ACTA ARITHMETICA
LXXXIII.4 (1998)

Imaginary quadratic fields with small odd class number

by

Steven Arno, M. L. Robinson and
Ferrell S. Wheeler (Bowie, Md.)

1. Introduction. Let $-d$ be the discriminant of an imaginary quadratic field with class number $h(-d)$. As is well known, Gauss [9] conjectured that $h(-d)$ tends to infinity with d. Hence, for fixed m, it was natural to ask for a complete list of negative fundamental discriminants $-d$ such that $h(-d) = m$. This problem is usually referred to as Gauss' class number problem or Gauss' class number m problem.

In 1934, Heilbronn [14] succeeded in proving Gauss' conjecture, thereby placing the class number problem on firm ground. The following year, Siegel [20] showed that for any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $h(-d) > c_\varepsilon d^{1/2-\varepsilon}$ as $d \to \infty$. Unfortunately, neither result was effective, and no further progress was made until the 1950's when Heegner [13] offered a solution for the class number 1 problem based on new ideas from the theory of modular functions. It is interesting to recall that Heegner's proof was generally discounted until the "gaps" in his argument were explained many years later (see [5], [8], [21], [24]). In the interim period, however, the first accepted proof of the class number 1 problem was given by Stark [23] in 1966. Shortly thereafter, Baker [2, 4] found another proof based on the theory of transcendental numbers. In 1971, Baker [3, 4] and Stark [26–28] independently resolved the class number 2 problem as well. However, there seemed to be little hope of generalizing these methods to solve higher class number problems.

In 1976, Goldfeld [10] presented a deep and entirely unexpected result which provided the framework for a general attack on the class number problem. He showed that if there exists a Weil curve whose associated L-function has a zero of at least the third order at $s = 1$, then for any $\varepsilon > 0$

1991 *Mathematics Subject Classification*: Primary 11R; Secondary 11Y, 11J.
Key words and phrases: binary quadratic forms, imaginary quadratic fields, class numbers, discriminants.

there exists an effectively computable constant c_ε such that

$$h(-d) > c_\varepsilon (\log d)^{1-\varepsilon}. \tag{1.1}$$

Of course, the utility of Goldfeld's result depended on finding an appropriate elliptic curve. And though one would expect to find a Weil curve with a high order zero at $s = 1$ based on the celebrated conjecture of Birch and Swinnerton-Dyer [5], technical difficulties kept things on hold for several years. Finally, in 1983 Gross and Zagier [11] were able to show that certain curves must have a zero of at least the third order at $s = 1$, thereby completing the attack of Goldfeld.

Goldfeld's proof was later simplified by Oesterlé [19], who provided, among other things, explicit constants for Goldfeld's theorem. As a result, Oesterlé was able to complete the class number 3 problem as well. The class number 4 problem was solved by the first author [1] who combined new techniques with the well-known methods of Stark [22], Montgomery–Weinberger [18], and Oesterlé [19]. In hindsight, [1] contains a prototype for the partitioning of minima (of reduced quadratic forms of discriminant $-d$), which plays a crucial role in this paper.

The central concern of this paper is the class number m problem for small, odd m. The aforementioned partitioning of minima enables us to significantly improve on earlier estimates. Our results are summarized in the following theorem.

Theorem 1. *For each odd integer m satisfying $5 \le m \le 23$, the class number m problem is solved. For each such m, a complete list of negative fundamental discriminants $-d$ for which $h(-d) = m$ can be found in Appendix* A.

Let $-d$ be the discriminant of an imaginary quadratic field with class number $h(-d)$. In Table 1 we present the number of fields satisfying $h(-d) = m$ and the largest such d for each odd m satisfying $1 \le m \le 23$.

Table 1. Upper bound on d satisfying $h(-d) = m$

m	# of d	max. d
1	9	163
3	16	907
5	25	2683
7	31	5923
9	34	10627
11	41	15667
13	37	20563
15	68	34483
17	45	37123
19	47	38707
21	85	61483
23	68	90787

The paper is organized as follows. In §2 we use Oesterlé's [19] explicit constants to produce a $d_3(m)$ such that if $d \geq d_3(m)$, then $h(-d) \neq m$. In §3 we provide the theoretical justification for the partitioning of minima mentioned above. In §4 we prove several technical lemmas concerning an auxiliary function which are needed in §5.

We divide §5 into three subsections, each of which rules out a certain range of fundamental discriminants. In §5.1, following the approach of Montgomery–Weinberger [18], we produce a $d_2(m)$ such that if $d_2(m) \leq d \leq d_3(m)$, then $h(-d) \neq m$. At the end of §5.1 we note that the methods of §5.1, even when pushed to their limits, do not reduce the range of admissible discriminants d to the point where a computationally intensive sieve (like the one introduced in §6) can be used to complete the class number m problem when $m > 7$. This shows that some new argument is necessary. In §5.2 we exploit the partitioning of minima introduced in §3 to produce a $d_1(m)$ such that if $d_1(m) \leq d \leq d_2(m)$, then $h(-d) \neq m$. When m is odd and $5 \leq m \leq 13$, $d_1(m)$ is small enough to allow the class number m problem to be completed with the computationally intensive sieve of §6. In §5.3 we use a more sophisticated version of the arguments in §5.2 to produce a $d_0(m)$ such that if $d_0(m) \leq d \leq d_1(m)$, then $h(-d) \neq m$. When m is odd and $15 \leq m \leq 23$, $d_0(m)$ is small enough to allow the class number m problem to be completed with the computationally intensive sieve of §6. We combine the partitioning of minima with the aforementioned sieve to complete the proof of Theorem 1 in §6.

Notation. Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the set of positive integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. Let p and q denote primes in $\mathbb{Z}^+$. The Kronecker symbol is denoted by either $\left(\frac{m}{n}\right)$ or $(m|n)$, depending on which is more convenient. As is customary, we let $\omega(n)$ denote the number of distinct prime divisors of n and $d(n)$ the total number of positive divisors of n. Finally, let $e(x) = e^{2\pi i x}$.

The number $-d$ denotes a negative fundamental discriminant or, equivalently, the discriminant of an imaginary quadratic number field. In other words, we have either $-d \equiv 1 \pmod 4$ and d is square-free, or $4 \mid d$ and $-d/4 \equiv 2$ or $3 \pmod 4$ and $d/4$ is square-free. χ_1 denotes the real primitive character with $\chi_1(n) = (-d|n)$.

A binary quadratic form

$$Q(x,y) = ax^2 + bxy + cy^2$$

of discriminant $-d = b^2 - 4ac$ is *reduced* if it satisfies either

$$-a < b \leq a < c \quad \text{or} \quad 0 \leq b \leq a = c \qquad (a,b,c \in \mathbb{Z}). \tag{1.2}$$

Note that this implies

$$a \le (d/3)^{1/2}. \tag{1.3}$$

Let

$$Q_d = \{Q(x,y) = ax^2 + bxy + cy^2 : b^2 - 4ac = -d,\ Q \text{ is reduced}\}$$

denote the finite set of reduced binary quadratic forms with discriminant $-d$. Note $h(-d) = |Q_d|$. The notation $\sum_{Q_d}$ denotes a sum over all $Q \in Q_d$ along with the associated coefficients a, b, and c.

The modified Bessel function of the second kind of order zero is given by

$$K_0(z) = \int_1^\infty e^{-(z/2)(t+t^{-1})}\frac{dt}{t} \qquad (\Re z > 0). \tag{1.4}$$

2. The high range. As mentioned in §1, Oesterlé [19] provided explicit constants for the c_ε in Goldfeld's inequality (1.1). Indeed, if $-d$ is a negative fundamental discriminant with class number $h(-d)$ and

$$\vartheta(d) = \prod_{p\,|\,d}{}^{*}\left(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right),$$

where the product is taken over all prime divisors p of d with the exception of the largest prime divisor, then

$$\vartheta(d)\log d \le Ch(-d)$$

where $C = 55$ if $(d, 5077) = 1$ and $C = 7000$ otherwise.

In order to evaluate $\vartheta(d)$ when $h(-d)$ is odd, recall that the Fundamental Theorem of Genera due to Gauss [9] implies that

$$2^{\omega(d)-1} \mid h(-d).$$

If $h(-d)$ is odd, then the preceding result implies that d has only one prime divisor. Since $-d$ is a discriminant, we see that $-d$ is either -4, -8, or $-p$ for some odd prime $p \equiv 3 \pmod 4$. It follows that when $h(-d)$ is odd, $\vartheta(d) = 1$. Since $h(-4) = h(-8) = 1$, we have

$$h(-d) \text{ is odd and } h(-d) > 1 \Rightarrow d \text{ is prime and } d \equiv 3 \pmod 4. \tag{2.1}$$

Hence, when $h(-d)$ is odd and $h(-d) > 1$, (2.1) implies that $(d, 5077) = 1$ (for 5077 is a prime equal to 1 modulo 4), and we have

$$\log d \le 55h(-d).$$

Thus, for $m \in \{5, 7, \ldots, 23\}$, we have $h(-d) > m$ if $d \ge d_3(m)$ where $d_3(m)$ is given in Table 2.

Table 2. $h(-d) \neq m$ for $d \geq d_3(m)$

m	$d_3(m)$
5	10^{120}
7	10^{168}
9	10^{215}
11	10^{263}
13	10^{311}
15	10^{359}
17	10^{407}
19	10^{454}
21	10^{502}
23	10^{550}

Note that when $h(-d)$ is even, d may be composite. This allows the possibility that a minimum a of a reduced form could satisfy $(a, d) > 1$. This introduces many technical difficulties, the least of which is a smaller value of $\vartheta(d)$. For these reasons, we confine our attention to the case where $h(-d)$ is odd.

3. Minima results. The coefficient a is referred to as the minimum of the form $ax^2 + bxy + cy^2 \in Q_d$, while the multiset of minima for a given negative fundamental discriminant $-d$ is denoted by

$$M_d = \{a : ax^2 + bxy + cy^2 \in Q_d\}.$$

From (1.2) it is easy to see that 1 occurs in M_d precisely once. We henceforth refer to $1 \in M_d$ as the *principal minimum* since 1 is the minimum of the principal form (i.e., either $x^2+(d/4)y^2$ or $x^2+xy+((d+1)/4)y^2$ depending on the parity of d). Furthermore, we note that all elements of M_d are positive by (1.2).

LEMMA 1. *If $a \in M_d$, then any positive divisor of a is also in M_d.*

Proof. Let $a_1 > 0$ be a divisor of a. If $a_1 = a$ there is nothing to prove. Hence, we may assume $a_1 \leq a/2$. It follows from (1.3) that

$$a_1 \leq (d/12)^{1/2}. \tag{3.1}$$

Let $ax^2 + bxy + cy^2 \in Q_d$ so that $b^2 - 4ac = -d$. It follows at once that the quadratic congruence $z^2 \equiv -d \pmod{4a_1}$ is solvable. Hence, there exists a b_1 such that $b_1^2 \equiv -d \pmod{4a_1}$, with $-2a_1 < b_1 \leq 2a_1$. Further, this range can be sharpened to ensure that $-a_1 < b_1 \leq a_1$ by replacing b_1 with $b_1 - 2a_1$ if $a_1 < b_1 \leq 2a_1$ and b_1 with $b_1 + 2a_1$ if $-2a_1 < b_1 \leq -a_1$. Let $c_1 = (b_1^2 + d)/(4a_1)$ and observe that $c_1 \geq d/(4a_1) > a_1$ using (3.1). From (1.2) we see that the quadratic form $Q = a_1x^2 + b_1xy + c_1y^2$ is reduced. ∎

Lemma 2. *Let a satisfy $2 \le a \le (d/4)^{1/2}$ and $\gcd(a,d) = 1$. Then $a \in M_d$ if and only if every prime divisor p of a satisfies $(-d|p) = 1$.*

Proof. First, assume that $a \in M_d$. Since $b^2 - 4ac = -d$, we see that

$$b^2 \equiv -d \pmod{4a}. \tag{3.2}$$

If $2 \mid a$, then we see at once that $b^2 \equiv -d \pmod 8$. We know that $-d$ must be odd because a is even and, by hypothesis, $\gcd(a,d) = 1$. Since 1 is the only odd square modulo 8, we have $d \equiv -1 \pmod 8$. Thus $(-d|2) = 1$. If p is any odd prime dividing a, it follows at once from (3.2) that $b^2 \equiv -d \pmod p$. Thus $(-d|p) = 1$, and the "only if" direction is proved.

Conversely, assume that $(-d|p) = 1$ for each prime divisor p of a. If p is odd, then the congruence $z^2 \equiv -d \pmod p$ is solvable, and by Hensel's lemma, the congruence $z^2 \equiv -d \pmod{p^\alpha}$ is solvable for all $\alpha \in \mathbb{Z}^+$. If $p = 2$, note that $(-d|2) = 1$ implies $d \equiv -1 \pmod 8$. Hence, the congruence $z^2 \equiv -d \pmod 8$ is solvable. Furthermore, it is well known that solutions with $z \equiv \pm 1 \pmod 8$ can be lifted to a solution of the congruence $z^2 \equiv -d \pmod{2^\alpha}$ for all $\alpha \in \mathbb{Z}^+$. Hence, by the Chinese Remainder Theorem, there exists a b such that the congruence $b^2 \equiv -d \pmod{4a}$ is solvable. Reasoning as in the proof of Lemma 1, there is in fact such a b with $-a < b \le a$. Let $c = (b^2 + d)/(4a)$. If $b \ne 0$, then $c > d/(4a) \ge a$ since $a \le (d/4)^{1/2}$ by hypothesis. If $b = 0$, then $c = d/(4a) \ge a$. In every case, $ax^2 + bxy + cy^2$ is reduced, and $a \in M_d$. ∎

Lemma 3. *Suppose $h(-d)$ is odd. If $a > 1$ and $ax^2 + bxy + cy^2 \in Q_d$, then $ax^2 - bxy + cy^2$ is a distinct member of Q_d.*

Proof. Since $a > 1$ and $a \in M_d$, we know that $h(-d) > 1$. Hence, (2.1) implies that d is prime. Since $ax^2 + bxy + cy^2$ is a reduced form, we know that $ax^2 - bxy + cy^2$ will be a distinct reduced form unless (i) $b = 0$, (ii) $b = a$, or (iii) $a = c$. If (i) is true, then $4ac = d$, contradicting the fact that d is prime. If (ii) is true, then $a^2 - 4ac = a(a - 4c) = -d$. Since $a > 1$ and d is prime, we have $a = d$, which is impossible since $a < (d/3)^{1/2}$. If (iii) is true, then $b^2 - 4a^2 = (b - 2a)(b + 2a) = -d$. Since $b \ge 0$ in case (iii) and d is prime, we know that $2a - b = a + (a - b) = 1$. This leads to a contradiction since $a > 1$ by hypothesis and $a \ge b$ by (1.3). ∎

Lemma 4. *Suppose $h(-d)$ is odd. Let $a \in \mathbb{Z}^+$ be odd and satisfy $a \le (d/4)^{1/2}$. If $a \in M_d$, then a appears in M_d exactly $2^{\omega(a)}$ times.*

Proof. If $a = 1$, it is easy to see from (1.2) that a appears precisely once in M_d. Thus, we may henceforth assume that $a > 1$. Of course, this implies that $h(-d) > 1$ as well. We want to count the number of integers b such that $ax^2 + bxy + ((b^2 + d)/(4a))y^2 \in Q_d$. By (1.2) this is just the number of integers b that satisfy $(b^2+d)/(4a) \in \mathbb{Z}$ with either $-a < b \le a < (b^2+d)/(4a)$ or $0 \le b \le a = (b^2+d)/(4a)$. Assume, for the moment, that $(b^2+d)/(4a) \in \mathbb{Z}$. Since $h(-d) > 1$, we see from (2.1) that $d \equiv 3 \pmod 4$. It follows at once that b is nonzero. Using the hypothesis, $a \le (d/4)^{1/2}$, we then deduce $a \le d/(4a) < (b^2+d)/(4a)$. It follows that we want to count the number of integers b that satisfy $b^2 \equiv -d \pmod{4a}$ with $-a < b \le a$.

From (2.1), we know that d is prime. Since $a \le (d/4)^{1/2} < d$, we see that $\gcd(a,d) = 1$. Let p be any prime divisor of a. By Lemma 2, we see that $(-d|p) = 1$. Since a is odd, we know that p is also. Hence, the congruence $z^2 \equiv -d \pmod p$ has exactly two solutions. By Hensel's Lemma, we know that the congruence $z^2 \equiv -d \pmod{p^\alpha}$ has exactly two solutions for all $\alpha \in \mathbb{Z}^+$. Also, since $d \equiv 1 \pmod 4$, we know that the congruence $z^2 \equiv -d \pmod 4$ has exactly two solutions. Thus by the Chinese Remainder Theorem, we see that there are precisely $2^{\omega(a)+1}$ integers b that satisfy both

$$b^2 \equiv -d \pmod{4a} \tag{3.3}$$

and $-2a < b \le 2a$. Now either both b and $b+2a$ satisfy (3.3) or neither do. Hence, the number of b that satisfy (3.3) with $-2a < b \le -a$ is equal to the number of b that satisfy (3.3) with $0 < b \le a$. A similar argument with b and $b-2a$ shows that the number of b that satisfy (3.3) with $a < b \le 2a$ is equal to the number of b that satisfy (3.3) with $-a < b \le 0$. It follows that exactly $2^{\omega(a)}$ integers b satisfy (3.3) with $-a < b \le a$, thereby proving Lemma 4. ∎

LEMMA 5. *If* $a > 1$, $a \in M_d$ *and* $(a,d) = 1$, *then* $a > (d/4)^{1/h(-d)}$.

Proof. Suppose $p \mid a$ and p is an odd prime. Then $b^2 \equiv -d \pmod p$ implies $(-d|p) = 1$. Also, if $2 \mid a$, then $b^2 \equiv -d \pmod 8$. Hence, $-d \equiv 1 \pmod 8$, so that $(-d|2) = 1$. Thus, if p is any prime dividing a we have $(-d|p) = 1$. This implies that p splits in $\mathbb{Q}(\sqrt{-d})$, $\langle p \rangle = \wp_1\wp_2$ with $\wp_1 \ne \wp_2$. Thus, $\wp_1^{h(-d)} = \langle \beta \rangle$ is a principal ideal, and $\beta \notin \mathbb{Z}$. It follows that $p^{h(-d)} = N(\wp_1^{h(-d)}) = N(\langle \beta \rangle) \ge d/4$. Since $a \ge p$, it follows that $a \ge (d/4)^{1/h(-d)}$. Note that $a > 1$ implies $h(-d) > 1$, so that $(d/4)^{1/h(-d)} \notin \mathbb{Z}$. The lemma follows. ∎

It follows from (1.3), (2.1) and Lemma 5 that

$$\left(\frac{d}{4}\right)^{1/h(-d)} < a \le \left(\frac{d}{3}\right)^{1/2} \qquad (a \in M_d\backslash\{1\},\ h(-d) > 1 \text{ is odd}). \tag{3.4}$$

To improve on these bounds, we separate the minima to a certain extent using the multiplicative structure of M_d developed in Lemmas 1–4.

Assume that $h(-d) > 1$ and $h(-d)$ is odd. From Lemma 3 we know that every $a \in M_d\backslash\{1\}$ appears an even number of times in M_d. Also, by (3.4) if $d > 2^{h(-d)+2}$, then 2 is not a minimum. It then follows by Lemma 1 that a must be odd if $a \in M_d$. In the remainder of this section assume that $d > 2^{h(-d)+2}$. Define the multiset M_d^* by

$$M_d^* = \{a : ax^2 + bxy + cy^2 \in Q_d, a > 1, b > 0\}. \tag{3.5}$$

From Lemma 3 we have

$$|M_d^*| = (h(-d) - 1)/2.$$

DEFINITION 1. A *partition for* d is a list of $(h(-d)-1)/2$ pairs of functions $(l_a(d), u_a(d))$, each increasing in d, corresponding to the $(h(-d)-1)/2$ elements in M_d^*.

DEFINITION 2. We say a partition for d *covers* M_d if

$$l_a(d) \le a \le u_a(d)$$

for every $a \in M_d^*$.

We begin with a simple example. Suppose that $h(-d) = 5$. Let p be the smallest nonprincipal member of M_d. By Lemma 1, p is prime. In past investigations of minima one generally used the result contained in Lemma 5, which states that all minima a, except $a = 1$, must satisfy $a > (d/4)^{1/5}$. In particular, the minima could simultaneously be small, each minimum lying close to the bound $(d/4)^{1/5}$. However, using Lemma 2 and Lemma 4, we see that if p is about $(d/4)^{1/5}$ in size, then there are exactly two reduced forms with minimum p and two reduced forms with minimum p^2 and one reduced form with minimum 1, which provides us with five reduced forms. This excludes the possibility that any other minimum is simultaneously small. Using similar reasoning it is not hard to see that if $h(-d) = 5$, then M_d is covered by one of the following three partitions:

1	$(d/4)^{1/5} \le p \le (d/4)^{1/4}$	$(d/4)^{2/5} \le p^2 \le (d/4)^{1/2}$
2	$(d/4)^{1/4} \le p \le (d/4)^{1/3}$	$(d/4)^{1/4} \le a \le (d/3)^{1/2}$
3	$(d/4)^{1/3} \le a \le (d/3)^{1/2}$	$(d/4)^{1/3} \le a \le (d/3)^{1/2}$

For each fixed value of $m = 5, 7, \ldots, 23$, a set of partitions covering all possible M_d is given in Appendix B. In these tables, p and q denote the first and second smallest prime minima in M_d, respectively, while a denotes a generic member of M_d, $v = d/4$ and $w = d/3$. In order to simplify our presentation of a covering partition, we now introduce some additional notation. The notation (3p.), for example, denotes that the inequalities for the first three powers of the prime are trivially inferred. Similarly, the notation (3), for example, means the inequality is to be listed three times. In this notation, the covering partitions for M_d when $h(-d)=5$ are given by

1	$v^{1/5} \le p \le v^{1/4}$ (2p.)	
2	$v^{1/4} \le p \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$
3	$v^{1/3} \le a \le w^{1/2}$ (2)	

In order to give a better idea of how the partitions in Appendix B were generated, we will go through the details of partition number 10 for class number 23 (Table B10). The partition is given by

10	$v^{1/10} \le p \le v^{1/8}$ (4p.)	$v^{3/16} \le q \le v^{1/4}$ (2p.)	$v^{23/80} \le pq \le v^{3/8}$ (2)
		$v^{31/80} \le p^2q \le v^{1/2}$ (2)	$v^{3/8} \le a \le w^{1/2}$

Assume the smallest nonprincipal minimum p satisfies $v^{1/10} \le p \le v^{1/8}$; then p^2, p^3, p^4 are all $\le (d/4)^{1/2}$, implying that $p, p^2, p^3, p^4 \in M_d$, which accounts for 8 minima. If the second smallest minimum q satisfies $q \le (d/4)^{3/16}$, then q, q^2, pq, pq^2, p^2q are $\le (d/4)^{1/2}$, implying that they are in M_d. But this accounts for 16 new minima, which would make 25 total. Thus, $v^{3/16} \le q$. Assume $q \le (d/4)^{1/4}$. Now, $q, q^2, pq, p^2q \le (d/4)^{1/2}$, so we have accounted for 21 minima. If any further minima a satisfy $a \le (d/4)^{3/8}$, then $pa \le (d/4)^{1/2}$ would imply that there are more than 23 minima. Therefore, the remaining two minima must satisfy $(d/4)^{3/8} \le a \le (d/3)^{1/2}$. All of the partitions in the tables of Appendix B are computed in this same fashion. By using a disjoint set of assumptions on the first and second largest prime minima p and q, these sets of partitions can be seen to cover all possible M_d.

4. Properties of an auxiliary function. In this section we prove two technical lemmas concerning the auxiliary function F defined by

$$F(x) = \sqrt{x} \sum_{n=1}^{\infty} d(n) K_0(nx) \quad (x > 0), \tag{4.1}$$

where $K_0(z)$ is defined in (1.4).

On several occasions in §5.2 and §5.3 we will need to compute accurate approximations to F at certain small arguments. For such purposes, the following crude generalization of [18, Lemma 7]) suffices.

LEMMA 6. *If $x > 0$, N is a nonnegative integer, and*

$$\Delta_N(x) = F(x) - \sqrt{x} \sum_{n=1}^{N} d(n) K_0(nx),$$

then

$$|\Delta_N(x)| \le \frac{2}{\sqrt{x}} (1 + \log(N + 1 + 2/x)) e^{-(N+1)x/2}$$

where the empty sum is understood to equal zero when $N = 0$.

Proof. Since $x > 0$, we see from (4.1) and (1.4) that

$$0 \le x^{-1/2}\Delta_N(x) \le \sum_{n=N+1}^{\infty} d(n) \int_1^{\infty} e^{-nxt/2}\frac{dt}{t} = \sum_{n=N+1}^{\infty} d(n) \int_{nx/2}^{\infty} e^{-u}\frac{du}{u}.$$

Using partial summation, we have

$$x^{-1/2}\Delta_N(x) \le \int_{(N+1)x/2}^{\infty} e^{-u}\Big(\sum_{n\le 2u/x} d(n)\Big)\frac{du}{u}.$$

For $y \ge 1$ we have

$$\sum_{n\le y} d(n) = \sum_{n\le y}\left\lfloor \frac{y}{n}\right\rfloor \le y\sum_{n\le y}\frac{1}{n} \le y(1+\log y),$$

so that

$$|x^{-1/2}\Delta_N(x)| \le \int_{(N+1)x/2}^{\infty} e^{-u}\frac{2u}{x}(1+\log(2u/x))\frac{du}{u}.$$

An integration by parts gives

$$|x^{-1/2}\Delta_N(x)| \le \frac{2}{x}\bigg((1+\log(N+1))e^{-(N+1)x/2} + \int_{(N+1)x/2}^{\infty} e^{-u}\frac{du}{u}\bigg).$$

Note that the function g, defined by

$$g(v) = \int_v^{\infty} e^{-u}\frac{du}{u} - e^{-v}\log(1+1/v),$$

is increasing for $v > 0$, and $\lim_{v\to\infty} g(v) = 0$. Hence,

$$\int_{(N+1)x/2}^{\infty} e^{-u}\frac{du}{u} \le e^{-(N+1)x/2}\log\left(1+\frac{2}{(N+1)x}\right),$$

and the lemma follows. ■

The next lemma will play an important role in §5.2 and §5.3.

Lemma 7. *$F(x)$ is a strictly decreasing function of x for $x > 0$.*

Proof. From (4.1) we have

$$F(x) = \sqrt{x}\sum_{n=1}^{\infty} d(n)K_0(nx) = \sqrt{x}\sum_{n=1}^{\infty}\sum_{d\,|\,n} K_0(nx) = \sqrt{x}\sum_{d=1}^{\infty}\sum_{m=1}^{\infty} K_0(dmx),$$

where K_0 is the modified Bessel function of the second kind of order zero given by (1.4). Hence, we get

$$F(x) = \sqrt{x}\int_1^\infty \Big(\sum_{d=1}^\infty\sum_{m=1}^\infty \exp\Big(-\frac{dmx}{2}(t+t^{-1})\Big)\Big)\frac{dt}{t}$$
$$= \int_1^\infty \Big(\sum_{d=1}^\infty \frac{\sqrt{x}}{\exp\big(\frac{d(t+t^{-1})}{2}x\big)-1}\Big)\frac{dt}{t}.$$

For $b>0$, define

$$f_b(x) = \frac{\sqrt{x}}{e^{bx}-1}.$$

Since $x\mapsto x/(e^{xb}-1)$ decreases on $(0,\infty)$, we know that $x\mapsto f_b(x)$ also decreases on $(0,\infty)$. Lemma 7 follows immediately. ∎

5. The medium range. Let χ be a real primitive character modulo k for some integer $k>1$. In the case where $-k$ is a negative fundamental discriminant we take $\chi(n) = (-k|n)$ (see [7, p. 40]). Define

$$A_d(s) = \sum_{Q_d} \chi(a)a^{-s}$$

and

$$P_k(s) = \prod_{p\mid k}(1-p^{-s}).$$

In 1966 Stark [22] exploited a formula for the zeta function of a quadratic number field (i.e., $\zeta(s)L(s,\chi_1)$) to show that if a tenth fundamental discriminant $-d$ of class number 1 existed, then $d>\exp(2.2\cdot 10^7)$. Later Stark [25] developed a formula for $L(s,\chi)L(s,\chi\chi_1)$ analogous to the formula for $\zeta(s)L(s,\chi_1)$. Montgomery and Weinberger [18] exploited this formula to obtain similar results for class numbers 2 and 3. Indeed, if $(k,d)=1$, then

$$\Big(\frac{k\sqrt{d}}{2\pi}\Big)^{s-1/2}\Gamma(s)L(s,\chi)L(s,\chi\chi_1) = T_d(s)+T_d(1-s)+U_d(s) \tag{5.1}$$

where

$$T_d(s) = \Big(\frac{k\sqrt{d}}{2\pi}\Big)^{s-1/2}\Gamma(s)\zeta(2s)P_k(2s)A_d(s),$$

$$U_d(s) = \frac{4\sqrt{\pi}}{k}\sum_{Q_d} a^{-1/2}\sum_{n=1}^\infty K_{s-1/2}\Big(\frac{\pi n\sqrt{d}}{ak}\Big)n^{s-1/2}V_Q(s,n), \tag{5.2}$$

and

$$V_Q(s,n) = \sum_{y\mid n} y^{1-2s}\,\Re\Big\{\sum_{j=1}^k \chi(Q(j,y))e\Big(\frac{jn}{ky}\Big)e\Big(\frac{bn}{2ak}\Big)\Big\}.$$

Let $s_0 = 1/2 + it_0$, with $t_0 > 0$, be a zero of $L(s,\chi)$. Substituting $s = s_0$ into (5.1) gives

$$T_d(s_0) + T_d(\overline{s}_0) = -U_d(s_0).$$

Applying the Schwarz Reflection Principle to T_d implies

$$2|T_d(s_0)|\cos(\arg T_d(s_0)) = -U_d(s_0),$$

which, in turn, gives

$$|\sin\arg(iT_d(s_0))| = \left|\frac{U_d(s_0)}{2T_d(s_0)}\right|. \tag{5.3}$$

The method for the middle range consists in showing that this equality is false for large intervals of d under the assumption that $h(-d)$ is some fixed odd integer. Indeed, for fixed k and t_0, define the constants

$$\xi_1 = t_0/2, \tag{5.4}$$

$$\xi_2 = t_0 \log\left(\frac{k}{2\pi}\right) + \arg\{i\Gamma(1/2+it_0)\zeta(1+2it_0)P_k(1+2it_0)\}, \tag{5.5}$$

$$\xi_3 = 2|\Gamma(1/2+it_0)\zeta(1+2it_0)P_k(1+2it_0)|. \tag{5.6}$$

To show (5.3) is false, all we need to show is that

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| > \frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|}. \tag{5.7}$$

5.1. *The range $d_2(m) \le d \le d_3(m)$*

LEMMA 8. *Let $t \in \mathbb{R}^+$. Suppose $m = h(-d)$ is odd and fixed. Then*

$$|A_d(1/2+it)| \ge 1 - \frac{m-1}{(d/4)^{1/(2m)}}. \tag{5.8}$$

Furthermore, if $d > \max\{4e^{2m}, 4(m-1)^{2m}\}$, then

$$|\arg A_d(1/2+it)| \le \frac{t(1-1/m)\log(d/4)}{(d/4)^{1/(2m)} - (m-1)}. \tag{5.9}$$

Proof. Both (5.8) and (5.9) are trivially true if $m = 1$, so assume $m > 1$. Using the lower bound in (3.4) gives

$$|A_d(1/2+it) - 1| = \Big|\sum_{Q_d, a\ne 1} \chi(a)a^{-1/2-it}\Big|$$

$$\le \sum_{Q_d, a\ne 1} a^{-1/2} \tag{5.10}$$

$$\le \frac{m-1}{(d/4)^{1/(2m)}}. \tag{5.11}$$

Now, (5.8) follows from (5.11) and the triangle inequality.

For the remainder of the proof assume $d > \max\{4e^{2m}, 4(m-1)^{2m}\}$. Since $d > 4(m-1)^{2m}$, we know from (5.11) that

$$|A_d(1/2+it) - 1| < 1,$$

so that $A_d(1/2+it)$ lies in the right half plane. Hence, when we write $\arg A_d(1/2+it)$ in (5.9) and below, we can, without loss of generality, assume we are dealing with the principal value of the argument. Let L be the line segment joining $1/2$ to $1/2+it$. Then an equation for L is given by $\ell(u) = 1/2 + iu$, $0 \le u \le t$. Since $d > 4(m-1)^{2m}$, we know by (5.8) that A_d does not vanish on L. Furthermore, $A_d(s)$ is an entire function of s. It follows [17, p. 218] that

$$\int_L \frac{A_d'(z)}{A_d(z)}\,dz = \int_0^t \frac{A_d'(1/2+iu)}{A_d(1/2+iu)} i\,du = \log A_d(1/2+iu)|_0^t.$$

Evaluating the right-hand side and taking imaginary parts yields

$$\begin{aligned}(5.12)\qquad |\arg A_d(1/2+it)| &= \left|\Im \int_0^t \frac{A_d'(1/2+iu)}{A_d(1/2+iu)} i\,du\right| \\ &\le t \max_{0\le u\le t}\left|\frac{A_d'(1/2+iu)}{A_d(1/2+iu)}\right|.\end{aligned}$$

Note that

$$\begin{aligned}(5.13)\qquad |A_d'(1/2+it)| &= \Big|\sum_{Q_d, a\ne 1} \chi(a) a^{-1/2-it}\log a\Big| \\ &\le \sum_{Q_d, a\ne 1} a^{-1/2}\log a.\end{aligned}$$

Now, $x^{-1/2}\log x$ is a decreasing function of x for $x > e^2$. Using (3.4) and the hypothesis $d > 4e^{2m}$, it then follows from (5.13) that

$$(5.14)\qquad |A_d'(1/2+it)| \le \frac{(1-1/m)\log(d/4)}{(d/4)^{1/(2m)}}.$$

Using (5.14) and (5.8) in (5.12) yields (5.9), and the lemma is proved. ∎

LEMMA 9. *Let $t \in \mathbb{R}$. If $k > 1$ is an odd square-free integer, then*

$$(5.15)\qquad |U_d(1/2+it)| \le \frac{4}{d^{1/4}} \sum_{r|k} \frac{3^{\omega(r)} 2^{\omega(k)-\omega(r)}}{r^{1/2}} \sum_{a\in M_d} F\left(\frac{\pi\sqrt{d}\,r^2}{ak}\right).$$

Proof. From the definition of U_d in (5.2) we deduce at once that

$$(5.16)\qquad |U_d(1/2+it)| \le \frac{4\sqrt{\pi}}{k} \sum_{Q_d} a^{-1/2} \sum_{n=1}^{\infty} K_0\left(\frac{\pi\sqrt{d}\,n}{ak}\right) |V_Q(1/2+it, n)|.$$

Using an argument of Weil [30, his inequality (5)], Montgomery and Weinberger [18, Lemma 7] have shown that

$$|V_Q(1/2+it,n)| \le 2^{\omega(k)}k^{1/2}\sum_{y|n}\prod_{p\,|\,(y,n/y,k)}\frac{p^{1/2}}{2}$$
$$= 2^{\omega(k)}k^{1/2}\sum_{r\,|\,k}2^{-\omega(r)}r^{1/2}\sum_{\substack{y\,|\,n\\(y,n/y,k)=r}}1,$$

because k is square-free. Since $(y,n/y,k)=r$ implies $r^2\,|\,n$, we have

$$|V_Q(1/2+it,n)| \le 2^{\omega(k)}k^{1/2}\sum_{\substack{r\,|\,k\\r^2\,|\,n}}2^{-\omega(r)}r^{1/2}d(n).$$

Inserting this into (5.16) gives

$$(5.17)\quad |U_d(1/2+it)| \le \frac{4\sqrt{\pi}}{k^{1/2}}\sum_{a\in M_d}\frac{1}{a^{1/2}}\sum_{r\,|\,k}2^{\omega(k)-\omega(r)}r^{1/2}\sum_{\substack{n=1\\r^2\,|\,n}}^{\infty}d(n)K_0\biggl(\frac{\pi\sqrt{d}\,n}{ak}\biggr).$$

Let $n=r^2m$ and note that $d(n)\le d(r^2)d(m)=3^{\omega(r)}d(m)$ since r is square-free. From (5.17) we have

$$|U_d(1/2+it)| \le \frac{4\sqrt{\pi}}{k^{1/2}}\sum_{a\in M_d}\frac{1}{a^{1/2}}\sum_{r\,|\,k}\left(\frac{3}{2}\right)^{\omega(r)}2^{\omega(k)}r^{1/2}\sum_{m=1}^{\infty}d(m)K_0\left(\frac{\pi\sqrt{d}\,r^2m}{ak}\right).$$

Applying definition (4.1) to the inner sum gives the result. ∎

Corollary. *Let $t\in\mathbb{R}$ and $m=h(-d)$. If $k>1$ is an odd square-free integer, then*

$$|U_d(1/2+it)| \le \frac{8k^{1/2}\log k}{3^{1/4}\pi^{1/2}d^{1/4}}(m-1+e^{-\pi(d^{1/2}-\sqrt{3})/(2k)})\prod_{p\,|\,k}(2+3p^{-3/2}).$$

Proof. Replace F in Lemma 9 with the upper bound given in Lemma 6 with $N=0$ to get

$$|U_d(1/2+it)| \le \frac{8k^{1/2}}{\pi^{1/2}d^{1/2}}\sum_{r\,|\,k}\left(\frac{3}{2}\right)^{\omega(r)}2^{\omega(k)}r^{-3/2}$$
$$\times\sum_{a\in M_d}a^{1/2}\left(1+\log\left(1+\frac{2ak}{\pi d^{1/2}r^2}\right)\right)e^{-\pi d^{1/2}r^2/(2ak)}.$$

In the inner sum, the $a = 1$ term is treated separately. When $a > 1$ we use the inequality $a \le (d/3)^{1/2}$. In both cases, we also use the inequalities $r \ge 1$, $d \ge 3$, and

$$\left(1 + \log\left(1 + \frac{2k}{\pi\sqrt{3}}\right)\right) e^{-\pi\sqrt{3}/(2k)} \le \log k \qquad (k \ge 2)$$

to finish the proof of the corollary. ■

For the remainder of §5.1, we assume there exists a discriminant $-d$ with the following properties:

(I) $h(-d) = m$, where $m \in \{5, 7, 9, \dots, 23\}$ is fixed;

(II) $d_2(m) \le d \le 10^{850}$, where $d_2(m)$ is given in Table 3 near the end of this subsection.

Our goal is to show that (5.7) is true for d with a suitable choice of k and s_0. For this purpose, we need a small zero, $s_0 = 1/2 + it_0$, of $L(s, \chi)$. Weinberger [31] has computed several such zeros, each corresponding to a different value of k. In this section, we use

$$k = 115147 \quad \text{and} \quad t_0 = 0.003157614$$

where the absolute error in t_0 is less than 10^{-8}, but we only make use of the first 4 significant places. From (5.4)–(5.6) we then have

$$\xi_1 = 0.001579, \quad \xi_2 = 0.02875, \quad \xi_3 = 555.8,$$

where these approximations are accurate to the number of places shown.

Since $m \le 23$ and $d \ge d_2(m) \ge 10^{63}$ by Table 3, the right-hand side of (5.8) is positive. Hence, letting $t = t_0$ in Lemma 8, we see from (5.8) that $|A_d(s_0)|$ does not vanish. Thus, using (5.8) and the corollary to Lemma 9 with $t = t_0$, we have

$$\frac{|U_d(s_0)|}{\xi_3 |A_d(s_0)|} \le R_2(d),$$

where

$$R_2(d) = \frac{8k^{1/2}\log k\,(m - 1 + e^{-\pi(d^{1/2} - \sqrt{3})/(2k)}) \prod_{p \mid k}(2 + 3p^{-3/2})}{\xi_3 3^{1/4}\pi^{1/2} d^{1/4}(1 - (m-1)/(d/4)^{1/(2m)})}.$$

Clearly $R_2(d)$ is decreasing in d so that

$$\frac{|U_d(s_0)|}{\xi_3 |A_d(s_0)|} \le R_2(d_2(m)) \tag{5.18}$$

since $d \ge d_2(m)$. Upper bounds for $R_2(d_2(m))$ are given in Table 3.

Since $5 \le m \le 23$ we have $\max\{4e^{2m}, 4(m-1)^{2m}\} = 4(m-1)^{2m} < 10^{63} < d_2(m)$. Hence, all of the hypotheses of Lemma 8 hold for $t = t_0$ and $m \in \{5, 7, \dots, 23\}$. We deduce from (5.9) that

$$|\arg A_d(s_0)| \le \alpha_2(d),$$

where

$$\alpha_2(d) = \frac{t(1-1/m)\log(d/4)}{(d/4)^{1/(2m)} - (m-1)}.$$

It is easy to see that $\alpha_2(d)$ is decreasing for $d \geq 4e$, so we certainly have

$$|\arg A_d(s_0)| \leq \alpha_2(d_2(m)) \tag{5.19}$$

since $d \geq d_2(m)$. Upper bounds for $\alpha_2(d_2(m))$ are given in Table 3.

Let $\beta_2(m)$ be defined by

$$\beta_2(m) = \xi_1 \log d_2(m) + \xi_2 - \alpha_2(d_2(m))$$

and $\gamma_2(m)$ be defined by

$$\gamma_2(m) = \xi_1 \log 10^{850} + \xi_2 + \alpha_2(d_2(m)).$$

Lower bounds for $\beta_2(m)$ and upper bounds for $\gamma_2(m)$ are given in Table 3. Using Table 3, the fact that $d_2(m) \leq d$, and (5.19), we have

$$0 < \beta_2(m) \leq \xi_1 \log d + \xi_2 + \arg A_d(s_0).$$

Using the fact that $d \leq 10^{850}$, (5.19), and Table 3, we have

$$\xi_1 \log d + \xi_2 + \arg A_d(s_0) \leq \gamma_2(m) < \pi.$$

Hence,

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| \geq \min\{|\sin \beta_2(m)|, |\sin \gamma_2(m)|\}. \tag{5.20}$$

Lower bounds for $|\sin \beta_2(m)|$ and $|\sin \gamma_2(m)|$ are given in Table 3.

From (5.20) and Table 3 we deduce that

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| > R_2(d_2(m)).$$

In light of (5.18), (5.7) follows immediately. Since (5.3) is false, we conclude that $h(-d) \neq m$ for $m \in \{5, 7, \ldots, 23\}$ and $d_2(m) \leq d \leq 10^{850}$. Note that from Table 2 in §2, we have certainly covered the range $d_2(m) \leq d \leq d_3(m)$.

Table 3. $h(-d) \neq m$ for $d_2(m) \leq d \leq 10^{850}$

m	$d_2(m)$	$R_2(d_2(m))$	$\alpha_2(d_2(m))$	$\beta_2(m)$	$\lvert\sin \beta_2(m)\rvert$	$\gamma_2(m)$	$\lvert\sin \gamma_2(m)\rvert$
5	10^{65}	$2.2 \cdot 10^{-14}$	$1.4 \cdot 10^{-7}$	0.264	0.26	3.121	0.020
7	10^{65}	$3.3 \cdot 10^{-14}$	$1.1 \cdot 10^{-5}$	0.264	0.26	3.121	0.020
9	10^{66}	$2.5 \cdot 10^{-14}$	$9.9 \cdot 10^{-5}$	0.268	0.26	3.121	0.020
11	10^{68}	$9.9 \cdot 10^{-15}$	$3.9 \cdot 10^{-4}$	0.275	0.27	3.121	0.020
13	10^{73}	$6.8 \cdot 10^{-16}$	$8.2 \cdot 10^{-4}$	0.293	0.28	3.121	0.020
15	10^{76}	$1.5 \cdot 10^{-16}$	$1.7 \cdot 10^{-3}$	0.303	0.29	3.122	0.019
17	10^{79}	$3.1 \cdot 10^{-17}$	$2.9 \cdot 10^{-3}$	0.312	0.30	3.124	0.017
19	10^{79}	$3.8 \cdot 10^{-17}$	$5.6 \cdot 10^{-3}$	0.310	0.30	3.126	0.015
21	10^{79}	$4.8 \cdot 10^{-17}$	$1.1 \cdot 10^{-2}$	0.304	0.29	3.132	0.009
23	10^{79}	$6.8 \cdot 10^{-17}$	$2.0 \cdot 10^{-2}$	0.295	0.29	3.141	$5.9 \cdot 10^{-4}$

The $d_2(m)$ listed in Table 3 are far too large to allow the completion of the class number m problem using the computationally intensive sieve in §6 to investigate the range $d < d_2(m)$. Pushing the preceding arguments to their limit, it is possible to produce a $d_2^*(m)$ ($< d_2(m)$) such that if $d_2^*(m) < d < d_3(m)$, then $h(-d) \neq m$. Approximate values of $d_2^*(m)$ are given in Table 4. Note, however, that for odd $m > 9$, $d_2^*(m)$ is also far too large to allow one to complete the class number problem by using a computationally intensive sieve. Some type of further argument is necessary. In §5.2 and §5.3, the idea of partitioning minima is introduced, which allows us to reduce the range of admissible discriminants so that a computationally intensive sieve can be used. It turns out that our new arguments are powerful enough to rule out the range $d_2^*(m) < d < d_2(m)$, obviating the need to push the arguments of this subsection to their tedious limits.

Table 4. Lower bounds on d obtained without partition estimates

m	$d_2^*(m)$	$4(m-1)^{2m}$
5	$4.5 \cdot 10^{12}$	$4.2 \cdot 10^{6}$
7	$4.4 \cdot 10^{14}$	$3.1 \cdot 10^{11}$
9	$2.2 \cdot 10^{18}$	$7.2 \cdot 10^{16}$
11	$7.0 \cdot 10^{23}$	$4.0 \cdot 10^{22}$
13	$8.7 \cdot 10^{29}$	$4.6 \cdot 10^{28}$
15	$2.1 \cdot 10^{36}$	$9.7 \cdot 10^{34}$
17	$8.4 \cdot 10^{42}$	$3.5 \cdot 10^{41}$
19	$5.3 \cdot 10^{49}$	$2.0 \cdot 10^{48}$
21	$5.0 \cdot 10^{56}$	$1.8 \cdot 10^{55}$
23	$6.8 \cdot 10^{63}$	$2.3 \cdot 10^{62}$

The last column of Table 4 underscores the fact that the methods in this subsection do not suffice when $m > 9$. This column arises from the fact that the denominator on the right-hand side of (5.9) must be positive. In other words, $d > 4(m-1)^{2m}$. Note that when $m > 9$, the last column of Table 4 precludes the use of the computationally intensive sieve in §6, no matter how sharp we make the other estimates in this subsection.

5.2. *The range* $d_1(m) \le d \le d_2(m)$

LEMMA 10. *Let* $t \in \mathbb{R}^+$. *Assume that* $m = h(-d)$ *is odd and the partition* $\mathcal{P}$ *covers* M_d *in the sense of Definition* 2. *Then*

$$|A_d(1/2+it)| \ge 1 - 2 \sum_{a \in M_d^*} l_a(d)^{-1/2}. \tag{5.21}$$

Furthermore, *if the right-hand side of* (5.21) *is positive*, *then*

$$|\arg A_d(1/2+it)| \le \frac{2t \sum_{a \in M_d^*} f(l_a(d))}{1 - 2\sum_{a \in M_d^*} l_a(d)^{-1/2}} \tag{5.22}$$

where f *is the nonincreasing function defined by*

$$f(x)=\begin{cases} 2/e & \text{for } 0\le x\le e^2,\\ x^{-1/2}\log x & \text{for } x>e^2.\end{cases}$$

Proof. The proof is identical to the proof of Lemma 8 except that inequality (3.4) is replaced with the partition inequalities $l_a(d)\le a\le u_a(d)$. ∎

LEMMA 11. *Let* $t\in\mathbb{R}$ *and* $m=h(-d)>1$. *Suppose that* m *is odd and the partition* $\mathcal{P}$ *covers* M_d *in the sense of Definition* 2. *If* k *is an odd prime, then*

$$|U_d(1/2+it)|\le \frac{24m(\pi\sqrt{3}\,k+2)}{d^{1/4}(\pi\sqrt{3})^{3/2}k^2}e^{-\pi\sqrt{3}k/2} + \frac{16\sqrt{k}(\pi\sqrt{d}+2k)}{\pi^{3/2}d}e^{-\pi\sqrt{d}/(2k)}+\frac{16}{d^{1/4}}\sum_{a\in M_d^*}F\left(\frac{\pi\sqrt{d}}{u_a(d)k}\right).$$

Proof. We use Lemma 9. In the outer sum of (5.15), $r=1$ or $r=k$. By Lemma 7, F is decreasing, so we can bound the inner summands from above by using upper bounds on a. When $r=k$ we use the general upper bound $a\le(d/3)^{1/2}$ of (3.4). When $r=1$ and $a>1$ we use the partition inequality $a\le u_a(d)$. Thus, we have

$$|U_d(1/2+it)| \le \frac{12m}{d^{1/4}k^{1/2}}F(\pi\sqrt{3}\,k)+\frac{8}{d^{1/4}}F\left(\frac{\pi\sqrt{d}}{k}\right)+\frac{16}{d^{1/4}}\sum_{a\in M_d^*}F\left(\frac{\pi\sqrt{d}}{u_a(d)k}\right).$$

Using Lemma 6 with $N=0$ and $x=\pi\sqrt{3}\,k$ and the inequality $\log(1+x)<x$ (valid for $x>0$), we have

$$|F(\pi\sqrt{3}\,k)|\le\frac{2}{\sqrt{\pi}\,3^{1/4}\sqrt{k}}\left(1+\frac{2}{\pi\sqrt{3}\,k}\right)e^{-\pi\sqrt{3}k/2}.$$

Similarly, for $x=\pi\sqrt{d}/k$, we obtain

$$|F(\pi\sqrt{d}/k)|\le\frac{2\sqrt{k}}{\sqrt{\pi}\,d^{1/4}}\left(1+\frac{2k}{\pi\sqrt{d}}\right)e^{-\pi\sqrt{d}/(2k)}.$$

Lemma 11 follows easily. ∎

For the remainder of §5.2, we assume there exists a discriminant $-d$ satisfying the following properties:

(I) $h(-d)=m$, where $m\in\{5,7,9,\dots,23\}$ is fixed;

(II) $d_1(m)\le d\le d_2(m)$, where $d_1(m)$ is given in Table 5 near the end of this subsection and $d_2(m)$ is given in Table 3.

Our goal is to show that (5.7) is true for d with a suitable choice of k and s_0. In a manner similar to §5.1, we let $s_0 = 1/2 + it_0$ and use

$$k = 17923 \quad \text{and} \quad t_0 = 0.030985799.$$

Weinberger [31] has shown that the error in t_0 is less than 10^{-8} but we only use the first 5 significant digits. From (5.4)–(5.6) we then have

$$\xi_1 = 0.01549, \quad \xi_2 = 0.2216, \quad \xi_3 = 57.1,$$

where all approximations are accurate to the number of places shown.

As in §3, let M_d denote the multiset of minima for $-d$. Turning to Appendix B, note that each entry in Table B$\frac{m-3}{2}$ (e.g., when $m = 15$, refer to Table B6) consists of $(m-1)/2$ pairs of functions $(l_a(d), u_a(d))$, each increasing in d, corresponding to the $(m-1)/2$ elements in M_d^*. In other words, each entry of Table B$\frac{m-3}{2}$ is a partition for d according to Definition 1 in §3. Since $d \geq d_1(m) \geq 2^{m+2}$ from Table 5, the arguments at the end of §3 show that there is some partition for d in Table B$\frac{m-3}{2}$ which covers M_d in the sense of Definition 2 in §3. We henceforth denote this partition by $\mathcal{P}_1$.

Applying the first part of Lemma 10 with $t = t_0$ and $\mathcal{P} = \mathcal{P}_1$, we have

$$(5.23) \qquad \begin{aligned} |A_d(s_0)| &\geq 1 - 2 \sum_{(l_a, u_a) \in \mathcal{P}_1} l_a(d)^{-1/2} \\ &\geq \min_{\mathcal{P}} \Big(1 - 2 \sum_{(l_a, u_a) \in \mathcal{P}} l_a(d)^{-1/2}\Big), \end{aligned}$$

where the minimum is henceforth understood to be over all partitions $\mathcal{P}$ for d occurring in Table B$\frac{m-3}{2}$. Since $l_a(d)$ is increasing in d, we replace d with $d_1(m)$ in the lower bound of (5.23) giving the new lower bound

$$(5.24) \qquad \min_{\mathcal{P}} \Big(1 - 2 \sum_{(l_a, u_a) \in \mathcal{P}} l_a(d_1(m))^{-1/2}\Big) > 0,$$

with the positivity following from direct computation.

From (5.23), (5.24), and Lemma 11 with $t = t_0$, $k = 17923$, and $\mathcal{P} = \mathcal{P}_1$, we have

$$\frac{|U_d(s_0)|}{\xi_3 |A_d(s_0)|} \leq R_1(d, \mathcal{P}_1),$$

where we define

$$\begin{aligned} R_1(d, \mathcal{P}) = \bigg\{ &\frac{24m(\pi\sqrt{3}\,k + 2)}{d^{1/4}(\pi\sqrt{3})^{3/2}k^2} e^{-\pi\sqrt{3}k/2} + \frac{16(\pi\sqrt{d} + 2k)}{\pi^{3/2} d} e^{-\pi\sqrt{d}/(2k)} \\ &+ \frac{16}{d^{1/4}} \sum_{(l_a, u_a) \in \mathcal{P}} F\Big(\frac{\pi\sqrt{d}}{u_a(d) k}\Big)\bigg\} \Big/ \Big\{\xi_3 \Big(1 - 2 \sum_{(l_a, u_a) \in \mathcal{P}} l_a(d)^{-1/2}\Big)\Big\}. \end{aligned}$$

Thus from (5.23) and (5.24) we have

$$\frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|} \leq \max_{\mathcal{P}} R_1(d, \mathcal{P}),$$

where the maximum is henceforth understood to be over all partitions $\mathcal{P}$ for d occurring in Table B$\frac{m-3}{2}$.

Note that for each u_a occurring in each of the partitions for d in Table B$\frac{m-3}{2}$, $\sqrt{d}/u_a(d)$ is a nondecreasing function of d. It follows from Lemma 7 that $F(\pi\sqrt{d}/(u_a(d)k))$ is a nonincreasing function of d. Therefore, the numerator of $R_1(d, \mathcal{P})$ is decreasing in d for any partition $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$. Thus, with the aid of (5.24), we have

$$\frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|} \leq \max_{\mathcal{P}} R_1(d_1(m), \mathcal{P}) \tag{5.25}$$

since $d \geq d_1(m)$. Upper bounds for $\max_{\mathcal{P}} R_1(d_1(m), \mathcal{P})$ can be found in Table 5. In order to produce these approximations, we need estimates for the functions F evaluated at small positive arguments. Such estimates are easily obtained by applying Lemma 6 with suitably large N.

Note that from (5.23) and (5.24) we know the right-hand side of (5.21) is positive for all partitions for d appearing in Table B$\frac{m-3}{2}$. Applying the second part of Lemma 10 with $t = t_0$ and $\mathcal{P} = \mathcal{P}_1$, we obtain

$$|\arg A_d(s_0)| \leq \alpha_1(d, \mathcal{P}_1),$$

where we define

$$\alpha_1(d, \mathcal{P}) = \frac{2t_0 \sum_{(l_a, u_a) \in \mathcal{P}} f(l_a(d))}{1 - 2\sum_{(l_a, u_a) \in \mathcal{P}} l_a(d)^{-1/2}}.$$

It follows from the rightmost inequality in (5.23) and (5.24) that

$$|\arg A_d(s_0)| \leq \max_{\mathcal{P}} \alpha_1(d, \mathcal{P}), \tag{5.26}$$

where the maximum is over all partitions $\mathcal{P}$ for d occurring in Table B$\frac{m-3}{2}$.

Now $\alpha_1(d, \mathcal{P})$ is decreasing in d for any fixed partition $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$. Hence,

$$|\arg A_d(s_0)| \leq \max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P}) \tag{5.27}$$

since $d \geq d_1(m)$. Table 5 contains upper bounds for $\max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P})$.

Let $\beta_1(m)$ be defined by

$$\beta_1(m) = \xi_1 \log d_1(m) + \xi_2 - \max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P})$$

and $\gamma_1(m)$ be defined by

$$\gamma_1(m) = \xi_1 \log d_2(m) + \xi_2 + \max_{\mathcal{P}} \alpha_1(10^{37}, \mathcal{P}).$$

Lower bounds for $\beta_1(m)$ and upper bounds for $\gamma_1(m)$ are given in Table 5. Using Table 5, the fact that $d_1(m) \le d$ and (5.27), we have

$$0 < \beta_1(m) \le \xi_1 \log d + \xi_2 + \arg A_d(s_0).$$

In the other direction, we claim that

$$\xi_1 \log d + \xi_2 + \arg A_d(s_0) \le \gamma_1(m) < \pi.$$

Note that from Tables 3 and 5 we have

$$d_1(m) < 10^{37} < d_2(m).$$

If $10^{37} < d$, our claim follows from the fact that $d \le d_2(m)$, (5.26) coupled with the fact that $\alpha_1(d, \mathcal{P})$ is decreasing in d for any fixed partition $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$, and Table 5. When $d \le 10^{37}$, note that by (5.27) we have

$$\xi_1 \log d + \xi_2 + \arg A_d(s_0) \le \xi_1 \log(10^{37}) + \xi_2 + \max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P}).$$

With the aid of Table 5, direct calculation shows that

$$\xi_1 \log(10^{37}) + \xi_2 + \max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P}) < \gamma_1(m).$$

Thus our claim is also true when $d \le 10^{37}$. Hence,

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| \ge \min\{|\sin \beta_1(m)|, |\sin \gamma_1(m)|\}.$$

From the preceding inequality and Table 5, we conclude that

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| > \max_{\mathcal{P}} R_1(d_1(m), \mathcal{P}).$$

In light of (5.25), (5.7) follows immediately. Since (5.3) is false, we conclude that $h(-d) \ne m$ for $m \in \{5, 7, \ldots, 23\}$ and $d_1(m) \le d \le d_2(m)$.

Table 5. $h(-d) \ne m$ for $d_1(m) \le d \le d_2(m)$

m	$d_1(m)$	$\max_{\mathcal{P}} R_1(d_1(m))$	$\max_{\mathcal{P}} \alpha_1(d_1(m), \mathcal{P})$	$\max_{\mathcal{P}} \alpha_1(10^{37}, \mathcal{P})$	$\beta_1(m)$	$\gamma_1(m)$
5	$3.6 \cdot 10^{11}$	0.555	0.041	$2.39 \cdot 10^{-4}$	0.592	2.542
7	$2.0 \cdot 10^{12}$	0.547	0.072	$1.89 \cdot 10^{-3}$	0.588	2.543
9	$7.9 \cdot 10^{12}$	0.519	0.134	$5.70 \cdot 10^{-3}$	0.547	2.583
11	$4.6 \cdot 10^{13}$	0.444	0.245	$1.15 \cdot 10^{-2}$	0.463	2.660
13	$4.9 \cdot 10^{14}$	0.291	0.448	$1.89 \cdot 10^{-2}$	0.297	2.846
15	$1.9 \cdot 10^{16}$	0.148	0.643	$2.77 \cdot 10^{-2}$	0.158	2.961
17	$1.2 \cdot 10^{18}$	0.062	0.794	$3.82 \cdot 10^{-2}$	0.072	3.079
19	$9.2 \cdot 10^{19}$	0.024	0.906	$5.06 \cdot 10^{-2}$	0.027	3.091
21	$7.7 \cdot 10^{21}$	0.009	0.991	$6.58 \cdot 10^{-2}$	0.010	3.107
23	$6.6 \cdot 10^{23}$	0.004	1.065	$8.47 \cdot 10^{-2}$	0.005	3.125

For $m \in \{5, 7, 9, 11, 13\}$, the $d_1(m)$ in Table 5 is small enough to allow the class number m problem to be completed by the computationally intensive sieve of §6. However, for $m \in \{15, 17, 19, 21, 23\}$, we have to resort to further refinements in §5.3.

5.3. *The range $d_0(m) \le d \le d_1(m)$*

LEMMA 12. *Let $t \in \mathbb{R}^+$. Assume that $m = h(-d)$ is odd and the partition $\mathcal{P}$ covers M_d in the sense of Definition 2. Let p be the smallest nonprincipal minimum in M_d. If l is a nonnegative integer such that the corresponding set $S_l = \{p, p^2, \ldots, p^l\} \subseteq M_d^*$ (where S_0 is understood to be the empty set), then*

$$|A_d(1/2 + it)| \ge \left|1 - 2\frac{1 - \lambda_p(t)^{l+1}}{1 - \lambda_p(t)}\right| - 2 \sum_{a \in M_d^* - S_l} l_a(d)^{-1/2}, \tag{5.28}$$

where $\lambda_p(t) = \chi(p)p^{-1/2-it}$. Furthermore, if $\Re A_d(1/2 + i\tau) > 0$ for $0 \le \tau \le t$ and the right-hand side of (5.28) is positive, then

$$|\arg A_d(1/2 + it)| \le \frac{2t(\log p|\sum_{j=1}^{l} j\lambda_p^j(t)| + \sum_a f(l_a(d)))}{\left|1 - 2\frac{1-\lambda_p^{l+1}(t)}{1-\lambda_p(t)}\right| - 2\sum_a l_a(d)^{-1/2}}, \tag{5.29}$$

where f is as defined in Lemma 10 and the sums are over all $a \in M_d^* - S_l$.

Proof. The proof uses arguments similar to those in the proofs of Lemmas 8 and 10. ■

For the remainder of §5.3, we assume there exists a discriminant $-d$ satisfying the following properties:

(I) $h(-d) = m$, where $m \in \{9, 11, \ldots, 23\}$ is fixed;

(II) $d_0(m) \le d \le d_1(m)$, where $d_0(m)$ is given in Table 6 near the end of this subsection and $d_1(m)$ is given in Table 5.

Our goal is to show that (5.7) is true for d with a suitable choice of k and s_0. As in §5.2, let $s_0 = 1/2 + it_0$ and use

$$k = 17923 \quad \text{and} \quad t_0 = 0.030986.$$

From (5.4)–(5.6) we then have

$$\xi_1 = 0.01549, \qquad \xi_2 = 0.2216, \qquad \xi_3 = 57.1,$$

where all approximations are accurate to the number of places shown.

For each partition $\mathcal{P}$ for d in Table B$\frac{m-3}{2}$, define $\ell_{\mathcal{P}}$ to be the number of powers of the smallest minima in M_d^* that can be shown to appear in $\mathcal{P}$ using the arguments of §3, if the number of such powers exceeds 2. Otherwise, set $\ell_{\mathcal{P}} = 0$. Denoting the smallest minimum in M_d^* by p, let $S_{\ell_{\mathcal{P}}} = \{p, p^2, \ldots, p^{\ell_{\mathcal{P}}}\}$ if $\ell_{\mathcal{P}} > 0$, and the empty set if $\ell_{\mathcal{P}} = 0$. Finally, let $\mathcal{P}^*$ denote the set of $(l_a(d), u_a(d))$ in $\mathcal{P}$ for which $a \notin S_{\ell_{\mathcal{P}}}$.

Since $d \geq d_0(m) > 2^{m+2}$ by Table 6, the arguments at the end of §3 show there is some partition in Table B$\frac{m-3}{2}$ which covers M_d in the sense of Definition 2 in §3. We henceforth denote this partiton by $\mathcal{P}_0$. Applying the first part of Lemma 12 with $t = t_0$, $\mathcal{P} = \mathcal{P}_0$, $l = \ell_{\mathcal{P}_0}$, and letting $\lambda_p = \lambda_p(t_0)$, we have

$$
\begin{aligned}
(5.30) \quad |A_d(s_0)| &\geq \left|1 - 2\frac{1-\lambda_p^{\ell_{\mathcal{P}_0}+1}}{1-\lambda_p}\right| - 2\sum_{(l_a,u_a)\in\mathcal{P}_0^*} l_a(d)^{-1/2} \\
&\geq \min_{\mathcal{P}}\left(\left|1 - 2\frac{1-\lambda_p^{\ell_{\mathcal{P}}+1}}{1-\lambda_p}\right| - 2\sum_{(l_a,u_a)\in\mathcal{P}^*} l_a(d)^{-1/2}\right),
\end{aligned}
$$

where the minimum is henceforth understood to be over all partitions $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$. Let $\lambda_q = \chi(q)q^{-1/2-it_0}$ and note that $l_a(d)$ and $u_a(d)$ are increasing in d. Note that we can find a further lower bound for the lower bound in (5.30) by evaluating the bound at all possible primes p. To this end, let

$$
\mathcal{Q} = \{q \text{ prime} : l_p(d_0(m)) \leq q \leq u_p(d_1(m)),\ (l_p, u_p) \in \mathcal{P}\}.
$$

Now, a new lower bound for (5.30) is given by

$$
(5.31) \qquad \min_{\mathcal{P}}\min_{q\in\mathcal{Q}}\left(\left|1 - 2\frac{1-\lambda_q^{\ell_{\mathcal{P}}+1}}{1-\lambda_q}\right| - 2\sum_{(l_a,u_a)\in\mathcal{P}^*} l_a(d_0(m))^{-1/2}\right) > 0,
$$

with the positivity following from a direct computation. From (5.30), (5.31), and Lemma 11 with $t = t_0$, $k = 17923$, and $\mathcal{P} = \mathcal{P}_0$, we have

$$
\frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|} \leq R_0(d, \mathcal{P}_0),
$$

where we define

$$
R_0(d,\mathcal{P}) = \frac{\frac{24m(\pi\sqrt{3}k+2)}{d^{1/4}(\pi\sqrt{3})^{3/2}k^2}e^{-\pi\sqrt{3}k/2} + \frac{16(\pi\sqrt{d}+2k)}{\pi^{3/2}d}e^{-\pi\sqrt{d}/(2k)} + \frac{16}{d^{1/4}}\sum_{\mathcal{P}} F\big(\frac{\pi\sqrt{d}}{u_a(d)k}\big)}{\xi_3\big(\min_{q\in\mathcal{Q}}\big(\big|1 - 2\frac{1-\lambda_q^{\ell_{\mathcal{P}}+1}}{1-\lambda_q}\big| - 2\sum_{\mathcal{P}^*} l_a(d_0(m))^{-1/2}\big)\big)}.
$$

Thus from (5.30) and (5.31) we have

$$
\frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|} \leq \max_{\mathcal{P}} R_0(d, \mathcal{P}),
$$

where the maximum is henceforth understood to be over all partitions $\mathcal{P}$ for d occurring in Table B$\frac{m-3}{2}$.

Note that for each u_a occurring in each of the partitions for d in Table B$\frac{m-3}{2}$, $\sqrt{d}/u_a(d)$ is a nondecreasing function of d. Let $\mathcal{P}$ be any of the partitions for d appearing in Table B$\frac{m-3}{2}$. It follows from Lemma 7 that $F(\pi\sqrt{d}/(u_a(d)k))$ is a nonincreasing function of d. Therefore, the numerator of $R_0(d,\mathcal{P})$ is decreasing in d for any partition $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$. Hence with the aid of (5.31) we have

$$\frac{|U_d(s_0)|}{\xi_3|A_d(s_0)|} \leq \max_{\mathcal{P}} R_0(d_0(m),\mathcal{P}). \tag{5.32}$$

Upper bounds for $\max_{\mathcal{P}} R_0(d_0(m),\mathcal{P})$ can be found in Table 6. In order to produce these approximations, we need estimates for the functions F evaluated at small positive arguments. Such estimates are easily obtained by applying Lemma 6 with suitably large N.

Note that from (5.30) and (5.31) we know the right-hand side of (5.28) is positive for all partitions for d appearing in Table B$\frac{m-3}{2}$. Suppose that $0 \leq \tau \leq t_0$. If $\ell_{\mathcal{P}_0} = 0$, then we have

$$\begin{aligned}|A_d(1/2+i\tau)-1| &\leq 2\sum_{(l_a,u_a)\in\mathcal{P}_0} l_a(d)^{-1/2}\\ &\leq \max_{\mathcal{P}} 2\sum_{(l_a,u_a)\in\mathcal{P}} l_a(d_0(m))^{-1/2} < 1,\end{aligned}$$

by a direct calculation. Hence, $\Re A_d(1/2+i\tau) > 0$ when $\ell_{\mathcal{P}_0} = 0$. On the other hand, suppose $\ell_{\mathcal{P}_0} \neq 0$. It follows from Table B$\frac{m-3}{2}$ and Table 5 that the smallest minimum $p \in M_d^*$ satisfies $p \leq (d_1(m)/4)^{1/6} \leq 7406$. Hence, $0 < 2t_0 \log p < \pi$, and it is not difficult to see that

$$\begin{aligned}&\Re A_d(1/2+i\tau)\\ &\quad\geq 1 - 2\sum_{j=1}^{\ell_{\mathcal{P}_0}} p^{-j/2} + \frac{2+2\cos(2t_0\log p)}{p} - 2\sum_{\mathcal{P}_0^*} l_a(d_0(m))^{-1/2}\\ &\quad\geq \min_{\mathcal{P}}\min_{q\in\mathcal{Q}}\left(1 - 2\sum_{j=1}^{\ell_{\mathcal{P}}} q^{-j/2} + \frac{2+2\cos(2t_0\log q)}{q} - 2\sum_{\mathcal{P}^*} l_a(d_0(m))^{-1/2}\right).\end{aligned}$$

It then follows by direct computation that $\Re A_d(1/2+i\tau) > 0$ for $0 \leq \tau \leq t_0$. Hence, we may apply the second part of Lemma 12 with $t = t_0$, $\mathcal{P} = \mathcal{P}_0$, and $l = \ell_{\mathcal{P}_0}$ to obtain

$$|\arg A_d(s_0)| \leq \frac{2t_0(\log p|\sum_{j=1}^{\ell_{\mathcal{P}_0}} j\lambda_p^j| + \sum_{(l_a,u_a)\in\mathcal{P}_0^*} f(l_a(d)))}{\left|1 - 2\frac{1-\lambda_p^{\ell_{\mathcal{P}_0}+1}}{1-\lambda_p}\right| - 2\sum_{(l_a,u_a)\in\mathcal{P}_0^*} l_a(d)^{-1/2}}.$$

Hence, since $l_a(d)$ and $u_a(d)$ are increasing in d, we have

$$|\arg A_d(s_0)| \leq \alpha_0(d, \mathcal{P}_0),$$

where we define

$$\alpha_0(d,\mathcal{P}) = \max_{q\in\mathcal{Q}} \left(\frac{2t_0(\log q|\sum_{j=1}^{\ell_{\mathcal{P}}} j\lambda_q^j| + \sum_{(l_a,u_a)\in\mathcal{P}^*} f(l_a(d)))}{\left|1 - 2\frac{1-\lambda_q^{\ell_{\mathcal{P}}+1}}{1-\lambda_q}\right| - 2\sum_{(l_a,u_a)\in\mathcal{P}^*} l_a(d)^{-1/2}} \right).$$

It follows from (5.30) and (5.31) that

$$|\arg A_d(s_0)| \leq \max_{\mathcal{P}} \alpha_0(d, \mathcal{P}).$$

Since f is nonincreasing and $l_p(d)$ is increasing in d, we know $\alpha_0(d, \mathcal{P})$ is decreasing in d for any fixed partition $\mathcal{P}$ for d appearing in Table B$\frac{m-3}{2}$. Hence,

$$|\arg A_d(s_0)| \leq \max_{\mathcal{P}} \alpha_0(d_0(m), \mathcal{P}) \tag{5.33}$$

since $d \geq d_0(m)$. Table 6 contains upper bounds for $\max_{\mathcal{P}} \alpha_0(d_0(m), \mathcal{P})$.

Let $\beta_0(m)$ be defined by

$$\beta_0(m) = \xi_1 \log d_0(m) + \xi_2 - \max_{\mathcal{P}} \alpha_0(d_0(m), \mathcal{P})$$

and $\gamma_0(m)$ be defined by

$$\gamma_0(m) = \xi_1 \log d_1(m) + \xi_2 + \max_{\mathcal{P}} \alpha_0(d_0(m), \mathcal{P}).$$

Lower bounds for $\beta_0(m)$ and upper bounds for $\gamma_0(m)$ are given in Table 6. Using Table 6, the fact that $d_0(m) \leq d$, and (5.33), we have

$$0 < \beta_0(m) \leq \xi_1 \log d + \xi_2 + \arg A_d(s_0).$$

Using the fact that $d \leq d_1(m)$, (5.33), and Table 6, we have

$$\xi_1 \log d + \xi_2 + \arg A_d(s_0) \leq \gamma_0(m) < \pi.$$

Hence,

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| \geq \min\{|\sin \beta_0(m)|, |\sin \gamma_0(m)|\}.$$

From the preceding inequality and Table 6, we conclude that

$$|\sin(\xi_1 \log d + \xi_2 + \arg A_d(s_0))| > \max_{\mathcal{P}} R_0(d_0(m), \mathcal{P}).$$

In light of (5.32), (5.7) follows immediately. Since (5.3) is false, we conclude that $h(-d) \neq m$ for $m \in \{9, 11, \ldots, 23\}$ and $d_0(m) \leq d \leq d_1(m)$.

Table 6. $h(-d) \neq m$ for $d_0(m) \leq d \leq d_1(m)$

m	$d_0(m)$	$\max_{\mathcal{P}} R_0(d_0(m))$	$\max_{\mathcal{P}} \alpha_0(d_0(m), \mathcal{P})$	$\beta_0(m)$	$\gamma_0(m)$
9	$6.4 \cdot 10^{12}$	0.553	0.089	0.589	0.771
11	$2.2 \cdot 10^{13}$	0.555	0.100	0.597	0.810
13	$4.2 \cdot 10^{13}$	0.556	0.116	0.591	0.862
15	$9.4 \cdot 10^{13}$	0.557	0.128	0.591	0.931
17	$1.9 \cdot 10^{14}$	0.556	0.138	0.592	1.005
19	$3.5 \cdot 10^{14}$	0.555	0.149	0.591	1.083
21	$6.5 \cdot 10^{14}$	0.542	0.173	0.576	1.176
23	$10.6 \cdot 10^{14}$	0.548	0.177	0.580	1.249

A comparison of Tables 5 and 6 shows that the gap between $d_0(m)$ and $d_1(m)$ is increasing rapidly as m increases from 9 to 23.

6. The low range. In this section, we complete the proof of Theorem 1, the statement of which appears in §1. Using the results of §2 and §5 (see Tables 5 and 6 in particular), it suffices to find all negative fundamental discriminants $-d$ with $h(-d) \in \{5, 7, \ldots, 23\}$ such that $d \leq 1.1 \cdot 10^{15}$.

To this end, we first consider the small discriminants $d \leq 7.5 \cdot 10^6$ for which an exhaustive search is employed. For each d in this range, we computed the class number by counting the number of reduced forms of discriminant $-d$. In other words, we searched for integers a, b, and c with $0 < a < (d/3)^{1/2}$ and $c = (b^2 - d)/(4a)$ such that either $-a < b \leq a < c$ or $0 \leq b \leq a = c$. This straightforward approach required only 32 minutes on a Cray C90, rendering further optimization unnecessary. A complete listing of the negative fundamental discriminants with odd class numbers m in the range $1 \leq m \leq 23$ that we found in this search is given in Appendix A. It is worth noting that Buell [6] had previously computed class numbers of imaginary quadratic number fields for $d \leq 4 \cdot 10^6$, and our results agree perfectly with his in this range. Furthermore, in recent unpublished work, Buell has independently verified our results up to $7.5 \cdot 10^6$, using our method of separating minima that was introduced in §3.

The largest value of d found in the above search was $d = 90787$. Thus, to complete the proof of Theorem 1 we need to show that there is no negative fundamental discriminant $-d$ with odd $h(-d) \leq 23$ in the range

$$7.5 \cdot 10^6 \leq d \leq 1.1 \cdot 10^{15}. \tag{6.1}$$

It is infeasible to directly check all d in the range (6.1). Instead we used partition-type information in the following form.

LEMMA 13. *If $d > 8$ and $h(-d) \leq 23$ is odd, then*

(i) $(-d|p) \neq 0$ *for all primes* $p < d$;
(ii) $(-d|p) = -1$ *for all primes* $p \leq (d/4)^{1/23}$;

(iii) $(-d|p) = -1$ *for all primes* $p \le (d/4)^{1/6}$ *with at most one exception*; *and*

(iv) $(-d|p) = -1$ *for all primes* $p \le (d/4)^{1/4}$ *with at most two exceptions.*

Proof. Item (i) follows directly from (2.1) and item (ii) follows from Lemmas 1 and 5. If two odd primes $p, q \le (d/4)^{1/6}$ satisfy $(-d|p) = 1$ and $(-d|q) = 1$, then $p, p^2, p^3, q, q^2, q^3, pq, p^2q, pq^2 \in M_d$, which implies $h(-d) \ge 25$ by Lemmas 1 and 4. Thus, item (iii) is true. Lastly, if three primes $p, q, r \le (d/4)^{1/4}$ satisfy $(-d|p) = (-d|q) = (-d|r) = 1$, then Lemmas 1 and 4 give $p, p^2, q, q^2, r, r^2, pq, pr, qr \in M_d$, which implies $h(-d) \ge 25$. This proves item (iv). ∎

Now, we can build up a substantially smaller set (than (6.1)) of possible d by using the Chinese Remainder Theorem on the residue requirements implicit in (i)–(iv) for a set of small primes. Consider an interval $d_0 \le d \le d_1$. Let p_i denote the ith prime number and choose k such that $m = 8\prod_{i=2}^{k} p_i > d_1$. Here, m is the Chinese Remainder Theorem modulus when constructing integers d based on the vector of Kronecker symbols $\langle(-d|p_i)\rangle_{1\le i\le k}$. Let

$$S_k(d) = \{\vec{\varepsilon} \in \{1, -1\}^k : \vec{\varepsilon} = \langle(-d|p_i)\rangle_{1\le i\le k} \text{ satisfies (i)–(iv)}\}$$

and for each $\vec{\varepsilon} \in \{1, -1\}^k$ let

$$D_{\vec{\varepsilon}} = \{0 \le d < m : \langle(-d|p_i)\rangle_{1\le i\le k} = \vec{\varepsilon}\}.$$

To search all possible d in $d_0 \le d \le d_1$ we use the Chinese Remainder Theorem to construct $D_{\vec{\varepsilon}}$ for each $\vec{\varepsilon} \in S_k(d_0)$. The requirement that $(-d|p) = -1$ (or $(-d|p) = 1$) implies that d is in one of $(p-1)/2$ residue classes mod p for odd primes p, and in one residue class mod 8 for $p = 2$. Thus, $|D_{\vec{\varepsilon}}| = \prod_{i=2}^{k}(p_i - 1)/2 \approx d_1/2^{k+3}$. Each $d \in \bigcup_{\vec{\varepsilon}\in S_k(d_0)} D_{\vec{\varepsilon}}$ is then checked using the necessary conditions (i)–(iv) for the primes $\{p_{k+1}, \ldots, p_l\}$ for l suitably chosen whereby no d satisfies the conditions. If l exists, then there is no fundamental discriminant $-d$ with $d_0 \le d \le d_1$ and $h(-d) \in \{1, 3, \ldots, 23\}$. We use this approach on (6.1) by dividing it up into 3 subintervals corresponding to $k = 10, 12, 13$.

First, consider $2.9 \cdot 10^{13} \le d \le 1.1 \cdot 10^{15}$. Take $k = 13$ so that $m = 1217001054108840 > 1.1 \cdot 10^{15}$. The bounds in (ii), (iii) and (iv) applied to $d_0 = 2.9 \cdot 10^{13}$ are 3, 137, and 1604, respectively. Therefore, $(-d|2) = (-d|3) = -1$ and at most one of the 11 primes in $\{p_3, \ldots, p_{13}\}$ satisfies $(-d|p) = 1$. Hence, $|S_{13}(d_0)| = 12$ resulting in at most $1.3 \cdot 10^{11}$ possible occurrences of d mod m. Using $l = 56$, these were eliminated in 95 minutes on a Cray C90. As a check on the accuracy of the computer program, we printed out the last holdout, namely $d = 123461955393043$. Note that $(-d|p) = -1$ for all primes $p \le p_{56} = 263$ except for $p = 19$, 179 and 263.

Next, consider $2.58 \cdot 10^{10} \le d \le 2.9 \cdot 10^{13}$. Take $k = 12$ so that $m = 29682952539240 > 2.9 \cdot 10^{13}$. The bounds in (ii), (iii) and (iv) applied to $d_0 = 2.58 \cdot 10^{10}$ are 2, 43, and 283, respectively. Therefore, $(-d|2) = -1$ and at most one of the 11 primes in $\{p_2, \ldots, p_{12}\}$ satisfies $(-d|p) = 1$. Hence, $|S_{12}(d_0)| = 12$ resulting in at most $6.5 \cdot 10^9$ possible occurrences of $d \bmod m$. Using $l = 51$, these were eliminated in 7.4 minutes on a Cray C90. Again as a check, note that for $d = 7647157072003$ we have $(-d|p) = -1$ for all primes $p \le p_{51} = 233$ except for $p = 43$, 67 and 233.

Lastly, consider $7.5 \cdot 10^6 \le d \le 2.58 \cdot 10^{10}$. Take $k = 10$ so that $m = 25878772920 > 2.58 \cdot 10^{10}$. The bounds in (ii), (iii) and (iv) applied to $d_0 = 7.5 \cdot 10^6$ are 1.8, 11, and 37, respectively. In this case, $|S_{10}(d_0)| = 46$ since among the first 10 primes there are 11 ways to have at most one exception and $5 \cdot 5 + \binom{5}{2}$ ways to have exactly two exceptions. In order to eliminate the $9.2 \cdot 10^7$ constructed values of d with primes greater than 37, we actually count the number of minima constructed with the primes satisfying $(-d|p) = 1$. Note that Lemmas 2 and 3 imply that a prime in the range $37 < p \le 1369 < (d_0/4)^{1/2}$ satisfying $(-d|p) = 1$ accounts for an additional 2 minima in M_d. When the count exceeds 23, the value of d can be eliminated.

Appendix A. Negative fundamental discriminants for class numbers $1, 3, 5, \ldots, 23$

Table A1. Values of d with $h(-d) = 1$

3	4	7	8	11	19	43	67	163

Table A2. Values of d with $h(-d) = 3$

23	31	59	83	107	139	211	283
307	331	379	499	547	643	883	907

Table A3. Values of d with $h(-d) = 5$

47	79	103	127	131	179	227	347	443
523	571	619	683	691	739	787	947	1051
1123	1723	1747	1867	2203	2347	2683		

Table A4. Values of d with $h(-d) = 7$

71	151	223	251	463	467	487	587	811
827	859	1163	1171	1483	1523	1627	1787	1987
2011	2083	2179	2251	2467	2707	3019	3067	3187
3907	4603	5107	5923					

Table A5. Values of d with $h(-d) = 9$

199	367	419	491	563	823	1087	1187	1291
1423	1579	2003	2803	3163	3259	3307	3547	3643
4027	4243	4363	4483	4723	4987	5443	6043	6427
6763	6883	7723	8563	8803	9067	10627		

Table A6. Values of d with $h(-d) = 11$

167	271	659	967	1283	1303	1307	1459
1531	1699	2027	2267	2539	2731	2851	2971
3203	3347	3499	3739	3931	4051	5179	5683
6163	6547	7027	7507	7603	7867	8443	9283
9403	9643	9787	10987	13003	13267	14107	14683
15667							

Table A7. Values of d with $h(-d) = 13$

191	263	607	631	727	1019	1451	1499
1667	1907	2131	2143	2371	2659	2963	3083
3691	4003	4507	4643	5347	5419	5779	6619
7243	7963	9547	9739	11467	11587	11827	11923
12043	14347	15787	16963	20563			

Table A8. Values of d with $h(-d) = 15$

239	439	751	971	1259	1327	1427	1567
1619	2243	2647	2699	2843	3331	3571	3803
4099	4219	5003	5227	5323	5563	5827	5987
6067	6091	6211	6571	7219	7459	7547	8467
8707	8779	9043	9907	10243	10267	10459	10651
10723	11083	11971	12163	12763	13147	13963	14323
14827	14851	15187	15643	15907	16603	16843	17467
17923	18043	18523	19387	19867	20707	22003	26203
27883	29947	32323	34483				

Table A9. Values of d with $h(-d) = 17$

383	991	1091	1571	1663	1783	2531	3323
3947	4339	4447	4547	4651	5483	6203	6379
6451	6827	6907	7883	8539	8731	9883	11251
11443	12907	13627	14083	14779	14947	16699	17827
18307	19963	21067	23563	24907	25243	26083	26107
27763	31627	33427	36523	37123			

Table A10. Values of d with $h(-d) = 19$

311	359	919	1063	1543	1831	2099	2339
2459	3343	3463	3467	3607	4019	4139	4327
5059	5147	5527	5659	6803	8419	8923	8971
9619	10891	11299	15091	15331	16363	16747	17011
17299	17539	17683	19507	21187	21211	21283	23203
24763	26227	27043	29803	31123	37507	38707	

Table A11. Values of d with $h(-d) = 21$

431	503	743	863	1931	2503	2579	2767
2819	3011	3371	4283	4523	4691	5011	5647
5851	5867	6323	6691	7907	8059	8123	8171
8243	8387	8627	8747	9091	9187	9811	9859
10067	10771	11731	12107	12547	13171	13291	13339
13723	14419	14563	15427	16339	16987	17107	17707
17971	18427	18979	19483	19531	19819	20947	21379
22027	22483	22963	23227	23827	25603	26683	27427
28387	28723	28867	31963	32803	34147	34963	35323
36067	36187	39043	40483	44683	46027	49603	51283
52627	55603	58963	59467	61483			

Table A12. Values of d with $h(-d) = 23$

647	1039	1103	1279	1447	1471	1811	1979
2411	2671	3491	3539	3847	3923	4211	4783
5387	5507	5531	6563	6659	6703	7043	9587
9931	10867	10883	12203	12739	13099	13187	15307
15451	16267	17203	17851	18379	20323	20443	20899
21019	21163	22171	22531	24043	25147	25579	25939
26251	26947	27283	28843	30187	31147	31267	32467
34843	35107	37003	40627	40867	41203	42667	43003
45427	45523	47947	90787				

Appendix B. Covering partitions for class numbers $5, 7, 9, \ldots, 23$. The following tables give a set of partitions covering all possible multisets M_d under the assumption that $h(-d) = m$ for $m = 5, 7, \ldots, 23$. Let p and q denote the first and second smallest prime minima in M_d, respectively, and let a denote a generic member of M_d. Let $v = d/4$ and $w = d/3$. The notation (np.) implies that inequalities for the first n powers of the prime are to be inferred. The notation (n) simply means the inequality is to be listed a total of n times.

Table B1. Partitions for class number 5

1	$v^{1/5} \le p \le v^{1/4}$ (2p.)	
2	$v^{1/4} \le p \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$
3	$v^{1/3} \le a \le w^{1/2}$ (2)	

Table B2. Partitions for class number 7

1	$v^{1/7} \le p \le v^{1/6}$	(3p.)	
2	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le a \le w^{1/2}$
3	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$ (2)
4	$v^{1/3} \le a \le w^{1/2}$	(3)	

Table B3. Partitions for class number 9

1	$v^{1/9} \le p \le v^{1/8}$	(4p.)	
2	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$
3	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le a \le w^{1/2}$ (2)
4	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$ (3)
5	$v^{1/3} \le a \le w^{1/2}$	(4)	

Table B4. Partitions for class number 11

1	$v^{1/11} \le p \le v^{1/10}$	(5p.)	
2	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$
3	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$ (2)
4	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le a \le w^{1/2}$ (3)
5	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$ (4)
6	$v^{1/3} \le a \le w^{1/2}$	(5)	

Table B5. Partitions for class number 13

1	$v^{1/13} \le p \le v^{1/12}$	(6p.)			
2	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{2/5} \le a \le w^{1/2}$		
3	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$ (2)		
4	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{3/8} \le pq \le v^{1/2}$ (2)
5	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$ (3)		
6	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)	$v^{1/3} \le pq \le v^{1/2}$ (2)
7	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$ (3)
8	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/3} \le a \le w^{1/2}$ (4)		
9	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$ (5)		
10	$v^{1/3} \le a \le w^{1/2}$	(6)			

Table B6. Partitions for class number 15

1	$v^{1/15} \le p \le v^{1/14}$	(7p.)					
2	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{5/12} \le a \le w^{1/2}$				
3	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{2/5} \le a \le w^{1/2}$	(2)			
4	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{1/4} \le a \le w^{1/2}$	(3)			
5	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)		$v^{7/24} \le pq \le v^{5/12}$	(2)
6	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/4} \le q \le v^{1/3}$			$v^{3/8} \le pq \le v^{1/2}$	(2)
						$v^{1/3} \le a \le w^{1/2}$	
7	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$	(4)			
8	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)		$v^{1/3} \le pq \le v^{1/2}$	(2)
						$v^{1/4} \le a \le w^{1/2}$	
9	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le q \le v^{1/3}$			$v^{1/4} \le a \le w^{1/2}$	(4)
10	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/3} \le a \le w^{1/2}$	(5)			
11	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$	(6)			
12	$v^{1/3} \le a \le w^{1/2}$	(7)					

Table B7. Partitions for class number 17

1	$v^{1/17} \le p \le v^{1/16}$	(8p.)				
2	$v^{1/16} \le p \le v^{1/14}$	(7p.)	$v^{3/7} \le a \le w^{1/2}$			
3	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{5/12} \le a \le w^{1/2}$	(2)		
4	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{3/10} \le a \le w^{1/2}$	(3)		
5	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{1/4} \le q \le v^{3/8}$		$v^{7/20} \le pq \le v^{1/2}$	(2)
					$v^{3/8} \le a \le w^{1/2}$	
6	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$	(4)		
7	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)	$v^{7/24} \le pq \le v^{5/12}$	(2)
					$v^{1/3} \le a \le w^{1/2}$	
8	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{3/8} \le pq \le v^{1/2}$	(2)
					$v^{1/3} \le a \le w^{1/2}$	(2)
9	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$	(5)		
10	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)	$v^{1/3} \le pq \le v^{1/2}$	(2)
					$v^{1/4} \le a \le w^{1/2}$	(2)
11	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$	(5)
12	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/3} \le a \le w^{1/2}$	(6)		
13	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$	(7)		
14	$v^{1/3} \le a \le w^{1/2}$	(8)				

Table B8. Partitions for class number 19

1	$v^{1/19} \le p \le v^{1/18}$	(9p.)					
2	$v^{1/18} \le p \le v^{1/16}$	(8p.)	$v^{7/16} \le a \le w^{1/2}$				
3	$v^{1/16} \le p \le v^{1/14}$	(7p.)	$v^{3/7} \le a \le w^{1/2}$	(2)			
4	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{1/3} \le a \le w^{1/2}$	(3)			
5	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{3/10} \le a \le w^{1/2}$	(4)			
6	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{1/4} \le q \le v^{3/8}$			$v^{7/20} \le pq \le v^{1/2}$	(2)
						$v^{3/8} \le a \le w^{1/2}$	(2)
7	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$	(5)			
8	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/6} \le q \le v^{1/4}$		(2p.)	$v^{7/24} \le pq \le v^{5/12}$	(2)
						$v^{1/3} \le a \le w^{1/2}$	(2)
9	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/4} \le q \le v^{1/3}$			$v^{3/8} \le pq \le v^{1/2}$	(2)
						$v^{1/4} \le a \le w^{1/2}$	(3)
10	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/3} \le a \le w^{1/2}$	(6)			
11	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/6} \le q \le v^{1/4}$		(2p.)	$v^{1/3} \le pq \le v^{1/2}$	(2)
						$v^{1/4} \le a \le w^{1/2}$	(3)
12	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/4} \le q \le v^{1/3}$			$v^{1/4} \le a \le w^{1/2}$	(6)
13	$v^{1/6} \le p \le v^{1/4}$	(2p.)	$v^{1/3} \le a \le w^{1/2}$	(7)			
14	$v^{1/4} \le p \le v^{1/3}$		$v^{1/4} \le a \le w^{1/2}$	(8)			
15	$v^{1/3} \le a \le w^{1/2}$	(9)					

Table B9. Partitions for class number 21

1	$v^{1/21} \le p \le v^{1/20}$	(10p.)					
2	$v^{1/20} \le p \le v^{1/18}$	(9p.)	$v^{4/9} \le a \le w^{1/2}$				
3	$v^{1/18} \le p \le v^{1/16}$	(8p.)	$v^{7/16} \le a \le w^{1/2}$	(2)			
4	$v^{1/16} \le p \le v^{1/14}$	(7p.)	$v^{5/14} \le a \le w^{1/2}$	(3)			
5	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{1/3} \le a \le w^{1/2}$	(4)			
6	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{1/4} \le q \le v^{3/10}$			$v^{1/3} \le pq \le v^{2/5}$	(2)
						$v^{5/12} \le p^2q \le v^{1/2}$	(2)
7	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{3/10} \le a \le w^{1/2}$	(5)			
8	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/16} \le q \le v^{1/4}$		(2p.)	$v^{23/80} \le pq \le v^{3/8}$	(2)
						$v^{31/80} \le p^2q \le v^{1/2}$	(2)
9	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{1/4} \le q \le v^{3/8}$			$v^{7/20} \le pq \le v^{1/2}$	(2)
						$v^{1/4} \le a \le w^{1/2}$	(3)
10	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$	(6)			

Table B9 (cont.)

11	$v^{1/8} \le p \le v^{1/6}$ (3p.)	$v^{1/6} \le q \le v^{1/4}$ (2p.)	$v^{7/24} \le pq \le v^{5/12}$	(2)
			$v^{1/4} \le a \le w^{1/2}$	(3)
12	$v^{1/8} \le p \le v^{1/6}$ (3p.)	$v^{1/4} \le q \le v^{1/3}$	$v^{3/8} \le pq \le v^{1/2}$	(2)
			$v^{1/4} \le a \le w^{1/2}$	(4)
13	$v^{1/8} \le p \le v^{1/6}$ (3p.)	$v^{1/3} \le a \le w^{1/2}$ (7)		
14	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/6} \le q \le v^{1/4}$ (2p.)	$v^{1/3} \le pq \le v^{1/2}$	(2)
			$v^{1/4} \le a \le w^{1/2}$	(4)
15	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/4} \le q \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$	(7)
16	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/3} \le a \le w^{1/2}$ (8)		
17	$v^{1/4} \le p \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$ (9)		
18	$v^{1/3} \le a \le w^{1/2}$ (10)			

Table B10. Partitions for class number 23

1	$v^{1/23} \le p \le v^{1/22}$	(11p.)				
2	$v^{1/22} \le p \le v^{1/20}$	(10p.)	$v^{9/20} \le a \le w^{1/2}$			
3	$v^{1/20} \le p \le v^{1/18}$	(9p.)	$v^{4/9} \le a \le w^{1/2}$	(2)		
4	$v^{1/18} \le p \le v^{1/16}$	(8p.)	$v^{3/8} \le a \le w^{1/2}$	(3)		
5	$v^{1/16} \le p \le v^{1/14}$	(7p.)	$v^{5/14} \le a \le w^{1/2}$	(4)		
6	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{9/28} \le pq \le v^{5/12}$	(2)
					$v^{11/28} \le p^2q \le v^{1/2}$	(2)
7	$v^{1/14} \le p \le v^{1/12}$	(6p.)	$v^{1/3} \le a \le w^{1/2}$	(5)		
8	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{1/5} \le q \le v^{3/10}$		$v^{17/60} \le pq \le v^{2/5}$	(2)
			$v^{11/30} \le p^2q \le v^{1/2}$	(2)	$v^{2/5} \le a \le w^{1/2}$	
9	$v^{1/12} \le p \le v^{1/10}$	(5p.)	$v^{3/10} \le a \le w^{1/2}$	(6)		
10	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/16} \le q \le v^{1/4}$	(2p.)	$v^{23/80} \le pq \le v^{3/8}$	(2)
			$v^{31/80} \le p^2q \le v^{1/2}$	(2)	$v^{3/8} \le a \le w^{1/2}$	
11	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{1/4} \le q \le v^{3/8}$		$v^{7/20} \le pq \le v^{1/2}$	(2)
					$v^{1/4} \le a \le w^{1/2}$	(4)
12	$v^{1/10} \le p \le v^{1/8}$	(4p.)	$v^{3/8} \le a \le w^{1/2}$	(7)		
13	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/6} \le q \le v^{1/4}$	(2p.)	$v^{7/24} \le pq \le v^{5/12}$	(2)
					$v^{1/4} \le a \le w^{1/2}$	(4)
14	$v^{1/8} \le p \le v^{1/6}$	(3p.)	$v^{1/4} \le q \le v^{1/3}$		$v^{3/8} \le pq \le v^{1/2}$	(2)
					$v^{1/4} \le a \le w^{1/2}$	(5)

Table B10 (cont.)

15	$v^{1/8} \le p \le v^{1/6}$ (3p.)	$v^{1/3} \le a \le w^{1/2}$ (8)	
16	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/6} \le q \le v^{1/4}$ (2p.)	$v^{1/3} \le pq \le v^{1/2}$ (2)
			$v^{1/4} \le a \le w^{1/2}$ (5)
17	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/4} \le q \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$ (8)
18	$v^{1/6} \le p \le v^{1/4}$ (2p.)	$v^{1/3} \le a \le w^{1/2}$ (9)	
19	$v^{1/4} \le p \le v^{1/3}$	$v^{1/4} \le a \le w^{1/2}$ (10)	
20	$v^{1/3} \le a \le w^{1/2}$ (11)		

Note added in proof. C. Wagner has independently solved the class number 5 and 7 problems (and the class number 6 problem as well) in a paper: *Class number* 5, 6 *and* 7, Math. Comp. 65 (1996), 785–800.

Acknowledgements. The authors would like to express their gratitude to the referee for suggesting several improvements. All of our numerical computations and explorations were done with the aid of *Mathematica* on a SPARCstation, with the exception of the C code written for the Cray C90 discussed in §6.

References

[1] S. Arno, *The imaginary quadratic fields of class number* 4, Acta Arith. 60 (1992), 321–334.
[2] A. Baker, *Linear forms in the logarithms of algebraic numbers. I*, Mathematika 13 (1966), 204–216.
[3] —, *Imaginary quadratic fields with class number* 2, Ann. of Math. 94 (1971), 139–152.
[4] —, *Transcendental Number Theory*, Cambridge Univ. Press, New York, 1975.
[5] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. Reine Angew. Math. 218 (1965), 79–108.
[6] D. A. Buell, *Class groups of quadratic fields. II*, Math. Comp. 48 (1987), 85–93.
[7] H. Davenport, *Multiplicative Number Theory*, 2nd ed., Grad. Texts in Math. 74, Springer, New York, 1980.
[8] M. Deuring, *Imaginäre quadratische Zahlkörper mit der Klassenzahl Eins*, Invent. Math. 5 (1968), 169–179.
[9] C. F. Gauss, *Disquisitiones Arithmeticae*, Yale Univ. Press, 1966.
[10] D. M. Goldfeld, *The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer*, Ann. Scuola Norm. Sup. Pisa 3 (1976), 623–663.
[11] B. Gross et D. Zagier, *Points de Heegner et derivées de fonctions L*, C. R. Acad. Sci. Paris 297 (1983), 85–87.
[12] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Oxford Univ. Press, London, 1968.

[13] K. Heegner, *Diophantische Analysis und Modulfunktionen*, Math. Z. 56 (1952), 227–253.

[14] H. Heilbronn, *On the class number in imaginary quadratic fields*, Quart. J. Math. Oxford Ser. 25 (1934), 150–160.

[15] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge Tracts in Math. 30, Cambridge Univ. Press, Cambridge, 1990.

[16] H. Kestelman, *Modern Theories of Integration*, Dover, New York, 1960.

[17] N. Levinson and R. M. Redheffer, *Complex Variables*, Holden-Day, San Francisco, 1970.

[18] H. L. Montgomery and P. J. Weinberger, *Notes on small class numbers*, Acta Arith. 24 (1974), 529–542.

[19] J. Oesterlé, *Nombres de classes des corps quadratiques imaginaires*, Sém. Bourbaki, 1983–1984, exp. 631.

[20] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper*, Acta Arith. 1 (1936), 83–86.

[21] —, *Zum Beweise des Starkschen Satzes*, Invent. Math. 5 (1968), 180–191.

[22] H. M. Stark, *On complex quadratic number fields with class number equal to one*, Trans. Amer. Math. Soc. 122 (1966), 112–119.

[23] —, *A complete determination of the complex quadratic fields of class number one*, Michigan Math. J. 14 (1967), 1–27.

[24] —, *On the "gap" in a theorem of Heegner*, J. Number Theory 1 (1969), 16–27.

[25] —, *L-functions and character sums for quadratic forms (II)*, Acta Arith. 15 (1969), 307–317.

[26] —, *A transcendence theorem for class number problems*, Ann. of Math. 94 (1971), 153–173.

[27] —, *On a transcendence theorem for class number problems II*, ibid. 96 (1972), 251–259.

[28] —, *On complex quadratic fields with class number two*, Math. Comp. 29 (1975), 289–302.

[29] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1966.

[30] A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204–220.

[31] P. J. Weinberger, *On small zeros of Dirichlet L-functions*, Math. Comp. 29 (1975), 319–328.

Center for Computing Sciences
17100 Science Drive
Bowie, Maryland 20715
U.S.A.
E-mail: arno@super.org
robinson@super.org
wheeler@super.org

Received on 13.3.1995
and in revised form on 15.12.1996 (2755)