\lfloor \frac{k-4}{4} \rfloor}\left(D_{k-4 \bmod 4}(x)\right)\right)}{1-x\left(f\left(f^{\lfloor \frac{k-4}{4} \rfloor}(D_{k-4 \bmod 4}(x))\right)-f\left(f^{\lfloor \frac{k-5}{4} \rfloor}(D_{k-5 \bmod 4}(x))\right)\right)}.
\end{aligned}
$$

The recurrence hypothesis implies

$$
D_{k+1}(x) = \frac{f(D_{k-4}(x))}{1-x\left(f(D_{k-4}(x))-f(D_{k-5}(x))\right)},
$$

and using Remark 1, we deduce

$$
\begin{aligned}
D_{k+1}(x) &= f\left(\frac{D_{k-4}(x)}{1-x\left(D_{k-4}(x)-D_{k-5}(x)\right)}\right) = f(D_{k-3}(x)) \\
&= f^{1+\lfloor \frac{k-3}{4} \rfloor}(D_{k-3 \bmod 4}(x)) = f^{\lfloor \frac{k+1}{4} \rfloor}(D_{k+1 \bmod 4}(x)).
\end{aligned}
$$

The induction is completed. $\square$

For instance, we have $D_{13}(x) = \frac{1-6\,x+10\,x^2-9\,x^4+2\,x^5+x^6}{1-7\,x+15\,x^2-5\,x^3-15\,x^4+9\,x^5+3\,x^6-x^7}$ and the first terms of its Taylor expansion are $1+x+2\,x^2+4\,x^3+9\,x^4+21\,x^5+51\,x^6+127\,x^7+323\,x^8+835\,x^9+2188\,x^{10}+5798\,x^{11}+15511\,x^{12}+41835\,x^{13}+113633\,x^{14}+310557\,x^{15}+853333\,x^{16}$.

Notice that, by taking limits in Theorem 4, we retrieve the continued fraction of the Motzkin numbers (P. Barry [3])

$$
D(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x^2}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x^2}{1-\cfrac{x}{\cdots}}}}}}}}.
$$

3.2 A constructive bijection

In this section, we exhibit a constructive bijection between $\mathcal{D}_n^{h,\geq}$ and the set $\mathcal{M}_n$ of Motzkin paths of length n, *i.e.*, lattice paths starting at $(0,0)$, ending at $(n,0)$, never going below the x-axis, consisting of up-steps $U=(1,1)$, down steps $D=(1,-1)$ and flat steps $F=(1,0)$. We set $\mathcal{M}=\cup_{n\geq 0}\mathcal{M}_n$.

Let us define recursively the map ϕ from $\mathcal{D}^{h,\geq}$ to $\mathcal{M}$ as follows. For $P\in\mathcal{D}^{h,\geq}$, we set

$$\phi(P)=\begin{cases}\epsilon & \text{if } P=\epsilon,\\ \phi(\alpha)F & \text{if } P=\alpha UD,\\ \phi(\alpha)\phi(\gamma)U\phi(\beta)D & \text{if } P=\alpha UU\beta D\gamma D.\end{cases}$$

Due to the recursive definition, the image by ϕ of a Dyck path of semilength n is a Motzkin path of length n. For instance, the images of $UDUDUD$, $UUDDUD$, $UUDUDD$, $UUUDDD$, $UUUUDDDDUUUDDUDD$ are respectively FFF, UDF, FUD, UFD and $UUDDFUFD$. We refer to Figure 4 for an illustration of this mapping.

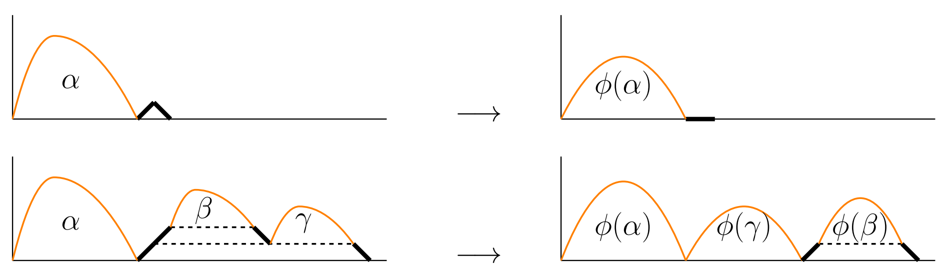

Figure 4: Illustration of the bijection between $\mathcal{D}_n^{h,\geq}$ and $\mathcal{M}_n$.

Moreover, we easily deduce the two following facts.

Fact 1 *If $\alpha,\beta\in\mathcal{D}_n^{h,\geq}$ and $\alpha\beta\in\mathcal{D}_n^{h,\geq}$, then we have $\phi(\alpha\beta)=\phi(\alpha)\phi(\beta)$.*

Fact 2 *We say that a peak UD is odd whenever the maximal sub-path of the form U^kD that contains it has an odd number of U-steps, i.e., k is odd. Then, ϕ transforms peaks (resp. odd peaks) of paths of $\mathcal{D}^{h,\geq}$ into peaks and flats (resp. flats) of Motzkin paths.*

Theorem 5 *The map $\phi:\ \mathcal{D}_n^{h,\geq}\to\mathcal{M}_n$ defined above is a bijection satisfying for any $P\in\mathcal{D}_n^{h,\geq}$,*

$$h(\phi(P))=\left\lfloor\frac{h(P)}{2}\right\rfloor.$$

Proof. We proceed by induction on n. Obviously, for $n = 1$, we have $\phi(UD) = F$ and $h(F) = 0 = \lfloor \frac{h(UD)}{2} \rfloor$. For $k \leq n$, we assume that ϕ is a bijection from $\mathcal{D}_k^{h,\geq}$ to $\mathcal{M}_k$ such that $h(\phi(P)) = \lfloor \frac{h(P)}{2} \rfloor$ for any $P \in \mathcal{D}_k^{h,\geq}$, and we prove the result for $n+1$.

Using the enumerative result of Theorem 3, it suffices to prove that ϕ is surjective from $\mathcal{D}_{n+1}^{h,\geq}$ to $\mathcal{M}_{n+1}$. So, let M be a Motzkin path in $\mathcal{M}_{n+1}$. We distinguish two cases: (i) $M = \sigma F$ with $\sigma \in \mathcal{M}_n$, and (ii) $M = \sigma U \pi D$ where σ and π are two Motzkin paths in $\mathcal{M}$.

(i) Using the recurrence hypothesis, there is $P \in \mathcal{D}_n^{h,\geq}$ such that $\sigma = \phi(P)$ and $h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor$. So, the Dyck path PUD belongs to $\mathcal{D}_{n+1}^{h,\geq}$ and $\phi(PUD) = \sigma F$ which proves that M belongs to the image by ϕ of $\mathcal{D}_{n+1}^{h,\geq}$. Moreover we have $h(\phi(PUD)) = h(\sigma F) = h(\sigma) = \lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(PUD)}{2} \rfloor$.

(ii) We suppose $M = \sigma U \pi D$. Let us define the longest suffix σ_s of σ (possibly empty) such that $\sigma_s \in \mathcal{M}$ and $h(\phi^{-1}(\sigma_s)) \leq 1 + h(\phi^{-1}(\pi))$ (σ_s exists since the empty path ϵ satisfies the inequality, and the recurrence hypothesis ensures the existence and the uniqueness of $\phi^{-1}(\sigma_s)$ and $\phi^{-1}(\pi)$). Let σ_r be the Motzkin path obtained from σ by deleting the suffix σ_s, and let $S \in \mathcal{D}_n^{h,\geq}$ (resp. $R \in \mathcal{D}_n^{h,\geq}$) such that $\phi(S) = \sigma_s$ and $h(\sigma_s) = \lfloor \frac{h(S)}{2} \rfloor$ (resp. $\phi(R) = \sigma_r$ and $h(\sigma_r) = \lfloor \frac{h(R)}{2} \rfloor$). Also there is $T \in \mathcal{D}_n^{h,\geq}$ such that $\phi(T) = \pi$ with $h(\pi) = \lfloor \frac{h(T)}{2} \rfloor$.

Due to the maximality of σ_s, σ_r is either empty or $1+h(T) < h(\phi^{-1}(\sigma_r \sigma_s))$. Using Fact 1 we obtain $h(\phi^{-1}(\sigma_r \sigma_s)) = h(RS) = h(R)$, and the last inequality can be written as $1 + h(T) < h(R)$.

- If $\sigma_r = \epsilon$ then the condition $h(S) \leq 1 + h(T)$ implies that $UUTDSD$ belongs to $\mathcal{D}_n^{h,\geq}$ and we have $\phi(UUTDSD) = \phi(S)U\phi(T)D = \sigma_s U \pi D = \sigma U \pi D$. Moreover, we have $h(\phi(UUTDSD)) = h(\phi(S)U\phi(T)D) = \max\{\phi(S), 1+\phi(T)\} = \max\{\lfloor \frac{h(S)}{2} \rfloor, 1+\lfloor \frac{h(T)}{2} \rfloor\}$. With $h(S) \leq 1+h(T)$, we deduce $h(\phi(UUTDSD)) = 1 + \lfloor \frac{h(T)}{2} \rfloor = \lfloor \frac{h(T)+2}{2} \rfloor = \lfloor \frac{h(UUTDSD)}{2} \rfloor$ as desired.

- If $\sigma_r \neq \epsilon$ then we have $h(S) \leq 1 + h(T) < h(R)$ which implies that the Dyck path $RUUTDSD$ belongs to $\mathcal{D}_{n+1}^{h,\geq}$. Moreover, we have $h(\phi(RUUTDSD)) = h(\phi(R)\phi(S)U\phi(T)D) = \max\{\phi(R), \phi(S), 1 + \phi(T)\} = \max\{\lfloor \frac{h(R)}{2} \rfloor, \lfloor \frac{h(S)}{2} \rfloor, 1 + \lfloor \frac{h(T)}{2} \rfloor\}$. From $h(S) \leq 1 + h(T) < h(R)$, we obtain $\lfloor \frac{h(S)+1}{2} \rfloor \leq \lfloor \frac{h(T)+2}{2} \rfloor \leq \lfloor \frac{h(R)}{2} \rfloor$ which induces that $h(\phi(RUUTDSD)) = \lfloor \frac{h(R)}{2} \rfloor = \lfloor \frac{h(RUUTDSD)}{2} \rfloor$ as desired.

The induction is completed. $\square$

Corollary 1 *Let P be a Dyck path in $\mathcal{D}_n^{h,\geq}$, $n \geq 1$. The Motzkin path $\phi(P)$ contains a flat step F at height $h(\phi(P))$ if and only if $h(P)$ is odd.*

Proof. We proceed by induction on n. For $n = 1$, we have $P = UD$ and $\phi(P) = F$ and the result holds since $h(P) = 1$ is odd. Assuming the result for $i \leq n$, we prove it for $n + 1$. We distinguish two cases: (i) $P = \alpha UD$ and (ii) $P = \alpha UU\beta D\gamma D$, where α, β and γ belong to $\mathcal{D}^{h,\geq}$.

(i) For $\alpha \neq \epsilon$, the recurrence hypothesis means that $\phi(\alpha)$ contains a flat step F at height $h(\phi(\alpha))$ if and only if $h(\alpha)$ is odd. Since we have $\phi(P) = \phi(\alpha)F$ and $h(P) = h(\alpha)$, $\phi(P)$ contains a flat step F a height $h(\phi(P)) = h(\phi(\alpha))$ if and only if $h(P)$ is odd.

(ii) If $P = \alpha UU\beta D\gamma D$ then we have $h(\alpha) \geq 2 + h(\beta) \geq 1 + h(\gamma)$ and $\phi(P) = \phi(\alpha)\phi(\gamma)U\phi(\beta)D$.

- As we have done in (i), a flat step F appears at height $h(\phi(P)) = h(\phi(\alpha))$ in $\phi(\alpha)$ if and only if $h(\alpha) = h(P)$ is odd.

- $\phi(P)$ contains a flat step at height $h(\phi(P))$ in $\phi(\beta)$ if and only if a flat step appears at height $h(\phi(\beta))$ in $\phi(\beta)$ and $h(\phi(P)) = h(\phi(\beta)) + 1$, *i.e.*, $\lfloor \frac{h(P)}{2} \rfloor = \lfloor \frac{h(\beta)}{2} \rfloor + 1$. Using the recurrence hypothesis, this is equivalent to $h(\beta)$ is odd, which means $2\lfloor \frac{h(P)}{2} \rfloor = h(\beta) + 1$. Since $h(P) \geq h(\beta) + 2$, this is equivalent to $h(P) = h(\beta) + 2$ and thus $h(P)$ is odd.

- Let us prove by contradiction that a flat step cannot occur at height $h(\phi(P))$ in $\phi(\gamma)$. Indeed, this should imply the following:

$$h(\phi(P)) = h(\phi(\gamma)) \geq 1 + h(\phi(\beta)).$$

  With Theorem 5 we obtain

$$\left\lfloor \frac{h(\gamma)}{2} \right\rfloor \geq 1 + \left\lfloor \frac{h(\beta)}{2} \right\rfloor.$$

  Due to the recurrence hypothesis, $h(\gamma)$ is odd, so

$$\left\lfloor \frac{1 + h(\gamma)}{2} \right\rfloor - 1 \geq \left\lfloor \frac{2 + h(\beta)}{2} \right\rfloor, \text{ and } \left\lfloor \frac{h(U\gamma D)}{2} \right\rfloor > \left\lfloor \frac{h(UU\beta DD)}{2} \right\rfloor,$$

  which implies

$$h(U\gamma D) > h(UU\beta DD).$$

  This last inequality contradicts $P \in \mathcal{D}^{h,\geq}$.

The induction is completed. $\square$

As a byproduct, we derive in Corollary 2 three generating functions for the number of Motzkin paths of height k according to a constraint on the maximal height of a flat. The two first results do not seem to appear in the literature, while the third one is presented in [4]. See Tables 4 and 5 for numerical data.

Corollary 2 *The generating function $\overline{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k and where there is a flat F at height k is*

$$\overline{M}_k(x) = D_{2k+1}(x) - D_{2k}(x).$$

The generating function $\widehat{M}_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k and where there is no flat F at height k is

$$\widehat{M}_k(x) = D_{2k}(x) - D_{2k-1}(x).$$

The generating function $M_k(x)$ where the coefficient of x^n is the number Motzkin paths in $\mathcal{M}_n$ of height exactly k is

$$M_k(x) = D_{2k+1}(x) - D_{2k-1}(x).$$

Proof. The proof is directly deduced from Theorem 5 and Corollary 1. $\square$

For instance, we have $\overline{M}_1(x) = \frac{x^3}{(1-2x)(1-x-x^2)}$, and the coefficient of x^n is given by $2^n - F(n+2)$ where $F(n)$ is the nth Fibonacci number. The first terms of its Taylor expansion are $x^3 + 3x^4 + 8x^5 + 19x^6 + 43x^7 + 94x^8 + 201x^9 + 423x^{10} + 880x^{11}$ (see sequence A008466 in [17]). Also, $\widehat{M}_1(x) = \frac{x^2}{(1-x-x^2)(1-x)}$ and the coefficient of x^n is $F(n) - 1$. The first terms of its Taylor expansion are $x^2 + 2x^3 + 4x^4 + 7x^5 + 12x^6 + 20x^7 + 33x^8 + 54x^9 + 88x^{10} + 143x^{11}$ (see sequence A000071 in [17]). Notice that the two sequences defined by $\sum\limits_{k\geq 0} \overline{M}_k(x)$ and $\sum\limits_{k\geq 0} \widehat{M}_k(x)$ do not appear in [17]. We leave open the question of finding closed forms for their generating functions.

Corollary 3 *The bivariate generating function $Q(x,y)$ where the coefficient of $x^n y^k$ is the number of Dyck paths of $\mathcal{D}_n^{h,\geq}$ containing exactly k peaks UD is*

$$\frac{1-x^2y+x^2-xy-\sqrt{x^4y^2-2x^4y+2x^3y^2+x^4-2x^3y+x^2y^2-2x^2y-2x^2-2xy+1}}{2x^2}.$$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1			1	3	8	19	43	94	201	423
2					1	5	19	62	187	536
3							1	7	34	137
4									1	9
$\sum$	1	1	2	4	10	25	64	164	424	1106

Table 4: Number of Motzkin paths of length n and height k with a flat at height k, $1 \leq n \leq 10$ and $0 \leq k \leq 4$.

$k\backslash n$	1	2	3	4	5	6	7	8	9	10
1		1	2	4	7	12	20	33	54	88
2				1	4	13	37	99	254	634
3						1	6	26	95	316
4								1	8	43
5										1
$\sum$	0	1	2	5	11	26	63	159	411	1082

Table 5: Number of Motzkin paths of length n and height k with no flats at height k, $1 \leq n \leq 10$ and $1 \leq k \leq 5$.

Proof. By Fact 2, the number of peaks UD in Dyck paths of $\mathcal{D}_n^{h,\geq}$ is equal to the number of peaks UD and flats F in Motzkin paths in $\mathcal{M}_n$. Since a non empty Motzkin path in $\mathcal{M}$ is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $Q(x, y)$ satisfies the functional equation $Q(x, y) = 1 + xyQ(x, y) + x^2yQ(x, y) + x^2(Q(x, y) - 1)Q(x, y)$. A simple calculation (with Maple for instance) completes the proof. $\square$

It is interesting to notice that $Q(x, y) - 1 = yQ'(x, y)$ where $Q'(x, y)$ is the bivariate generating function where the coefficient of $x^n y^k$ is the number of Motzkin n-paths with k weak valleys (see A110470 in [17]).

As a consequence, we deduce in Corollary 4 the popularity of peaks in $\mathcal{D}_n^{h,\geq}$.

Corollary 4 *The generating function where the coefficient of x^n is the total* *number of peaks in all Dyck paths of* $\mathcal{D}_n^{h,\geq}$ *is*

$$\frac{1 - x^2 - (1+x)\sqrt{1 - 2\,x - 3\,x^2}}{2x\sqrt{1 - 2\,x - 3\,x^2}}.$$

Proof. Using Corollary 3, the result is given by $\frac{\partial Q(x,y)}{\partial y}\rfloor_{y=1}$. $\square$

The popularity of peaks in $\mathcal{D}_n^{h,\geq}$, $n \geq 1$, is given by the sequence A025566 in [17], which is the first difference of the sequence A005773 that counts the directed animals of size n. The first terms are $x + 3\,x^2 + 8\,x^3 + 22\,x^4 + 61\,x^5 + 171\,x^6 + 483\,x^7 + 1373\,x^8 + 3923\,x^9 + 11257\,x^{10}$.

Corollary 5 *The bivariate generating function* $R(x, y)$ *where the coefficient of* $x^n y^k$ *is the number of Dyck paths of* $\mathcal{D}_n^{h,\geq}$ *containing exactly* k *odd peaks* UD *is*

$$\frac{1 - xy - \sqrt{1 - 2\,xy - 4\,x^2 + x^2y^2}}{2x^2}.$$

Proof. By Fact 2, the number of odd peaks on $\mathcal{D}_n^{h,\geq}$ is also the number of flats on $\mathcal{M}_n$. Since a non empty Motzkin path in $\mathcal{M}$ is either αF or $\alpha U \beta D$ where $\alpha, \beta \in \mathcal{M}$, the generating function $R(x, y)$ satisfies the functional equation $R(x, y) = 1 + xyR(x, y) + x^2R(x, y)^2$. A simple calculation (with Maple for instance) completes the proof. $\square$

Corollary 6 *The generating function where the coefficient of* x^n *is the total number of odd peaks in all Dyck paths of* $\mathcal{D}_n^{h,\geq}$ *is*

$$\frac{1 - x - \sqrt{1 - 2\,x - 3\,x^2}}{2x\sqrt{1 - 2\,x - 3\,x^2}}.$$

Proof. Using Corollary 5, the result is given by $\frac{\partial R(x,y)}{\partial y}\rfloor_{y=1}$. $\square$

The popularity of odd peaks is given by the sequence A005717 in [17], where the nth term is the nth Motzkin number divided by n. The first terms are $x + 2\,x^2 + 6\,x^3 + 16\,x^4 + 45\,x^5 + 126\,x^6 + 357\,x^7 + 1016\,x^8 + 2907\,x^9 + 8350\,x^{10}$.

4 Final remarks

In this paper, we study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. For future research, it would be interesting to investigate other statistics on Dyck paths such that number of peaks, valleys, zigzag or double rises, etc. More generally, can we extend this study for other combinatorial objects such as Motzkin and Ballot paths or permutations?

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References

[1] E. Barcucci, A. Del Lungo, S. Fezzi, and R. Pinzani. Nondecreasing Dyck paths and q-Fibonacci numbers. *Discrete Math.*, 170(1997), 1-3, 211-217.

[2] J.-L. Baril and A. Petrossian. Equivalence classes of Dyck paths modulo some statistics. *Discrete Math.*, 338(2015), 4, 655-660.

[3] P. Barry. On sequences with $\{-1, 0, 1\}$ Hankel transforms. *preprint*, 2012, available electronically at https://arxiv.org/abs/1205.2565.

[4] C. Brennan and A. Knopfmacher. The height of two types of generalised Motzkin paths. *J. of Statistical Planning and Inference*, 143 (2013), 2112-2120.

[5] D. Callan. Two bijections for Dyck path parameters. *preprint*, 2004, available electronically at http://www.arxiv.org/abs/math.CO/0406381.

[6] E. Czabarka, R. Florez and L. Junes. Some enumeration on non-decreasing Dyck paths. *Electronic Journal of Combinatorics*, 22(1)(2015).

[7] A. Denise and R. Simion. Two combinatorial statistics on Dyck paths. *Discrete Math.*, 137(1995), 1-3, 155-176.

[8] E. Deutsch. Dyck path enumeration. *Discrete Math.*, 204(1999), 167-202.

[9] T. Mansour. Statistics on Dyck paths. *J. Integer Sequences*, 9(2006), 06.1.5.

[10] T. Mansour. Counting peaks at height k in a Dyck path. *J. Integer Sequences*, 5(2002), 02.1.1.

[11] T. Mansour, M. Shattuck. Counting Dyck paths according to the maximum distance between peaks and valleys. *J. Integer Sequences*, 15(2012), 12.1.1.

[12] D. Merlini, R. Sprugnoli, M.C. Verri. Some statistics on Dyck paths. *J. Statis. Plann. Inference*, 101(2002), 211-227.

[13] A. Panayotopoulos, A. Sapounakis. On the prime decomposition of Dyck paths. *J. Combin. Math. Combin. Comput.*, 40(2002), 33-39.

[14] P. Peart, W.J. Woan. Dyck paths with no peaks at height k. *J. Integer Sequences*, 4(2001), 01.1.3.

[15] A. Sapounakis, I. Tasoulas, P. Tsikouras. Counting strings in Dyck paths. *Discrete Math.*, 307(23)(2007), 2909-2924.

[16] A. Sapounakis, I. Tasoulas, P. Tsikouras. Some strings in Dyck paths. *Australasian J. Combin.*, 39(2007), 4972.

[17] N.J.A. Sloane: The On-line Encyclopedia of Integer Sequences, available electronically at http://oeis.org.

[18] Y. Sun. The statistic "number of udu's" in Dyck paths. *Discrete Math.*, 287(2004), 177-186.

[19] R.P. Stanley. Enumerative Combinatorics, Vol. 2, Cambridge University Press, 1999.