A Motzkin filter in the Tamari lattice

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Abstract

The Tamari lattice of order n can be defined on the set $\mathcal{T}_n$ of binary trees endowed with the partial order relation induced by the well-known rotation transformation. In this paper, we restrict our attention to the subset $\mathcal{M}_n$ of Motzkin trees. This set appears as a filter of the Tamari lattice. We prove that its diameter is $2n-5$ and that its radius is $n-2$. Enumeration results are given for join and meet irreducible elements, minimal elements and coverings. The set $\mathcal{M}_n$ endowed with an order relation based on a restricted rotation is then isomorphic to a ranked join-semilattice recently defined in [2]. As a consequence, we deduce an upper bound for the rotation distance between two Motzkin trees in $\mathcal{T}_n$ which gives the exact value for some specific pairs of Motzkin trees.

Keywords: Lattice; ideal; filter; binary tree; Motzkin; Tamari; diameter.

1 Introduction

Interpreting associativity as a leftward substitution rule on parenthesizations leads to what is known as a Tamari lattice [9, 13, 14]. This partial order on a Catalan set first appeared in 1951 in Dov Tamari's thesis at the Sorbonne in Paris [27]. The Tamari order was originally defined as a partial order on parenthesizations, but it can also be understood as an order on binary trees endowed with the well-known rotation operation occurring among other in computer science. Quite a number of important papers have been published on the topic in many areas such as algebra, combinatorics, physics. However, among this plentiful literature, there are only a few studies related to specific subsets of the Tamari lattice. For instance, the paper [17] highlights a Boolean sublattice of the Tamari lattice. More recently, it has been proved that the subset of balanced binary trees is closed by interval in the Tamari lattice [10]. The subset of Motzkin words has also been studied whenever this subset is endowed with the Tamari partial order on parenthesizations [2].

In this paper, we tackle the problem by studying how the rotation transformation acts on the subset $\mathcal{M}_n$ of Motzkin trees of order n which are binary trees such that the internal nodes whose left subtree is a leaf also have a leaf as their right subtree. These trees are in bijection with Motzkin paths, which explains their name.

In Section 3, we show that $\mathcal{M}_n$ is a filter in the Tamari lattice $\mathcal{T}_n$ of binary trees of order n. We compute the diameter and the radius of $\mathcal{M}_n$. In Section 4, enumeration results are given for join and meet irreducible elements, minimal elements and coverings. In Section 5, we endow the set $\mathcal{M}_n$ with a partial order based on a restricted rotation transformation, and we prove that this poset is isomorphic to a ranked join-semilattice presented in a recent paper of the authors [2]. As a consequence, we deduce an upper bound for the rotation distance between two Motzkin trees in $\mathcal{T}_n$ which gives the exact value of the classical distance rotation for some specific pairs of Motzkin trees. This result suggests that $\mathcal{M}_n$ is better behaved than the Tamari lattice regarding the rotation distance and the diameter.

2 Definition and notations

The Tamari lattice $\mathcal{T}_n$ of order n is defined on the set of binary rooted ordered trees with n internal nodes and thus $n+1$ leaves (see [9, 13, 14, 28]). In this lattice, a tree T' covers a tree T when it is obtained from it by a left-rotation (see Figure 1).

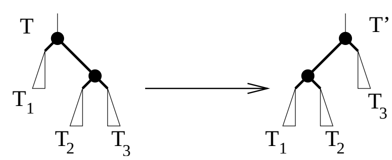

Figure 1: The left-rotation transformation on binary trees.

Now, we introduce the Polish notation of binary trees that will be convenient later for the proofs. An internal node of a binary ordered tree admits a left and a right subtree. The *prefix order* on a binary tree is defined recursively by visiting the root and then the left subtree and the right subtree. The *infix order* is defined recursively by visiting the left subtree, the root and the right subtree. The Polish (or linear) notation of T is obtained by reading T in prefix order and replacing each internal node (resp. each leaf) with $\bigcirc$ (resp. with $\Box$). The left-rotation transformation $\longrightarrow$ on a tree T can be viewed on the Polish notation of trees as the elementary transformation $\bigcirc T_1 \bigcirc T_2 T_3 \longrightarrow \bigcirc \bigcirc T_1 T_2 T_3$ where T_1, T_2 and T_3 are the Polish notations of three subtrees of T. For instance, $\bigcirc\bigcirc\Box\bigcirc\Box\Box\bigcirc\Box\Box$ is obtained from $\bigcirc\Box\bigcirc\bigcirc\Box\Box\bigcirc\Box\Box$ by a left-rotation. The inverse transformation $\longleftarrow$ will be called a right-rotation and the transitive closure of the left-rotation will be denoted $\stackrel{*}{\longrightarrow}$.

The rotation transformation has been widely studied using weight sequences of binary trees introduced in [15]. Some of our proofs consist in switching from one of the three representations to the other (tree, Polish notation and weight sequence). So we provide the definition of the *weight sequence* of $T \in \mathcal{T}_n$ (*w*-sequence for short).

Given $T \in \mathcal{T}_n$, the *weight* of T is the number of its leaves, *i.e.* $n+1$. The w-sequence of $T \in \mathcal{T}_n$ is $w_T = w_T(1)w_T(2)\dots w_T(n)w_T(n+1)$, where $w_T(i)$ is the weight of the largest subtree of T whose last leaf is the ith leaf of T in prefix order. For convenience, we do not use the last value $w_T(n+1)$ which is always equal to $n+1$. Two distinct trees cannot have the same w-sequence (see [15]).

Proposition 1 *(Theorem 1 in [15]) A necessary and sufficient condition for an integer sequence w of length n to be the w-sequence of a tree in $\mathcal{T}_n$ is $1 \leq w(i) \leq i$ for all $i \in [n]$, and if $j \in [i - w(i) + 1, i]$ then $i - w(i) \leq j - w(j)$.*

Proposition 2 *(Lemma 2 in [21]) Given $T \in \mathcal{T}_n$ with w-sequence $w_T = w_T(1)w_T(2)\dots w_T(n)$, then the tree obtained by performing a left-rotation on the kth internal node in infix order (if it possible) has the w-sequence $w_T(1)w_T(2)\dots w_T(k-1)(w_T(k) + w_T(k - w_T(k)))w_T(k+1)\dots w_T(n)$.*

For example, the left-rotation that transforms ○□ ○ ○□□ ○ □□ into ○□ ○ ○ ○ □□□□ corresponds to changing the w-sequence 1121 into the w-sequence 1123.

Theorem 1 *(Theorem 2 in [15]) Given T and T' in $\mathcal{T}_n$, we have:*

$$T \xrightarrow{*} T' \iff w_T(i) \leq w_{T'}(i) \text{ for all } i \in [n].$$

We define the rotation distance $d(T, T')$ between two binary trees $T, T' \in \mathcal{T}_n$ as the minimum number of left- and right-rotations needed to transform T into T' (see [12, 22]). Previous works on rotation distance have focused on approximation algorithms [1, 4, 18]. However, there remains today an open problem whether the rotation distance can be computed in polynomial time.

3 The Motzkin filter $\mathcal{M}_n$

Let $\mathcal{M}_n$ be the set of Motzkin trees with n internal nodes, *i.e* binary trees where all internal nodes that have a leaf □ as their left subtree also have a leaf □ as their right subtree. Equivalently, Motzkin trees are the ones whose Polish notation does not contain any occurrence of ○□○. It is well known that this set is enumerated by the nth term of the Motzkin sequence A001006 in [23] (see for instance [5, 26]). For example, $\mathcal{M}_4 = \{$○ ○ ○ ○ □□□□□, ○ ○ ○□ ○ □□□□, ○○○□□ ○ □□□, ○○ □□ ○○□□□$\}$. We refer to [8] and [24] for other combinatorial classes enumerated by the Motzkin numbers. See Figure 2 for an illustration of a Motzkin tree. For readability, binary trees undermentioned will be sometimes illustrated without leaves (see Figure 3).

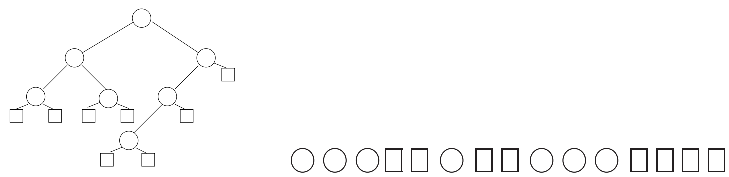

Figure 2: A Motzkin tree, with w-sequence 1214123, and its Polish notation.

Due to the definition of the left-rotation (see Figure 1), the set $\mathcal{M}_n$ is closed under the left-rotation transformation, *i.e.* any left-rotation on a Motzkin tree creates a tree that belongs to $\mathcal{M}_n$. Notice that this property means that $\mathcal{M}_n$ is a *filter* in $\mathcal{T}_n$ which is the notion dual to that of an *ideal* ([6, 11]). Hence the following proposition holds.

Proposition 3 *The poset $\mathcal{M}_n$ is a filter of $\mathcal{T}_n$ whose maximum element $\mathbf{1}$ has the w-sequence $w_{\mathbf{1}} = 123\dots n$. Thus $(\mathcal{M}_n, \xrightarrow{*})$ is a join-semilattice.*

Let $\delta(T, T')$ be the rotation distance in $\mathcal{M}_n$ between two Motzkin trees T and T', *i.e.* the minimum number of left- and right-rotations needed to transform T into T' by passing through Motzkin trees in $\mathcal{M}_n$. Obviously, we have $d(T, T') \leq \delta(T, T')$ where d is the classical rotation distance in $\mathcal{T}_n$. The *diameter* $\delta(\mathcal{M}_n)$ of $\mathcal{M}_n$ is the maximum distance in $\mathcal{M}_n$ among all pairs of Motzkin trees. The *radius* $rad(\mathcal{M}_n)$ of $\mathcal{M}_n$ is the minimum (on $T \in \mathcal{M}_n$) of the greatest distance of T from any other Motzkin tree (see [7]). The following theorems give the diameter and the radius of $\mathcal{M}_n$. Notice that computing the diameter $d(\mathcal{T}_n)$ of the Tamari lattice was for many years an open problem. Recently, Pournin has proved that $d(\mathcal{T}_n) = 2n - 6$ for $n > 10$ using only combinatorial arguments [20].

Lemma 1 *For all $n \geq 3$, the diameter of $\mathcal{M}_n$ satisfies*

$$\delta(\mathcal{M}_n) \leq 2n - 5.$$

Proof. The proof uses the same general idea as that of Lemma 2 in [22]. We build a path between two arbitrary objects by transforming them into a canonical object. The proof is also similar in the way this transformation is done, since we cluster the leaves of the trees to the right of the Polish notation, while in [22], they increase the incidence of a given vertex in two triangulations. We proceed by induction on $n \geq 3$. For $n = 3$, it is clear that $\delta(\mathcal{M}_3) = 1 \leq 2 \cdot 3 - 5$. Now let us assume that $\delta(\mathcal{M}_k) \leq 2k - 5$ for all $k < n$, and let us prove that $\delta(\mathcal{M}_n) \leq 2n - 5$. Let T be a Motzkin tree in $\mathcal{M}_n$ and $\ell(T) \leq n$ be the number of internal nodes before its first leaf in its Polish notation. If $\ell(T) < n$ then there necessarily exists a rotation on T that increases by one the number $\ell(T)$. Thus, we can produce the maximum tree $\mathbf{1}$ with $\ell(\mathbf{1}) = n$ by performing $n - \ell(T)$ rotations. Then, given any two trees T and T' we can convert T into T' in $2n - \ell(T) - \ell(T')$ rotations via $\mathbf{1}$. Thus, we have $\delta(T, T') \leq 2n - \ell(T) - \ell(T')$.

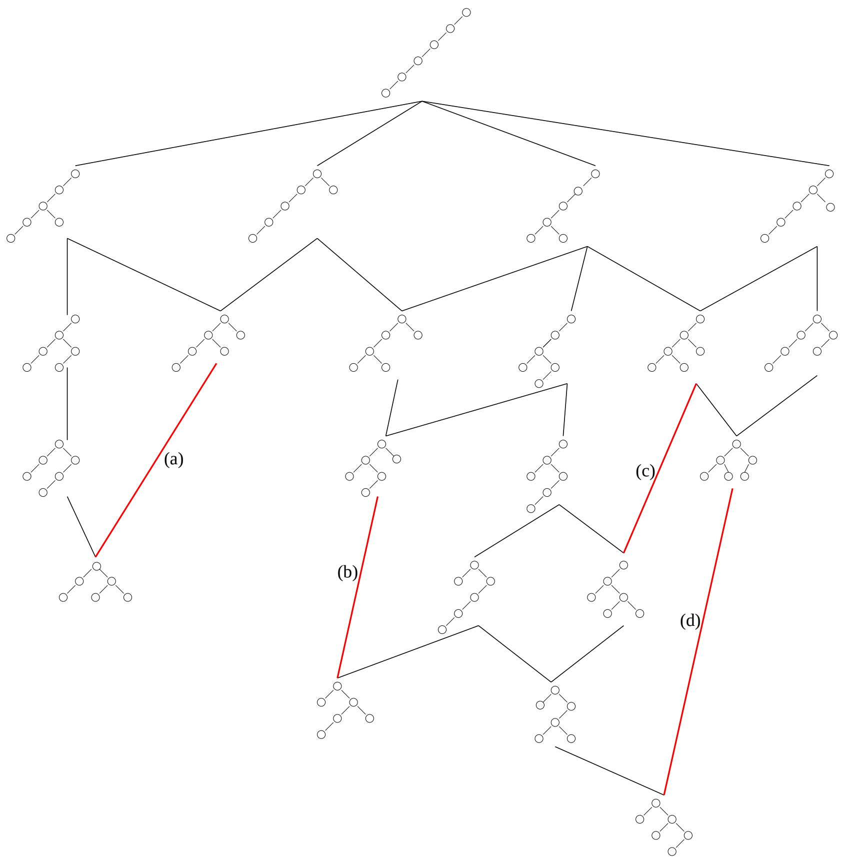

Figure 3: The Motzkin semilattice $\mathcal{M}_6$.

Due to the fact that T and T' belong to $\mathcal{M}_n$, we have $\ell(T) \geq 2$ and $\ell(T') \geq 2$. In the case where $\ell(T) = 2$ and $\ell(T') = 2$, T and T' have the same left subtree ◯□□. Therefore, the distance between T and T' is the distance between the two right subtrees of T and T' that lie in $\mathcal{M}_{n-2}$. The induction hypothesis implies $\delta(T, T') \leq 2(n-2) - 5 = 2n - 9 \leq 2n - 5$.

In the case where $\ell(T) > 2$ or $\ell(T') > 2$, we have $\ell(T) + \ell(T') \geq 5$ and we have $\delta(T, T') \leq 2n - 5$.

According to the two previous cases, we deduce $\delta(\mathcal{M}_n) \leq 2n - 5$ which completes the induction. □

We obtain the lower bounds using the same general argument and a similar construction as in [20]. In particular, we exhibit a pair of Motzkin trees with n internal nodes and show they are at distance $2n-5$ using a map ϕ (which works as the deletions from [20]) that removes an internal node from a Motzkin tree, and that removes the rotations involving this node from any path within the graph of the semilattice $\mathcal{M}_n$. Lemma 4 of this paper corresponds to Corollary 1 from [20].

Now let us define the transformation ϕ from $\mathcal{T}_n$ to $\mathcal{T}_{n-1}$ such that $\phi(T)$ is obtained from T by replacing the last internal node in infix order with its left subtree (its right subtree being necessarily a leaf). Notice that whenever the left subtree of the last node of T is a leaf, then $\phi(T)$ is obtained from T by replacing the last node with a leaf. For instance, if $T =$ ○ ○ □□ ○ ○□□ ○ □□ then $\phi(T) =$ ○ ○ □□ ○ ○□□□, and if $T' =$ ○ ○ □□ ○ ○□□ ○ ○□□□ then $\phi(T') =$ ○ ○ □□ ○ ○□□ ○ □□. In terms of w-sequences, we have $w_T = 12121$, $w_{\phi(T)} = 1212$ and $w_{T'} = 121212$, $w_{\phi(T')} = 12121$. See Figure 4 for an illustration of ϕ.

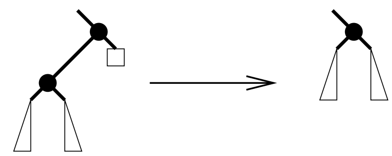

Figure 4: The transformation ϕ.

Lemma 2 *Let $T \in \mathcal{T}_n$ and $w(1)w(2)\dots w(n)$ be its w-sequence. Then, the w-sequence of $\phi(T)$ is $w(1)w(2)\dots w(n-1)$. Therefore, if $T \in \mathcal{M}_n$ then $\phi(T) \in \mathcal{M}_{n-1}$.*

Proof. For $1 \leq i \leq n-1$, the transformation ϕ does not modify the largest subtree of T whose last leaf is the ith leaf of T in prefix order. Thus, the w-sequence of $\phi(T)$ is $w(1)w(2)\dots w(n-1)$ and $\phi(T)$ belongs to $\mathcal{M}_{n-1}$. □

Lemma 3 *Let T and T' be two Motzkin trees in $\mathcal{M}_n$ such that T' is obtained from T by a rotation involving the last internal node of T in infix order. Then, we have $\phi(T) = \phi(T')$.*

Proof. Let us assume that T' is obtained from T by a left-rotation (resp. right-rotation) involving the last internal node of T in infix order. Then, the w-sequence of T' is obtained from that of T by increasing (resp. decreasing) the last value $w_T(n)$. With Lemma 2, we deduce that the w-sequences of $\phi(T)$ and $\phi(T')$ are the same, and thus $\phi(T) = \phi(T')$. □

Lemma 4 *Let $T = T_0, T_1, \dots, T_k = T'$ be a shortest path in $\mathcal{M}_n$ between T and T'. Let $p \geq 0$ be the number of (left or right) rotations involving the last internal node in infix order. Then, we have*

$$\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p.$$

Proof. According to Lemma 2, $\phi(T) = \phi(T_0), \phi(T_1), \dots, \phi(T_k) = \phi(T')$ is a path in $\mathcal{M}_{n-1}$ between $\phi(T)$ and $\phi(T')$, provided one removes duplicates from this sequence. Two consecutive trees in the sequence are then indeed related by a rotation, which follows from Lemma 2 and from Proposition 2. With Lemma 3, there are p pairs (T_i, T_{i+1}) such that $\phi(T_i) = \phi(T_{i+1})$. Thus, the length of the previous path between $\phi(T)$ and $\phi(T')$ is $\delta(T, T') - p$, which implies $\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p$. □

Theorem 2 *For $n \geq 3$, we have $\delta(\mathcal{M}_n) = 2n - 5$.*

Proof. Considering Lemma 1, it suffices to exhibit a family of pairs of Motzkin trees $T, T' \in \mathcal{M}_n$, $n \geq 3$, satisfying $\delta(T, T') = 2n - 5$. For n even, $n \geq 4$, we define T and T' by their weight sequences $w_T = 121212\ldots12$ and $w_{T'} = 1231212\ldots121$. For n odd, $n \geq 3$, T and T' are defined by $w_T = 121212\ldots121$ and $w_{T'} = 1231212\ldots12$.

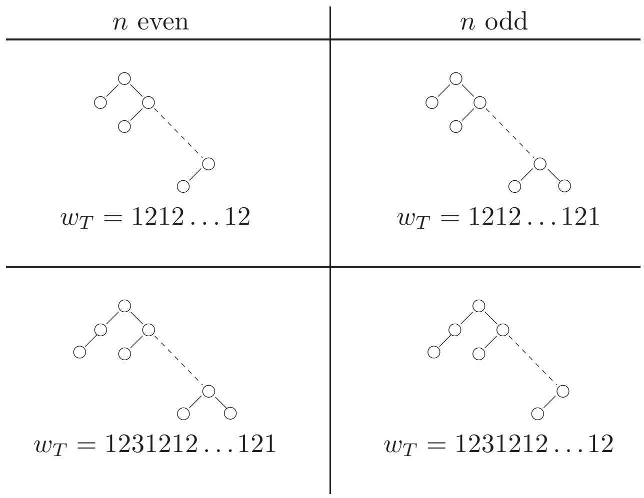

Figure 5: The Motzkin trees T and T' in the proof of Theorem 2

We proceed by induction on $n \geq 3$. It is straightforward to verify that $\delta(121, 123) = 1 = 2 \cdot 3 - 5$ and $\delta(1212, 1231) = 3$. Therefore the cases $n = 3$ and $n = 4$ hold. Let us assume that $\delta(T, T') = 2k - 5$ for all k, $3 \leq k < n$, and let us prove that $\delta(T, T') = 2n - 5$ whenever T and T' belong to $\mathcal{M}_n$.

Exchanging T and T' according to the parity of n (if needed), we assume that $w_T = \ldots12$ and $w_{T'} = \ldots121$. Let $T = T_0, T_1, \ldots, T_k = T'$ be a shortest path in $\mathcal{M}_n$ between T and T'. Let p be the number of rotations in this path that involve the last internal node in infix order. Lemma 4 induces that $\delta(\phi(T), \phi(T')) \leq \delta(T, T') - p$. By Lemma 2, we use the induction hypothesis and we deduce $\delta(T, T') \geq 2(n - 1) - 5 + p = 2n - 7 + p$. Now, let us prove that $p \geq 2$. Indeed, a path in $\mathcal{M}_n$ between $w_T = \ldots12$ and $w_{T'} = \ldots121$ necessarily moves the last value $w_T(n) = 2$ of w_T. We distinguish two cases: (i) the first rotation r involving the last node of a tree in the path increases the value $w_T(n)$, and (ii) the first rotation r involving the last node of a tree in the path moves the last value $w_T(n) = 2$ into one. We will prove that case (i) is the only possibility.

In the case (i), it is clear that we need at least one more rotation in order to decrease to one the last value. Thus, we necessarily have $p \geq 2$.

In the case (ii), whenever we decrease $w_T(n) = 2$ to one, it is necessary to have $w_T(n-1) \neq 1$ (otherwise the obtained tree would not be a Motzkin tree). Thus, the path contains a rotation before r that moves the value $w_T(n-1) = 1$. However the only possibility to move it, is that $w_T(n) \neq 2$ (see the characterization of a w-sequence in Proposition 1), which means that $w_T(n) = 2$ must be changed before. This case does not occur since r was the first rotation moving the last value.

Hence, we have $p \geq 2$ and we deduce $\delta(T, T') \geq 2(n-1) - 5 + p \geq 2n - 5$ which completes the induction. □

Theorem 3 *For $n \geq 3$, we have $rad(\mathcal{M}_n) = n - 2$.*

Proof. It is clear that (see for instance [7]), we have the inequality

$$rad(\mathcal{M}_n) \leq \delta(\mathcal{M}_n) \leq 2 \cdot rad(\mathcal{M}_n).$$

Using Theorem 2, we deduce that the radius of $\mathcal{M}_n$ is at least $n-2$. On the other hand, we consider the Motzkin tree defined in the proof of Theorem 2 with the weight sequence $w_T = 121212\ldots$ Since the distance between a Motzkin tree and the tree $\mathbf{1} = 123\ldots n$ is the Hamming distance of their w-sequences (see Section 3 in [16] for instance), we have $\delta(1212\ldots, 1234\ldots n) = n-2$ and $n-2$ is the maximum distance between the tree $\mathbf{1} = 1234\ldots n$ and any Motzkin tree. Thus, the radius of $\mathcal{M}_n$ is at most $n-2$. □

4 Enumeration results for $(\mathcal{M}_n, \xrightarrow{*})$

In this part, we present several enumeration results for some specific elements of the semilattice $(\mathcal{M}_n, \xrightarrow{*})$. Given $T \in \mathcal{M}_n$, we denote T_L (resp. T_R) its left (resp. right) subtree, *i.e.* $T = \bigcirc T_L T_R$.

Proposition 4 *The generating function for the number of minimal elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{1 + x - x^3}{1 - x^2 - x^3}.$$

For $0 \leq n \leq 12$, the first values are $1, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16$ (see Padovan sequence A000931 in [23]).

Proof. A minimal element T in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree where any right-rotation creates a tree that does not belong in $\mathcal{M}_n$. Given $T = \bigcirc T_L T_R \in \mathcal{M}_n$, then T_L and T_R are necessarily minimal elements. Moreover, the right-rotation involving the root of T necessarily creates a tree that does not lie in $\mathcal{M}_n$. This means that the right subtree of T_L is necessarily a leaf. By induction, T_L does not contain any right subtree not reduced to a leaf. In the case where T_L contains at least three internal nodes, then the right-rotation involving its root creates a Motzkin tree. Therefore, the only two possibilities are either $T_L = \bigcirc\square\square$ or $T_L = \bigcirc\bigcirc\square\square\square$.

Let $A(x)$ be the generating function for the number of minimal elements in $\mathcal{M}_n$ for $n \geq 0$. Then, we have the functional equation $A(x) = x^2A(x) + x^3(A(x)-1)+1+x$ which gives $A(x) = \frac{1+x-x^3}{1-x^2-x^3}$. □

Recall that $T \in \mathcal{M}_n$ is a join (resp. meet) irreducible element if $T = T_1 \vee T_2$ (resp. $T = T_1 \wedge T_2$) implies $T = T_1$ or $T = T_2$. Since the set $\mathcal{M}_n$ is finite, join (resp. meet) irreducible elements are elements that have a unique lower (resp. upper) cover.

Proposition 5 *For $n \geq 1$, the meet irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ are enumerated by the triangular numbers $\frac{(n-2)(n-1)}{2}$.*

For $1 \leq n \leq 12$, the first values are $0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55$ (see A000217 in [23]).

Proof. A meet irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree T where only a single left-rotation can be performed. Thus, only one internal node can be a right child which means that the weight sequence of T is necessarily of the form $w_T = 12\ldots\ell 12\ldots k(k+\ell+1)\ldots n$ where $k \geq 1$ is the number of internal nodes of the unique subtree T' of T whose root is a right child, and $\ell \geq 2$ is the number of internal nodes of the left subtree of the unique node having an internal node as right child. Finally, the number of such trees is given by $\sum_{k=1}^{n-2}\sum_{\ell=2}^{n-k} 1 = \frac{(n-2)(n-1)}{2}$. □

Proposition 6 *The generating function for the number of join-irreducible elements in $(\mathcal{M}_n, \xrightarrow{*})$ is given by*

$$\frac{x^3(1+x+2x^2+3x^3+3x^4+x^5)}{(1-x^2-x^3)^3}.$$

For $1 \leq n \leq 12$, the first values are $0, 0, 1, 1, 5, 9, 18, 34, 58, 100, 164, 265$.

Proof. A join-irreducible element in $(\mathcal{M}_n, \xrightarrow{*})$ is a Motzkin tree T on which only one right-rotation is possible. Let $B(x)$ be the generating function for the number of join irreducible elements in $\mathcal{M}_n$. The Polish notation of the only one join irreducible in $\mathcal{M}_3$ is $\bigcirc\bigcirc\bigcirc\square\square\square\square$. Now we assume $n \geq 4$. Given $T = \bigcirc T_L T_R$ and $A(x)$ be the generating function for the number of minimal Motzkin trees (see Proposition 4).

Case 1: if T_R is a leaf then the right-rotation involving the root provides a Motzkin tree, which implies that T_L is necessarily a minimal element of weight at least four. Thus, the corresponding generating function is $x(A(x)-1-x-x^2)$.

Case 2: if T_R is a minimal Motzkin tree of weight at least two, then T_L is either (i) a minimal tree of weight at least four, or (ii) a join irreducible element whose right subtree is a leaf. Indeed, in sub-case (i), the unique possible right-rotation is the one at the root of T. The generating function for the case (i) is $x(A(x)-1)(A(x)-1-x-x^2)$. For the case (ii), the unique right-rotation is the one that can be performed in T_L. So, a join irreducible element with a leaf as right subtree has necessarily a minimal left subtree. Thus, the number of Motzkin trees satisfying (ii) is given by the generating function $x^2(A(x)-1)(A(x)-1-x)$.

Case 3: if T_R is a join irreducible Motzkin tree of weight at least two, then T_L is either $\bigcirc\Box\Box$ or $\bigcirc\bigcirc\Box\Box\Box$. Indeed, if T_L is minimal then the right subtree of T_L must be a leaf (otherwise the right-rotation at the root of T would transform T into a Motzkin tree). So, the corresponding generating function is $x^2B(x)+x^3B(x)$. Finally we have the following functional equation that gives the result:

$$\begin{aligned} B(x) = \quad & x^3 + x(A(x)-1-x-x^2) + x(A(x)-1)(A(x)-1-x-x^2) + \\ & x^2(A(x)-1)(A(x)-1-x) + x^2B(x) + x^3B(x), \end{aligned}$$

where $A(x)$ is given in Proposition 4. $\square$

Proposition 7 *The generating function for the number of coverings in* $(\mathcal{M}_n, \stackrel{*}{\longrightarrow})$ *is given by*

$$\frac{(1-x)(1-2x-x^2-(1-x)\sqrt{1-2x-3x^2})}{2x(1-2x-x^2)}.$$

For $1 \leq n \leq 12$, the first values are $0, 0, 1, 3, 10, 30, 88, 252, 712, 1992, 5537, 15323$.
Proof. Let c_n be the number of coverings in $\mathcal{M}_n$, $C(x)$ be the associated generating function and $M(x) = \sum_{i\geq 0} M_i x^i$ be the generating function for the number of Motzkin trees.

Since we have $c_0 = c_1 = c_2 = 0$, we assume $n \geq 3$. Given $T = \bigcirc T_L T_R$, we distinguish three cases.

Case 1: T_R is a leaf. Thus, there are c_{n-1} possible left-rotations in T_L. The corresponding generating function is $xC(x)$.

Case 2: if $T_R = \bigcirc\Box\Box$. There are M_{n-2} left-rotations involving the root of T and c_{n-2} possible rotations in T_L. The corresponding generating function is $x^2M(x) + x^2C(x)$.

Case 3: if the weight of T_R is at least three. Let i, $1 \leq i \leq n-3$, be the number of internal nodes of T_L. There are $c_i + c_{n-i-1}$ left-rotations in the two subtrees T_L and T_R, and M_iM_{n-i-1} left-rotations involving the root of T. Varying i from 1 to $n-3$, there are $\sum_{i=1}^{n-3}(c_i + c_{n-i-1}) + \sum_{i=1}^{n-3} M_iM_{n-i-1}$ possible left-rotations. Since $\sum_{i=1}^{n-3} M_iM_{n-i-1} = M_n - M_{n-1} - M_{n-2}$ (see [25] for instance), the corresponding generating function is $\frac{x^2C(x)}{1-x} + \frac{x^3C(x)}{1-x} + M(x) - xM(x) - x^2M(x) - x$.

Considering all cases, we have the following functional equation and the result is deduced:

$$C(x) = xC(x) + x^2C(x) + x^2\frac{C(x)}{1-x} + x^3\frac{C(x)}{1-x} + M(x) - xM(x) - x,$$

where $M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$. $\square$

5 A ranked Motzkin poset included in $(\mathcal{M}_n, \xrightarrow{*})$

Let $\mathcal{W}$ be the set of Motzkin words, *i.e.* the language over $\{(,)\}$ defined by the grammar $S \rightsquigarrow \lambda|(SS)$, and $\mathcal{W}_n$ be the set of Motzkin words of length $2n$ (with n open and n close parentheses). From a Motzkin word in $\mathcal{W}_n$ we can associate a binary tree in $\mathcal{T}_n$ where its Polish notation is obtained by replacing each open (resp. close) parenthesis with $\bigcirc$ (resp. $\square$), and by adding $\square$ at the end. For instance, the Polish notation of the associated tree of the Motzkin word $(()(()))$ is $\bigcirc\bigcirc\square\bigcirc\bigcirc\square\square\square\square$.

Let $\mathcal{MW}_n$ be the set of binary trees in $\mathcal{T}_n$ associated to the Motzkin words belonging to $\mathcal{W}_n$. Since a Motzkin word is obtained from the rule $S \rightsquigarrow \lambda|(SS)$, the Polish notation of its associated tree is either of the form (i) $\bigcirc T_L\square$ or (ii) $T = \bigcirc T_L T_R$ where T_L and T_R lie in some sets $\mathcal{MW}_k$ for $k < n$, and such that T_R satisfies (i). Actually, the set $\mathcal{MW}_n$ consists of the mirrors of binary trees whose Polish notation has no three consecutive internal nodes.

In [2], we investigate the rotation transformation $\rightarrow$ on the set $\mathcal{MW}_n$. We have proved that $(\mathcal{MW}_n, \xrightarrow{*})$ is a ranked join-semilattice.

In this part, we construct an isomorphism between $(\mathcal{MW}_{n-1}, \xrightarrow{*})$ and $(\mathcal{M}_n, \overset{*}{\rightarrowtail})$ where $\rightarrowtail$ is the restricted left-rotation defined by

$$\bigcirc T_1 \bigcirc T_2 \square \rightarrowtail \bigcirc\bigcirc T_1 T_2 \square$$

where T_1, T_2 are the Polish notations of some subtrees. Notice that in [4], the authors study on binary trees an analogous restricted rotation defined by $\bigcirc\square\bigcirc T_2 T_3 \rightarrowtail \bigcirc\bigcirc\square T_2 T_3$.

Let ψ be the map from $\mathcal{MW}_{n-1}$ to $\mathcal{T}_n$ defined by the following recursive rule. For $T = \bigcirc T_L T_R \in \mathcal{MW}_{n-1}$, we define

$$\psi(T) = \chi(\bigcirc T\square),$$

where $\chi(\bigcirc T\square)$ is recursively defined by

$$\chi(\bigcirc\bigcirc T_L T_R\square) = \bigcirc\chi(\bigcirc T_L\square)\chi(T_R),$$

anchored with $\chi(\square) = \square$ and $\chi(\bigcirc\square\square) = \bigcirc\square\square$.

Less formally, $\psi(T)$ is obtained from $\bigcirc T\square$ by performing the following process: for all nodes x and y such that y is the left child of x, the right subtree of y is moved into the right subtree of x.

For example, if $T = \bigcirc\bigcirc\square\bigcirc\bigcirc\bigcirc\square\square\bigcirc\square\square\square\square$ then $\psi(T) = \chi(\bigcirc T\square) = \bigcirc\bigcirc\bigcirc\square\square\bigcirc\bigcirc\bigcirc\square\square\square\bigcirc\square\square\square$ (see Figure 6).

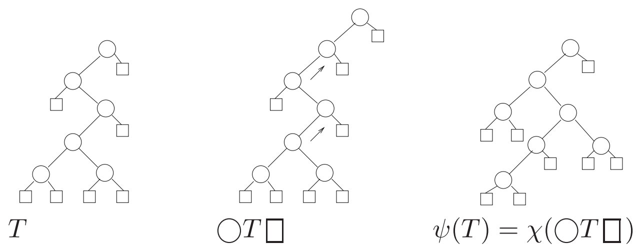

Figure 6: The bijection ψ.

Lemma 5 *The map ψ is a bijection from $\mathcal{MW}_{n-1}$ to $\mathcal{M}_n$.*

Proof. By the recursive definition of ψ, it is straightforward to see that $\psi(T)$ does not contain any pattern $\bigcirc\square\bigcirc$. Thus, we have $\psi(\mathcal{MW}_{n-1}) \subseteq \mathcal{M}_n$. Moreover, from the recursive definition of χ, ψ is necessarily injective. Indeed, for any T, T' such that $T = \bigcirc T_L T_R$, $T' = \bigcirc T'_L T'_R$ and $\psi(T) = \psi(T')$, we have $\bigcirc\chi(\bigcirc T_L \square)\chi(T_R) = \bigcirc\chi(\bigcirc T'_L \square)\chi(T'_R)$. Using the induction hypothesis, we obtain $T'_L = T_L$, $T'_R = T_R$ and thus, $T = T'$. The two sets $\mathcal{MW}_{n-1}$ and $\mathcal{M}_n$ being enumerated by the Motzkin numbers (see [5]), we deduce that ψ is a bijection from $\mathcal{MW}_{n-1}$ to $\mathcal{M}_n$. Notice that the bijections described in [5] induce a different isomorphism between $\mathcal{M}_n$ and the set $\mathcal{MW}_{n-1}$. $\square$

Theorem 4 *The two join-semilattices $(\mathcal{MW}_{n-1}, \stackrel{*}{\longrightarrow})$ and $(\mathcal{M}_n, \stackrel{*}{\rightarrowtail})$ are isomorphic.*

Proof. According to Lemma 5, it suffices to prove that the map ψ transports the rotation transformation $\longrightarrow$ between two trees in $\mathcal{MW}_{n-1}$ onto the restricted rotation $\rightarrowtail$ in $\mathcal{M}_n$, and *vice versa*. Let $T, T' \in \mathcal{MW}_{n-1}$ be so that T' is obtained from T by a left-rotation. It is worth noticing that a left-rotation between two trees of $\mathcal{MW}_{n-1}$ is a restricted rotation between these two trees. Since the rotation transformation is a local transformation, we will consider T and T' near the node involved by the rotation. Therefore, we give arguments using $T = \ldots \bigcirc \bigcirc A \bigcirc \bigcirc B \square \square C \ldots$ and $T' = \ldots \bigcirc \bigcirc \bigcirc A \bigcirc B \square \square C \ldots$ (see the trees on the top of Figure 7).

We have $\psi(T) = \ldots \bigcirc \chi(\bigcirc A \square) \bigcirc \chi(\bigcirc B \square) \square \ldots$ and $\psi(T') = \ldots \bigcirc \bigcirc \chi(\bigcirc A \square)\chi(\bigcirc B \square)\square \ldots$. Setting $A' = \chi(\bigcirc A \square)$ and $B' = \chi(\bigcirc B \square)$, we recognize the restricted rotation $\ldots \bigcirc A' \bigcirc B' \square \ldots \rightarrowtail \ldots \bigcirc \bigcirc A' B' \square \ldots$ (see Figure 7 for an illustration of this proof). This argument still remains available for the converse *mutatis mutandis*. Finally, the map ψ transports the rotation transformation $T \longrightarrow T'$ where $T, T' \in \mathcal{MW}_{n-1}$ into the restricted rotation $\psi(T) \rightarrowtail \psi(T')$ where $\psi(T)$ and $\psi(T')$ belong to $\mathcal{M}_n$. $\square$

The image of the semilattice $(\mathcal{MW}_5, \stackrel{*}{\longrightarrow})$ by the map ψ can be viewed in Figure 3 by not taking into account the four rotations labeled (a), (b), (c) and (d).

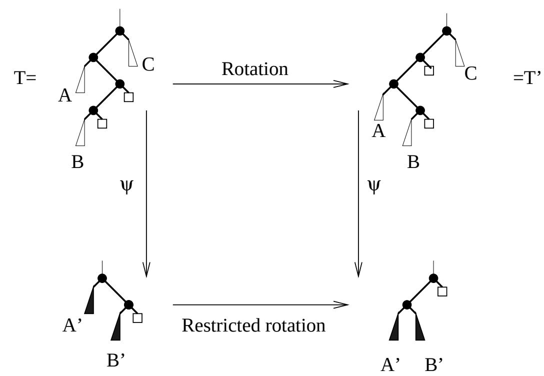

Figure 7: An illustration for the proof of Theorem 4.

In [2], the authors compute the length $\rho(T,T')$ of a shortest path between T and T' in $\mathcal{MW}_{n-1}$. The following corollary provides an upper bound for the rotation distance d in $\mathcal{T}_n$ (and also in $\mathcal{M}_n$).

Corollary 1 *Given T and T' in $\mathcal{M}_n$, we have*

$$d(T,T') \leq \delta(T,T') \leq \rho(\psi^{-1}(T),\psi^{-1}(T')).$$

Since computing the rotation distance d in $\mathcal{T}_n$ is a difficult problem, our upper bounds are valuable, especially because they are sometimes sharp and because ρ can be computed easily. Indeed, the bounds give the exact value of the classical distance rotation d for some specific pairs of Motzkin trees. For example, if n is even, $n \geq 4$, then we define T and T' by their w-sequences $w_T = 121212\ldots121n$ and $w_{T'} = 121212\ldots12$. If n is odd, $n \geq 3$, then T and T' are defined by $w_T = 121212\ldots12n$ and $w_{T'} = 121212\ldots121$. A simple calculation proves that $d(T,T') = \lfloor\frac{n-1}{2}\rfloor = \delta(T,T') = \rho(\psi^{-1}(T),\psi^{-1}(T'))$.

6 Others research directions

Motzkin trees are in bijection with trees where internal nodes have one or two children. How the rotation operation can be described on these trees?

Motzkin trees can be defined as binary trees whose Polish notation avoids the pattern ○□○ (or equivalently, a certain binary tree pattern). Is there a criterion to decide, for a given set of patterns P, if the set $\mathcal{M}_n^P$ of binary trees avoiding P form a subposet (resp. a sublattice, a join-semilattice, a meet-semilattice) of the Tamari lattice of order n?

Recently, some studies have focused on m-Tamari lattices which generalize the classical Tamari lattices for trees where internal nodes are of arity $m+1$ (see the survey paper of Bergeron [3] for any m, and [19] for ternary trees). Is there a generalization of Motzkin trees and the results of this paper for trees where internal nodes are of arity $m+1$?

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