Descriptions of the Algorithms

with several illustrations by the author

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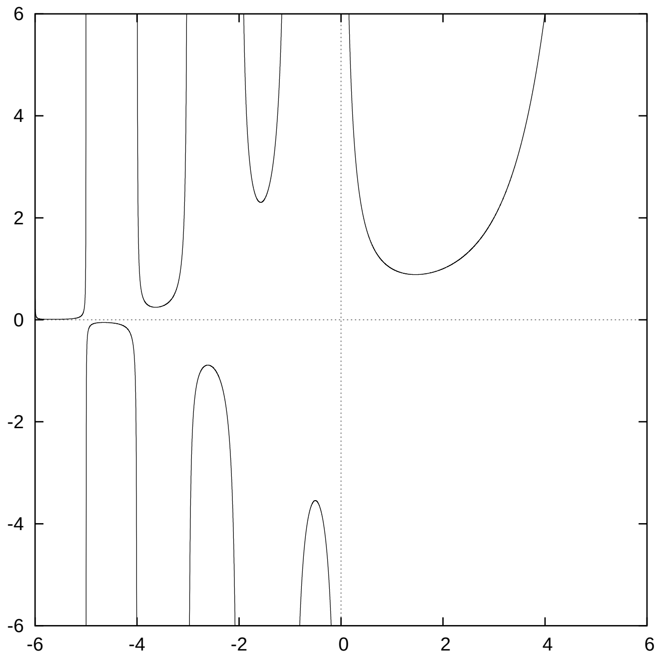

Contents

$\pi\rho o\lambda\epsilon\gamma\omega\mu\epsilon\nu\alpha$

It's easy to solve the halting problem with a shotgun. :-)
[Larry Wall,"<199801151836.KAA14656@wall.org>"]

Prepared with a heavily used copy of Abramowitz&Stegun, a painfully slow internet connection, a freshly decalcified coffee machine with several pounds of carefully grained beans in slightly smeary de-aired packages and a well cleaned mug with dirty pictures inside...

Introduction

Nec sic incipies, ut scriptor cyclicus olim:
"Fortunam Priami cantabo et nobile bellu".
Quid dignum tanto feret hic promissor hiatu?
Parturient montes, nascetur ridiculus mus.[1]
[QUINTUS HORATIUS FLACCUS,"De Arte Poetica", 138]

The documentation of the functions is sparse, it rarely describes the algorithms used, only the usage of the functions. These notes are meant to fill the gap and describe the underlying algorithms formally. Normal mathematic expressions have been used most of the time, but the three or four string functions need more legibility and are described in Z[2].

Additionally some real world examples are given together with some code listings[3].

Please be aware that the algorithms listed here are the algorithms used in the program and are not necessarily the same as in your textbooks! The algorithms in these textbooks have educational purpose and are build with a pedagogical goal in mind[4]. The algorithms here have been chosen to give a numerical result with a certain precision and moreso with respect to the special intricacies of a modern computer and ECMA-script[46] and the many different implementations of them.

Some algorithms can be implemented verbatim like Heron's formula for the area of a triangle

$$A = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4} \tag{1}$$

[1]Be not your opening fierce, in accents bold,
Like the rude ballad-monger's chaunt of old;
"The fall of Priam, the great Trojan King!
Of the right noble Trojan War, I sing!"
Where ends this Boaster, who, with voice of thunder,
Wakes Expectation, all agape with wonder?
The mountains labour! hush'd are all the spheres!
And, oh ridiculous! a mouse appears.
(Translation by George Colman, London, 1783)

[2]Rules are from J. M. Spivey's book[100] but will be changed to rules according to the standard ISO/IEC 13568:2002 [47] in the near future.

[3]*Exempla sunt odiosa* said Schopenhauer, so don't expect too many of such gems.

[4]At least they say so when they apply for government subsidies.

This is implemented as

```
Math.triangleAreaHeron = function(a,b,c){
  var one   = a+b+c;
  var two   = a+b-c;
  var three = b+c-a;
  var four  = c+a-b;
  return (one*two*three*four)/4;
};
```

The formula has been parted for better legibility but it is the verbatim translation of Heron's formula to ECMA-script.

On the other side there are some occasions where a literal translation is not possible or not optimal. The former holds for every function of the real line or above if we assume the the number of possible steps of a Turing machine is at most countable. The latter is the case in the implementation of the partial harmonic function H_n^1 for example

$$H_n^1 = \sum_{i+1}^{n} \frac{1}{i} \tag{2}$$

This formula works well for small n up to about 1 000 but for larger values of n the naïve algorithm loses a lot of precision because of the division of 1 by more and more larger numbers. The individual losses at every division step accumulates. Only slowly, but they add up until the point where a simple asymptotic series not only suffices but is also more precise.

Christoph Zurnieden

Chapter 1

Special Data Types

1.1 Matrix

The `Matrix` class handles dense complex matrices numerically. Several operations are implemented. The basic operations are described in chapter 3.1 and the operations for linear algebra in chapter 5.

1.1.1 Usage of the Class

Instantiation

A new matrix can be installed by generating a new instance of the `Matrix` class or via a specially formated string. The `Matrix` class offers several special matrixes and suffers a bit from *featuritis bombasti* but is nevertheless still useful. The format of the string to build a matrix might be familiar to some.

```
var s = "[1 + 2.3i,3 + 4i;2 + 0.1i,-9 - -9i]";
var m = s.parseMatrix();
```

The lines above produce the following 2×2 matrix

$$\begin{bmatrix} 1 + 2.3i & 3 + 4i \\ 2 + 0.1i & -9 - -9i \end{bmatrix}$$

The individual elements a_{ij} of the matrix can be reached directly. The only difference is that the counting starts at 0. To get the element a_{22} the following lines are necessary.

```
var s = "[1 + 2.3i,3 + 4i;2 + 0.1i,-9 - -9i]";
var m = s.parseMatrix();
alert(m.a[1][1]); // gives "-9 - -9i"
```

The special matrices offered are (all values are complex numbers, imaginary parts are omitted if they are 0):

- The identity matrix I_n is a $n \times n$ matrix with the entries in the main diagonal set to 1 and 0 otherwise.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- The exchange matrix E_n is the $n \times n$ identity matrix rotated $-\frac{\pi}{2}$ (90 degrees counter-clockwise)

$$E_n = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & 0 \\ 1 & \dots & 0 & 0 \end{bmatrix}$$

- The Hilbert matrix H_n is a $n \times n$ matrix constructed with the formula $a_{ij} = \frac{1}{(i+j-1)}$

$$H_n = \begin{bmatrix} \frac{1}{(1+1-1)} & \frac{1}{(1+2-1)} & \dots & \frac{1}{(1+n-1)} \\ \frac{1}{(2+1-1)} & \frac{1}{(2+2-1)} & \dots & \frac{1}{(2+n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(n+1-1)} & \frac{1}{(n+2-1)} & \dots & \frac{1}{(n+n-1)} \end{bmatrix}$$

- The Lehmer matrix L_n is a $n \times n$ matrix constructed with the formula

$$a_{ij} = \begin{cases} \frac{i}{j} & j \geq i \\ \frac{j}{i} & \text{otherwise} \end{cases}$$

- The upper shift matrix U_n is a $n \times n$ matrix with the superdiagonal set to 1.

$$U_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The lower shift matrix U_n is a $n \times n$ matrix with the subdiagonal set to 1.

$$L_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The "empty" matrix $\emptyset$ is a 2×2 matrix with all elements set to `Number.NaN` except a_{22} which is set to the string `"empty"`

$$\emptyset = \begin{bmatrix} \texttt{Number.NaN} & \texttt{Number.NaN} \\ \texttt{Number.NaN} & \texttt{"empty"} \end{bmatrix}$$

- The zero matrix is a $n \times n$ matrix with all elements set to $0 + 0i$

1.2 Complex

1.2.1 i

The `Complex` class handles the complex numbers numerically. Because the real numbers are a proper subset of the complex numbers the complex set is also called the complex plane.

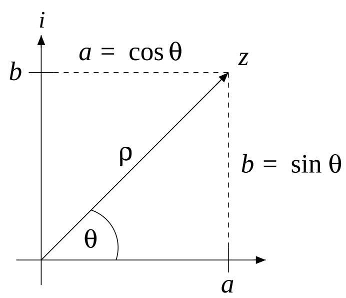

Figure 1.1: $z = a + bi = \rho e^{i\theta}$

One of the complex numbers is the number i which is the solution to $i^2 = -1$[1]. The most common notation of a complex number z with a real part $\Re z = a$ and an imaginary part of $\Im z = b$ is $z = (a + bi)$. This is the vector from $(0, 0)$ to (a, b) on the plane. These Cartesian coordinates can also be represented as polar coordinates (ρ, θ) where ρ is the distance from the origin and θ the angle between the vector and the x-axis. The common notation is $\rho e^{\theta i}$. From the identity $e^{ix} = \cos x + i \sin x$ follows that a direct conversion between the Cartesian and the polar form of a complex number z is possible with

$$\begin{aligned} \Re z &= \rho\cos\theta \\ \Im z &= \rho\sin\theta \end{aligned} \tag{1.1}$$

Both forms have their pros and cons but generally the Cartesian form is more usefully for addition and the polar form for multiplication. The polar form is used only internally in this program; the notation used for in- and output is the Cartesian form $(x + yi)$.

[1] It is not defined as $i = sqrt{-}1$ even if that might seem logic.

Two conversion functions are implemented to convert the different forms back and forth, namely `pol2cart` to convert form polar form to Cartesian form and `cart2pol` for the other direction. Polar to Cartesian (`pol2cart`)

$$\begin{aligned} \Re z &= \rho\cos\theta \\ \Im z &= \rho\sin\theta \end{aligned} \tag{1.2}$$

Cartesian to polar (`cart2pol`)

$$\begin{aligned} \Re z &= |z| \\ \Im z &= \operatorname{atan2}(\Im z, \Re z) \end{aligned} \tag{1.3}$$

Because the real line is a proper subset of the complex plane all complex operations are mere extensions to the operations on real numbers and thus can be used as such if the imaginary part is zero. This is usefully for operations that are not defined for any real number but for complex ones (for example with some inputs to `Math.asin()`).

1.2.2 Usage of the Class

Instantiation

A new complex number can be installed by generating a new instance of the `Complex` class. The following lines produce complex numbers, all of the same value $0 + 0i$:

```
var i = new Complex();
var j = new Complex(0,0);
var n = 0;
var k = n.toComplex();
var l = n.toComplex(0);
var s = "0 + 0i";
var m = s.toComplex();
```

1.3 Vector

The `Vector` class handles complex vectors numerically.

Chapter 2

Constants

A short list of more or less useful constants is also included. All constants have been rounded when more than 37 decimal digits were available. Most of the mathematical constants have been calculated by the author with 100 decimal digits of precision but almost all[1] can be found on the net. Some of the constants implemented have been omitted if the source is obvious and the description in the source-code sufficient. With some exceptions.

2.1 Mathematical Constants

2.1.1 Airy functions

From [1], pp. 446 ff.:

The Airy functions are the solutions to the differential equation

$$w'' - zw = 0 \tag{2.1}$$

Pairs of linearly independent solutions are

$$\begin{aligned} &\mathrm{Ai}(z), \mathrm{Ai}(z) \\ &\mathrm{Ai}(z), \mathrm{Ai}(ze^{2\pi\frac{1}{3}}) \\ &\mathrm{Ai}(z), \mathrm{Ai}(ze^{-2\pi\frac{1}{3}}) \end{aligned} \tag{2.2}$$

[1]Most probably all, but the author has not looked up all of them, so it has to be "almost all"

The ascending series

$$\mathrm{Ai}(z) = c_1 f(z) - c_2 g(z) \tag{2.3}$$

$$\mathrm{Ai}(z) = \sqrt{3}\,(c_1 f(z) - c_2 g(z)) \tag{2.4}$$

$$f(z) = 1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!} + \frac{1\cdot 4\cdot 7}{9!} + \cdots \tag{2.5}$$

$$= \sum_0^\infty 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!}$$

$$g(z) = z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \frac{2\cdot 5\cdot 8}{10!}z^1 0 + \cdots \tag{2.6}$$

$$= \sum_0^\infty 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$\left(\alpha + \frac{1}{3}\right)_0 = 1$$

$$3^k \left(\alpha + \frac{1}{3}\right)_0 = (3\alpha+1)(3\alpha+4)\cdots(3\alpha+3k-2) \qquad \alpha \text{ arbitrary}; k = 1, 2, 3, \ldots$$

$$c_1 = \mathrm{Ai}(0) = \frac{\mathrm{Ai}(0)}{\sqrt{3}} = \frac{3^{-\frac{2}{3}}}{\Gamma(\frac{2}{3})} \tag{2.7}$$

$$c_2 = -\mathrm{Ai}'(0) = \frac{\mathrm{Ai}'(0)}{\sqrt{3}} = \frac{3^{-\frac{1}{3}}}{\Gamma(\frac{1}{3})} \tag{2.8}$$

2.1.2 Apéry's Constant ($\zeta(3)$)

2.1.3 $\operatorname{atan} \frac{1}{2}$

The arcus tangent function at $\frac{1}{2}$.

The value has been calculated via the continuous fraction

$$\operatorname{atan} z = \cfrac{z}{1 + \cfrac{z^2}{1 + \cfrac{4z^2}{3 + \cfrac{9z^2}{5 + \cfrac{16z^2}{7 + \cfrac{n^2 z^2}{(n+1) + \cdots}}}}}} \tag{2.9}$$

Please see the section about trigonometry in section 3.3

2.1.4 Artins Constant

For the description see E. Artin's collected papers in [6]. A short oversight is at [117]

$$c = \prod_{k=1}^{\infty} \left[1 - \frac{1}{p_k (p_k - 1)} \right] \qquad p_k \text{ is the } k^{\text{th}} \text{ prime} \tag{2.10}$$

2.1.5 Backhouse's Constant

Let $P(x)$ be a power series whose coefficients n are the primes p_n and $p_0 = 1$.

$$P(x) = \sum_{k=0}^{\infty} p_k x^k \qquad k \in \mathbb{N} \tag{2.11}$$

$$= 1 + 2x + 3x^2 + 5x^3 + 7x^4 + 11x^5 + 13x^6 + 17x^7 \cdots$$

Let $Q(x)$ be

$$Q(x) = \frac{1}{P(x)} \tag{2.12}$$

$$= \sum_{k=0}^{\infty} q_k x^k \tag{2.13}$$

$$= 1 - 2x + x^2 - x^3 + 2x^4 - 3x^5 + 7x^6 + \cdots \tag{2.14}$$

N. Backhouse's conjecture:

$$B = \lim_{n \to \infty} \left| \frac{q_{n+1}}{q_n} \right| \tag{2.15}$$

$$= 1.4560749 \ldots \tag{2.16}$$

2.1.6 Bernstein's Constant

If $E_n(f)$ is the error of the best approximation to a real function $f(x)$ on the interval $[-1, 1]$ by real polynomials of degree at most n and $\alpha(x) = |x|$ then Bernstein showed in [11] that

$$0.267 \cdots < \lim_{n \to \infty} n E_{2n}(\alpha) < 0.286 \ldots \tag{2.17}$$

Many people had refined the underlying theory and augmented the numeric approximation of the constant $\beta(\frac{1}{2})$. The number used here has been calculated on the principles shown in [20](section 2).

Bernstein himself established the upper and lower bounds 25 years later in [10] as

$$\beta(\alpha) < \frac{\Gamma(2\alpha) \, |sin(\pi\alpha)|}{\pi} \qquad \text{for} \quad a > 0 \tag{2.18}$$

$$\frac{\Gamma(2\alpha)\,|sin(\pi\alpha)|}{\pi}\left(1-\frac{1}{2\alpha-1}\right) < \beta(\alpha) \qquad \text{for} \quad a > \frac{1}{2} \tag{2.19}$$

2.1.7 Brun's Constant

The sum of the reciprocals of all odd twin-primes.

$$B_2 = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots \tag{2.20}$$

Calculated with the above algorithms with the exception from the general way insofar that it had been calculated to only 10 decimal digits precision because the author had no access to the necessary computing power at the time of calculation.

The second constant which is also called Brun's Constant is based on the same algorithm as above but over the so called "Cousin Primes" $p, p+4$, so the above formula can be rewritten as

$$\begin{aligned} B_4 = &\left(\frac{1}{7}+\frac{1}{11}\right)+\left(\frac{1}{13}+\frac{1}{17}\right)+\left(\frac{1}{19}+\frac{1}{23}\right)+\left(\frac{1}{37}+\frac{1}{41}\right) \\ &+\left(\frac{1}{43}+\frac{1}{47}\right)+\cdots+\left(\frac{1}{2^{64}-2289}+\frac{1}{2^{64}-2285}\right)+\ldots \end{aligned} \tag{2.21}$$

The formula above converges *very* slowly.

2.1.8 Cahen's Constant

With a_k the k^{th} term of the Sylvester sequence

$$e_n = 1+\prod_{i=0}^{n-1} e_i = e_{n-1}^2 - e_{n-1} + 1 \tag{2.22}$$

Cahen's constant is defined in [19] as

$$C = \sum_{k=0}^{\infty} \frac{(-1)^k}{a_k - 1} \tag{2.23}$$

The constant has been calculated by the formulas given in [95].

2.1.9 Catalan's Constant

Catalan's constant is named in honor of E. C. Catalan (1814–1894)[21]. One of the possible formulas had been given by J. W. L. Glaisher in [34] as

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \tag{2.24}$$

The formula used to compute the numerical approximation was

$$G = \frac{\pi}{8} \ln \left(2 + \sqrt{3}\right) + \sum_{k=0}^{\infty} \frac{k!^2}{(2k)!(2k+1)^2} \tag{2.25}$$

2.1.10 Champernown's constant

Champernown's constant C is build by concatenating the positive natural integers. With $\odot$ the concatenation operator and $n \in \mathbb{N}$ the decimal representation of Champernown's constant is

$$C = 0.1 \odot \bigodot_{n=2}^{\infty} \approx 0.1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12\ldots \tag{2.26}$$

The resulting real is transcendental and a simple normal number in base 10. A normal number is a real number with its digits showing a uniform distribution in all bases and a simple normal number is a number in base b with its digits appearing with probability $\frac{1}{b}$.

2.1.11 Continued Fraction Constant

In a posting to the Math–fun list ([40]) Bill Gosper said:

> By strange coincidence, the information in a typical continued fraction term is very nearly one decimal digit—actually
>
> $$c = \frac{1}{6} \frac{\pi^2}{\ln(2)\ln(10)} \tag{2.27}$$

2.1.12 Conway's Constant

Conway's Constant describes the rate of growth of the number of digits in the *look–and–say* sequence. This sequence is an integer sequence with the term $n+1$ "describing" the term n. Starting with $n_0 = 1$:

$$n_0 = 1 \tag{2.28}$$
$$n_1 = 11 \quad \text{"one" 1} \tag{2.29}$$
$$n_2 = 21 \quad \text{"two" 1s} \tag{2.30}$$
$$n_3 = 1211 \quad \text{"one" 2, "one" 1} \tag{2.31}$$
$$\ldots \tag{2.32}$$

It is the unique real root of the polynomial

$$\begin{aligned}0 = {} & x^{71} - x^{69} - 2x^{68} - x^{67} + 2x^{66} + 2x^{65} + x^{64} - x^{63} - x^{62} - x^{61} - x^{60} \\ & - x^{59} + 2x^{58} + 5x^{57} + 3x^{56} - 2x^{55} - 10x^{54} - 3x^{53} - 2x^{52} + 6x^{51} + 6x^{50} \\ & + x^{49} + 9x^{48} - 3x^{47} - 7x^{46} - 8x^{45} - 8x^{44} + 10x^{43} + 6x^{42} + 8x^{41} \\ & - 5x^{40} - 12x^{39} + 7x^{38} - 7x^{37} + 7x^{36} + x^{35} - 3x^{34} + 10x^{33} + x^{32} - 6x^{31} \\ & - 2x^{30} - 10x^{29} - 3x^{28} + 2x^{27} + 9x^{26} - 3x^{25} + 14x^{24} - 8x^{23} - 7x^{21} \\ & + 9x^{20} + 3x^{19} - 4x^{18} - 10x^{17} - 7x^{16} + 12x^{15} + 7x^{14} + 2x^{13} - 12x^{12} \\ & - 4x^{11} - 2x^{10} + 5x^{9} + x^{7} - 7x^{6} + 7x^{5} - 4x^{4} + 12x^{3} - 6x^{2} + 3x - 6\end{aligned} \tag{2.33}$$

The above polynomial is not a mere approximation of the constant but the closed form. For detailed descriptions see J. H. Conway's articles in [22] and [23].

2.1.13 Copeland-Erdős' Constant

Copeland-Erdős' constant is a variation of Champernown's constant: not the positive integers are concatenated but the positive primes.

2.1.14 cos 1

The cosine of 1 had been calculated with

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{2.34}$$

2.1.15 cosh 1

The cosine hyperbolicus of 1 had been calculated with

$$\begin{aligned}\cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \cosh(iz)\end{aligned} \tag{2.35}$$

2.1.16 $\sqrt[3]{2}$

The cube root of 2 had been calculated with the "long hand" method and the help of a pencil and several perforated sheets of paper to 101 digits accuracy by the author while he was suffering from the symptoms of a visit to a restaurant with surprisingly bad hygienics.

2.1.17 $\sqrt[3]{3}$

The cube root of 3 had been calculated with Perl's implementation of big integer and big floating point numbers and Newton's method. All steps were done with fractions $\frac{p}{q}$ to make use of the absolute precision of integer arithmetic. Only the last step was done by computing the division out to get a decimal representation.

2.1.18 Dubois-Raymond Constant

One of the remarkable numbers to be found in [59] is the second Dubois-Raymond constant

$$C = \frac{e^2 - 7}{2} \tag{2.36}$$

2.1.19 Euler-Mascheroni Constant γ

This constant, also known as *Euler's constant*, is defined as the limit of

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n \right] \tag{2.37}$$

The identity with the harmonic numbers

$$\gamma = \lim_{n\to\infty} (H_n - \ln n) \tag{2.38}$$

makes this constant useful for computing several related functions like the harmonic function itself in section 4.10.4.

2.1.20 Embree-Trefethen Constant

The Embree-Trefethen constant is the generalized form of Viswanath's constant described in 2.1.71. For the recurrence

$$x_{n+1} = x_n \pm \beta x_{n-1} \qquad \text{with } a_0 = 0, a_1 = 1, P(\text{sign}) = \frac{1}{2} \tag{2.39}$$

a limit exists for almost all values of β

$$\sigma(\beta) = \lim_{n\to\infty} |x|_{\frac{1}{n}} \tag{2.40}$$

The critical value β^* such that $\sigma(\beta^*) = 1$ is sometimes called the Embree-Trefethen constant because of[24].

Viswanath's constant can be found at $\sigma(1)$.

2.1.21 Erdős-Borwein Constant

The Erdős-Borwein constant, named after Paul Erdős and Peter Borwein is the sum of the reciprocals of the Mersenne numbers.

$$E_B = \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \tag{2.41}$$

2.1.22 e

The constant e, also known as Napier's constant, is the base of the natural logarithm.

The most common descriptions are

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{2.42}$$

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \tag{2.43}$$

The equation 2.42 is due to [**?**].

There is a nice, albeit non-simple continued fraction representation of e

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}} \tag{2.44}$$

2.1.23 Gompertz' constant

Gompertz constant

$$G \equiv \int_0^{\infty} \frac{e^{-u}}{1+u}\,\mathrm{d}\,u \tag{2.45}$$

$$= -e\,\mathrm{Ei}(-1) \tag{2.46}$$

$\mathrm{Ei}(x)$ is the exponential integral.A simple continued fraction has been found by Stieltjes[101]

$$G = \cfrac{1}{2 - \cfrac{1^2}{4 - \cfrac{2^2}{6 - \cfrac{3^2}{8 - \cdots}}}} \tag{2.47}$$

2.1.24 Feigenbaum Constants

Theory

The Feigenbaum constants describe the ratios in a bifurcation diagram. The constant δ is a universal constant for functions approaching chaos via period doubling, discovered by Mitchel Feigenbaum in [26] with the iteration

$$f(x) = 1 - \mu |x|^r \tag{2.48}$$

With μ_n the point of the 2^n-cycle, μ_∞ the value of μ_n at ∞ and under the assumption of geometric convergence

$$\lim_{n\to\infty} \mu_\infty - \mu_n = \frac{g}{\delta^n} \tag{2.49}$$

Solving for δ with g constant and $\delta > 1$

$$\delta = \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} \tag{2.50}$$

The Feigenbaum constant α is the separation of adjacent elements from one doubling to the next

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} \tag{2.51}$$

with d_n the value of the element closest to zero in the 2^n-cycle [2].

The constants given in the implementation for δ and α are for the logistic map

$$f(x) = 1 - \mu |x|^2 \tag{2.52}$$

The other constants, b, c, d, were given by [17] in an email to Simon Plouffe[3]. With $f(x)$ and $g(x)$ even functions $f(0) = g(0) = 1$, δ as large as possible and

$$\frac{g(\alpha x)}{\alpha} = g(g(x)) \tag{2.53}$$

$$\frac{\delta f(\alpha x)}{\alpha} = g'\left(g\left(x\right)\right) f\left(x\right) + f\left(g\left(x\right)\right) \tag{2.54}$$

together with

$$g(b) = 0 = \frac{1}{g(c+di)} \qquad \text{with } \{b, c, d\} \in \mathbb{R}^+ \tag{2.55}$$

while $\{b, c^2 + b^2\}$ are minimal. With κ the order of the nearest singularity and z approaching zero

$$\frac{1}{g(c+di+z)} = O(z^\kappa) \tag{2.56}$$

[2] vid. [16, 86] for both constants

[3] With Keith Briggs, David Bailey and Steven Finch listed as the recipients of a quite pale carbon copy

~~Funny Pictures~~ Plotting Bifurcation Diagrams for Highly Pedagogical Aims

To get a Bifurcation diagram[4] of the recurrence formula given in [37]

$$x_{n+1} = \rho x_n (1 - x_n) \tag{2.57}$$

by on-board means[5] two small program-listings might be helpful.

At first a standard[45] C program.

```c
#include <stdlib.h>
#include <stdio.h>

int main(int argc, char **argv){
  double x = 0.0;
  double r = 0.0;
  // start must be between 0(zero) and 1(one)
  double start=0.5;
  int i=0;

  for(r = 0; r < 4; r += 0.001){
    x = start;
    for(;i<500;i++) {
      x = r * x * (1 - x);
      /*
        Discard the first 450 points because the
        iterations need some time to settle on a
        fixed point.
       */
      if(i > 450){
        printf("%f\t%f\n",r,x);
      }
    }
  }
  exit(EXIT_SUCCESS);
}
```

then the listing for `Gnuplot`

```
# Postscript files will get _very_ large!
#set term postscript enhanced "Helvetica" 12
#set output "bifurc.ps"

# Default is PNG with black ticks and red data
# points
set terminal png

# Gnuplot will not overwrite (at least with version
# 3.7p2) so the file listed here has to be deleted
```

[4] A logistic map, to be a bit more exact

[5] At least on-board means of most Unix distributions.

```
# before another run of Gnuplot
set output "bifurc.png"

set nokey
set nomxtics
set nomytics

# upper and lower limits
set yrange[0:1]

# value needed to have a single 'dot' for the default
# sizes
set pointsize .05

# start from the lower left
set origin 0,0

# multiplicator for size
# default size for postscript is 10x7 inches
# default size for PNG is 640x480 pixel
set size 1,1

set xlabel "r"
# the letter 'x' is rotated 90 degrees counterclockwise
set ylabel "x"

# 'bifurcation.out' is the name of the outputfile of the
# data generating program. Please change accordingly if
# necessary.
plot 'bifurcation.out' using 1:2 title "Bifurcation"
```

Assuming the existence a standard Unix shell, the GNU-compiler suit GCC, the above listing in the file `bifurcation.c` and the `Gnuplot` script in the file `bifurcation.plot`

```
gcc -std=c99 -W -Wall -o bifurcation bifurcation.c
./bifurcation > bifurcation.out
gnuplot bifurcation.plot
IMAGE_VIEWER_OF_CHOICE bifurcation.png
```

The two variables `x` and `r` in the C-listing are the variables x_n and ρ from the equation 2.57. More variables to play around with are the iterations of the loops, the limit of the discarding and the sizes of the `Gnuplot` script.

2.1.25 Fibonacci Factorial Constant

The Fibonacci factorial constant F is the infinite product

$$F = \prod_{k=1}^{\infty} \left(1 - a^k\right) \tag{2.58}$$

with

$$a = -\frac{1}{\phi^2} \tag{2.59}$$

and ϕ the Golden Ration

$$\phi = \frac{1 + \sqrt{5}}{2} \tag{2.60}$$

2.1.26 Fransén-Robinson Constant

The Fransén-Robinson constant F is defined (vid. [29, 31, 30]) by the integral

$$F = \int_0^{\infty} \frac{\mathrm{d}\,x}{\Gamma(x)} \tag{2.61}$$

2.1.27 Froda's Constant

Froda's constant is simply 2^e. The interesting thing is that he tried to prove its irrationality in [32]. It is unknown to the author if the prove holds.

2.1.28 Gibbs Constant $Si(\pi)$

The Gibbs- or Wilbraham-Gibbs constant G' is the sine-integral with the upper limit π

$$G' = \int_0^{\infty} \operatorname{sinc}\theta \,\mathrm{d}\,\theta \tag{2.62}$$

$$= \operatorname{Si}(\pi) \tag{2.63}$$

There are several functions gathered under the hood of the name "sine-integral". The variation used here is

$$\operatorname{Si}(z) = \int_0^{z} \frac{\sin t}{t} \,\mathrm{d}\,t \tag{2.64}$$

From equations 2.62 and 2.64 it is evident that the function $\operatorname{sinc} x$[6] could be defined with

$$\operatorname{sinc}(x) = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{otherwise} \end{cases} \tag{2.65}$$

[6] *sine cardinal* with its full name

2.1.29 Gauss-Kuzmin-Wirsing Constant

With $F_n(x)$ the Gauss-Kuzmin distribution and $\Psi(0) = \Psi(1) = 0$

$$\lim_{n\to\infty} \frac{F_n(x) - \ln(1+x)}{-\lambda} = \Psi(x) \tag{2.66}$$

Here λ is the Gauss-Kuzmin-Wirsing constant ([124]).

Biggs ([12]) computed the constant with the help of the matrix

$$\begin{aligned} M_{jk} = \frac{(-i)^j}{j!(-2)^k} \sum_{i=0}^{k} \binom{k}{i} (-2)^i (i+2)_j \\ \left[\zeta(i+j+1)\left(2^{i+j+2}-1\right) - 2^{i+j+2}\right] \end{aligned} \tag{2.67}$$

with $0 \le j, k \le n$, $(x)_n$ the raising factorial (Pochhammer symbol) and $\zeta(x)$ is Riemann's ζ-function. [119] gives an example:

$$M_{22} = \begin{bmatrix} (\pi^2 - 8) & 7\zeta(3) - \frac{1}{4}\pi^2 - 6 \\ 16 - 15\zeta(3) & 7\zeta(3) - \frac{1}{2}\pi^2 + 40 \end{bmatrix} \tag{2.68}$$

The constant is the negative of the absolute value of the second largest Eigenvalue of that matrix.

2.1.30 Glaisher-Kinkelin Constant

The Glaisher-Kinkelin constant can be defined memorizable by means of Riemann's ζ-function([114, 54])

$$A = e^{\frac{1}{12} - \zeta'(-1)} \tag{2.69}$$

It can ([35, 36]) also be defined by means of the K-function4.7.9

$$A = \lim_{n\to\infty} \frac{K(n+1)}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{\frac{-n^2}{4}}} \tag{2.70}$$

And by means of the G-function4.7.10

$$\frac{e^{\frac{1}{12}}}{A} = \lim_{n\to\infty} \frac{G(n)}{n^{\frac{n^2}{2}-\frac{1}{12}} (2\pi)^{\frac{n}{2}} e^{-3\frac{n^2}{4}}} \tag{2.71}$$

2.1.31 Golden Ratio

The Golden Ratio ϕ

$$\phi = \frac{1+\sqrt{5}}{2} \tag{2.72}$$

The triangle described by the edges A,B and C in figure 2.1.31 is also called a *Golden Gnomon*. The ratio of the lengths of the lines $a = \overline{BC}$ and $b = \overline{AB}$ is equal to the *Golden Ratio*

$$\phi = \frac{a}{b} \tag{2.73}$$

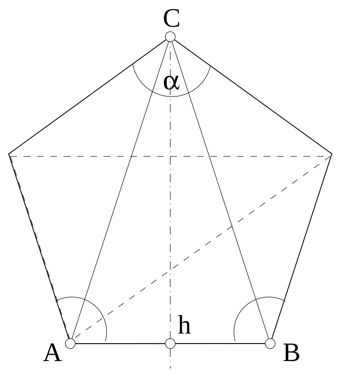

Figure 2.1: Pentagon with Golden Gnomon

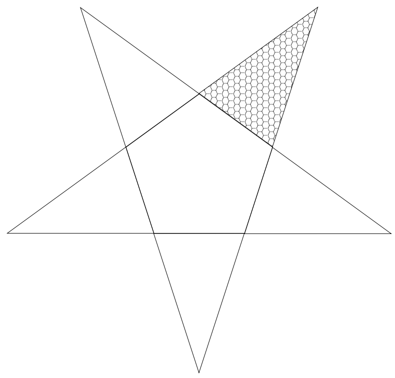

Figure 2.2: Pentagram with Golden Gnomon

Thus the angle α is in radians

$$\alpha = 2\sin^{-1}\left(\frac{b}{2a}\right) \tag{2.74}$$

$$= \frac{1}{5}\pi \tag{2.75}$$

or 36 degrees.

Mirroring the triangle at $\overline{AB}$ and copying and rotating 36 degrees results in the pentagram in figure 2.1.31

Rotating the triangle at the point s 36 degrees a sufficient number of times gives a decagon with a side length of $\overline{AB}$

The infinite series for the *Golden Ratio* is according to [**?**]

$$\phi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}} \tag{2.76}$$

The continued fraction is very simple to memorize

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}} \tag{2.77}$$

The *Golden Ratio* has relations with many other functions, for example with the Fibonacci numbers (with F_n the n^{th} Fibonacci number)

$$\phi = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{F_n F_{n+1}} \tag{2.78}$$

which follows from the continued fraction in equation 2.77

$$x_n = 1 + \frac{1}{x_{n-1}} \tag{2.79}$$

with $x_1 = 1$ and has the obvious solution

$$x_n = \frac{F_{n+1}}{F_n} \tag{2.80}$$

so

$$\phi = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \tag{2.81}$$

Equation 2.81 has been used to calculate the *Golden Ratio* because it is possible to calculate it with rationals up to the last point where one `Big_Float` division is necessary.

2.1.32 Golomb's Constant

The Golomb constant [38], also known as the Golomb-Dickman constant, is the limit of the ratio

$$\lambda = \lim_{n\to\infty} = \frac{a_n}{n} \tag{2.82}$$

where a_n is the expected length of the longest cycle in a uniformly distributed random permutation of a set $\mathbf{S}$ with $\#\mathbf{S} = n$.

An approximation for a_n[7] as shown in [82]

$$a_n = \phi^{2-\phi} n^{\phi-1} + \mathcal{O}\left(\frac{n^{\phi-1}}{\ln n}\right) \tag{2.83}$$

where ϕ denotes the *Golden Ration* $\frac{1+\sqrt{5}}{2}$.

2.1.33 Grothendieck's Majorant

Grothendieck's majorant [41]

$$g = \frac{\pi}{2\ln(1+\sqrt{2})} \tag{2.84}$$

2.1.34 Hadamard-de la Valle-Poussin Constant

More prominently known as the Meissel-Mertens ([71]) or prime reciprocal constant it is defined by the infinite sum

$$B_1 = \gamma + \sum_{k=1}^{\infty} \left(\ln\left(1 - p_k^{-1}\right) + \frac{1}{p_k}\right) \tag{2.85}$$

[7] The sequence is known as the Golomb sequence or Silverman's sequence

where γ is the Euler-Mascheroni constant and p_k is the k^{th} prime, or by the limit

$$B_1 = \lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p} - \ln\ln x\right) \tag{2.86}$$

A fast converging series is according to [55]

$$B_1 = \gamma + \sum_{n=2}^{\infty}\frac{\mu(n)}{n}\ln\left(\zeta\left(n\right)\right) \tag{2.87}$$

where $\zeta(n)$ is Rieman's ζ-function and $\mu(x)$ is the Möbius function.

2.1.35 Hafner-Sarnak-McCurley Constant

The Hafner-Sarnak-McCurley constant is the average probability $P(n)$ that the determinants of two $n \times n$ integer matrices are relatively prime.

$$P(n) = \prod_{k=1}^{\infty}\left(1 - \left(1 - \prod_{j=1}^{n}\left(1 - p_n^{-j}\right)\right)^2\right) \tag{2.88}$$

p_k is the k^{th} prime.

With $P(1)$ as the average probability that two random integers are relatively prime

$$P(1) = \frac{6}{\pi^2} \tag{2.89}$$

As this is obviously the inverse of $\zeta(2) = \frac{\pi^2}{6}$ another, exponentially[8] converging equation has been found by [28]

$$\sigma \equiv \lim_{n\to\infty} P(n) = \prod_{k=2}^{\infty}\zeta(k)^{-a_k} \tag{2.90}$$

2.1.36 Hard-Hexagon Entropy Constant

The hard square hexagon constant κ_h is given by

$$\kappa_h = \lim_{n\to\infty}\left(G\left(n\right)\right)^{\frac{1}{n^2}} \tag{2.91}$$

[8] at $\approx 0.57^n$

κ_h is algebraic ([8, 49])

$$\kappa_h = \kappa_1 \kappa_2 \kappa_3 \kappa_4 \tag{2.92}$$

$$\kappa_1 = 4^{-1} 3^{\frac{4}{5}} 11^{\frac{-5}{12}} c^{-2} \tag{2.93}$$

$$\kappa_2 = \left(1 - \sqrt{1-c} + \sqrt{2 + c + 2\sqrt{1 + c + c^2}}\right)^2 \tag{2.94}$$

$$\kappa_3 = \left(-1 - \sqrt{1-c} + \sqrt{2 + c + 2\sqrt{1 + c + c^2}}\right)^2 \tag{2.95}$$

$$\kappa_4 = \left(\sqrt{1-a} + \sqrt{2 + a + 2\sqrt{1 + a + a^2}}\right)^{-\frac{1}{2}} \tag{2.96}$$

$$a = \frac{-124}{363} 11^{\frac{1}{3}} \tag{2.97}$$

$$b = \frac{2501}{11979} 33^{\frac{1}{2}} \tag{2.98}$$

$$c = \left(\frac{1}{4} + \frac{3}{8} a \left((b+1)^{\frac{1}{3}} - (b-1)^{\frac{1}{3}}\right)\right)^{\frac{1}{3}} \tag{2.99}$$

This can be summarized to be the unique positive root of

$$\begin{aligned}
\kappa_h &= 25937424601z^{24} + 2013290651222784z^{22} \\
&+ 2505062311720673792z^{20} + 797726698866658379776z^{18} \\
&+ 7449488310131083100160z^{16} + 2958015038376958230528z^{14} \\
&- 7240567028564916161 7408z^{12} + 10715544815044338804 3264z^{10} \\
&- 7122080944140040588 4928z^{8} - 7334749118363010387 1488z^{6} \\
&+ 97143135277377575190528z^{4} - 32751691810479015985152
\end{aligned} \tag{2.100}$$

The $G(n)$ in equation 2.91 is the number of arrays with no adjacent 1s in a binary $n \times n$-matrix M

$$M = \begin{bmatrix} 1 & 0 & 1 & \dots \\ 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.101}$$

The adjacent elements are the set of some $a_{ij} = 1$

$$\{a_{i,j}\} \times \{a_{i+i,j}, a_{i,j+1}, a_{i+1,j+1}\} \tag{2.102}$$

The number $G(n)$ is also the number of configurations of non-attacking kings on a $n \times n$ hexagonal chessboard. A detail of such a board with two possible combinations is shown in figure 2.3, the possibles moves of a king according some of the most common rules is shown in figure 2.4.

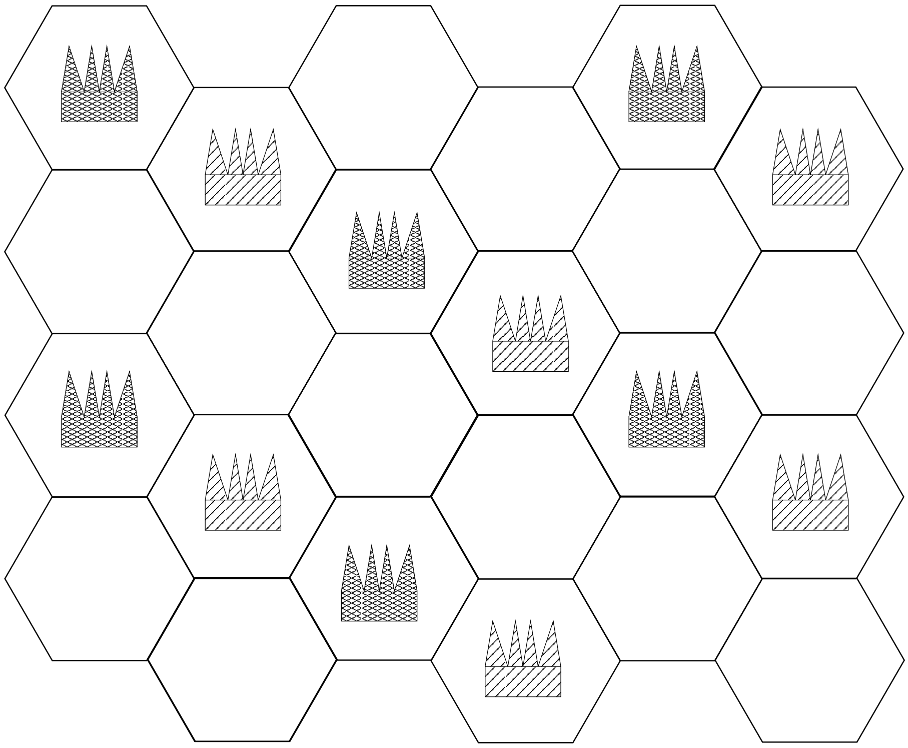

Figure 2.3: Hexagonal Chessboard

2.1.37 Khintchine Constant

The Khintchine constant ([52]) is the limit of the geometric mean $G_n()x$ of the partial quotients a_n of a continued fraction for $n \to \infty$

$$G_n(x) = (a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}} \tag{2.103}$$

The exact value is difficult to compute, see for example [7] for examples. One is

$$K = \exp\left(\frac{1}{\ln 2}\sum_{n=2}^{\infty}\frac{H'_{2n-1}\left[\zeta(2n)-1\right]}{n}\right) \tag{2.104}$$

Here, $\zeta(z)$ is Riemann's ζ-function and H'_n is an alternating harmonic number.

Not all numbers are equal of course, so some real x exist for which $\lim_{n\to\infty G_n(x)\neq K}$, for example e, $\sqrt{2}$, $\sqrt{3}$.

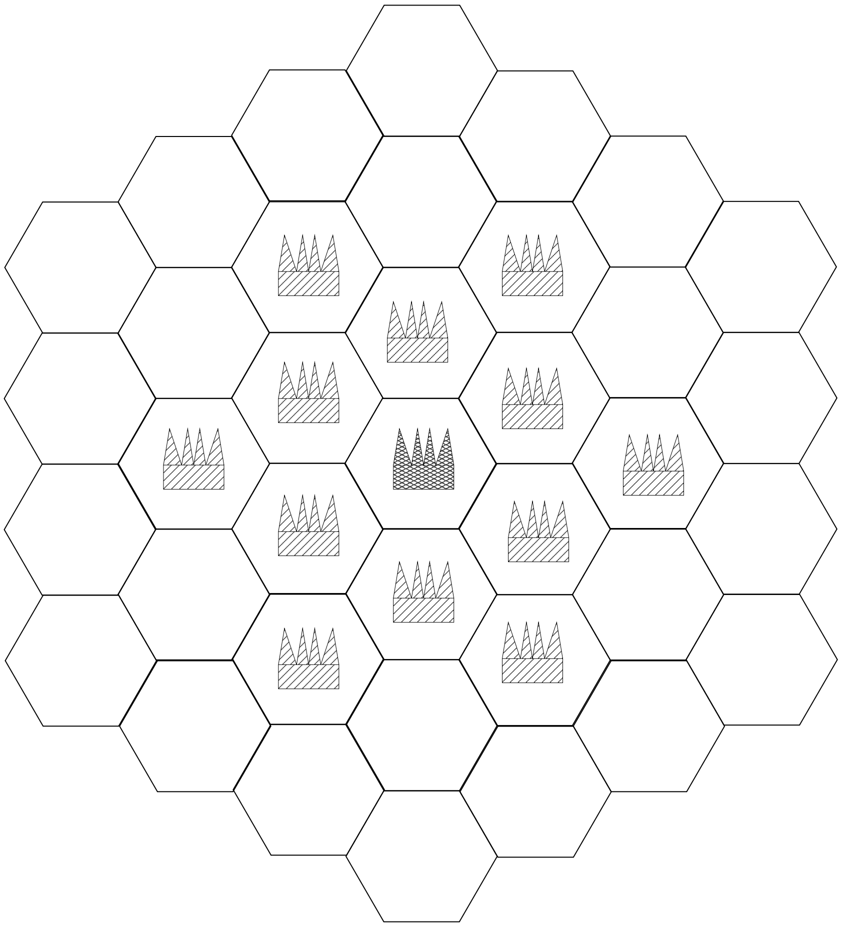

Figure 2.4: Moves of a king on a hexagonal Chessboard

2.1.38 Khintchine's Harmonic Mean

Khintchine's Harmonic mean is a variation of Khintchine's constant described in 2.1.37. It is described by the integral

$$K_{-1} = \lim_{n\to\infty} \frac{n}{a_1^{-1} + a_2^{-1} + a_2^{-1} + \cdots + a_n^{-1}} \tag{2.105}$$

2.1.39 Komornik-Loreti Constant

The Komornik-Loreti constant is the value of q described by

$$1 = \sum_{n=1}^{\infty} \frac{t_k}{q^k} \tag{2.106}$$

with t_k the Thue-Morse sequence. This constant is the smallest number in $[1, 2]$ for which a unique q-development of the form

$$1 = \sum_{n=1}^{\infty} \epsilon_i q^{-i} \tag{2.107}$$

exists ([56]).

The constant is also the unique positive real root of

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{q^{2^k}}\right) = \left(1 - \frac{1}{q}\right)^{-1} - 2 \tag{2.108}$$

The constant is transcendental ([2]).

2.1.40 Second Order Landau-Ramanujan Constant

If $S(x)$ is the number of positive integers $\leq x$ which can be expressed as a sum of two squares, then the following limit exists [58].

$$K_L = lim_{n\to\infty} \frac{\sqrt{\ln x}}{x} S(x) \tag{2.109}$$

The sums of squares of the first ten positive integers that are expressible as the sum of two squares

$$1 = 0^2 + 1^2 \tag{2.110}$$
$$2 = 1^2 + 1^2 \tag{2.111}$$
$$4 = 0^2 + 2^2 \tag{2.112}$$
$$5 = 1^2 + 2^2 \tag{2.113}$$
$$8 = 2^2 + 2^2 \tag{2.114}$$
$$9 = 0^2 + 3^2 \tag{2.115}$$
$$10 = 1^2 + 3^2 \tag{2.116}$$
$$13 = 2^2 + 3^2 \tag{2.117}$$
$$16 = 0^2 + 4^2 \tag{2.118}$$
$$18 = 3^2 + 3^2 \tag{2.119}$$

So $S(1) = 1$, $S(2) = 2$, $S(4) = 3$, $S(5) = 4$, $S(8) = 5$, $S(9) = 6$, $S(10) = 7$, $S(13) = 8$, $S(16) = 9$, $S(18) = 10\ldots$

Ramanujan exchanged the lower bound for the series 0 with a variable A. The according equation

$$S(x) = K_{LR} \int_A^x \frac{\mathrm{d}\,t}{\sqrt{\ln t}} + \theta(x) \tag{2.120}$$

The constant K_{LR} is the first order Landau-Ramanujan constant.[**?**] A fast converging formula had been given in [28]

$$K_{LR} = \frac{1}{sqrt2} \prod_{n=1}^{\infty} \left[\left(1 - \frac{1}{2^{2^n}}\right) \frac{\zeta(2^n)}{\beta(2^n)} \right]^{\frac{1}{2^{n+1}}} \tag{2.121}$$

where $\zeta(x)$ is Riemann's ζ-function and $\beta(x)$ is Dirichlet's β-function.

The second order Landau-Ramanujan constant C is the limit

$$\lim_{n\to\infty} \frac{(\ln x)^{\frac{3}{2}}}{K_{LR}} \left(S(x) - \frac{K_{LR}}{\sqrt{\ln x}} \right) = C \tag{2.122}$$

2.1.41 Laplace Limit Constant

The Laplace Limit constant is the point at which Laplace's formula for Kepler's equation starts to diverge. It is the unique real root of

$$f(x) = \frac{y \exp\left(\sqrt{1+x^2}\right)}{1+\sqrt{1+x^2}} \tag{2.123}$$

2.1.42 Lehmer Constant

The Lehmer constant occurs in the Lehmer cotangent expansion ([60])

$$x = \cot\left(\sum_{n=0}^{\infty} (-1)^n \cot^{-1} c_n \right) \tag{2.124}$$

where c_n is the recurrence

$$c_n = c_{n+1}^2 + c_n + 1 \qquad \text{with } n \geq 1 \tag{2.125}$$

2.1.43 Lemniscate Constant

With the arc length of a lemniscate12.6 with $a = 1$ being

$$s = \frac{1}{\sqrt{2\pi}} \left(\Gamma\left(\frac{1}{4}\right) \right)^2 \tag{2.126}$$

the Lemniscate constant is ([1])

$$L = \frac{1}{2} s \tag{2.127}$$

Other constants exists under this name:

First Lemniscate Constant With L the lemniscate constant as described in equation 2.127, the second lemniscate constant is ([59])

$$L_1 = \frac{1}{2} L \tag{2.128}$$

Second Lemniscate Constant With $M = \frac{1}{G}$ and G the Gauss constant**??** the number

$$L_2 = \frac{1}{2}M \tag{2.129}$$

is sometimes called the second lemniscate constant([105].

2.1.44 Lengyel Constant

With L the partition lattice of a set $\{a_0, a_1, \ldots, a_n\}$ the maximum element $E_{max} = \{\{a_0, a_1, \ldots, a_n\}\}$ and the minimum element $E_{min} = \{\{\{a_0\}, \{a_1\}, \ldots, \{a_n\}\}$, the number Z_n denoting the number of chains $\mathbf{C}$ with $\mathbf{C} \subseteq L \wedge \{E_{max}, E_{min}\} \in \mathbf{C}$ satisfies the recurrence relation

$$Z_n = \sum_{k=1}^{n-1} s(n,k) Z_k \text{with } Z_1 = 1 \tag{2.130}$$

The quotient

$$r(n) = \frac{Z_n (2\ln 2)^n n^{1+\frac{\ln 2}{3}}}{(n!)^2} \tag{2.131}$$

is bound between two constants as n approaches infinity ([62]).

2.1.45 Lévy constant

In a continued fraction representation of a number x the nth root of the denominator q_n of the nth convergent asymptotically approaches a constant when n approaches $+\infty$.

$$\lim_{n\to\infty} q_n^{(1/n)} = e^{\left(\pi^2/12\ln 2\right)} \tag{2.132}$$

With the exception of the set of x of measure zero[63, 61]. Plouffe ([**?**]) called the exponent of $\frac{\pi^2}{12\ln 2}$ the Khinchin-Lévy constant.

2.1.46 Madelung's Constant

In determine the energy of a single ion in a crystal the constant M in the equation

$$E = -\frac{z^2 e^2 M}{4\pi\epsilon_o r_0} \tag{2.133}$$

is called Madelung constant. Different crystals have different geometric arrangements, so Madelungs constant depends on the orientation

$$M = \sum_k (\pm)_k \frac{r_0}{r_k} \tag{2.134}$$

Madelung constants for cubic lattice sums are defined by ([67])

$$b_n(2s) = \sum_{k_1,\ldots,k_n=-\infty}^{\infty}{}' \frac{(-1)^{k_1+\cdots+k_n}}{\left(k_1^2 + \cdots + k_n^2\right)^s} \tag{2.135}$$

Where the prime indicates that the summation over $(0, \ldots, 0)$ is excluded.

For a three dimensional table salt crystal (NaCl)

$$b_3(1) = \sum_{k_1,k_2,k_3=-\infty}^{\infty}{}' \frac{(-1)^{k_1+k_2+k_3+1}}{\sqrt{k_1^2 + k_2 + k_3^2}} \tag{2.136}$$

M

Tyagi has found ([108]) a fast converging sum

$$M = -\frac{1}{6} - \frac{\ln 2}{4\pi} - \frac{4\pi}{3} + \frac{1}{2\sqrt{2}} + \frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{\pi^{\frac{3}{2}}\sqrt{2}} - 2 \tag{2.137}$$

$$\sum_{k_1,k_2,k_3=-\infty}^{\infty}{}' \frac{(-1)^{k_1+k_2+k_3}}{\sqrt{k_1^2 + k_2 + k_3^2}\left[exp\left(8\pi\sqrt{k_1^2 + k_2 + k_3^2}\right) - 1\right]} \tag{2.138}$$

The constant factor 2.137 of the equation is good for ten decimal digits on its own, without the following summation.

The other possible packing for a crystal is hexagonal, for example cesium chloride (CsCl). The formula for the hexagonal lattice sum $h_2(2)$ has a closed form.

$$h_s(2) = \pi \ln 3\sqrt{3} \tag{2.139}$$

The Madelung constant used in the implementation is that of $b_3(1)$ and the calculation has been done with the equation 2.137.

2.1.47 Magata's constant

Let **S** be the data set $\{(1, 2), (2, 3), (3, 5), (4, 7), \ldots, (n, p_n)\}$ with p_n denoting the nth prime. With polynomial fit of degree $n - 1$

$$c_0 + c_1(x-1) + c_2(x-1)(x-2) + c_3(x-1)(x-2)(x-3) + \cdots + c_n(x-1)\cdots(x-n) \tag{2.140}$$

the sum of the coefficients approach a constant when n approaches ∞ ([68]).

2.1.48 Meissel-Mertens constant

See 2.1.34.

2.1.49 Niven's constant

Let the prime factorization of a number $x \in \mathbb{N} \setminus \{0\}$ be described by

$$x = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \tag{2.141}$$

then the two functions

$$h(x) = \min(n_1, n_2, \ldots, n_k) \quad (2.142)$$
$$H(x) = \max(n_1, n_2, \ldots, n_k) \quad (2.143)$$
$$\text{with } H(1) = h(1) = 1 \quad (2.144)$$

have the following properties ([76]):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} h(x) = 1 \quad (2.145)$$

$$\lim_{n\to\infty} \frac{\sum_{x=1}^{n} h(x) - n}{\sqrt{x}} = \frac{\zeta\left(\frac{3}{2}\right)}{\zeta(3)} \quad (2.146)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} H(x) = C \quad (2.147)$$

The constant C in equation 2.147 is the number known as the Niven constant and has the value

$$C = 1 + \left(\sum_{k=2}^{\infty} \left[1 - \frac{1}{\zeta(k)} \right] \right) \quad (2.148)$$

2.1.50 Reciprocal of the One-Ninth-Constant

See 2.1.51.

2.1.51 One-Ninth-Constant

The One-Ninth-constant is based on a conjecture later proven to be false.

In the beginning was a proof by Schönhage ([93]) that

$$\lim_{n\to\infty} (\lambda_{0,n}) = \frac{1}{3} \quad (2.149)$$

with $\lambda_{m,n}$ Chebychev constants. The conjecture was

$$\Lambda = \lim_{n\to\infty} (\lambda_{n,n})^{1/n} = \frac{1}{9} \quad (2.150)$$

The naming of constants follows some weird rules, so this constant was named the "one-ninth constant" and, by the same logic, its reciprocal is sometimes known as "Varga's constant".

A first hint, that $\Lambda \neq \frac{1}{9}$ was given numerically ([107]), a formal disprove followed only two years later ([39]). An exact value of Λ is given by ([69])

$$\Lambda = \exp \left| -\frac{\pi K\left(\sqrt{1-c^2}\right)}{K(c)} \right| \tag{2.151}$$

with K the complete elliptic integral of the first kind and c a solution to

$$K(k) = 2E(k) \tag{2.152}$$

with E the complete elliptic integral of the second kind.

Another name for this constant has been proposed by Varga ([109]): Halphen constant. Halphen computed the root ([42]) of the equation

$$\sum_{k=0}^{\infty} (2k+1)^2 (-x)^{k(k+1)/2} = 0 \tag{2.153}$$

With $x \in (0, 1)$ the unique solution is indeed Λ ([70])

2.1.52 Paris Constant

For the recursion

$$\phi_n = \sqrt{1 + \phi_{n-1}} \qquad \text{for } n \geq 2 \wedge \phi_1 = 1 \tag{2.154}$$

Paris had proved in [81] that ϕ_n approaches the *Golden Ratio* ϕ at a constant rate.

$$\lim_{n\to\infty} \frac{(\phi - \phi_n)(2\phi)^n}{2} = C \tag{2.155}$$

So ([27])

$$C = \prod_{n=2}^{\infty} \frac{2\phi}{\phi + \phi_n} \tag{2.156}$$

2.1.53 Parking Rényi Constant

This constant answers the question of how much place is wasted by randomly parking cars in a street. Because this is a theoretical constant the cars have unit length, the length of the street is a real number and the cars do not overlap nor are they allowed to push. So within the closed interval $[0, x]$ with $x > 1$ the mean number $M(x)$ of cars that can park on that street is described by ([87])

$$M(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 + \frac{2}{x-1} \int_0^{x-1} M(y)\,\mathrm{d}y & \text{for } x \geq 1 \end{cases} \tag{2.157}$$

The mean density $m = \lim_{x\to\infty} \frac{M(x)}{x}$, Rényi's parking constant, can be described by

$$\int_0^\infty \exp\left(-2\int_0^x \frac{1-e^{-y}}{y}\right) \mathrm{d}\, x \tag{2.158}$$

While the inner integral is $\gamma + \Gamma(0,x) + \ln x$ with $\Gamma(x,y)$ the incomplete Γ-function or $\gamma + \mathrm{Ei}(-x) + \ln x$ with $\mathrm{Ei}(x)$ the exponential integral, no other form exist for the outer one. Inserting that in equation 2.158 gives

$$m = e^{-2\gamma} \int_0^\infty \frac{e^{-2\Gamma(0,x)}}{x^2} \tag{2.159}$$

$$= e^{-2\gamma} \int_0^\infty \frac{e^{-2\,\mathrm{Ei}(-x)}}{x^2} \tag{2.160}$$

The above holds in one dimension only, but [80] conjectured that for two dimensions

$$\lim_{x,y\to\infty} \frac{M(x,y)}{xy} = m^2 \tag{2.161}$$

which is not yet proven nor disproven[9].

2.1.54 Smallest Pisot-Vijayaraghavan Number

> A Pisot number is a positive algebraic integer greater than 1 all of whose conjugate elements have absolute value less than 1. A real quadratic algebraic integer greater than 1 and of degree 2 or 3 is a Pisot number if its norm is equal to ± 1. *[120]*

The smallest Pisot number θ_1, also known as the Plastic constant, is the positive root of

$$x^3 - x - 1 = 0 \tag{2.162}$$

as shown by [92] and proved by [96] The second smallest Pisot number, found by [96], is the positive root of

$$x^4 - x^3 - 1 = 0 \tag{2.163}$$

He also showed that θ_1 and θ_2 are isolated and that the positive roots of the following polynomials are also Pisot numbers

$$x^n \left(x^2 - x - 1\right) + x^2 - 1 \qquad \text{for } n \in \{\mathbb{N} \setminus \{0\}\} \tag{2.164}$$

$$x^n - \frac{x^{n+1} - 1}{x^2 - 1} \qquad \text{for } n \in \{\mathbb{N} \setminus \{0,1,2\}\} \wedge \text{odd}\, n \tag{2.165}$$

$$x^n - \frac{x^{n-1} - 1}{x^2 - 1} \qquad \text{for } n \in \{\mathbb{N} \setminus \{0,1,2\}\} \wedge \text{odd}\, n \tag{2.166}$$

$$\tag{2.167}$$

[9] At least not known to the author at the date given on the front page of this article.

The numbers have been named by [91] because of the closely related works of [83, 111] about $\text{frac}(x) \equiv x - \lfloor x \rfloor$

2.1.55 Plastic Constant

See 2.1.54. As a sidenote: with P the plastic constant the circumference of a *Snub Icosidodecadodecahedron* with $a = 1$ is

$$\frac{1}{2}\sqrt{\frac{2P-1}{P-1}} \tag{2.168}$$

2.1.56 Porter constant

Porter's constant ([85])

$$C = \frac{6\ln 2}{\pi^2}\left[3\ln 2 + 4\gamma - \frac{24}{\pi}\zeta'(2) - 2\right] - \frac{1}{2} \tag{2.169}$$

Example

With $T(m,n)$ the number of steps to compute $\gcd(m,n)$ by means of the Euklidian algorithm and $T(m,0) = 0$ if $m \geq 0$ then the value of $T(m.n)$ is defined by the reccurence formula

$$T(m,n) = \begin{cases} 1 + T(n, m \mod n) & \text{for } m \geq n \\ 1 + T(n,m) & \text{for } m < n \end{cases} \tag{2.170}$$

With fixed n and randomly choosen m the average number of steps is ([55])

$$T(n) = \frac{1}{n}\sum_{0\leq m<n} T(m,n) \tag{2.171}$$

and it has been shown by [78] that

$$T(n) = \frac{12\ln 2}{\pi}\left[\ln n - \sum_{d|n}\frac{\Lambda(d)}{d}\right] + C + \frac{1}{n}\sum_{d|n}\phi(d)\,\mathcal{O}(d^{-1/6+\epsilon}) \tag{2.172}$$

with $\Lambda(d)$ the Mangoldt-function, $\phi(d)$ the totient-function and C the Porter constant.

2.1.57 Sum of the Product of the Inverse of Primes

This constant is probably better described as the sum of the reciprocals of the *primorials*[10].

[10] Which rhymes with *factorials* not with *primordials*. As said elsewhere in this article: the name-finding in mathematics is not always fully comprehensible.

A primorial is defined as the product of all primes p_k up to a given prime p_n.

$$p_n\# = \prod_{k=1}^{n} p_k \tag{2.173}$$

More generally with a number n instead of the prime p_n

$$n\# = \prod_{k=1}^{\pi(n)} p_k \tag{2.174}$$

where $\pi(n)$ is the prime counting function which has the numbers of primes up to the limit n.

The limit of the reciprocals of the primorials is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} p_n\# \approx 0.7052301717918\ldots \tag{2.175}$$

which is implemented here as the *sum of the product of the inverse of primes*.

This limit ([90]) might be of additional interest

$$\lim_{n\to\infty} (p_n\#)^{1/p_n} = e \tag{2.176}$$

where e is the base of the natural logarithm.

2.1.58 Rabbit Constant

The rabbit constant has been named aptly: it is a result of the miraculous giftedness of rabbits to grow up and reproduce. Despite the biological mechanism of the reproduction of rabbits the rabbit sequence starts with one rabbit; a young one even, not able to reproduce before growing older. This single pre-pubescent rabbit of unknown sex shall be denoted 0. A rabbit in legal age shall be denoted 1. With the two mappings $0 \to 1$ for a rabbit growing up and $1 \to 10$ for multiplicating bunnies. Following the timeline we get $0 \to 1 \to 10 \to 101 \to 10110\cdots$. Written as a binary fraction gives $0.1011010110110\ldots_2$ which is called the Rabbit constant. The implementation gives the decimal representation $0.70980344\ldots$.

The Rabbit constant is related to the Fibonacci sequence by the continued fraction representation of the constant

$$\left[0, 2^0, 2^{F_1}, 2^{F_2}, 2^{F_3}, \ldots\right] \tag{2.177}$$

where F_n are Fibonacci numbers ([4, 33, 94]).

2.1.59 Ramanujan-Soldner Constant

The root of the logarithmic integral $\mathrm{li}(x) = 0$ is also known as the Ramanujan-Soldner constant μ[11]. With the logarithmic integral defined as the Cauchy principal value

$$\mathrm{li}(x) = \mathrm{PV}\int_0^x \frac{\mathrm{d}\,t}{\ln t} \tag{2.178}$$

$$= \lim_{\epsilon\to 0+}\left[\int_0^{1-\epsilon} \frac{\mathrm{d}\,t}{\ln t}\int_{1+\epsilon}^x \frac{\mathrm{d}\,t}{\ln t}\right] \tag{2.179}$$

and μ the identity follows ([113, 75])

$$\mathrm{PV}\int_0^x \frac{\mathrm{d}\,t}{\ln t} = \int_\mu^x \frac{\mathrm{d}\,t}{\ln t} \qquad \text{for } x > \mu \tag{2.180}$$

2.1.60 Reciprocal Fibonacci Constant

The reciprocal Fibonacci constant is exactly what its name implies: the sum of the reciprocals of the Fibonacci numbers F_n

$$P_F = \sum_{n=1}^{\infty}\frac{1}{F_n} \tag{2.181}$$

This constant was proved to be irrational by [5].

2.1.61 Reciprocal Prime Constant

See 2.1.34.

2.1.62 Robbins' Constant

This constant is the average distance of two randomly chosen points inside a unit cube. More exact ([89])

$$\Delta(3) = \frac{1}{105}\left[4 + 17\sqrt{2} - 6\sqrt{3} + 21\ln\left(1+\sqrt{2}\right) + 42\ln\left(2+\sqrt{3}\right) - 7\pi\right] \tag{2.182}$$

A useful identity is probably

$$\ln\left(1+\sqrt{2}\right) = \operatorname{asinh} 1 \tag{2.183}$$

It might be of interests to the more claustrophobic readers that the average distance of two randomly chosen points on *different* faces of the cube is [14, 15]

$$\Delta_F(3) = \frac{1}{75}\left[1 + 17\sqrt{2} - 6\sqrt{3} + 21\ln\left(1+\sqrt{2}\right) + 42\ln\left(2+\sqrt{3}\right) - 7\pi\right] \tag{2.184}$$

[11] But c is also found here and there

The ratio of these two is

$$\Delta_F(3) = \frac{7}{5}\Delta(3) \tag{2.185}$$

2.1.63 Smallest Known Salem Number

The smallest Salem number is the largest root of

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \tag{2.186}$$

2.1.64 Sierpiński Constant

The Sierpińsky constant K can be described by ([97])

$$K = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{r_2(k)}{k} - \pi\ln n\right] \tag{2.187}$$

where $r_2(x)$ is the number of ways to represent the number x as

$$x = a^2 + b^2 \qquad \text{for } \{a, b\} \in \mathbb{N} \tag{2.188}$$

2.1.65 sin 1

The sine of 1 had been calculated with

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{2.189}$$

2.1.66 sinh 1

The sine hyperbolicus of 1 had been calculated with

$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= -i\sinh(iz)\end{aligned} \tag{2.190}$$

2.1.67 Traveling Salesman Constant

The length of a self-avoiding space-filling curve through a set of n points.

$$\lambda = \lim_{m\to\infty}\frac{L_m}{\sqrt{n_m}} \tag{2.191}$$

$$= \frac{4\left(1 + 2\sqrt{2}\right)\sqrt{51}}{153} \tag{2.192}$$

with L_m the curve length at the mth iteration and n_m the size of the point set ([77])

2.1.68 Tribonacci Constant

The Tribonacci sequence is one generalization of the Fibonacci sequence

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad \text{with } T_1 = T_2 = 1, T_3 = 2 \text{ and } n \geq 4 \tag{2.193}$$

It has a corresponding constant, the positive root of the polynomial

$$0 = x^3 - x^2 - x - 1 \tag{2.194}$$

Continuing with this technique[12]

Polynome	**Constant**	**Name**
$0 = x^2 - x - 1$	$\frac{1}{2}\left(1+\sqrt{5}\right)$	Fibonacci
$0 = x^3 - x^2 - x - 1$	$\frac{1}{3}\left[1+\sqrt[3]{19-3\sqrt{33}}+\sqrt[3]{19+3\sqrt{33}}\right]$	Tribonacci
$0 = x^4 - \cdots - x - 1$	≈ 1.927561975	Tetranacci
$0 = x^5 - \cdots - x - 1$	≈ 1.965948236	Pentanacci
$0 = x^6 - \cdots - x - 1$	≈ 1.983582843	Hexanacci
$0 = x^7 - \cdots - x - 1$	≈ 1.991964196	Heptanacci
$0 = x^n - x^{n-1} - \cdots - x - 1$	2	n-anacci

2.1.69 The A.G.M of 1 and $\frac{1}{\sqrt{2}}$

The Gauss constant G is the reciprocal of the arithmetic-geometric mean of 1 and $\sqrt{2}$

$$G = \frac{1}{M\left(1, \sqrt{2}\right)} \tag{2.195}$$

$$= \frac{\sqrt{2}}{\pi} K\left(\frac{1}{\sqrt{2}}\right) \tag{2.196}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 \tag{2.197}$$

where $K(x)$ is the complete elliptic integral of the first kind and $\Gamma(z)$ the Γ-function.

A series, converging quite fast is given by [27]

$$G = 2^{\frac{5}{4}} e^{\frac{-\pi}{3}} \left(\sum_{n=-\infty}^{\infty} (-1)^n e^{-2n\pi(3n+1)}\right)^2 \tag{2.198}$$

The constant

$$\frac{M}{\sqrt{2}} \tag{2.199}$$

is called the *ubiquitous constant* in some articles ([99, 27])

[12] The names of the constants are due to [121]

2.1.70 Universal Parabolic Constant

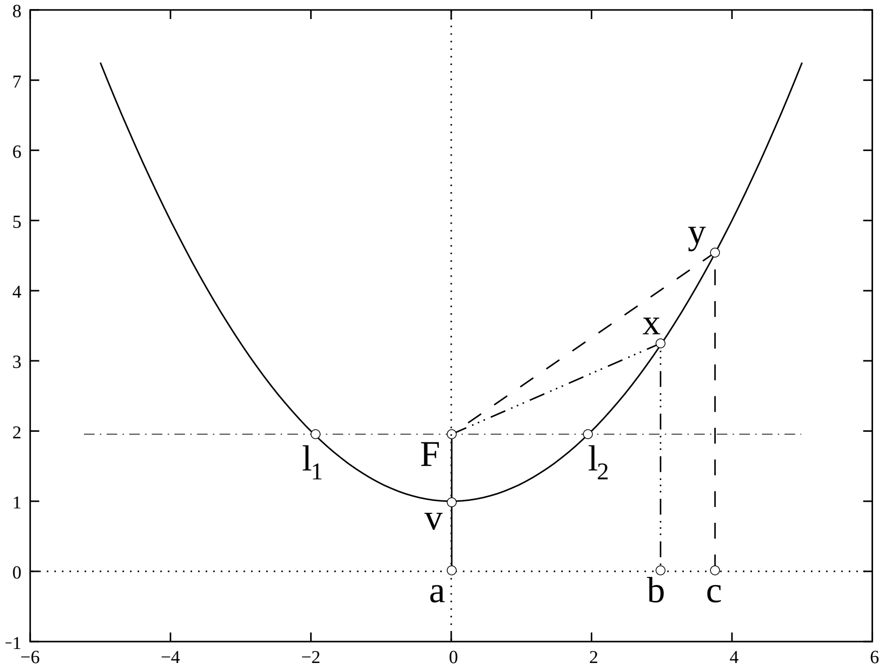

Figure 2.5: Parabola $\frac{1}{4}x^2 + 1$

The universal parabolic constant P is the ratio between the length of the line $\overline{l_1 l_2}$, the *latus rectum* and the length of the segment of the parabola $\overline{l_1 v l_2}$ in figure 2.5. The exact value is

$$P = \sqrt{2} + \ln\left(1 + \sqrt{2}\right) \tag{2.200}$$

$$= \sqrt{2} + \operatorname{asinh} 1 \tag{2.201}$$

2.1.71 Viswanath's Constant

The Viswanath constant is a special form of the Embree-Trefethen constant described in section 2.1.20.

For the recurrence

$$a_n = \pm a_{n-1} \pm a_{n-2} \qquad \text{with } a_0 = 0, a_1 = 1, P(\text{sign}) = \frac{1}{2} \tag{2.202}$$

exists almost surely the limit ([112])

$$\lim_{n\to\infty} \sqrt[n]{|a|} \tag{2.203}$$

2.1.72 Weierstrass Constant

The Weierstrass constant w is defined as the value $\frac{1}{2}\sigma(1;1,i)$ of Weierstrass' sigma function $\sigma(z;\omega_1,\omega_2)$. It has the closed form ([115, 116])

$$w = \frac{2^{\frac{5}{4}}\sqrt{\pi}e^{\frac{\pi}{8}}}{\Gamma^2\left(\frac{1}{4}\right)} \tag{2.204}$$

2.1.73 Some ζ Values

Some of the more useful $\zeta(n)$ values of mostly odd n are implemented. For more information about Riemann's ζ-function and the numerical evaluation of it see section 4.10.1.

2.2 Physical Constants

This section lacks a lot of bibliographical links, but it is difficult to get the hands on the original works. Most of the older books are only available as abridged translations[13].

2.2.1 Astronomical Unit

The astronomical unit AU is the mean distance between Earth and sun. More formal: the radius of an unperturbed circular orbit a massless object would revolve about the sun in $\frac{2\pi}{k}$ days. The Gaussian constant k is defined exactly as 0.01720209895 in this case. ([74])

2.2.2 Avogadro Constant

The Avogadro constant N_A is the number of atoms in 0.012 kg of C^{12} and the current value is ([79])

$$6.02214179 \times 10^{23}\mathrm{mol}^{-1} \pm 0.3 \times 10^{17} \tag{2.205}$$

2.2.3 Boltzmann Constant

The Boltzmann constant k describes the relation between the macroscopic temperature and the microscopic particle energy. It is the ratio of the gas constant R and the Avogadro constant N_A

$$k = \frac{R}{N_A} \tag{2.206}$$

[13]And sometimes bad translations! The author has read a translation of some work of Newton which could be clearly judged as wrong even without knowning the original text at all—the mathematics were glaring wrong.

2.2.4 Candela

The definition of a candela[14]

> The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hz and that has a radiant intensity in that direction of $\frac{1}{683}$ watt per steradian.

A wax candela emits about one candela hence the now historic name.

2.2.5 Dielectric Constants

The dielectric constant ϵ_r is the ratio of the static permitivity of the material ϵ_s and the electric constant ϵ_0

$$\epsilon_r = \frac{\epsilon_s}{\epsilon_0} \tag{2.207}$$

The values given in the implementation are the values of ϵ_s.

2.2.6 Dirac Constant

The Dirac constant $\hbar$ is related to Planck's constant h by the ratio

$$\hbar = \frac{h}{2\pi} \tag{2.208}$$

2.2.7 Gas Constant

The gas constant is related to the Boltzmann constant but measures the energy of one mol of particles instead of single particles.

2.2.8 Weight of One Mol Water

The Avogadro constant times the *average* atomic mass of water–about 18 grams. The word "average" is very important because the composition of water can vary between $H_2^1O^{16}$ and $H_2^2O^{18}$ or even $H_2^3O^{16}$!

2.2.9 Speed of Light

The speed of light is fixed at $299\,792\,458 \frac{\text{m}}{\text{s}}$ to have a steady point in spacetime to hang up the physicist's hat.

More formal: the meter is defined as[15]

> The metre is the length of the path traveled by light in vacuum during a time interval of $1/299\,792\,458$ of a second.

[14] 16^{th} CGPM 1979, resolution 3
[15] 17^{th} CGPM 1983, resolution 1

So it follows that the speed of light is exactly $299\,792\,458\,\frac{\mathrm{m}}{\mathrm{s}}$. The second is defined as[16]

> The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

This definition is not complete, so at the 1997 meeting of the CIPM it was a made clear that:

> This definition refers to a caesium atom at rest at a temperature of 0 K.

2.2.10 Light Year

The year in this implementation is defined to be 365.25 days with 24 hours in a day, 60 minutes in an hour and 60 seconds in a minute. A lightyear is the length a photon travels in one year, so the length of a lightyear might differ from the numbers in other implementations.

2.2.11 Magnetic Permeability of the Vacuum

The magnetic permeability of the vacuum μ_0 follows from the definition of the Ampère and is therefore defined as

$$\mu_0 \stackrel{\text{def}}{=} 4\pi \times 10^7 \frac{\mathrm{N}}{\mathrm{A}^2} \tag{2.209}$$

2.2.12 Newton's Gravity Constant

Newton's gravity constant is the proportional constant G in

$$F = G\frac{m_1 m_2}{r^2} \tag{2.210}$$

where F is the force, m_1 and m_s are masses > 0 and r is the distance between the masses.

2.2.13 Parsec

Parsec is short for *parallax second*.

$$P = U_A \cdot \cot\left(\frac{2\pi 1''}{360}\right) \tag{2.211}$$

$$= U_A \cdot \cot\left(\frac{2\pi 0.00027778}{360}\right) \tag{2.212}$$

with U_A the astronomical unit.

[16] 13^{th} CGPM 1967/68, resolution 1

2.2.14 Planck Constant

The Planck constant h is the ratio between the energy of a photon E and its frequency ν

$$h = \frac{E}{\nu} \tag{2.213}$$

2.2.15 Seconds in a Year

Because the term *year* is ambiguous and variable[17] this constant is offered in this implementation and is the number of seconds in a year of 365.25 days of 86 400 seconds per day.

2.2.16 Stefan-Boltzmann constant

Also known as Stefan's constant it is exactly

$$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} \tag{2.214}$$

with h Planck's constant, k Boltzmann constant and c the speed of light. It is the constant in the Stefan-Boltzmann law of energy flux of a black-body

$$\Phi = \sigma T^4 \tag{2.215}$$

with T the absolute temperature of the black-body.

2.2.17 Luminosity of the Sun

Implemented here is the defined value[18].

2.2.18 Electric Constant

The electric constant ϵ_0 is defined as

$$\epsilon_0 \stackrel{\text{def}}{=} \frac{1}{\mu_0 c^2} \tag{2.216}$$

with μ_0 the magnetic constant and c the speed of light.

2.2.19 Wien's displacement constant

Wien's displacement constant b is the ratio between the wavelength of the emission peak of a black-body and the temperature of this black-body

$$b = \lambda T \tag{2.217}$$

With λ the emission's peak wavelength and T the absolute temperature of the black-body.

[17] Sidereal year? Tropical year? Which planet are you even talking about?
[18] The sun is a slightly variable star, so the luminosity varys

Chapter 3

Common Functions

3.1 Basic operations

The basic functions are defined here as the mathematical functions in the ECMA-standard with the exception of the trigonometric functions which have their own section. This program extends these functions to the complex plane with the restriction that ECMA-script in its current version does not allow overloading of all functions, so the syntax for the complex arithmetic is different.

3.1.1 Operations on complex numbers

Equality

Two complex numbers z_1 and z_2 are equal if and only if the real and the imaginary parts are equal.

$$z_1 = z_2 \qquad \text{iff} \qquad \Re z_1 = \Re z_2 \wedge \Im z_1 = \Im z_2 \tag{3.1}$$

Addition

The addition of complex numbers in their Cartesian notation works like vector addition because, well, they are vectors.

$$z_1 + z_2 = z_3 = \begin{cases} \Re z_3 & = \Re z_1 + \Re z_2 \\ \Im z_3 & = \Im z_1 + \Im z_2 \end{cases} \tag{3.2}$$

```
var z_1 = new Complex(2.3, -3.4);
var z_2 = new Complex(-7.3, 3.8);
var z_3 = z_1.add(z_2);
```

Subtraction works accordingly (z_1`.sub(`z_2`)`).

Multiplication

There are mainly two different ways two multiply two complex numbers: the one with the Cartesian notation and the one with the polar form. The polar form is easier to calculate manually but lacks precision when used with Gaussian integers[1] because of the use of e.

The algorithm for the Cartesian form $z = (\Re z + \Im z)$

$$z_1 \cdot z_2 = z_3 = \begin{cases} \Re z_3 & = \Re z_1 \cdot \Re z_2 - \Im z_1 \cdot \Im z_2 \\ \Im z_3 & = \Im z_1 \cdot \Re z_2 + \Re z_1 \cdot \Im z_2 \end{cases} \tag{3.3}$$

The algorithm for the polar form $z = \rho e^{\theta i}$

$$z_1 \cdot z_2 = z_3 = \rho_{z_1} \rho_{z_2} \left(e^{\theta_{z_1} + \theta_{z_2}} \right) \tag{3.4}$$

The algorithm used in the program is the Cartesian because of the reasons listed above.

Division

The rules for division are obviously the same as for multiplication and there are mainly two ways to divide. The algorithm for the Cartesian form $z = (\Re z + \Im z)$

$$\frac{z_1}{z_2} = z_3 = \begin{cases} \Re z_3 & = \dfrac{(\Re z_1 \cdot \Re z_2 + \Im z_1 \cdot \Im z_2)}{((\Re z_2)^2 + (\Im z_2)^2)} \\ \\ \Im z_3 & = \dfrac{(\Im z_1 \cdot \Re z_2 + \Re z_1 \cdot \Im z_2)}{((\Re z_2)^2 + (\Im z_2)^2)} \end{cases} \tag{3.5}$$

The algorithm for the polar form $z = \rho e^{\theta i}$

$$\frac{z_1}{z_2} = z_3 = \frac{\rho_{z_1}}{\rho_{z_2}} \left(e^{\theta_{z_1} - \theta_{z_2}} \right) \tag{3.6}$$

[1] Gaussian integers are complex numbers with both the real and integer part integer

Actually, the algorithm used for division is a third one. It is slightly less precise (about one decimal digit) but avoids overflow

$$\frac{z_1}{z_2} = z_3 = \begin{cases} |\Im z_1| \geq |\Im z_2| & \begin{cases} \Re z_3 & = \dfrac{1}{\left(\Re z_2 + \Im z_2 \frac{\Im z_2}{\Re z_2}\right)} \left(\Re z_1 + \Im z_1 \dfrac{\Im z_2}{\Re z_2}\right) \\ \Im z_3 & = \dfrac{1}{\left(\Re z_2 + \Im z_2 \frac{\Im z_2}{\Re z_2}\right)} \left(\Im z_1 - \Re z_1 \dfrac{\Im z_2}{\Re z_2}\right) \end{cases} \\ \text{otherwise} & \begin{cases} \Re z_3 & = \dfrac{1}{\left(\Re z_2 \frac{\Re z_2}{\Im z_2} + \Im z_2\right)} \left(\Re z_1 \dfrac{\Re z_2}{\Im z_2} + \Im z_1\right) \\ \Im z_3 & = \dfrac{1}{\left(\Im z_2 \frac{\Re z_2}{\Im z_2} + \Im z_2\right)} \left(\Re z_1 \dfrac{\Re z_2}{\Im z_2} - \Re z_1\right) \end{cases} \end{cases} \tag{3.7}$$

Power

To get the power of complex numbers is not so easy but solvable, even with complex exponents.

In case the exponent is a real number x it is easy

$$z^x = e^{x \ln z} \tag{3.8}$$

When the exponent is complex too, it is still not that complicated. $\arg(z)$ is $theta$ from the polar form and thus $\arg z = \operatorname{atan2}(\Im z, \Re z)$.

$$z_1^{z_2} = a = \begin{cases} \Re a & = e^{\ln |z_1|} \Re z_2 - \Im z_2 \cdot \arg z_1 \\ \Im a & = \ln |z_1| \Im z_2 + \Re z_2 \cdot \arg z_1 \end{cases} \tag{3.9}$$

Both algorithms return the complex number in its polar form, so this program converts the results to the Cartesian form before returning them.

Roots

To get roots of complex numbers use the `Complex.pow()` function with rational exponents (for example `z.pow(.5)` for the square root). Only one branch is returned.

Exponential e^x

The exponential works like the power function but is implemented as follows

$$e^z = a = \begin{cases} \Re a & = e^{\Re z} \\ \Im a & = \Im z \end{cases} \tag{3.10}$$

This returns the result in polar form. The program converts it to Cartesian form before it returns the result.

Logarithm

The logarithm of a complex number has infinite results, so this program returns the principal branch only.

$$\ln z = a = \begin{cases} \Re z = \Im z = 0 & \begin{cases} \Re a &= -\infty \\ \Im a &= -\infty \end{cases} \\ \\ \Im z = 0 & \begin{cases} \Re a &= \ln \Re z \\ \Im a &= 0 \end{cases} \\ \\ \text{otherwise} & \begin{cases} \Re a &= \ln |z| \\ \Im a &= \arg z \end{cases} \end{cases} \tag{3.11}$$

Conjugate

The conjugate $\bar{z}$ of the complex number z is

$$\bar{z} = \begin{cases} \Re z &= \quad \Re z \\ \Im z &= -\Im z \end{cases} \tag{3.12}$$

The following identity is also of interest

$$z \cdot \bar{z} = |z|^2 = \rho^2 \tag{3.13}$$

Inverse

There are two different inverses for complex numbers implemented in the program: the multiplicative inverse $\operatorname{inv} z$ such that $z \cdot \operatorname{inv} z = 1$ and the additive inverse $-z$ such that $z + -z = 0$. The multiplicative inverse

$$\operatorname{inv} z = a = \begin{cases} \Re a &= \dfrac{\Re z}{((\Re z + \Im z)(\Re z - \Im z))} \\ \Im a &= -\dfrac{\Im z}{((\Re z + \Im z)(\Re z - \Im z))} \end{cases} \tag{3.14}$$

The additive inverse

$$-z = a = \begin{cases} \Re a &= -\Re z \\ \Im a &= -\Im z \end{cases} \tag{3.15}$$

Norm

The norm $|z|$ of the complex number z is defined as the distance to the origin which can be calculated with Pythagoras' simple formula $c^2 = a^2 + b^2$. By the definition of the polar form ρ is this distance, so we get the following identities

$$\begin{aligned} |z| &= \rho \\ &= \sqrt{(\Re z)^2 + (\Im z)^2} \\ &= \sqrt{(\Re z + \Im z)(\Re z - \Im z)} \\ &= \sqrt{z \cdot \bar{z}} \end{aligned} \tag{3.16}$$

To avoid overflow and too much impreciseness $|z|$ is implemented in this program as

$$|z| = \begin{cases} |\Re z| & \Im z = 0 \\ |\Im z| & \Re z = 0 \\ 0 & \Re z = \Im z = 0 \\ \Re z \sqrt{\left(1 + \left(\frac{\Im z}{\Re z}\right)^2\right)} & \text{otherwise} \end{cases} \tag{3.17}$$

3.1.2 Operations on Matrices

Matrices are implemented here in two different ways:

1. With ECMA-script arrays and real numbers. This is from the first implementation of matrices, not very elegant but hopefully better comprehensible than the new implementation that uses in parts code that was ported from hardly legible handoptimized Fortran code. The algorithms used for this kind of matrix implementation are more or less verbatim translations from the textbook[2].

2. With ECMA-script arrays, complex numbers and a slightly more object-oriented design.

The matrix operations described in this section are based on the second implementation if not noted otherwise.

Equality

Several definitions of equality are possible for matrices. The most common definition, also the one implemented here, defines two matrices to be equal if they are equal in size, elements and position of elements. Given two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1l} \\ b_{21} & b_{22} & \dots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kl} \end{bmatrix} \tag{3.18}$$

[2] The textbook in question is most probably [25] as the author concludes from a comment in the original code.

these matrices are equal in size, elements and position of elements if

$$A = B \iff \forall a_{ij} \in A \exists b_{kl} \in B\{b_{kl} | a_{ij} = b_{kl} \wedge i = k \wedge j = l \wedge \{i, j, k, l\} \in \mathbb{N}\} \tag{3.19}$$

The equation 3.19 is simplified—matrices are not common sets of course—but basically correct[3].

Addition

The addition of two matrices of the same size and form is done by adding the elements of the same positions.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1j} + b_{1l} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2j} + b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} + b_{k1} & a_{i2} + b_{k2} & \dots & a_{ij} + b_{kl} \end{bmatrix} \tag{3.20}$$

Subtraction

The subtraction of two matrices is implemented by multiplying one matrix with -1 and adding both.

$$A - B = A + (B \cdot (-1)) \tag{3.21}$$

Multiplication

The multiplication of two matrices is a bit more complicated.

matrix A times a number n That is the simplest algorithm in this group: just multiply every element with the number.

$$A \cdot n = \begin{bmatrix} a_{11} \cdot n & a_{12} \cdot n & \dots & a_{1j} \cdot n \\ a_{21} \cdot n & a_{22} \cdot n & \dots & a_{2j} \cdot n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} \cdot n & a_{i2} \cdot n & \dots & a_{ij} \cdot n \end{bmatrix} \tag{3.22}$$

matrix A times matrix B The product of two matrices is only defined if $j = k$, that is if A has as many columns as B has rows. The algorithm itself is nearly as simple as matrix addition.

$$c_{il} = \sum_{n=1}^{j} a_{in} b_{nl} \tag{3.23}$$

[3] Hopes the author.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \tag{3.24}$$

$$C = AB \tag{3.25}$$

$$c_{11} = a_{11}b11 + a_{12}b21 + a_{13}b31 \tag{3.26}$$

$$c_{12} = a_{11}b12 + a_{12}b22 + a_{13}b32 \tag{3.27}$$

$$c_{21} = a_{21}b11 + a_{22}b21 + a_{23}b31 \tag{3.28}$$

$$c_{22} = a_{22}b12 + a_{22}b22 + a_{23}b32 \tag{3.29}$$

Matrix multiplication is not commutative!

$$AB \neq BA \qquad \text{for most } A, B \tag{3.30}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \tag{3.31}$$

$$AB = \begin{bmatrix} 4 & 11 \\ 10 & 23 \end{bmatrix} \tag{3.32}$$

$$BA = \begin{bmatrix} 5 & 8 \\ 16 & 22 \end{bmatrix} \tag{3.33}$$

The conditions such that $AB = BA$ are

- if both matrices are equal under the conditions described in equation 3.19
- if all elements of one or both of two square matrices are zero
- if the size of at least one matrix is one (only one element)

Division

The division of two matrices $\frac{A}{B}$ is implemented as the product of A and the inverse of B. For an explanation of the *matrix inverse* see 26

Rotation

One of the sometimes ridiculed but nevertheless useful operations is the rotation of the matrix 90 degrees anti-clockwise.

$$\text{rot}_{90} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{13} & a_{23} \\ a_{12} & a_{22} \\ a_{11} & a_{21} \end{bmatrix} \tag{3.34}$$

Main Diagonal

The main diagonal of a matrix is described for a square matrix by

$$\mathrm{diag}(A) = \{a_{11}, a_{22}, \ldots, a_{ii}\} \tag{3.35}$$

Trace

The trace is the sum of the elements on the main diagonal. For a 3×3 matrix A it is defined as

$$\mathrm{trace}(A) = a_{11} + a_{22} + a_{33} \tag{3.36}$$

Norm, 2-Norm, Rank

These values are calculated by doing a singular value decomposition of the matrix. The singular value decomposition is sadly implemented for real matrices only. The 2-norm of a matrix is sometimes called the condition and notated in equations for a matrix A as $\mathrm{cond}(A)$

Determinant

The determinant for square matrices is determined differently depending on the size of the matrix.

1×1 The determinant of a constant is the constant itself

2×2 Direct way

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} \tag{3.37}$$

3×3 By following Sarrus' rule strictly

$$\det(A) = a_{11}a_{22}a_{33} \tag{3.38}$$
$$+ a_{12}a_{23}a_{31} \tag{3.39}$$
$$+ a_{13}a_{21}a_{32} \tag{3.40}$$
$$- a_{13}a_{22}a_{31} \tag{3.41}$$
$$- a_{12}a_{21}a_{33} \tag{3.42}$$
$$- a_{11}a_{23}a_{32} \tag{3.43}$$

$n \times n$ **with** $n > 3$ This is done by means of the LU-decomposition in the new implementation of matrices and by building and adding all subdeterminants in a computationaly very expensive recursion in the old one.

For further explanations see the chapter about linear algebra on page 97.

Adjoint

The adjoint of a matrix is implemented in quite a slow but generally working way for square matrices only. For a square matrix A

$$a_{ij} = (-1)^{i+j}\det(S_{i-1,j-1}) \tag{3.44}$$

where the matrix S_{kl} is build by the algorithm

$$s_{ij} = \begin{cases} a_{ij} & \text{if } i \leq k \wedge j \leq l \\ a_{i+1,j} & \text{if } i > k \wedge j \leq l \\ a_{i+1,j+1} & \text{if } i > k \wedge j > l \\ a_{i,j+1} & \text{if } i \leq k \wedge j > l \end{cases} \tag{3.45}$$

There exist much faster algorithms for special kinds of matrices. They might be implemented later, but an interpreted language is rarely useful for computational exhaustive algorithms like the manipulation of large matrices if that language is not optimized for it. ECMA-script is not very optimal for that usage.

Transpositions

The transposition is implemented in two ways: as a normal transposition and, the matrix element are complex numbers after all, as a conjugate transpose.

$$\text{normal transpose} \qquad a_{ij} = a_{ji} \tag{3.46}$$

$$\text{conjugated normal transpose} \qquad \Re a_{ij} = \Re a_{ji}, \Im a_{ij} = -\Im a_{ji} \tag{3.47}$$

Inverse

The inverse of a matrix is implemented, like the matrix adjoint, in quite a slow but generally working way for square matrices only.

$$\text{inv}(A) = \text{trans}(\text{adj}(A))\frac{1}{\det(A)} \qquad \text{if } \det(A) \neq 0 \tag{3.48}$$

3.2 Geometric Functions

3.2.1 Triangle

The area A of a triangle, given the lengths of all three sides a, b, c, is according to Heron's formula

$$A = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4} \tag{3.49}$$

The area of a triangle, given the lengths of two sides a, b and the angel α between them, is

$$A = \frac{ab\sin\alpha}{2} \tag{3.50}$$

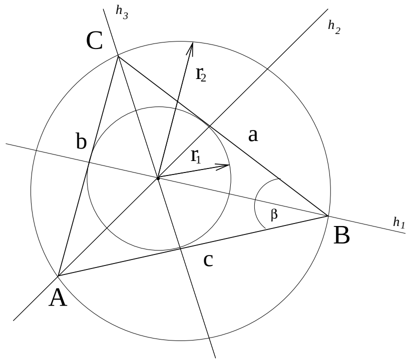

Figure 3.1: A general triangle

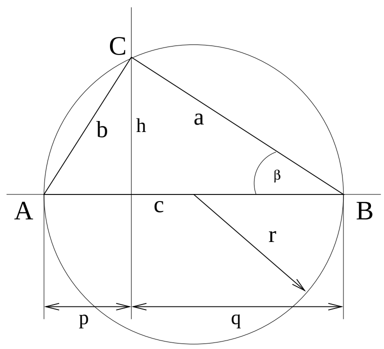

Figure 3.2: A right-angled triangle

The area of a right-angled triangle, given the lengths of the two catheti a, b is

$$A = \frac{ab}{2} \tag{3.51}$$

The area of a triangle, given the length of one side s and the height h is

$$A = \frac{sh}{2} \tag{3.52}$$

3.2.2 Polygon

Some functions for the gardeners among the readers have been implemented; mainly a way to get equilateral polygons more easily than the old way with a pointy stick and a piece of knotty yarn. The number of edges is restricted to 3, 4, 5, 6, 8, and 10 for the two functions `Math.encircledPolygonsSide(edges,side)` and `Math.encircledPolygonsRadius(edges,radius)`. The first one calculates the polygon from the number of edges and the length of one side, the second one with the help of the number of edges and the radius of the circumcircle. It is probably the second formula that is more useful in the dirt-business.

The functions return a couple of values computed from the given arguments. For `Math.encircledPolygonsSide(edges,side)` these are the angel $alpha$, the radii of both the circumcircle r_2 and the incircle r_1, and the area[4] A.

[4]The area is needed to calculate the volume of crap necessary to harvest the biggest potatoes.

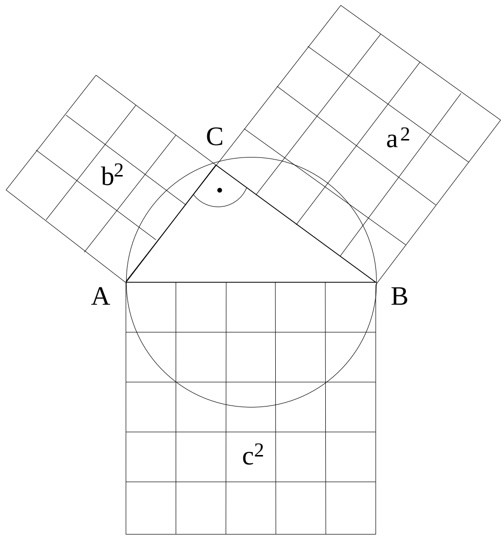

Figure 3.3: $\Pi\upsilon\theta\alpha\gamma\acute{o}\rho\alpha\varsigma$

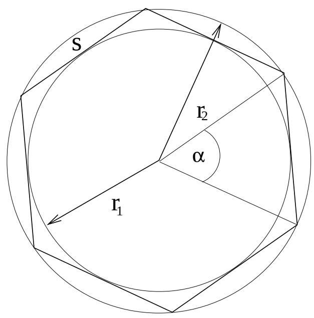

Figure 3.4: Equilateral hexagon with in- and circumcircle

s is the length of one side and n the number of edges

$$\left.\begin{aligned} \alpha &= 120 \\ r_2 &= \frac{s}{3}\sqrt{3} \\ r_1 &= \frac{s}{6}\sqrt{3} \\ A &= \frac{s^2}{4}\sqrt{3} \end{aligned}\right\} n = 3, \quad \left.\begin{aligned} \alpha &= 90 \\ r_2 &= \frac{s}{2}\sqrt{2} \\ r_1 &= \frac{s}{2} \\ A &= s^2 \end{aligned}\right\} n = 4, \quad \left.\begin{aligned} \alpha &= 72 \\ r_2 &= \frac{s}{10}\sqrt{50 + 10\sqrt{5}} \\ r_1 &= \frac{s}{10}\sqrt{25 + 10\sqrt{5}} \\ A &= \frac{s^2}{4}\sqrt{25 + 10\sqrt{5}} \end{aligned}\right\} n = 5 \tag{3.53}$$

$$\left.\begin{aligned} \alpha &= 60 \\ r_2 &= s \\ r_1 &= \frac{s}{2}\sqrt{3} \\ A &= \frac{3s^2}{4}\sqrt{3} \end{aligned}\right\} n = 6, \quad \left.\begin{aligned} \alpha &= 45 \\ r_2 &= \frac{s}{2}\sqrt{4 + 2\sqrt{2}} \\ r_1 &= \frac{s}{2}\left(\sqrt{2} + 1\right) \\ A &= 2s^2\left(\sqrt{2} + 1\right) \end{aligned}\right\} n = 8, \quad \left.\begin{aligned} \alpha &= 36 \\ r_2 &= \frac{s}{2}\left(\sqrt{5} + 1\right) \\ r_1 &= \frac{s}{2}\sqrt{5 + 2\sqrt{5}} \\ A &= \frac{5s^2}{2}\sqrt{5 + 2\sqrt{5}} \end{aligned}\right\} n = 10 \tag{3.54}$$

c is the circumference of the polygon

$$\left.\begin{aligned}\alpha &= 120\\ s &= r_2\sqrt{3}\\ c &= 3r_2\sqrt{3}\\ r_1 &= \frac{r_2}{2}\\ A &= \frac{3r_2^2}{4}\sqrt{3}\end{aligned}\right\} n = 3, \quad \left.\begin{aligned}\alpha &= 90\\ s &= r_2\sqrt{2}\\ c &= 4r_2\sqrt{2}\\ r_1 &= \frac{r_2}{2}\sqrt{2}\\ A &= 2r_2^2\end{aligned}\right\} n = 4, \quad \left.\begin{aligned}\alpha &= 72\\ s &= \frac{r_2}{2}\sqrt{10 - 2\sqrt{5}}\\ c &= \frac{5r_2}{2}\sqrt{10 - 2\sqrt{5}}\\ r_1 &= \frac{r_2}{4}\left(\sqrt{5} + 1\right)\\ A &= \frac{5r_2}{8}\sqrt{10 + 2\sqrt{20}}\end{aligned}\right\} n = 5 \tag{3.55}$$

$$\left.\begin{aligned}\alpha &= 60\\ s &= r_2\\ c &= 6r_2\\ r_1 &= \frac{r_2}{2}\sqrt{3}\\ A &= \frac{3r_2}{2}\sqrt{3}\end{aligned}\right\} n = 6, \quad \left.\begin{aligned}\alpha &= 45\\ s &= r_2\sqrt{2 - \sqrt{2}}\\ c &= 8r_2\sqrt{2 - \sqrt{2}}\\ r_1 &= \frac{r_2}{2}\sqrt{2 + \sqrt{2}}\\ A &= 2r_2^2\sqrt{2}\end{aligned}\right\} n = 8, \quad \left.\begin{aligned}\alpha &= 36\\ s &= \frac{r_2}{2}\left(\sqrt{5} - 1\right)\\ c &= 5r_2\left(\sqrt{5} - 1\right)\\ r_1 &= \frac{r_2}{4}\sqrt{10 + 2\sqrt{5}}\\ A &= \frac{5r_2}{4}\sqrt{10 - 2\sqrt{5}}\end{aligned}\right\} n = 10 \tag{3.56}$$

A couple of more general formulas for equilateral polygons are also implemented.

$$c = nr_2 \sin\left(\frac{\pi}{n}\right) \tag{3.57}$$

$$c = 2nr_1 \tan\left(\frac{\pi}{n}\right) \tag{3.58}$$

$$A = nr_2^2 \sin\left(\frac{2\pi}{n}\right)\frac{1}{2} \tag{3.59}$$

$$A = nr_1^2 \tan\left(\frac{\pi}{n}\right) \tag{3.60}$$

$$s = 2r_2 \sin\left(\frac{2\pi}{n}\right) \tag{3.61}$$

3.2.3 Circle

"Circle" is the common name of the set of all points with the same distance from a fixed point c. The following formulas are for a circle on a plane.

Circumference

The circumference g of a circle can be calculated in a lot of ways. One of these paths is

$$g = 2r\pi \tag{3.62}$$

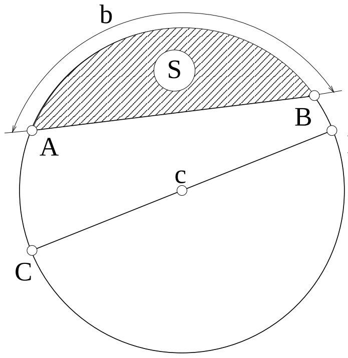

Figure 3.5: Secant and Segment

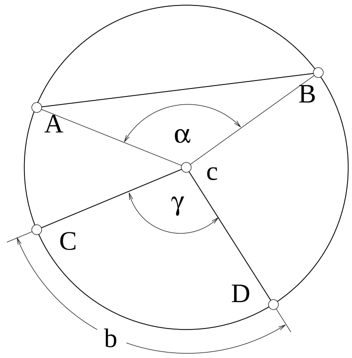

Figure 3.6: Segment and Sector

Another one is by means of Riemann's ζ-function to avoid the endless discussions about the exact value of π.

$$g = 2r\left|\sqrt{6\zeta(2)}\right| \tag{3.63}$$

Partial Circumference

The length b of the bow in the hatched part of figure 3.5 or the cake piece in figure 3.6 is calculated as follows

$$b = r\alpha \tag{3.64}$$

This might look curious to some, because the common formula for such a bow is

$$b = \frac{2\pi r\alpha}{360°} \tag{3.65}$$

The dimension of $360°$ are degrees with 360 of them to build a full circle. A unit circle, a circle with a radius of 1, has a circumference g of $g = 2 \cdot r \cdot \pi = 2 \cdot 1 \cdot \pi = 2\pi$. Replacing $360°$ with 2π gives

$$b = \frac{2\pi r\alpha}{2\pi} \tag{3.66}$$

$$= \frac{\cancel{2\pi} r\alpha}{\cancel{2\pi}} \tag{3.67}$$

The dimension of the angel α changes from degrees to radians and the value according to the formula given in 3.131.

Area

The area A of a circle

$$A = r^2\pi \tag{3.68}$$

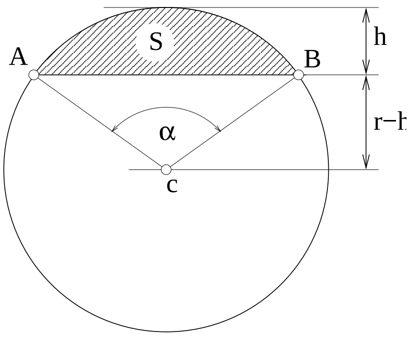

Figure 3.7: Segment

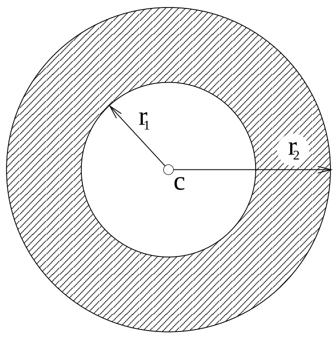

Figure 3.8: Ring

Secant

The secant s of circle is any finite line with its two endpoints laying on the circle. It is the line $\overline{AB}$ in figures 3.6 and 3.7 and it's length s is

$$s = 2r \sin \alpha \tag{3.69}$$

Area between Secant and Circle

The area meant is the hatched area in figure 3.7 limited by the line $\overline{AB}$ and the part of the circle. It's size S is

$$S = \frac{r^2}{2\,(\alpha - \sin \alpha)} \tag{3.70}$$

Segment of the Circle (cake piece)

The area A of the segment bounded by the points CD on the circle and the legs $\overline{cC} = \overline{cD} = r$ of the angle γ in figure 3.6 is

$$A = \frac{r^2 \gamma}{\pi} \tag{3.71}$$

Ring

The area of the ring—the hatched area of figure 3.8—is of size

$$A = \pi(r_1 + r_2)(r_2 - r_1) \tag{3.72}$$

3.2.4 Ellipse

Area

The area of an ellipse can be calculated in several different ways based on several different measures. The two kinds mainly used are the calculations A_1

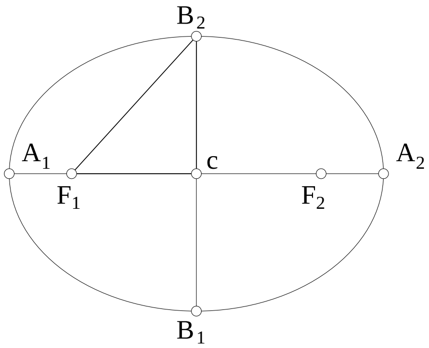

Figure 3.9: Ellipse with foci at F_1 and F_2

based on the length of the two axes $a_1 = \overline{cF_n}$ and $a_2 = \overline{cB_n}$, and A_2 based on the length of the two diameters $d_1 = \overline{cA_n}$ and $d_1 = \overline{cB_n}$. See figure 3.9 for the details.

$$A_1 = a_1 a_2 \pi \tag{3.73}$$

$$A_2 = d_1 d_2 \frac{\pi}{4} \tag{3.74}$$

3.2.5 Sphere

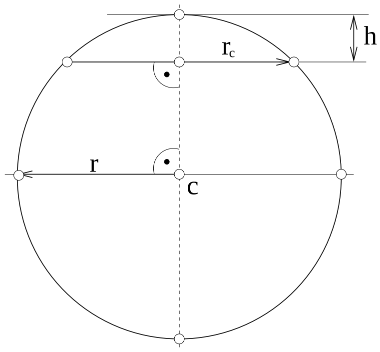

Figure 3.10: Cap

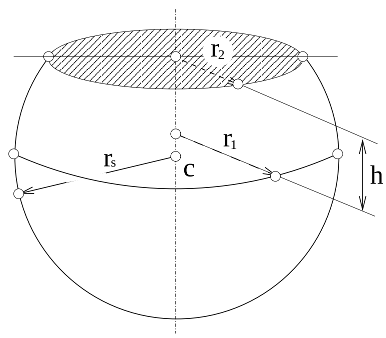

Figure 3.11: Slice

Volumes

The volume V_s of a sphere

$$V_s = \frac{4}{3}\pi r^3 \tag{3.75}$$

The volume V_c of a cap of a sphere (figure: 3.10)

$$V_c = \frac{h\pi}{6}\left(3r_c^2 + h^2\right) \tag{3.76}$$

$$= \frac{h^2\pi}{3}(3r - h) \tag{3.77}$$

Areas

The area A of a sphere

$$A = 4\pi r^2 \tag{3.78}$$

3.2.6 Torus

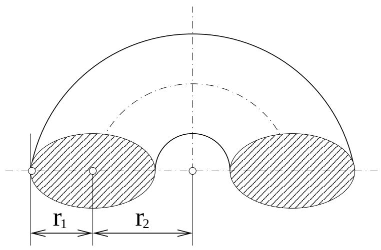

Figure 3.12: Torus

The Volume V of a torus

$$V = 2\pi^2 r_1^2 r_2 \tag{3.79}$$

The area A of a torus

$$A = 4\pi^2 r_1 r_2 \tag{3.80}$$

3.2.7 Cone

The cones here are always cones with a circle as the base.

Volume

The volume V of a slanted cone (figure: 3.14)

$$V = \frac{1}{3}\pi r^2 h \tag{3.81}$$

The mantel area A_m of a straight cone (figure: 3.13)

$$A_m = \pi r s \tag{3.82}$$

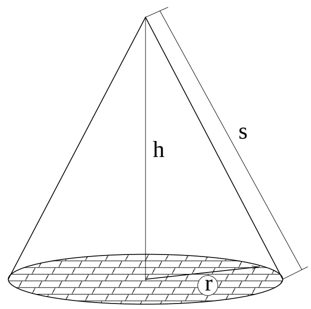

Figure 3.13: A straight cone

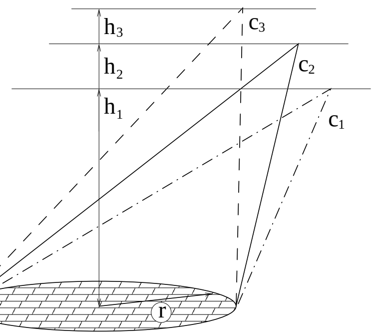

Figure 3.14: Three slanted cones

The area A of a straight cone (figure: 3.13)

$$A = A_c \cdot A_m \tag{3.83}$$
$$= r^2\pi \cdot \pi r s \tag{3.84}$$
$$= r^3\pi^3 s \tag{3.85}$$

The side s of a straight cone (figure: 3.13)

$$s = \sqrt{r^2 + h^2} \tag{3.86}$$

The height h of a straight cone (figure: 3.13)

$$h = \sqrt{s^2 - r^2} \tag{3.87}$$

3.2.8 Miscellaneous Geometric Figures

Lamé Curve

The `gnuplot` script to produce figure 3.15 was:

```
set terminal fig big fontsize 12 metric;
set output "superellipse.fig";
set parametric;
set trange[-2*pi:2*pi];
set xrange[-3.2:3.2];set yrange[-3.2:3.2];
y(t,n)=b*sgn(sin(t))*abs(sin(t))**(2/n);
x(t,n)=a*sgn(cos(t))*abs(cos(t))**(2/n);
a=2;b=3;plot\
x(t,3),y(t,3),x(t,6.1),y(t,6.1),x(t,2.5),y(t,2.5),\
x(t,2),y(t,2),x(t,1),y(t,1),x(t,.5),y(t,.5),x(t,.01),\
y(t,.01),x(t,5.6),y(t,.56);
```

The Lamé curve has been named after the Mathematician Gabriel Lamé who described it first in[57].

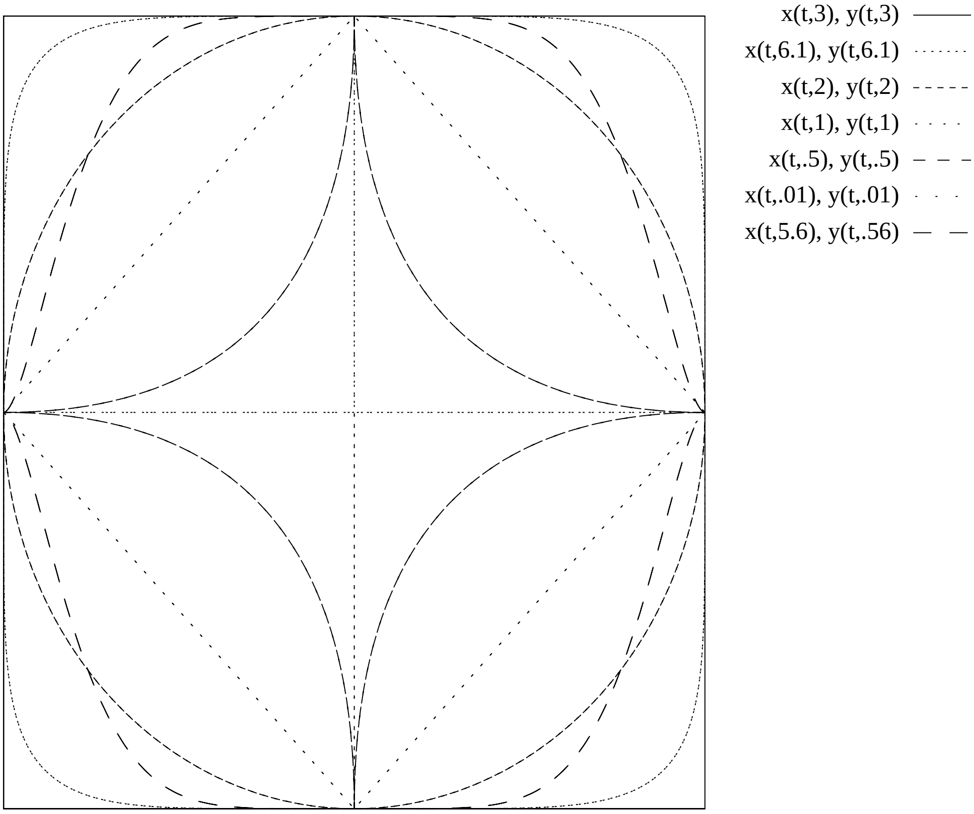

Figure 3.15: Lamé Curves

The equation for the Cartesian coordinates system

$$1 = \left|\frac{x}{a}^r\right| + \left|\frac{y}{b}^r\right| \tag{3.88}$$

The parameter generating function, useful for plotting the curve

$$x = a\cos^{\frac{2}{r}} t \tag{3.89}$$

$$y = b\sin^{\frac{2}{r}} t \tag{3.90}$$

$$\tag{3.91}$$

The area of the curve is

$$A = \frac{4^{1-1/n} ab\sqrt{\pi}\,\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} \tag{3.92}$$

The formula used in the program differs slightly from equation 3.92

$$A = \frac{4ab\,\Gamma\left(1+\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)} \tag{3.93}$$

3.3 Trigonometric Functions

A couple of basic trigonometric functions have been implemented. Some for real arguments that are not in the standard ECMA-script and all of those known to the author for complex arguments. The functions taking complex arguments are also usefully for the trigonometric functions that expand to the complex plane. For example `Math.asin(2)` will yield `NaN` because it is not defined on the real line but on the complex plane whereas `alert(new Complex(2,0) .asin())` gives the expected $\approx 1.570796 - 1.316958i$.

3.3.1 `Complex.prototype.sin`

The sinus for complex numbers $(x + yi)$.

$$\sin(x + yi) = z = \begin{cases} \Re z & = \sin(x)\cosh(yi) \\ \Im z & = \cos(x)\sinh(yi) \end{cases} \tag{3.94}$$

3.3.2 `Complex.prototype.cos`

The cosines for complex numbers $(x + yi)$.

$$\cos(x + yi) = z = \begin{cases} \Re z & = \cos(x)\cosh(yi) \\ \Im z & = -\sin(x)\sinh(yi) \end{cases} \tag{3.95}$$

3.3.3 `Complex.prototype.tan`

The tangent for complex numbers $(x + yi)$.

$$\tan(x + yi) = \frac{\sin(x + yi)}{\cos(x + yi)} \tag{3.96}$$

3.3.4 `Complex.prototype.asin`

The arcus sinus for complex numbers $z = (x + yi)$.

$$\operatorname{asin} z = -i \ln\left(zi + \sqrt{1 - z^2}\right) \tag{3.97}$$

3.3.5 `Complex.prototype.acos`

The arcus cosines for complex numbers $z = (x + yi)$.

$$\operatorname{acos} z = \frac{\pi}{2} + i \ln\left(zi + \sqrt{1 - z^2}\right) \tag{3.98}$$

3.3.6 `Complex.prototype.atan`

The arcus tangent for complex numbers $z = (x + yi)$.

$$\operatorname{acos} z = \frac{i}{2} \ln(1 - zi) - \ln(1 + zi) \tag{3.99}$$

3.3.7 `Math.cot`

The cotangent for real numbers x.

$$\cot x = \frac{1}{\tan x} \tag{3.100}$$

3.3.8 `Complex.prototype.cot`

The cotangent for complex numbers $(x + yi)$.

$$\cot(x + yi) = \frac{1}{\tan(x + yi)} \tag{3.101}$$

3.3.9 `Math.acot`

The arcus cotangent for real numbers x.

$$\operatorname{acot} x = \frac{\pi}{2} - \operatorname{atan}(x) \tag{3.102}$$

3.3.10 `Complex.prototype.acot`

The arcus cotangent for complex numbers $(x + yi)$.

$$\operatorname{acot}(x + yi) = \frac{\pi}{2} - \operatorname{atan}(x + yi) \tag{3.103}$$

3.3.11 `Complex.prototype.acoth`

The area cotangent hyperbolicus for complex numbers $(x + yi)$.

$$\operatorname{acoth}(x + yi) = \operatorname{atanh} \frac{1}{(x + yi)} \tag{3.104}$$

3.3.12 `Math.sec`

The secant for real numbers x.

$$\sec x = \frac{1}{\cos x} \tag{3.105}$$

3.3.13 `Complex.prototype.sec`

The secant for complex numbers $(x + yi)$

$$\sec(x + yi) = \frac{1}{\cos(x + yi)} \tag{3.106}$$

3.3.14 `Complex.prototype.sech`

The secant hyperbolicus for complex numbers $(x + yi)$

$$\operatorname{sech}(x + yi) = \frac{1}{\cosh(x + yi)} \tag{3.107}$$

3.3.15 `Complex.prototype.asec`

The arcus secant for complex numbers $(x + yi)$

$$\sec(x + yi) = \operatorname{acos}\frac{1}{(x + yi)} \tag{3.108}$$

3.3.16 `Complex.prototype.asech`

The area secant hyperbolicus for complex numbers $(x + yi)$

$$\operatorname{sech}(x + yi) = \operatorname{acosh}\frac{1}{(x + yi)} \tag{3.109}$$

3.3.17 `Math.csc` and `Math.cosec`

The cosecant for real numbers x.

$$\csc x = \frac{1}{\sin x} \tag{3.110}$$

3.3.18 `Complex.prototype.csc`

The cosecant for complex numbers $(x + yi)$

$$\csc(x + yi) = \frac{1}{\sin(x + yi)} \tag{3.111}$$

3.3.19 `Complex.prototype.csch`

The cosecant for complex numbers $(x + yi)$

$$\operatorname{csch}(x + yi) = \frac{1}{\sinh(x + yi)} \tag{3.112}$$

3.3.20 `Complex.prototype.acsc`

The arcus cosecant for complex numbers $(x + yi)$

$$\operatorname{acsc}(x+yi) = \operatorname{asin}\frac{1}{(x+yi)} \tag{3.113}$$

3.3.21 `Complex.prototype.acsch`

The area cosecant hyperbolicus for complex numbers $(x + yi)$

$$\operatorname{acsch}(x+yi) = \operatorname{asinh}\frac{1}{\operatorname{asinh}(x+yi)} \tag{3.114}$$

3.3.22 `Math.sem`

The semiversus for real numbers x.

$$\operatorname{sem} x = \sin^2\left(\frac{1}{2}\right) \tag{3.115}$$

3.3.23 `Math.asem`

The arcus semiversus for real numbers x.

$$\operatorname{asem} x = 2\operatorname{asin}\left(\sqrt{x}\right) \tag{3.116}$$

3.3.24 `Math.atan2`

The two-argument arcus tangent for real numbers x.

$$\operatorname{atan2}(x_1, x_2) = \begin{cases} \operatorname{atan}\frac{x_2}{x_1} & x_2 > 0 \\ \operatorname{atan}\frac{x_2}{x_1} + \pi & x_2 < 0, y \geq 0 \\ \operatorname{atan}\frac{x_2}{x_1} + \pi & x_2 < 0, y < 0 \\ \frac{\pi}{2} & x_2 = 0, x_1 > 0 \\ -\frac{\pi}{2} & x_2 = 0, x_1 < 0 \\ 0 & x_2 = 0, x_1 = 0 \end{cases} \tag{3.117}$$

3.3.25 `Math.cosh`

The cosines hyperbolicus for real numbers x.

$$\cosh x = \frac{1}{2}\left(e^{|x|} + \frac{1}{e^{|x|}}\right) \tag{3.118}$$

3.3.26 `Math.sinh`

The sinus hyperbolicus for real numbers x.

$$\sinh x = \begin{cases} x & x = 0 \\ -\frac{1}{2}\left(e^{|x|} - \frac{1}{e^{|x|}}\right) & x < 0 \\ \frac{1}{2}\left(e^{|x|} - \frac{1}{e^{|x|}}\right) & x > 0 \end{cases} \tag{3.119}$$

3.3.27 `Math.tanh`

The tangent hyperbolicus for real numbers x.

$$\tanh x = \begin{cases} x & x = 0 \\ -\left(1 - \frac{2}{e^{2|x|}+1}\right) & x < 0 \\ 1 - \frac{2}{e^{2|x|}+1} & x > 0 \end{cases} \tag{3.120}$$

3.3.28 `Complex.prototype.tanh`

The tangent hyperbolicus for complex numbers $z = (x + yi)$

$$\tanh z = \frac{\sinh z}{\cosh z} \tag{3.121}$$

3.3.29 `Math.coth`

The cotangent hyperbolicus for real numbers x.

$$\coth x = \begin{cases} x & x = 0 \\ -\left(1 + \frac{2}{e^{2|x|}-1}\right) & x < 0 \\ 1 + \frac{2}{e^{2|x|}-1} & x > 0 \end{cases} \tag{3.122}$$

3.3.30 `Complex.prototype.coth`

The cotangent hyperbolicus for complex numbers $(x + yi)$

$$\coth(x + yi) = \frac{\cosh(x + yi)}{\sinh(x + yi)} \tag{3.123}$$

3.3.31 `Math.acosh`

The arcus cosines hyperbolicus for real numbers x.

$$\operatorname{acosh} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1 \tag{3.124}$$

3.3.32 `Complex.prototype.acosh`

The area sinus hyperbolicus for complex numbers $z = (x + yi)$. This version has been implemented because of overflow. The simplified version $\operatorname{acosh} z = \ln(\sqrt{(z^2 - 1)})$ is more elegant but obviously suffers from overflow if z^2 is too large.

$$\operatorname{acosh} z = \ln\left(z + \left(\sqrt{z-1}\sqrt{z+1}\right)\right) \tag{3.125}$$

3.3.33 `Math.asinh`

The arcus sinus hyperbolicus for real numbers x.

$$\operatorname{asinh} x = \begin{cases} x & x = 0 \\ -\ln\left(-x + \sqrt{x^2+1}\right) & x < 0 \\ \ln\left(x + \sqrt{x^2+1}\right) & x > 0 \end{cases} \tag{3.126}$$

3.3.34 `Complex.prototype.asinh`

The area sinus hyperbolicus for complex numbers $z = (x + yi)$.

$$\operatorname{asinh} z = \ln \sqrt{z + (1 + z^2)} \tag{3.127}$$

3.3.35 `Math.atanh`

The arcus tangent hyperbolicus for real numbers x.

$$\operatorname{atanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 \le x \le 1 \tag{3.128}$$

3.3.36 `Complex.prototype.atanh`

The area tangent hyperbolicus for complex numbers $z = (x + yi)$.

$$\operatorname{atanh} z = \frac{1}{2}\Bigg(\ln(1+z) - \ln(1-z)\Bigg) \tag{3.129}$$

3.3.37 `Math.acoth`

The arcus tangent hyperbolicus for real numbers x.

$$\operatorname{acoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad |x| \ge 1 \tag{3.130}$$

3.3.38 Conversions

Conversions for different dimensions of angles.

Degrees d to Radians r:

$$r = \frac{2\pi d}{360} \tag{3.131}$$

Radians r to Degrees d:

$$d = \frac{360r}{2\pi} \tag{3.132}$$

Grad g to Radians r:

$$r = \frac{2\pi g}{400} \tag{3.133}$$

Radians r to Grad g:

$$g = \frac{400r}{2\pi} \tag{3.134}$$

Degrees d to grad g:

$$g = \frac{400d}{360} \tag{3.135}$$

Grad g to degrees d:

$$d = \frac{360d}{400} \tag{3.136}$$

Conversions between different coordinate systems.

Cartesian to spherical:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{3.137}$$

$$\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{3.138}$$

$$\alpha = \operatorname{atan} \frac{y}{x} \tag{3.139}$$

Cartesian to spherical (unit):

$$e_r = \sin(\theta)\,(e_x \cos\alpha + e_y \sin\alpha) + e_z \cos\theta \tag{3.140}$$

$$e_\theta = \cos(\theta)\,(e_x \cos\alpha + e_y \sin\alpha) + e_z \sin\theta \tag{3.141}$$

$$e_\alpha = -e_x \sin\alpha + e_y \cos\alpha \tag{3.142}$$

Cylinder to Cartesian:

$$x = \rho\cos\alpha \tag{3.143}$$

$$y = \rho\sin\alpha \tag{3.144}$$

$$z = z \tag{3.145}$$

Cylinder to Cartesian (unit):

$$e_x = e_\rho \cos\alpha - e_\alpha \sin\alpha \tag{3.146}$$
$$e_y = e_\rho \sin\alpha + e_\alpha \cos\alpha \tag{3.147}$$
$$e_z = e_z \tag{3.148}$$

Sphere to Cartesian:

$$x = r \cdot \cos\alpha \cdot \sin\theta \tag{3.149}$$
$$y = r \cdot \sin\alpha \cdot \cos\theta \tag{3.150}$$
$$z = r \cdot \cos\alpha \tag{3.151}$$

Sphere to Cartesian (unit):

$$x = e_r \sin\theta \cos\alpha - e_\alpha \sin\alpha + e_\theta \cos\theta \cos\alpha \tag{3.152}$$
$$y = e_r \sin\theta \sin\alpha + e_\alpha \cos\alpha + e_\theta \sin\theta \cos\alpha \tag{3.153}$$
$$z = e_r \cos\theta - e_\theta \sin\theta \tag{3.154}$$

Cylinder to sphere:

$$r = \sqrt{\rho^2 + z^2} \tag{3.155}$$
$$\theta = \operatorname{atan} \frac{\rho}{z} \tag{3.156}$$
$$\alpha = \alpha \tag{3.157}$$

Cylinder to sphere (unit):

$$r = e_\rho \sin\theta + e_z \cos\theta \tag{3.158}$$
$$\theta = e_\rho \cos\theta - e_z \sin\theta \tag{3.159}$$
$$\alpha = e_\alpha \tag{3.160}$$

Sphere to cylinder:

$$\rho = r \sin\theta \tag{3.161}$$
$$\alpha = \alpha \tag{3.162}$$
$$z = r \cos\theta \tag{3.163}$$

Sphere to cylinder (unit):

$$\rho = e_r \sin\theta + e_\theta \cos\theta \tag{3.164}$$
$$\alpha = e_\alpha \tag{3.165}$$
$$z = e_r \cos\theta - e_\theta \sin\theta \tag{3.166}$$

Chapter 4

Special Functions

This chapter contains *large* parts of [1].

4.1 Dilogarithm

The dilogarithm or Spence's integral for $n = 2$ [1]

$$f(x) = -\int_1^x \frac{\ln t}{t-1}\,\mathrm{d}\,t \tag{4.1}$$

A series expansion

$$f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2} \qquad \text{for } 2 \geq x \geq 0 \tag{4.2}$$

4.2 Exponential Integral

4.3 Kummer's Confluent Hypergeometric Function

4.4 Fresnel Integrals

$$\mathrm{C}(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}\,t \tag{4.3}$$

$$\mathrm{S}(z) = \int_0^z \cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}\,t \tag{4.4}$$

[1] From [1] p. 1004

Variations of these functions

$$C_1(z) = \sqrt{\frac{\pi}{2}} \int_0^z \cos t^2 \, \mathrm{d}\, t \tag{4.5}$$

$$C_2(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos t}{\sqrt{t}} \, \mathrm{d}\, t \tag{4.6}$$

$$S_1(z) = \sqrt{\frac{\pi}{2}} \int_0^z \sin t^2 \, \mathrm{d}\, t \tag{4.7}$$

$$S_2(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin t}{\sqrt{t}} \, \mathrm{d}\, t \tag{4.8}$$

Relations to the original integrals

$$C(x) = C_1\left(x\sqrt{\frac{\pi}{2}}\right) = C_2\left(x^2 \frac{\pi}{2}\right) \tag{4.9}$$

$$S(x) = S_1\left(x\sqrt{\frac{\pi}{2}}\right) = S_2\left(x^2 \frac{\pi}{2}\right) \tag{4.10}$$

Series expansions

$$C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n}}{(2n)!(4n+1)} z^{4n+1} \tag{4.11}$$

$$\begin{aligned} C(z) &= \cos\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1} \\ &+ \sin\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3} \end{aligned} \tag{4.12}$$

$$S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{2})^{2n+1}}{(2n+1)!(4n+3)} z^{4n+3} \tag{4.13}$$

$$\begin{aligned} S(z) &= -\cos\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3} \\ &+ \sin\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1} \end{aligned} \tag{4.14}$$

There are also series expansions by means of the Bessel function $J_{n+1/2}(z)$ for the variations C_2 and S_2

$$C_2(z) = J_{\frac{1}{2}}(z) + J_{\frac{5}{2}}(z) + J_{\frac{8}{2}}(z) + \cdots \tag{4.15}$$

$$S_2(z) = J_{\frac{3}{2}}(z) + J_{\frac{7}{2}}(z) + J_{\frac{11}{2}}(z) + \cdots \tag{4.16}$$

The symmetries

$$\mathrm{C}(-z) = -\,\mathrm{C}(z) \qquad \mathrm{S}(-z) = -\,\mathrm{S}(z) \tag{4.17}$$

$$\mathrm{C}(zi) = i\,\mathrm{C}(z) \qquad \mathrm{S}(zi) = -i\,\mathrm{S}(z) \tag{4.18}$$

$$\mathrm{C}(\overline{z}) = \overline{\mathrm{C}(z)} \qquad \mathrm{S}(\overline{z}) = \overline{\mathrm{S}(z)} \tag{4.19}$$

The values at infinity are $\frac{1}{2}$ for both integrals $\mathrm{C}(z)$ and $\mathrm{S}(z)$.

4.5 Bessel Functions

The notation of the different Bessel functions follow that of [1] explained at page 358[2].

[2] [1] notes that the function $Y_\nu(z)$ is often denoted as $N_\nu(z)$ by physicists and European workers and lists a small list of other variations.

Aldis, Airey

$$\mathrm{G}_n(z) \text{ for } -\frac{1}{2}\pi\,\mathrm{Y}_n(z) \tag{4.20}$$

$$\mathrm{K}_n(z) \text{ for } (-)^n\,\mathrm{K}_n(z) \tag{4.21}$$

Clifford

$$\mathrm{C}_n(x) \text{ for } x^{-in}\,\mathrm{J}_n\left(2\sqrt{x}\right) \tag{4.22}$$

Gray, Mathews and MacRobert

$$\mathrm{Y}_n(z) \text{ for } \frac{1}{2}\pi\,\mathrm{Y}_n(z) + (\ln 2 - \gamma)\,\mathrm{J}_n(z) \tag{4.23}$$

$$\overline{\mathrm{Y}}_n(z) \text{ for } \pi^{\nu\pi i}\sec(\nu\pi)\,\mathrm{Y}_\nu(z) \tag{4.24}$$

$$\mathrm{G}_\nu(z) \text{ for } \frac{1}{2}\pi i\,\mathrm{H}^{(1)}_\nu(z) \tag{4.25}$$

Jahnke, Emde and Lösch

$$\Lambda_\nu(z) \text{ for } \Gamma(\nu+1)(\frac{1}{2}z)^{-\nu}\,\mathrm{J}_\nu(z) \tag{4.26}$$

Jeffreys

$$\mathrm{H}\,s_\nu(z) \text{ for } \mathrm{H}^{(1)}_\nu(z) \tag{4.27}$$

$$\mathrm{H}\,i_\nu(z) \text{ for } \mathrm{H}^{(2)}_\nu(z) \tag{4.28}$$

$$\mathrm{K}\,h_\nu(z) \text{ for } (\frac{2}{\pi})\,\mathrm{K}_\nu(z) \tag{4.29}$$

$$\tag{4.30}$$

Heine

$$\mathrm{K}_n(z) \text{ for } -\frac{1}{2}\pi\,\mathrm{Y}_n(z) \tag{4.31}$$

Neuman

$$\mathrm{Y}^n(z) \text{ for } \frac{1}{2}\pi\,\mathrm{Y}_n(z) + (\ln 2 - \gamma)\,\mathrm{J}_n(z) \tag{4.32}$$

Whittaker and Watson

$$\mathrm{K}_\nu(z) \text{ for } \cos(\nu\pi)\,\mathrm{K}_\nu(z) \tag{4.33}$$

4.5.1 Differential Equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d} z^2} + z \frac{\mathrm{d} w}{\mathrm{d} z} + w(z^2 - \nu^2) = 0 \tag{4.34}$$

has as solutions the Bessel functions of the first kind $\mathrm{J}_{\pm\nu}(z)$, of the second kind $\mathrm{Y}_\nu(z)$[3] and of the third kind $\mathrm{H}^{(1)}_\nu(z)$ and $\mathrm{H}^{(2)}_\nu(z)$[4].

For fixed ν, $z \to 0$ and small arguments

$$\mathrm{J}_\nu(z) \approx \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \qquad \text{with } \nu \notin \{\mathbb{Z}^- \setminus \{0\}\} \tag{4.35}$$

$$\begin{aligned} \mathrm{Y}_0(z) &\approx -i\,\mathrm{H}^{(1)}_0(z) \\ &\approx i\,\mathrm{H}^{(2)}_0(z) \\ &\approx \frac{2}{\pi}\ln z \end{aligned} \tag{4.36}$$

$$\begin{aligned} \mathrm{Y}_\nu(z) &\approx -i\,\mathrm{H}^{(1)}_\nu(z) \\ &\approx i\,\mathrm{H}^{(2)}_\nu(z) \\ &\approx -(\frac{1}{\pi})\Gamma(\nu)(\frac{1}{2})^{-\nu} \qquad \text{with } \Re\nu > 0 \end{aligned} \tag{4.37}$$

[3]also known as the Weber function
[4]also known as the Hankel functions

4.5.2 Ascending Series

$$\mathrm{J}_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)}{k!\,\Gamma(\nu+k+1)} \tag{4.38}$$

$$\begin{aligned}\mathrm{Y}_n(z) = &-\frac{(\frac{1}{2}z)^-n}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(\frac{1}{4}z^2)^k \\ &+\frac{2}{\pi}\ln(\frac{1}{2}z)\,\mathrm{J}_n(z) \\ &-\frac{(\frac{1}{2}z)^n}{\pi}\sum_{k=0}^{\infty}(\psi\,(k+1)+\psi\,(nk+1))\,\frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}\end{aligned} \tag{4.39}$$

$$\mathrm{J}_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \cdots \tag{4.40}$$

$$\begin{aligned}\mathrm{Y}_0(z) = &\frac{2}{\pi}\left(\ln\left(\frac{1}{2}z\right)+\gamma\right)\mathrm{J}_0(z) \\ &+\frac{2}{\pi}\left[\frac{\frac{1}{4}z^2}{(1!)^2} - \left(1+\frac{1}{2}\right)\frac{\left(\frac{1}{4}z^2\right)^2}{(2!)^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\left(\frac{1}{4}z^2\right)^3}{(3!)^2} - \cdots\right]\end{aligned} \tag{4.41}$$

4.5.3 Relation to the Gamma-function

$$\mathrm{J}_\nu(z)\,\mathrm{J}_\mu(z) = (\frac{1}{2}z)^{\nu+\mu}\sum_{k=0}^{\infty}\frac{(-)^k\Gamma(\nu+\mu+2k+1)(\frac{1}{4}z^2)^k}{\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\nu+\mu+k+1)k!} \tag{4.42}$$

4.5.4 Wronskians

$$\mathrm{W}(\mathrm{J}_\nu(z),\mathrm{J}_{-\nu}(z)) = \mathrm{J}_{\nu+1}(z)\,\mathrm{J}_{-\nu}(z) + \mathrm{J}_\nu(z)\,\mathrm{J}_{-(\nu+1)}(z) \tag{4.43}$$

$$= -2\sin\left(\frac{\nu\pi}{\pi z}\right) \tag{4.44}$$

$$\mathrm{W}(\mathrm{J}_\nu(z),\mathrm{Y}_\nu(z)) = \mathrm{J}_{\nu+1}(z)\,\mathrm{Y}_\nu(z) - \mathrm{J}_\nu(z)\,\mathrm{Y}_{\nu+1}(z) \tag{4.45}$$

$$= \frac{2}{\pi z} \tag{4.46}$$

$$\mathrm{W}(\mathrm{H}_\nu^{(1)}(z),\mathrm{H}_\nu^{(2)}(z)) = \mathrm{H}_{\nu+1}^{(1)}(z),\mathrm{H}_\nu^{(2)}(z) - \mathrm{H}_\nu^{(1)}(z),\mathrm{H}_{\nu+1}^{(2)}(z) \tag{4.47}$$

$$= -\frac{4i}{\pi z} \tag{4.48}$$

4.5.5 Asymptotic Expansions for Large Arguments

Principal Asymptotic Forms

For ν fixed and $x \to \infty$

$$\mathrm{J}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + e^{|\Im z|}\, \mathcal{O}\left(|z|^{-1}\right)\right) \qquad \text{for } |\arg z| < \pi \tag{4.49}$$

$$\mathrm{Y}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) + e^{|\Im z|}\, \mathcal{O}\left(|z|^{-1}\right)\right) \qquad \text{for } |\arg z| < \pi \tag{4.50}$$

$$\mathrm{H}^{(1)}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \qquad \text{for } -\pi < \arg z < 2\pi \tag{4.51}$$

$$\mathrm{H}^{(2)}_\nu(z) \approx \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \qquad \text{for } -2\pi < \arg z < \pi \tag{4.52}$$

Hankel's Expansion

With $x = z - \pi(\frac{1}{2}\nu + \frac{1}{4})$

$$\mathrm{J}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\mathrm{P}(\nu, z) \cos x - \mathrm{Q}(\nu, z) \sin x\right) \qquad \text{for } |\arg z| < \pi \tag{4.53}$$

$$\mathrm{Y}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\mathrm{P}(\nu, z) \sin x - \mathrm{Q}(\nu, z) \cos x\right) \qquad \text{for } |\arg z| < \pi \tag{4.54}$$

$$\mathrm{H}^{(1)}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\mathrm{P}(\nu, z) + i\, \mathrm{Q}(\nu, z)\right) e^{ix} \qquad \text{for } -\pi < \arg z < 2\pi \tag{4.55}$$

$$\mathrm{H}^{(2)}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(\mathrm{P}(\nu, z) - i\, \mathrm{Q}(\nu, z)\right) e^{-ix} \qquad \text{for } -2\pi < \arg z < \pi \tag{4.56}$$

With $\mu = 4\nu^2$

$$\mathrm{P}(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{\nu, 2k}{(2z)^{2k}} \tag{4.57}$$

$$= 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} - \cdots \tag{4.58}$$

$$P(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{\nu, 2k}{(2z)^{2k}} \tag{4.59}$$

$$= \frac{\mu - 1}{8z} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \cdots \tag{4.60}$$

If ν is real and non-negativ and z is positive, the remainder after k terms in the expansion of $P(\nu, z)$ does not exceed the $(k+1)^{\text{th}}$ term in absolute value and is of the same sign, provided that $k > \frac{1}{2}\nu - \frac{1}{4}$. The same is true of $Q(\nu, z)$, provided that $k > \frac{1}{2}\nu - \frac{3}{4}$.

Asymptotic Expansions of Derivatives

With the conditions and notation of the preceding subsection

$$J'_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(-R(\nu, z)\sin x - S(\nu, z)\cos x\right) \qquad \text{for } |\arg z| < \pi \tag{4.61}$$

$$Y'_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(R(\nu, z)\cos x - S(\nu, z)\sin x\right) \qquad \text{for } |\arg z| < \pi \tag{4.62}$$

$$H^{(1)'}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(i\,R(\nu, z) - S(\nu, z)\right) e^{ix} \qquad \text{for } -\pi < \arg z < 2\pi \tag{4.63}$$

$$H^{(2)'}_\nu(z) = \sqrt{\frac{2}{\pi z}} \left(-i\,R(\nu, z) - S(\nu, z)\right) e^{-ix} \qquad \text{for } -2\pi < \arg z < \pi \tag{4.64}$$

$$(R)(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{4\nu^2 + 16k^2 - 1}{4\nu^2 - (4k-1)^2} \frac{(\nu, 2k)}{(2z)^{2k}} \tag{4.65}$$

$$= 1 - \frac{(\mu-1)(\mu+15)}{2!(8z)^2} + \cdots \tag{4.66}$$

$$(S)(\nu, z) = \sum_{k=0}^{\infty} (-1)^k \frac{4\nu^2 + 4(2k+1)^2 - 1}{4\nu^2 - (4k+1)^2} \frac{(\nu, 2k+1)}{(2z)^{2k+1}} \tag{4.67}$$

$$= \frac{\mu + 3}{8z} - \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3} + \cdots \tag{4.68}$$

Modulus and Phase

For real ν and positive x

4.6 Beta Functions

4.6.1 Beta function $\mathrm{B}(a, b)$

The Beta function, also known as the Euler integral of the first kind, is defined by the integral

$$\mathrm{B}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,\mathrm{d}\,x \qquad \text{for} \quad \{a, b\} \in \mathbb{R}^+ \setminus \{0\} \tag{4.69}$$

The Beta function can be extended to the complex plane by analytic continuation. One of the possible results is also the identity used in the implementation[5].

$$\mathrm{B}(a, b) = \frac{\Gamma a \Gamma(b)}{\Gamma(a+b)} \qquad \text{for} \quad \{a, b, a+b\} \in \mathbb{C} \setminus \{0, \mathbb{Z}^-\} \tag{4.70}$$

From the identity above it is obvious that the function is symmetric

$$\mathrm{B}(a, b) = \mathrm{B}(b, a) \tag{4.71}$$

Some interesting identities:

$$\binom{n}{k} = \frac{1}{(n+1)\,\mathrm{B}(n-k+1, k+1)} \tag{4.72}$$

$$\mathrm{B}(a, b)\,\mathrm{B}(a+b, 1-b) = \frac{\pi}{x \sin(\pi x)} \tag{4.73}$$

4.6.2 Incomplete Beta Function

The incomplete beta function is the generalization of the beta function and is, like the incomplete gamma function, defined by the indefinite integral

$$\mathrm{B}(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}\,t \tag{4.74}$$

From which follows

$$\mathrm{B}(1; a, b) = \mathrm{B}(a, b) \tag{4.75}$$

The incomplete beta function is related to the hypergeometric function

$$\mathrm{B}(x; a, b) = a^{-1} x^a\,\mathrm{F}(a, 1-b; a+1; x) \tag{4.76}$$

The implementation of the incomplete beta function is by means of the regularized beta function.

[5] With the exception that the current implementation of the Gamma-function is for $\mathbb{R} \setminus \{0, \mathbb{Z}^-\}$ only.

4.6.3 Regularized Incomplete Beta Function

The regularized incomplete beta function

$$\mathrm{I}_x(a,b) = \frac{\mathrm{B}(x;a,b)}{B(a,b)} \tag{4.77}$$

It is implemented with the continued fraction given at[122].

$$\mathrm{I}_z(a,b) = \frac{z^a(1-z)^b}{a\,\mathrm{B}(a,b)} \cfrac{1}{1+\cfrac{r(1)}{1+\cfrac{r(2)}{1+\cfrac{r(3)}{1+\cfrac{r(4)}{1+\cfrac{r(5)}{1+\cdots}}}}}} \tag{4.78}$$

$$r(2n+1) = -\frac{z(a+n)(a+b+n)}{(a+2n)(a+2n+1)}$$

$$r(2n) = \frac{zn(b-n)}{(a+2n-1)(a+2n)}$$

4.7 Gamma Functions

4.7.1 Γ-function

The Γ-function is the generalized factorial $x!$.

$$\Gamma(z+1) = z! \qquad \text{for} \quad z-1 \in \mathbb{N}^+ \setminus \{0\} \tag{4.79}$$

It is defined for all $z \in \mathbb{C} \setminus \{0, \mathbb{Z}^-\}$ by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-1}\,\mathrm{d}\,t \tag{4.80}$$

The picture on the frontpage is a plot of the values of $\Gamma(x)$ with $x \in \mathbb{R} \wedge -6 \leq x \leq 6$. It is the result of the following `Gnuplot`-script

```
set terminal postscript eps enhanced solid "Helvetica"
    14
set output "gammafunction.eps"
set zeroaxis
set nokey
set parametric
set size ratio -1 1,1
set trange[-6:6]
set xrange[-6:6];set yrange[-6:6]
set samples 15000
```

```
f(x) = (-pi)/(x*gamma(abs(x))*sin(pi*x))
plot t,(t<0)?f(t):gamma(t)
```

The $\Gamma(z)$-function is implemented here with the common Lanczos approximation[6].

$$\Gamma(z) = \frac{\sqrt{2\pi}\,(z+7-0.5)^{z-0.5}}{e^{z+7-0.5}}\left(p_0 \sum_{n=1\ldots8} \frac{p_n}{z+n}\right) \tag{4.81}$$

The values of $p_0 \ldots p_8$

p_0	0.99999999999980993
p_1	676.5203681218851
p_2	−1259.1392167224028
p_3	771.32342877765313
p_4	−176.61502916214059
p_5	12.507343278686905
p_6	−0.13857109526572012
p_7	9.9843695780195716e − 6
p_8	1.5056327351493116e − 7

Negative non-integer values of z are computed by the way of the identity

$$\Gamma(-z) = \frac{-\pi}{z\Gamma(z)\sin \pi z} \tag{4.82}$$

The logarithm of the Γ-function is implemented as

$$\begin{aligned}\Gamma(z) &= (a-0.5)\ln(a+7-0.5) - (a+7-0.5) \\ &\quad + \ln\left(\sqrt{2\pi}\left(p_0 \sum_{n=1\ldots8} \frac{p_n}{z+n}\right)\right)\end{aligned} \tag{4.83}$$

And the logarithm of the Γ-function of the negative non-integer values by

$$\Gamma(-z) = \ln(|\Gamma(z)|) \tag{4.84}$$

Viktor T. Thot gave in [106] a simple way to produce arbitrary precision values for $\Gamma(z)$ with $z \in \mathbb{C}$ with the following formulas[7]

$$\ln\Gamma(z+1) = \ln \mathbf{ZP} + (z+0.5)\ln(z+g+0.5) - (z+g+0.5) \tag{4.85}$$

With $\mathbf{Z}$ the row vector

$$\mathbf{Z} = \left[1 \frac{1}{z+1} \frac{1}{z+2} \cdots \frac{1}{z+n-1}\right] \tag{4.86}$$

[6]The algorithm has been implemented for rationals only at the time of this writing.
[7]C++-code can be found at the webpage linked from [106]

$\mathbf{P}$ the product of the $n \times n$ matrices $\mathbf{D}, \mathbf{B}, \mathbf{C}$ and the column vector $\mathbf{F}$

$$\mathbf{B}_{ij} = \begin{cases} 1 & \text{if} \quad i = 0 \\ -1^{j-i}\binom{i+j-1}{j-1} & \text{if} \quad i > 0, j \geq i \\ 0 & \text{otherwise} \end{cases} \tag{4.87}$$

$$\mathbf{C}_{ij} = \begin{cases} \frac{1}{2} & \text{if} \quad i = j = 0 \\ 0 & \text{if} \quad j > i \\ -1^{i-j} \sum_{k=0}^{i} \binom{2i}{2k}\binom{k}{k+j-i} & \text{otherwise} \end{cases} \tag{4.88}$$

$$\mathbf{D}_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j = 0 \\ -1 & \text{if} \quad i = j = 1 \\ \dfrac{\mathbf{D}_{i-1,j-1} 2\,(2i-1)}{i-1} & \text{otherwise} \end{cases} \tag{4.89}$$

$$\mathbf{F}_i = \frac{(2i)!\, e^{i+g+0.5}}{i!\, 2^{2i-1}\,(i+g+0.5)^{i+0.5}} \tag{4.90}$$

The typical error can be calculated with

$$\mathbf{E} = \mathbf{CF} \tag{4.91}$$

$$|\epsilon| = \left| \frac{\pi}{2\sqrt{2e}} \left(e^g \sqrt{\pi} - \sum_{i=0}^{n-1} -1^j (E)_j \right) \right| \tag{4.92}$$

Another way are the polynomial approximations for values $0 \leq x \leq 1$ with the coefficients as listed in [1] p.257.

$$\Gamma(x+1) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x) \qquad |\epsilon(x)| \leq 5 \times 10^{-5} \tag{4.93}$$

$$\begin{aligned} a_1 &= -.57486\,46 & a_4 &= .42455\,49 \\ a_2 &= .95123\,63 & a_5 &= -.10106\,78 \\ a_3 &= -.69985\,88 \end{aligned} \tag{4.94}$$

And with a slightly smaller error

$$\Gamma(x+1) = 1 + b_1 x + b_2 x^2 + \cdots + b_8 x^8 + \epsilon(x) \qquad |\epsilon(x)| \leq 3 \times 10^{-7} \tag{4.95}$$

$$\begin{aligned} b_1 &= -.57719\,1652 & b_5 &= -.75670\,4078 \\ b_2 &= .98820\,5891 & b_6 &= .48219\,9394 \\ b_3 &= -.89705\,6937 & b_7 &= -.19352\,7818 \\ b_4 &= .91820\,6857 & b_8 &= .03586\,8343 \end{aligned} \tag{4.96}$$

4.7.2 Incomplete Γ-function

The incomplete Γ-function is defined by the integral

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-1} \, \mathrm{d}\, t \tag{4.97}$$

It is implemented with Kummer's confluent hypergeometric function $M(a,b;z)$ (see section 4.3).

$$\gamma(a,x) = a^{-1} x^a e^{-z} M(1, a+1; x) \tag{4.98}$$

The implementation in some of the earlier versions was with the asymptotic series from [106] which uses the raising factorial, also known as the Pochhammer symbol.

$$\gamma(a,x) = x^a e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{a(a+1)\ldots(a+n)} \tag{4.99}$$

$$= x^a e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(a+n+1)\Gamma(a)} \tag{4.100}$$

4.7.3 Regularized Γ-function

There are two regularized Γ-functions P and Q

$$P(a,x) = \frac{\gamma(a,x)}{\Gamma(a)} \tag{4.101}$$

$$Q(a,x) = \frac{\Gamma(a,x)}{\Gamma(a)} \tag{4.102}$$

$$= 1 - P(a,x)$$

where $\Gamma(a,x)$ is the upper incomplete Γ-function defined by the integral

$$\Gamma(a,x) = \int_x^{\infty} t^{a-1} e^{-1} \, \mathrm{d}\, t \tag{4.103}$$

such that

(4.104)

$$\Gamma(a) = \Gamma(a,x) + \gamma(a,x) \tag{4.105}$$

4.7.4 Digamma $\psi_0(x)$ or F

The digamma function is the logarithmic derivative of the Γ-function.

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}\, x} = \frac{\Gamma'(x)}{\Gamma(x)} \tag{4.106}$$

It is implemented here with the harmonic series for small integer values of x and an asymptotic expansion otherwise. γ is the Euler-Mascheroni constant, $k = x - 1$, and L is the largest number representable. It is in case of IEEE's double precision used in ECMA-script $L = 2^{1024} - 1$.

$$\psi_0(x) = \begin{cases} \sum_{n=1}^{n=k} \frac{1}{n} - \gamma & k \in \mathbb{N} \wedge k < 10^3 \\ \ln(k) + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} + \frac{1}{240k^8} & k < \frac{\sqrt[8]{L}}{240} \\ \ln(k) + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} & k < \frac{\sqrt[6]{L}}{252} \\ \ln(k) + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} & k < \frac{\sqrt[4]{L}}{120} \\ +\infty & \text{otherwise} \end{cases} \tag{4.107}$$

The asymptotic expansion is based on the approximation given in [1] p.259

$$\psi(z) \approx \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} \qquad \text{for } z \to \infty \text{ in } |\arg z| < \pi \tag{4.108}$$

B_n is the n^{th} Bernoulli number.

A slightly more precise approximation is

$$\begin{aligned} \psi(k) \approx \ln k &- \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} + \frac{1}{240k^8} - \frac{1}{132k^{10}} + \frac{691}{32760k^{12}} \\ &- \frac{1}{12k^{14}} + \frac{3617}{8160k^{16}} - \frac{43867}{14364k^{18}} + \frac{174611}{6600k^{20}} - \frac{77683}{276k^{22}} \\ &+ \frac{236364091}{65520k^{24}} - \frac{657931}{12k^{26}} + \frac{3392780147}{3480k^{28}} - \frac{1723168255201}{85932k^{30}} \\ &+ \frac{7709321041217}{16320k^{32}} - \frac{151628697551}{12k^{34}} + \frac{26315271553053477373}{69090840k^{36}} \\ &- \frac{154210205991661}{12k^{38}} + \frac{261082718496449122051}{541200k^{40}} \end{aligned} \tag{4.109}$$

It might be of interest that a solution with elementary functions exists for rational numbers ($x \in \mathbb{Q}$).

$$\begin{aligned} \psi\left(\frac{m}{k}\right) &= -\gamma - \ln(2k) - \frac{\pi}{2}\cot\left(\frac{m\pi}{k}\right) \\ &+ 2\sum_{n=1}^{\lfloor (k-1)/2 \rfloor} \cos\left(\frac{2\pi nm}{k}\right) \ln\left(\sin\left(\frac{n\pi}{k}\right)\right) \end{aligned} \tag{4.110}$$

All numbers in a computer are $\in \mathbb{Q}$, so the above formula seems to be a fitting algorithm but the solution for some x is not in $\mathbb{Q}$ anymore, so any direct implementation would suffer from lost of precision.

4.7.5 Polygamma $\psi_n(x)$

The polygamma function $\psi_n(x)$ is the n^{th} derivative of the digamma function $\psi_0(x)$.

$$\psi_n(x) = \frac{\mathrm{d}^n\, \psi_0(z)}{\mathrm{d}\, x} \tag{4.111}$$

It is implemented here, for positive integer orders only, with Hurwitz' ζ-function.

$$\psi_n(x) = \begin{cases} \psi_0(x) & n = 0 \\ (-1)^{n+1}\Gamma(n+1)\zeta(n+1, x) & n \in \mathbb{N}^+ \end{cases} \tag{4.112}$$

4.7.6 Double Factorial

The double factorial $x!!$ is implemented here as follows

$$x!! = \begin{cases} \text{undefined} & \text{odd}(n) \in \mathbb{Z}^- \\ 1 & n = -1, n = 0 \\ \exp\left(\ln\Gamma(\frac{n+1}{2} + 0.5) + \frac{1}{\ln 2}\frac{n+1}{2} - \ln(\sqrt{\pi})\right) & \text{odd}(\lfloor |n| \rfloor) \in \mathbb{Z}^- \\ \exp\left(\frac{1}{\ln 2}\frac{n}{2} + \ln\Gamma\left(\frac{n}{2} + 1\right)\right) & \text{otherwise} \end{cases} \tag{4.113}$$

4.7.7 Hyperfactorial

The logarithm of the hyperfactorial $H(n)$ is implemented as

$$H(n) = \begin{cases} \text{not defined here} & n \notin \mathbb{N}^+ \\ 1 & n = 0 \\ \sum_{i=1}^{n} i \ln(i) \end{cases} \tag{4.114}$$

4.7.8 Raising Factorial (Pochhammer Symbol)

The raising factorial or Pochhammer symbol $P_n(a)$ is implemented in three different ways:

By Product

$$P_n(a) = \prod_{i=0}^{n} (a + i) \tag{4.115}$$

By the Γ-Function

$$P_n(a) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{4.116}$$

By the $\ln\Gamma$**-Function**

$$\ln P_n(a) = (\ln\Gamma(a+n)) - (\ln\Gamma(a)) \tag{4.117}$$

4.7.9 K-Function

The definition of the K-function used here is $K(n) = H(n-1)$ where $H(n)$ is the hyperfactorial and $n \in \mathbb{N}^+$.

$$K(n) = 0^0 1^1 2^2 3^3 \cdots (n-1)^{n-1} \tag{4.118}$$

The hyperfactorial function of the implementation returns the logarithm of the hyperfactorial and so does the implementation of the K-function in the implementation.

4.7.10 Barne's G-Function

Barne's G-Function is defined as

$$G(n) = \frac{[\Gamma(n)]^{n-1}}{K(n)} \begin{cases} 1 & \text{if } n = 0 \\ 0!1!2! \cdots (n-2)! & \text{if } n > 0 \end{cases} \tag{4.119}$$

Where the square brackets denote the rounding to the nearest integer.

It is implemented with logarithmic calculations

$$\ln G(n) = \begin{cases} \text{not defined} & n \notin \mathbb{N} \\ ln 1 & n = 0, n = 1, n = 2 \\ ((n-1)\ln\Gamma(n)) - \ln K(n) & \end{cases} \tag{4.120}$$

with $K(n)$ the K-function at the positive integer n.

4.8 Error Function

The error function is defined by the integral [8]

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, \mathrm{d}\, t \tag{4.121}$$

[8] From [1] p. 297 ff.

The complemetary error function

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} \,\mathrm{d}\, t \qquad (4.122)$$
$$= 1 - \operatorname{erf} z$$

with the restriction that $\arg t \to \alpha$ with $|\alpha| < \frac{\pi}{4}$ as $t \to \infty$ and $\alpha = \frac{\pi}{4}$ if $\Re t^2$ is bounded to the left.

Another useful integral

$$w(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} \,\mathrm{d}\, t \right) \qquad (4.123)$$
$$= e^{-z^2} \operatorname{erfc}(-zi)$$

Several series expansions are possible

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \qquad (4.124)$$

$$= \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdots (2n+1)} z^{2n+1} \qquad (4.125)$$

$$w(z) = \sum_{n=0}^{\infty} \frac{(zi)^n}{\Gamma\left(\frac{n}{2}+1\right)} \qquad (4.126)$$

Some useful symmetries

$$\operatorname{erf}(-z) = -\operatorname{erf} z \qquad (4.127)$$

$$\operatorname{erf} \overline{z} = \overline{\operatorname{erf} z} \qquad (4.128)$$

$$w(-z) = 2e^{-z^2} - w(z) \qquad (4.129)$$

$$w(\overline{z}) = \overline{w(-z)} \qquad (4.130)$$

Continued fractions

$$2e^{z^2} \int_z^\infty e^{-t^2} \,\mathrm{d}\, t = \cfrac{1}{z + \cfrac{\frac{1}{2}}{z + \cfrac{1}{z + \cfrac{\frac{3}{2}}{z + \cfrac{2}{z + \cdots}}}}} \qquad \text{for } \Re z > 0 \qquad (4.131)$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} \,\mathrm{d}\, t}{z - t} = \cfrac{1}{z - \cfrac{\frac{1}{2}}{z - \cfrac{1}{z - \cfrac{\frac{3}{2}}{z - \cfrac{2}{z - \cdots}}}}} \qquad \text{for } \Im z \neq 0 \qquad (4.132)$$

With $x_k^{(n)}$ the zeros and $H_k^{(n)}$ the weight factors of the Hermite polynomials the formula in 4.132 can be described as

$$\frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{H_k^{(n)}}{z - x_k^{(n)}} \tag{4.133}$$

The error function is related to the confluent hypergeometric function (see section 4.3)

$$\begin{aligned} \operatorname{erf} z &= \frac{2z}{\sqrt{\pi}} \mathrm{M}\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) \\ &= \frac{2z}{\sqrt{\pi}} e^{-z^2} \mathrm{M}\left(1, \frac{3}{2}, -z^2\right) \end{aligned} \tag{4.134}$$

Which leads to the implementations for $x \in \mathbb{R}$

$$\operatorname{erf} x = \operatorname{sign}(x) \frac{\gamma\left(\frac{1}{2}, x^2\right)}{\sqrt{\pi}} \tag{4.135}$$

$$\operatorname{erfc} z = 1 - \operatorname{erf} x \tag{4.136}$$

Two other approximations for $x \in \mathbb{Q}_0^+$

$$\operatorname{erf} x = 1 - \frac{1}{\left(1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\right)^4} + \epsilon(x) \qquad \text{with } |\epsilon(x)| \le 5 \times 10^{-4} \tag{4.137}$$

$$\begin{aligned} a_1 &= .27839\,3 & a_2 &= .23038\,9 \\ a_3 &= .00097\,2 & a_4 &= .07810\,8 \end{aligned} \tag{4.138}$$

$$\operatorname{erf} x = 1 - \frac{1}{\left(1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6\right)^6} + \epsilon(x) \qquad \text{with } |\epsilon(x)| \le 3 \times 10^{-7} \tag{4.139}$$

$$\begin{aligned} a_1 &= .07052\,30784 & a_2 &= .04228\,20123 \\ a_3 &= .00927\,05274 & a_4 &= .00015\,20143 \\ a_5 &= .00027\,65672 & a_6 &= .00004\,30638 \end{aligned} \tag{4.140}$$

4.9 Generalized Laguerre Function

4.10 ζ (Riemann, Hurwitz), H_n^m

4.10.1 Riemann's ζ-Function

For the right halfplane by means of the Dirchlet η-function

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{4.141}$$

For the entire complex plane without $s = 1$ ([43])

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s} \tag{4.142}$$

For even $n \in \mathbb{N}$ and $n \geq 2$

$$\zeta(n) = \frac{2^{n-1} |B_n| \pi^n}{(n)!} \tag{4.143}$$

and for odd $n \in \mathbb{N}$ and $n \geq 1$

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{4.144}$$

where B_n are Bernoulli numbers.

Derivatives

The derivative of the Riemann ζ-function is defined for the right half-plane by

$$\zeta'(s) = -\sum_{n_0 \in \{1,2\}}^{\infty} \frac{\ln n}{n^s} \tag{4.145}$$

For $2n \in \mathbb{N}$ and $n \geq 1$

$$\zeta'(-2n) = \frac{(-1)^n \zeta(2n+1)(2n)!}{2^{2n+1} \pi^{2n}} \tag{4.146}$$

4.10.2 Hurwitz' ζ-Function

4.10.3 General Partial Harmonic Function H_n^m

The general partial harmonic function H_n^m is implemented with the summation formula

$$H_n^m = \sum_{i=1}^{n} \frac{1}{i^m} \tag{4.147}$$

4.10.4 Partial Harmonic Function H_n^1

The partial harmonic function H_n^1 is implemented in two ways: with simple summation up to $n = 1\,000$ and an asymptotic series approximation otherwise.

$$H_n = \begin{cases} \sum^n \frac{1}{i} & \text{with} \quad n < 1\,000 \\ \gamma + \ln(n) + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} & \text{with} \quad n \geq 1\,000 \end{cases} \tag{4.148}$$

That limit is not as arbitrary as it seems to be. With $f(n)$ the summation function in ECMA-script, $g(n)$ a high precision calculation[9] and $h(n)$ the asymptotic approximation as described above.

$$\left.\begin{aligned} f(n) &= 7.485\,470\,860\,550\,344\,163\,2 \\ g(n) &= 7.485\,470\,860\,550\,343\,275\,0 \\ h(n) &= 7.485\,470\,860\,550\,344\,912\,656\,518\,204\,333\ldots \end{aligned}\right\} n = 1\,000 \tag{4.149}$$

$$\left.\begin{aligned} f(n) &= 12.090\,146\,129\,863\,429\,181 \\ g(n) &= 12.090\,146\,129\,863\,281\,743 \\ h(n) &= 12.090\,146\,129\,863\,427\,947\,363\,219\,363\,504\ldots \end{aligned}\right\} n = 100\,000 \tag{4.150}$$

14 decimal digits of precision for $n \in \mathbb{N}$ seem sufficient.

[9] With GNU's `bc(1)` with the summation formula and `scale=100`

Chapter 5

Linear Algebra (Matrices)

5.1 Matrix Decompositions

5.1.1 Eigen Decomposition

5.1.2 LU Decomposition

5.1.3 QR Decomposition

5.1.4 Singular Value Decomposition

5.2 Systems of Equations

5.2.1 Determinant

Chapter 6

Sets

The set arithmetic used here is based on the common Zermel-Fraenklin axioms. If $\mathbf{A}, \mathbf{B}$ are subsets of a universe $\mathbf{U}$ than the basic operations are the equality, union, intersection, difference, power, and the Cartesian product.

Sets are implemented in this program as simple arrays.

6.1 Equality $\mathbf{A} = \mathbf{B}$

Two sets $\mathbf{A}, \mathbf{B}$ are equal if both sets have the same elements.

$$\forall \mathbf{A}, \mathbf{B}, x(x \in \mathbf{A} \iff x \in \mathbf{B}) \implies (\mathbf{A} = \mathbf{B}) \tag{6.1}$$

6.2 Union $\mathbf{A} \cup \mathbf{B}$

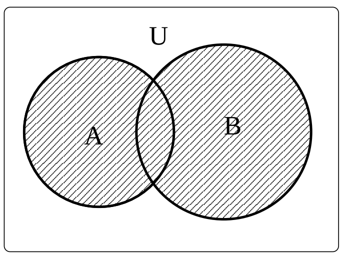

Figure 6.1: $\mathbf{A} \cup \mathbf{B}$

The union of the two sets $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \cup \mathbf{B} = \{x | x \in \mathbf{A} \vee x \in \mathbf{B}\} \tag{6.2}$$

The implementation of the union concatenates both arrays, sorts the result and removes all non-unique entries. That is admittedly not the fastest way to do it.

6.3 Intersection $\mathbf{A} \cap \mathbf{B}$

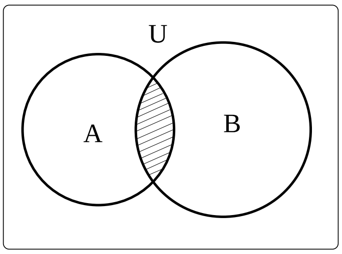

Figure 6.2: $\mathbf{A} \cap \mathbf{B}$

The intersection of the two sets $\mathbf{A}, \mathbf{B}$ is defined as

$$\mathbf{A} \cap \mathbf{B} = \{x | x \in \mathbf{A} \wedge x \in \mathbf{B}\} \tag{6.3}$$

Some properties of union and intersection are

$$\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A} \tag{6.4}$$
$$\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A} \tag{6.5}$$
$$(\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C} = \mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) \tag{6.6}$$
$$(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) \tag{6.7}$$
$$\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{B} \cup \mathbf{C}) \tag{6.8}$$
$$\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{B} \cap \mathbf{C}) \tag{6.9}$$

6.4 Difference $\mathbf{A} \setminus \mathbf{B}$

Both variations of the set difference make no difference operationally, so it is implemented with a single function: `Math.setminus(a,b)`.

6.4.1 Absolute Complement

The absolute complement, "complement" for short, $\complement_{\mathbf{U}}\mathbf{A}$ of the set $\mathbf{A} \subseteq \mathbf{U}$ can be defined as

$$\complement_{\mathbf{U}}\mathbf{A} = \mathbf{U} \setminus \mathbf{A} \tag{6.10}$$

The notation of the absolute complement for a fixed universe $\mathbf{U}$ can be also $\complement\mathbf{A}$ or $\mathbf{A}^C$.

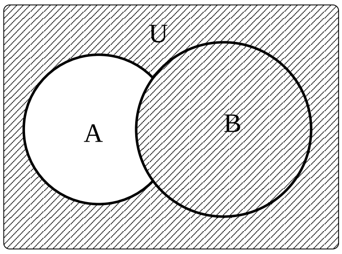

Figure 6.3: $\complement_{\mathbf{U}} \mathbf{A}$

De Morgan's Law for $\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{U}$ and other identities

$$\complement(\mathbf{A} \cup \mathbf{B}) = \complement\mathbf{A} \cap \complement\mathbf{B} \tag{6.11}$$
$$\complement(\mathbf{A} \cap \mathbf{B}) = \complement\mathbf{A} \cup \complement\mathbf{B} \tag{6.12}$$
$$\mathbf{A} \cup \complement\mathbf{A} = \mathbf{U} \tag{6.13}$$
$$\mathbf{A} \cap \complement\mathbf{A} = \varnothing \tag{6.14}$$
$$\complement\varnothing = \mathbf{U} \tag{6.15}$$
$$\complement\mathbf{U} = \varnothing \tag{6.16}$$
$$\mathbf{A} \subseteq \mathbf{B} \implies \complement\mathbf{B} \subseteq \complement\mathbf{A} \tag{6.17}$$
$$\complement\complement\mathbf{A} = \mathbf{A} \tag{6.18}$$

6.4.2 Relative Complement

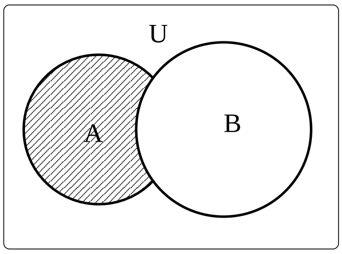

Figure 6.4: $\mathbf{A} \setminus \mathbf{B}$

The relative complement, "difference" or "set difference" for short, $\mathbf{A} \setminus \mathbf{B}$[1] can be defined as

$$\mathbf{A} \setminus \mathbf{B} = \{x | x \in \mathbf{A} \wedge x \notin \mathbf{B}\} \tag{6.19}$$

[1]sometimes written with a minus sign instead: $\mathbf{A} - \mathbf{B}$

De Morgan's Law and other identities

$$\mathbf{A} \setminus (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{A} \setminus \mathbf{C}) \tag{6.20}$$
$$\mathbf{A} \setminus (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \setminus \mathbf{B}) \cap (\mathbf{A} \setminus \mathbf{C}) \tag{6.21}$$
$$\mathbf{A} \setminus (\mathbf{B} \setminus \mathbf{C}) = (\mathbf{C} \cap \mathbf{A}) \cup (\mathbf{A} \setminus \mathbf{B}) \tag{6.22}$$
$$\mathbf{A} \cap (\mathbf{B} \setminus \mathbf{C}) = (\mathbf{B} \cap \mathbf{A}) \setminus \mathbf{C} \tag{6.23}$$
$$\mathbf{A} \cup (\mathbf{B} \setminus \mathbf{C}) = (\mathbf{B} \cup \mathbf{A}) \setminus (\mathbf{C} \setminus \mathbf{A}) \tag{6.24}$$

The relative and the absolute complement are related as follows

$$\mathbf{A} \setminus \mathbf{B} = \mathbf{A} \cap \complement\mathbf{B} \tag{6.25}$$
$$\complement(\mathbf{A} \setminus \mathbf{B}) = \complement\mathbf{A} \cup \mathbf{B} \tag{6.26}$$

6.4.3 Symmetric Difference

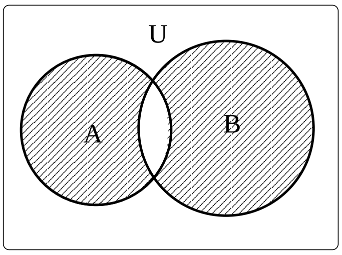

Figure 6.5: Symmetric difference

The symmetric difference of two sets $\mathbf{A}, \mathbf{B}$ can be defined as

$$\begin{aligned}
\mathbf{A} \Delta \mathbf{B} &= (\mathbf{A} \setminus \mathbf{B}) \cup (\mathbf{B} \setminus \mathbf{A}) && (6.27)\\
&= (\mathbf{A} \cup \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{B}) && (6.28)\\
&= \Big\{ x \Big| \big((x \in \mathbf{A}) \vee (x \in \mathbf{B})\big) \wedge \neg\big((x \in \mathbf{A}) \wedge (x \in \mathbf{B})\big) \Big\} && (6.29)
\end{aligned}$$

The last equation resembles the binary operator `XOR`. To make it more legible

$$x \dot{\vee} y = (x \vee y) \wedge \neg(x \wedge y) \tag{6.30}$$

The notation for this operation varies but $x\,\mathrm{XOR}\,y$ and $x \oplus y$ can be found more often than others like the $x \dot{\vee} y$ for example, the one used here.

The `XOR` operation is heavily used in cryptography where the following properties are more than willing and able to brake the security of a lot of implementations. Exchanging the Δ-sign with $\oplus$ for legibility we get

$$\mathbf{X} \oplus \mathbf{Y} = \mathbf{Y} \oplus \mathbf{X} \tag{6.31}$$

$$(\mathbf{X} \oplus \mathbf{Y}) \oplus \mathbf{Z} = \mathbf{X} \oplus (\mathbf{Y} \oplus \mathbf{Z}) \tag{6.32}$$

$$(\mathbf{X} \oplus \mathbf{Y}) \oplus (\mathbf{Y} \oplus \mathbf{Z}) = \mathbf{X} \oplus \mathbf{Z} \tag{6.33}$$

$$\mathbf{X} \oplus \mathbf{X} = \varnothing \tag{6.34}$$

$$\mathbf{X} \cap (\mathbf{Y} \oplus \mathbf{Z}) = (\mathbf{X} \cap \mathbf{Y}) \oplus (\mathbf{X} \cap \mathbf{Z}) \tag{6.35}$$

6.5 Power

The power set is not implemented directly in this program because the function `Array.prototype.subsets` does not add the empty set to the output[2]. For the sake of completeness: the axiom of the power set is

$$\mathcal{P}(\mathbf{A}) = \forall \mathbf{A} \exists y \forall z \; z \in y \iff (\forall t \; t \in x \to t \in \mathbf{A}) \tag{6.36}$$

6.5.1 Families and other Collections

The subsets of the power set form so called "families" or "collections" of sets, for example the set of all disjoint sets of some power set. The common notation is to typeset the names of these collections in Fraktura[3].

For any family $\mathfrak{A}$

$$\mathfrak{A} \subseteq \mathcal{P}(\mathbf{A}) \tag{6.37}$$

Example: let $\mathbf{A}$ be a small set of integers $\mathbf{A} = \{2, 3, 4\}$. The power set is then

$$\mathcal{P}(\mathbf{A}) = \{\varnothing, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\} \tag{6.38}$$

The set $\mathfrak{A}$ of all subsets of $\mathbf{A}$ that have an even prime as a member is

$$\mathfrak{A} = \{\{2\}, \{2,3\}, \{2,4\}, \{2,3,4\}\} \tag{6.39}$$

6.6 Cartesian Product $\mathbf{A} \times \mathbf{B}$

The cartesian product of two sets $\mathbf{A}, \mathbf{B}$ is a set of ordered pairs where each pair consists of one element from $\mathbf{A}$ and one from $\mathbf{B}$

$$\mathbf{A} \times \mathbf{B} = \{(a, b) \in \mathcal{P}(\mathbf{A} \cup \mathbf{B}) | a \in \mathbf{A} \wedge b \in \mathbf{B}\} \tag{6.40}$$

[2] It is difficult to express the empty set in computerprograms, even "nothing" has to have an address

[3] The `\mathfrac{letter}` command from the $\mathcal{AMS}$-LaTeX packet has been used here. The font is called `eufrak`—Euler Fraktur alphabet.

6.6.1 Binary Relation

The implementation of the function to build a cartesian product has the ability to implement a binary relation $a\mathcal{R}b$ easily and the appropriatly named function `Math.timesFunction(a,b,f)` takes as it's third argument the name of a function that shall act as the relation[4].

The simplest function would be a test for equality and acting accordingly:

```
var a = [1,3,5,7,9];
var b = [2,4,6,8,10];
Array.prototype.stripEmptyElements = function(){
  var len = this.length;
  var ret = new Array();
  for(var i=0;i<len;i++){
    if(xtypeof(this[i]) == 'array'){
      ret.push(this[i]);
    }
  }
  return ret;
};
function gatherPrimePairs(x,y){
  if(x < 0x80000000 && y < 0x80000000){
    var a = Math.factor(x);
    var b = Math.factor(y);
    if(xtypeof(a) == 'array' && xtypeof(b) == 'array'){
      if(a.length == 1 && b.length == 1){
        return [x,y];
        }
    }
  }
}
var ret = Math.timesFunction(a,b,gatherPrimePairs);
alert(ret.stripEmptyElements().join("\n"))
```

These relations are the base of current computer programs. So here is a short refresher of the basics.

- The domain
$$\operatorname{dom}\mathcal{R} = \{x|\exists y, (x,y) \in \mathcal{R}\} \tag{6.41}$$
That would be the allowed input for a computerprogram. This includes the empty set with the obvious question if no input at all should be considered an empty set—or more, like the macro `NULL` in the C-language[45]. The solution used in almost all cases: a program takes either no input at all—or only the empty set, to be a bit overly correct—and ignores any input given, uses some default values if no input has been given at all or it stops immediately.

[4] It should be noted that the cartesian product is a binary relation, too

- The range
$$\operatorname{ran}\mathcal{R} = \{y|\exists x, (x,y) \in \mathcal{R}\} \tag{6.42}$$
For a computer program: the output it is able to give.

- The field
$$\operatorname{field}\mathcal{R} = \operatorname{dom}\mathcal{R} \cup \operatorname{ran}\mathcal{R} \tag{6.43}$$
The fields a current computer program can use are finite subsets of $\mathbb{Q}$.

- The inverse
$$\mathcal{R}^{-1} = \{(y,x)|(x,y) \in \mathcal{R}\} \tag{6.44}$$
That is the "back" in "back&forth" it is the "forth".

- The image of a set $\mathbf{A}$
$$\mathcal{R}[\mathbf{A}] = \{y \in \operatorname{ran}\mathcal{R}|\exists x \in \mathbf{A}, (x,y) \in \mathcal{R}\} \tag{6.45}$$

- The image of a set $\mathbf{A}$ under the inverse of a relation $\mathcal{R}$, also known as the "preimage"
$$\mathcal{R}^{-1}[\mathbf{A}] = \{x \in \operatorname{dom}\mathcal{R}|\exists y \in \mathbf{A}, (x,y) \in \mathcal{R}\} \tag{6.46}$$

So a relation $\mathcal{R}$ is a relation on $\mathbf{A}$ if
$$\operatorname{field}\mathcal{R} \subseteq \mathbf{A} \tag{6.47}$$

It is desirable to keep $\#\mathbf{A}$, the number of elements of $\mathbf{A}$, as small as possible and moreso that field $\mathcal{R} = \mathbf{A}$ or the program might not be fully testable. It will definitly not be fully testable if $|\mathbf{A}| \geq |\mathbb{N}|$ even with infinite Turing machines [5].

Complicated looking formulas for which a simple example is advised. Consider $f(x) = x^2$ which maps the real line to itself $f : \mathbb{R} \to \mathbb{R}$.

- The domain is $\mathbb{R}$
- The range is $\mathbb{R}^+$
- The field is $\mathbb{R}$
- The inverse is $f^{-1}(x) = \sqrt{x}$. That means that[6]

$$f^{-1}(\{4\}) = \{-2, 2\} \tag{6.48}$$
$$f^{-1}((1,2)) = (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \tag{6.49}$$
$$f^{-1}(\{-1\}) = \varnothing \tag{6.50}$$

[5]Under the assumption, of course that the number of steps of a Turing machine is at most countable.

[6]The parentheses anotate a range of numbers, in the case of $(1,2)$ it is the set $\{1,\ldots,2\} \notin \mathcal{P}(\mathbb{N})$.

So domain, range and field of $f^{-1}(x)$ are

$$\operatorname{dom} f^{-1} = \mathbb{R}^+ \tag{6.51}$$
$$\operatorname{ran} f^{-1} = \mathbb{R} \tag{6.52}$$
$$\operatorname{field} f^{-1} = \mathbb{R}^+ \cup \mathbb{R} = \mathbb{R} \tag{6.53}$$

- The image is the "result" of $f(x)$ for every $x \in \operatorname{dom} f$; it is the range of f or the image of dom f.

$$\operatorname{im}\{2,4\} = f(\{2,4\}) = \{4,16\} \tag{6.54}$$
$$\operatorname{im} f = \mathbb{R}^+ \tag{6.55}$$

- The preimage is, by the definition of the image and the definition of the inverse, the image of ran f.

$$\operatorname{im}^{-1}\{4,16\} = f^{-1}(\{4,16\}) = \{-4,-2,2,4\} \tag{6.56}$$
$$\operatorname{im}^{-1} f = \mathbb{R} \tag{6.57}$$

With a function $f : \mathbf{X} \to \mathbf{Y}$, the sets $\{A_1, A_2, \ldots, A_n\} \subseteq \mathbf{X}$ and $\{B_1, B_2, \ldots, B_n\} \subseteq \mathbf{Y}$ the following (in)equalities should look familiar to the reader.

$$f(A_n \cup A_m) = f(A_n) \cup f(A_m) \tag{6.58}$$
$$f^{-1}(B_n \cup B_m) = f^{-1}(B_n) \cup f^{-1}(B_m) \tag{6.59}$$

but

$$f(A_n \cap A_m) \subseteq f(A_n) \cap f(A_m) \tag{6.60}$$
$$f^{-1}(B_n \cap B_m) = f^{-1}(B_n) \cap f^{-1}(B_m) \tag{6.61}$$

From the example with $f(x) = x^2$ it should be obvious that $f(x)$ is not always simply the inverse of $f^{-1}(x)$ for every x

$$f(f^{-1}(B_n)) \subseteq B_n \tag{6.62}$$
$$f^{-1}(f(A_n)) \supseteq A_n \tag{6.63}$$

The average computer language designer has problems with the empty set as described above, so to avoid it the mapping f shall be extended to $f : \mathbb{C} \to \mathbb{C}$ because the square root is well defined for every $x \in \mathbb{C}$.

$$\operatorname{dom} f = \mathbb{C} \tag{6.64}$$
$$\operatorname{dom} f^{-1} = \mathbb{C} \tag{6.65}$$
$$\operatorname{ran} f = \mathbb{C} \tag{6.66}$$
$$\operatorname{ran} f^{-1} = \mathbb{C} \tag{6.67}$$

Chapter 7

Combinatorics

Chapter 8

Statistical Functions

It is easy to lie with statistics. It is hard to tell the truth without statistics. [ANDREJS DUNKELS]

He uses statistics as a drunken man uses lampposts—for support rather than illumination. [ANDREW LANG]

8.1 Distributions

Some basic algorithms for the most common distributions are implemented; least the probability function and the distribution function, but also the inverse of the distribution function for almost all of the offered distributions.

8.1.1 Beta Distribution

The beta distribution is based on the beta function

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{8.1}$$

Probability Function

The probability function $P_B(x,a,b)$ is implemented for $a > 0, b > 0, 0 < x < 1$

$$P_B(x,a,b) = x^{a-1}\frac{(1-x)^(b-1)}{B(a,b)} \tag{8.2}$$

Distribution Function

The distribution function $D_B(x,a,b)$ is implemented for $a > 0, b > 0, x > 0$

$$D_B(x,a,b) = \begin{cases} 1 & \text{if } a > 0 \wedge b > 0 \wedge x \geq 1 \\ I(x,a,b) & \text{otherwise} \end{cases} \tag{8.3}$$

where $I(x, a, b)$ is the regularized beta function.

8.1.2 Binomial

Probability Function

The probability function $P_b(n, k, p)$ is implemented in two different ways for $n \in \mathbb{N}, 0 \leq p \leq 1$.

$$P_b(n, k, p) = \binom{n}{k} p^k (1-p)^{n-k} \tag{8.4}$$

where $\binom{n}{k}$ is the binomial coefficient calculated with factorials, following the textbook. The other implementations uses the logarithm of the binomial coefficient calculated with the log-Gamma function.

$$\ln \binom{n}{k} = \ln \Gamma(n+1) - (\ln \Gamma(k+1) + \ln \Gamma(n-k+1)) \tag{8.5}$$

This second implementation returns the logarithm of the binomial probability function $\ln P_b(n, k, p)$

$$\ln P_b(n, k, p) = \ln \binom{n}{k} + k \ln p + (n-k) \ln(1-p) \tag{8.6}$$

Distribution Function

The distribution function $D_b(n, k, p)$ for $0 \leq p \leq 1$ is implemented with the help of the regularized beta function.

$$D_b(n, k, p) = I(1-p, k - \lfloor n \rfloor, 1 + \lfloor n \rfloor) \tag{8.7}$$

8.1.3 Cauchy

Probability Function

The probability function $P_C(x, \lambda, \sigma)$ for $\sigma \geq 0$ is

$$P_C(x, \lambda, \sigma) = \frac{1}{(1 + \frac{x-\lambda}{\sigma}^2)\pi\sigma} \tag{8.8}$$

Distribution Function

The distribution function $D_C(x, \lambda, \sigma)$ for $\sigma \geq 0$ is

$$D_C(x, \lambda, \sigma) = \frac{1}{2} + \frac{\operatorname{atan}\left(\frac{x-\lambda}{\sigma}\right)}{\pi} \tag{8.9}$$

Inverse of the Distribution Function

The inverse of the distribution function $D_C^{-1}(x, \lambda, \sigma)$ for $\sigma \geq 0, 0 \leq x \leq 1$ can be represented in closed form

$$D_C^{-1}(x, \lambda, \sigma) = \begin{cases} -\infty & \text{if } x = 0 \\ +\infty & \text{if } x = 1 \\ \lambda - \sigma \cot(x\pi) & \text{else} \end{cases} \tag{8.10}$$

8.1.4 F-Distribution

Probability Function

The probability function $P_F(x, m, n)$ for $x > 0, m > 0, n > 0$ is implemented as

$$P_F(x, m, n) = \exp\left(\left(\frac{m}{2} - 1\right) \ln\left(\frac{mx}{n}\right) - \frac{m+n}{2} \ln\left(1 + \frac{mx}{n}\right)\right) \frac{\left(\frac{m}{n}\right)}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \tag{8.11}$$

Distribution Function

The distribution function $D_F(x, m, n)$ for $x > 0, m > 0, n > 0$ is implemented as

$$P_F(x, m, n) = \begin{cases} 1 & \text{if } x = +\infty \\ 1 - I\left(\frac{1}{1+(mx/n)}, \frac{n}{2}, \frac{m}{2}\right) & \text{else} \end{cases} \tag{8.12}$$

where $I(x; a, b)$ is the regularized beta function.

Inverse of the Distribution Function

The inverse of the distribution function $D_F^{-1}(x, m, n)$ for $0 \leq x \leq 1, m > 0, n > 0$ can be represented in closed form

$$D_F^{-1}(x, m, n) + \begin{cases} +\infty & \text{if } x = 1 \\ \dfrac{1}{D_B^{-1}\left(1 - x, \frac{n}{2}, \frac{m}{2}\right) \frac{n}{m}} & \text{else} \end{cases} \tag{8.13}$$

where D_B^{-1} is the inverse of the beta distribution function.

8.1.5 Geometric

Probability Function

The probability function $P_G(x, p)$ for $0 \leq p \leq 1, x \in \mathbb{N}$ is implemented as

$$P_G(x, p) = \begin{cases} 1 & \text{if } x = +\infty \wedge p = 0 \\ p(1-p)^x & \text{else} \end{cases} \tag{8.14}$$

Distribution Function

The distribution function $D_G(x,p)$ for $0 \le p \le 1, x \in \mathbb{N}$ is implemented as

$$D_G(x,p) = \begin{cases} 1 & \text{if } x = +\infty \wedge p = 0 \\ 1 - (1-p)^{x+1} & \text{else} \end{cases} \tag{8.15}$$

Inverse of the Distribution Function

The inverse of the distribution function $D_G^{-1}(x,p)$ for $0 \le x \le 1, 0 \le p \le 1$ can be represented in closed form

$$D_G^{-1}(x,p) = \begin{cases} +\infty & \text{if } x = 1 \\ \max\left(\left\lceil \frac{\ln(1-x)}{\ln(1-p)} \right\rceil - 1, 0\right) & \text{else} \end{cases} \tag{8.16}$$

where $\max(a,b)$ denotes the maximum value of the two variables a, b.

8.1.6 Hypergeometric

Probability Function

The probability function $P_H(x,m,t,n)$ for $\{m,n,t\} \in \mathbb{N}, n \ne 0, m \le t, n \le t$ is implemented as

$$P_H(x,m,t,n) = \begin{cases} 0 & \text{if } x \notin \mathbb{N} \wedge x > m \wedge n < x \wedge (n-x) > (t-m) \\ \frac{\binom{m}{x}\binom{t-m}{n-x}}{\binom{t}{n}} & \text{else} \end{cases} \tag{8.17}$$

Distribution Function

The distribution function $D_H(x,m,t,n)$ for $\{m,n,t\} \in \mathbb{N}, n \ne 0, m \le t, n \le t$ is implemented by means of the distribution function of the discrete distribution.

$$D_H(x,m,t,n) = D_D\left(x, (i)_{i=1}^n, P_H(i,m,t,n)_{i=1}^n\right) \tag{8.18}$$

This might be one of the cases, where the code of the implementation is more readable as the underlying algorithm.

Inverse Distribution Function

The inverse distribution function is similar to the distribution function except that it uses the inverse distribution function of the discrete distribution instead of the normal distribution function.

The inverse distribution function $D_H^{-1}(x,m,t,n)$ for $\{m,n,t\} \in \mathbb{N}, n \ne 0, m \le t, n \le t$ is implemented by means of the distribution function of the discrete distribution.

$$D_H^{-1}(x,m,t,n) = D_D^{-1}\left(x, (i)_{i=1}^n, P_H(i,m,t,n)_{i=1}^n\right) \tag{8.19}$$

8.2 Means

Let S be a sequence $\{a_i\}_{i=1}^{n}$ of data points, for example the content of the author's ashtray after he gave up cigarette smoking—a handful of coins of different values:

$$n = 8 \tag{8.20}$$

$$\{a_1, a_2, \ldots, a_n\} = \{(0.01, 5), (0.02, 8), (0.05, 3), (0.10, 20), (0.20, 15), (0.50, 21), (1.00, 4), (2.00, 3)\} \tag{8.21}$$

Such a sequence can be described in several ways.

8.2.1 Range

The range $\text{ran}(S)$ of a dataset S is the difference between the smallest and the largest element.

$$\text{ran}(S) = \max(S) - \min(S) \tag{8.22}$$

8.2.2 Harmonic Mean

The harmonic mean $H(S)$ is defined by

$$H(S) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \tag{8.23}$$

8.2.3 Geometric Mean

The geometric mean $G(S)$ is defined by

$$G(S) = \left(\prod_{i=1}^{n} a_i \right) \frac{1}{n} \tag{8.24}$$

With only two elements in the sequence ($n = 2$) the geometric mean is related to the arithmetic mean $A(S)$ and harmonic mean $H(S)$

$$G(S) = \sqrt{A(S)H(S)} \tag{8.25}$$

As every mean is a special case of the general power mean, the geometric mean is the case M_0 of the power mean. The geometric mean is, the name implies it, describable by geometric means and is thus one of the three Pythagorean means.

The following might be of additional interest ([44])

$$G(a_1 + c, a_2 + c, \ldots, a_n + c) > c + G(a_1, a_2, \ldots, a_n) \tag{8.26}$$

8.2.4 Root-Mean-Square

The root-mean-square $\mathrm{RMS}(S)$ is defined for a discrete distribution by

$$\mathrm{RMS}(S) = \sqrt{\left(\sum_{i=1}^{n} a_i^2\right)\frac{1}{n}} \tag{8.27}$$

and for a continuous distribution by

$$\mathrm{RMS}(S) = \sqrt{\frac{\int P(a)a^2 \,\mathrm{d}\,a}{\int P(a)\,\mathrm{d}\,a}} \tag{8.28}$$

It has been shown by [44] that for a positive constant c

$$\mathrm{RMS}(a_1 + c, a_2 + c, \ldots, a_n + c) < c + \mathrm{RMS}(a_1, a_2, \ldots, a_n) \tag{8.29}$$

8.2.5 Arithmetic Mean

The arithmetic mean is probably the most common mean, taught in primary school and called the average. The arithmetic mean $A(S)$ is defined by

$$A(S) = \left(\sum_{i=1}^{n} a_i\right)\frac{1}{n} \tag{8.30}$$

Similar to the geometrical mean [44] showed

$$A(a_1 + c, a_2 + c, \ldots, a_n + c) = c + A(a_1, a_2, \ldots, a_n) \tag{8.31}$$

8.2.6 Logarithmic Arithmetic Mean

The logarithmic arithmetic mean $A_L(S)$ is defined by

$$A_L(S) = \left(\sum_{i=1}^{n} \ln a_i\right)\frac{1}{n} \tag{8.32}$$

8.2.7 Power Mean

The power mean is the generalized algorithm for means. Every other mean can be derived from the power mean. It is defined by

$$M_p(S) = \left(\frac{1}{n}\prod_{i=1}^{n} a_i^p\right)\frac{1}{p} \qquad \text{with } a_k \geq 0 \tag{8.33}$$

If we include maximum and minimum as means and denote the power mean as M_p the special cases of the power mean are

$\mathbf{M_p}$	**Common Symbol**	**Name of Mean**
$M_{-\infty}$	min	Minimum
M_{-1}	H	Harmonic Mean
M_0	G	Geometric Mean
M_1	A	Arithmetic Mean
M_2	RMS	Root-Mean-Square
M_∞	max	Maximum

8.2.8 Arithmetic Geometric Mean

This mean is actually a limit. The recurrence formulae for $\mathrm{AGM}(a, b)$ are

$$a_{n+1} = \frac{1}{2}(a_n + b_n) \qquad \text{with } a_0 = a \tag{8.34}$$

$$b_{n+1} = \sqrt{a_n b_n} \qquad \text{with } b_0 = b \tag{8.35}$$

$$\tag{8.36}$$

The mean is reached when $a_n = b_n$ but *in praxi* the iterating takes forth until $a_n = b_n + c$ or $a_n + c = b_n$ where c is an arbitrary small constant—the desired precision for the numerical solution.

Some combinations of a and b have a closed form for $\mathrm{AGM}(a, b)$ for example the famous Gauss constant. The Gauss constant is the reciprocal of $\mathrm{AGM}(1, \sqrt{2})$

$$\frac{1}{\mathrm{AGM}(1, \sqrt{2})} = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{2\pi^{3/2}\sqrt{2}} \tag{8.37}$$

The derivative

$$\frac{\partial}{\partial b}\mathrm{AGM}(a, b) = \frac{\pi}{8\frac{a-b}{a+b}b} \cdot \frac{(a = b)\,\mathrm{E}(\frac{a-b}{a+b}) - 2b\,\mathrm{K}(\frac{a-b}{a+b})}{\left[\mathrm{K}(\frac{a-b}{a+b})\right]^2} \tag{8.38}$$

where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind and $\mathrm{E}(x)$ is that of the second kind.

For the special case of $\mathrm{AGM}(1, b)$ a series expansion exist

$$\mathrm{AGM}(1, b) = -\frac{\pi}{2\ln\left(\frac{1}{4}b\right)} + \frac{\pi\left[1 + \ln\left(\frac{1}{4}b\right)\right]b^2}{8\left[\ln\left(\frac{1}{4}b\right)\right]^2} + \mathcal{O}\left(b^4\right) \tag{8.39}$$

Some properties of $\mathrm{AGM}(a, b)$

$$c\,\mathrm{AGM}(a, b) = \mathrm{AGM}(ca, cb) \tag{8.40}$$

$$\mathrm{AGM}(a, b) = \mathrm{AGM}\left(\frac{1}{2}(a + b), \sqrt{ab}\right) \tag{8.41}$$

$$\mathrm{AGM}\left(1, \sqrt{1 - x^2}\right) = \mathrm{AGM}(1 + x, 1 - x) \tag{8.42}$$

$$\mathrm{AGM}(1, b) = \frac{1 + b}{2}\,\mathrm{AGM}\left(1, \frac{2\sqrt{b}}{1 + b}\right) \tag{8.43}$$

8.2.9 Geometric Harmonic Mean

This mean is the limit of the recurrence formulae

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad \text{with } a_0 = a \tag{8.44}$$

$$b_{n+1} = \sqrt{a_n b_n} \qquad \text{with } b_0 = b \tag{8.45}$$

It is implemented in a slightly different way

$$a_{n+1} = \frac{2}{\frac{1}{a_n} + \frac{1}{b_n}} \qquad \text{with } a_0 = H(S) \tag{8.46}$$

$$b_{n+1} = \sqrt{a_n b_n} \qquad \text{with } b_0 = G(S) \tag{8.47}$$

with the sum $\frac{1}{a_n} + \frac{1}{b_n} \neq 0$, $H(S)$ the harmonic mean and $G(S)$ the geometric mean. The mean is reached when $a_n = b_n$ but *in praxi* the iterating takes forth until $a_n = b_n + c$ or $a_n + c = b_n$ where c is an arbitrary small constant—the desired precision for the numerical solution. The geometric harmonic mean $G_H(S)$ is related to the arithmetic geometric mean AGM by

$$G_H(S) = \lim_{n \to \infty} a_n = \frac{1}{\text{AGM}\left(\frac{1}{a_0}, \frac{1}{b_0}\right)} \tag{8.48}$$

8.2.10 Arithmetic Harmonic Mean

The arithmetic harmonic mean is identical to the geometric mean. The recurrence formulae are

$$a_{n+1} = \frac{1}{2}\left(a_n + b_n\right) \qquad \text{with } a_0 = a \tag{8.49}$$

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad \text{with } b_0 = b \tag{8.50}$$

Calculating the limits $a, b \to \infty$ gives

$$A_H(S) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \sqrt{ab} \tag{8.51}$$

which is indeed the geometric mean.

8.2.11 Weighted Geometric Mean

The weighted geometric mean $G_\omega(S)$ puts a value $\omega_n \in \mathbb{N}$ to each element of the sequence a_n.

$$G_\omega(S) = \frac{\sum_{i=1}^{n} \omega_n \ln a_n}{\sum_{i=1}^{n} \omega_n} \tag{8.52}$$

with $\sum_{i=1}^{n} \omega_n = 1$

8.2.12 Weighted Arithmetic Mean

The weighted arithmetic mean $A_\omega(S)$ puts a value $\omega_n \in \mathbb{N}$ to each element of the sequence a_n.

$$A_\omega(S) = \frac{\sum_{i=1}^{n} \omega_n a_n}{\sum_{i=1}^{n} \omega_n} \tag{8.53}$$

with $\sum_{i=1}^{n} \omega_n = 1$

8.2.13 Weighted Harmonic Mean

The weighted harmonic mean $H_\omega(S)$ puts a value $\omega_n \in \mathbb{N}$ to each element of the sequence a_n.

$$H_\omega(S) = \frac{\sum_{i=1}^{n} \frac{\omega_n}{a_n}}{\sum_{i=1}^{n} \omega_n} \tag{8.54}$$

with $\sum_{i=1}^{n} \omega_n = 1$

8.2.14 Pythagorean Means

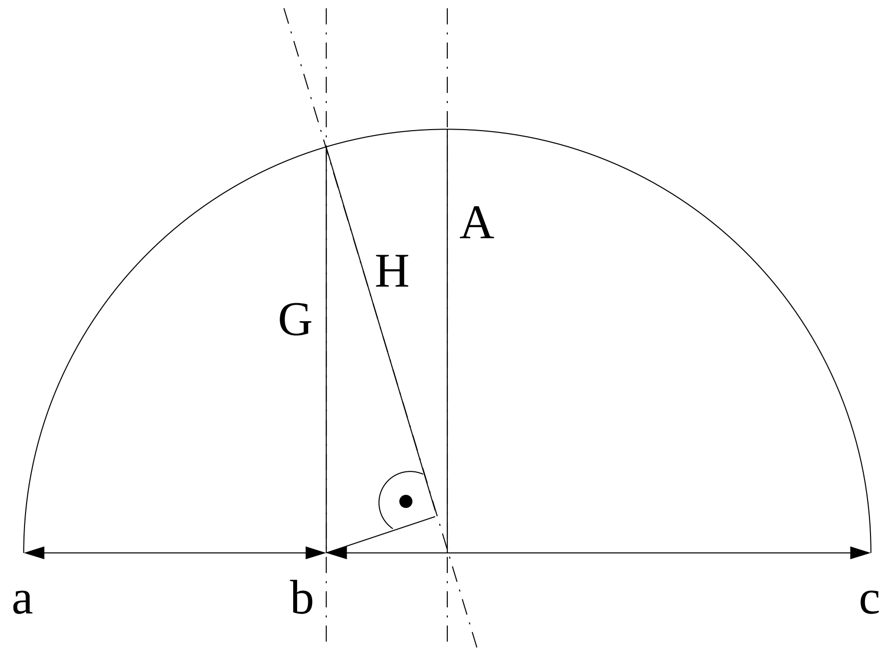

Figure 8.1: $A(a,b) \geq G(a,b) \geq H(a,b)$

The group of the arithmetic mean $A(S)$, geometric mean $G(S)$ and harmonic mean $H(S)$ are also called the Pythagorean means. The relation between them in case of positive arguments $a, b > 0$ is

$$A(S) \geq G(S) \geq H(S) \tag{8.55}$$

The inequality is shown geometrically in figure 8.1.

8.2.15 Median

The median $\mu_{1/2}$ is the value in the middle of a sorted sequence S with length n and defined by

$$\mu_{1/2} = \begin{cases} \lfloor a_{n/2} \rfloor & for \ \mathrm{odd}(n) \\ \frac{a_{n/2}+a_{(n+1)/2}}{2} & for \ \mathrm{even}(n) \wedge n \neq 0 \end{cases} \tag{8.56}$$

The efficiency of the median depends on the sample size $N = 2n + 1$ by the way of ([50])

$$\lim_{N \to \infty} \frac{4n}{\pi(2n+1)} = \frac{2}{\pi} \tag{8.57}$$

8.2.16 Mode

The mode is the number of occurances of one value. In the case of the change in the ashtray the mode is either $(0.50, 21)$ (unimodal) or all of the elements (multimodal[1]). Only the unimodal mode is implemented directly.

8.2.17 Variance

The bias corrected sample variance σ^2 is defined by

$$\sigma = \frac{\sum_{i=1}^{n} \left(a_n - A\left(S\right)\right)^2}{n-1} \tag{8.58}$$

where $A(S)$ is the arithmetic mean.

8.2.18 Logarithm of the Variance

The logarithm of the bias corrected sample variance σ_l^2 is defined by

$$\sigma_l = \frac{\sum_{i=1}^{n} \left(\ln a_n - A\left(S\right)\right)^2}{n-1} \tag{8.59}$$

where $A(S)$ is the arithmetic mean

[1]That is, of course, not the exclusive name for "all elements occur the same number of times", only to distinguish from unimodal. There are distinct names for special values of multimodality in the literature like "bimodal", "trimodal" and so on.

8.2.19 Co-Variance

The co-variance $\sigma^2(a,b)$ between two sample sequences S_1, S_2 is defined by

$$\sigma(S_1,S_2) = \frac{1}{n-1}\sum_{i=1}^{n}\left(a_n - A\left(S_1\right)\right)\left(b_n - A\left(S_2\right)\right) \tag{8.60}$$

where $A(S)$ is the arithmetic mean.

8.2.20 Standard Deviation

The standard deviation is the square root of the variance. Implemented here is the standard deviation with the bias corrected variance.

$$\sigma = +\sqrt{\sigma^2} \tag{8.61}$$

The standard deviation is used to calculate the confidence interval CI by

$$x_{CI} = \sqrt{2}\,\mathrm{erf}^{-1}(CI) \tag{8.62}$$

Some values ([123])

Range	**CI**
σ	0.6826895
2σ	0.9544997
3σ	0.9973002
4σ	0.9999366
5σ	0.9999994

To calculate the range of the standard deviation from the confidence interval

$$n = \sqrt{2}\,\mathrm{erf}^{-1}(CI) \tag{8.63}$$

Some values ([123])

CI	**Range**
0.800	$\pm 1.28155\sigma$
0.900	$\pm 1.64485\sigma$
0.950	$\pm 1.95996\sigma$
0.990	$\pm 2.57583\sigma$
0.995	$\pm 2.80703\sigma$
0.999	$\pm 3.29053\sigma$

8.2.21 Logarithm of the Standard Deviation

The logarithmic standard deviation is the square root of the logarithmic variance. Implemented here is the logarithmic standard deviation with the bias corrected logarithmic variance.

$$\sigma_l = +\sqrt{\sigma_l^2} \tag{8.64}$$

8.2.22 Average Deviation

Also known as mean deviation σ_m it is defined by

$$\sigma_m = \frac{1}{2} \sum_{i=1}^{n} |a_i - A(S)| \tag{8.65}$$

where $A(S)$ is the arithmetic mean.

8.2.23 Geometric Standard Deviation

The geometric standard deviation σ_g

$$\sigma_g = \exp \left(\sqrt{ \frac{1}{n} \sum_{i=1}^{n} (\ln a_i - \ln G(S))^2 } \right) \tag{8.66}$$

where $G(S)$ is the geometric mean.

8.2.24 Skewness

The skewness β is defined differently in different sources, the variation implemented is

$$\beta = \frac{\sum_{i=1}^{n} (a_i - A(S))^3}{(n-1)\, A^3(S)} \tag{8.67}$$

where $A(S)$ is the arithmetic mean.

8.2.25 Kurtosis

The kurtosis is measured in two different ways: one is called "kurtosis proper" β_2 and the other "kurtosis excess" γ_2 ([1]). The difference is 3[2]. It is implemented as

$$\beta_2 = \frac{\sum_{i=1}^{n} (a_i - A(S))^4}{(n-1)\, A^4(S)} \tag{8.68}$$

$$\gamma_2 = \frac{\sum_{i=1}^{n} (a_i - A(S))^4}{(n-1)\, A^4(S)} - 3 \tag{8.69}$$

8.2.26 Other Means

Other means, beside the obvious "money" and "blunt object" exists. One of these is the Stolarsky mean S_n ([102])

$$S_r(a,b) = \left(\frac{a^r - b^r}{r(a-b)} \right)^{\frac{1}{r-1}} \tag{8.70}$$

[2]Yes, really!

This mean is not implemented but the following ECMA-script code might give an idea.

```
function stolarsky(a,b,r){
  // Sanity checks omitted here
  var nominator   = Math.pow(a,r)-Math.pow(b,r);
  var denominator = r*(a-b);

  return Math.pow(nominator/denominator,1/(r-1));
};
```

The Stolarsky mean is derived from the mean value theorem

$$\exists\xi \in [x,y] f'(\xi) = \frac{f(x)-f(y)}{x-y} \tag{8.71}$$

Solving for ξ gives

$$\xi = f'^{-1}\left(\frac{f(x)-f(y)}{x-y}\right) \tag{8.72}$$

Or in a more generalized form for $n+1$ variables with the nth derivative

$$S_r(a_1,a_2,\ldots,a_{n+1}) = f^{n^{-1}}\left(n! f\left(a_1,a_2,\ldots,a_{n+1}\right)\right) \tag{8.73}$$

Some relations to other means

Stolarsky Mean	**Related Mean**
$\lim_{r\to-\infty} S_r(a,b)$	Minimum
$S_{-1}(a,b)$	Geometric mean
$\lim_{r\to 0} S_r(a,b)$	Logarithmic mean ($f(x) = \ln x$)
$S_{\frac{1}{2}}(a,b)$	Power mean with exponent $\frac{1}{2}$
$\lim_{r\to 1} S_r(a,b)$	Identric[3] mean ($f(x) = x \ln x$)
$S_2(a,b)$	Arithmetic mean
$\lim_{r\to+\infty} S_r(a,b)$	Maximum

And as we are at it, the identric mean $I(a,b)$ mentioned in the table 8.2.26 is defined by

$$I(a,b) = \frac{1}{e}\sqrt[a-b]{\frac{a^a}{b^b}} \tag{8.74}$$

The identric mean is not implemented but this short ECMA-script should give an idea.

```
function identric(a,b){
  // Sanity checks omitted here
  var nominator   = Math.pow(a,a);
  var denominator = Math.pow(b,b);
  var reciprocalE = 1/Math.E;

```

```
    return reciprocalE * Math.pow(nominator/denominator,a
        -b);
};
```

Generalization to more then the mere two variables follows, like the generalization of the Stolarsky mean, by the mean value theorem for divided differences.

The identric mean has caught some interest from [3, 18], just to name two.

Chapter 9

Physics Functions

9.1 Astronomy

9.2 Mechanics

9.3 Quantum Mechanics

9.4 Thermodynamics

9.5 Electric

9.5.1 Capacitance of a Cylinder Capaciator (Coax-cable)

Chapter 10

String Functions

10.1 Comparing

10.1.1 Similarity (Ratcliff/Obershelp)

10.1.2 Difference (Levenshtein)

10.2 Sampling

10.3 Mixing

Chapter 11

Helper Functions

11.1 Lists (Arrays)

Chapter 12

Miscellaneous Functions

12.1 $\mathbb{N}$

12.1.1 Size of a Bloomfilter

12.1.2 Happy Numbers

A positive integer is a happy number if the sum of the squares of its digits—iterated—reaches 1.

Let s be a concatenation $d_1 \cdots d_n$ of decimal digits $d_n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then the reccurence formula

$$d'_1 \dots d'_{n'} = d_1^2 + \cdots d_n^2 \tag{12.1}$$

reaches one of the points $\{0, 1, 4, 16, 20, 37, 42, 58, 89, 145\}$([84]). For $s = 97$ the calculation is

$$130 = 9^2 + 7^2 \tag{12.2}$$
$$10 = 1^2 + 3^2 + 0^2 \tag{12.3}$$
$$1 = 1^2 + 0^2 \tag{12.4}$$

so 97 is a happy number, but

$$162 = 9^2 + 9^2 \tag{12.5}$$
$$41 = 1^2 + 6^2 + 2^2 \tag{12.6}$$
$$17 = 4^2 + 1^2 \tag{12.7}$$
$$50 = 1^2 + 7^2 \tag{12.8}$$
$$25 = 5^2 + 0^2 \tag{12.9}$$
$$29 = 2^2 + 5^2 \tag{12.10}$$
$$85 = 2^2 + 9^2 \tag{12.11}$$
$$89 = 8^2 + 5^2 \tag{12.12}$$

So 99 is not a happy number and therefore called, admittedly not very inventive, an unhappy number. These unhappy numbers never reach 1 but end in periodical sequences. Carrying on with the example 99 at the state 89

$$145 = 8^2 + 9^2 \tag{12.13}$$
$$42 = 1^2 + 4^2 + 5^2 \tag{12.14}$$
$$20 = 4^2 + 2^2 \tag{12.15}$$
$$4 = 2^2 + 0^2 \tag{12.16}$$
$$16 = 4^2 \tag{12.17}$$
$$37 = 1^2 + 6^2 \tag{12.18}$$
$$58 = 3^2 + 7^2 \tag{12.19}$$
$$89 = 5^2 + 8^2 \tag{12.20}$$

the period is

$$\mathbf{89} \to 42 \to 20 \to 4 \to 16 \to 37 \to 58 \to \mathbf{89} \to 42 \to 20 \to 4 \to \cdots \tag{12.21}$$

The numbers listed in the set $\{4, 16, 20, 37, 42, 58, 89, 145\}$ end all in a periodical sequence when used as a startvalue for formula 12.1. Only the numbers 0 and 1 from the set $\{0, 1, 4, 16, 20, 37, 42, 58, 89, 145\}$ consolidate. The number 0 is excepted from the rule because the only number that consolidates at 0 is 0 itself.

Happy Bases

Happy number exist in other bases, too. There are even happy bases like base-2 and base-4, where all numbers are happy.

12.1.3 Roman Numbers ↔ Arabic Numbers

The main difference between the base-5 Roman system and the base-10 Arabic system is the lack of a symbol for zero in the Roman system.

The implementation is restricted to positive integers between 1 and 6 000

12.1.4 Factorizing

12.2 ℤ

12.3 ℚ

12.3.1 Greatest Common Denominator

12.3.2 Least Common Multiple

12.3.3 Basic Operations

12.4 ℝ

12.4.1 Rounding & Truncating

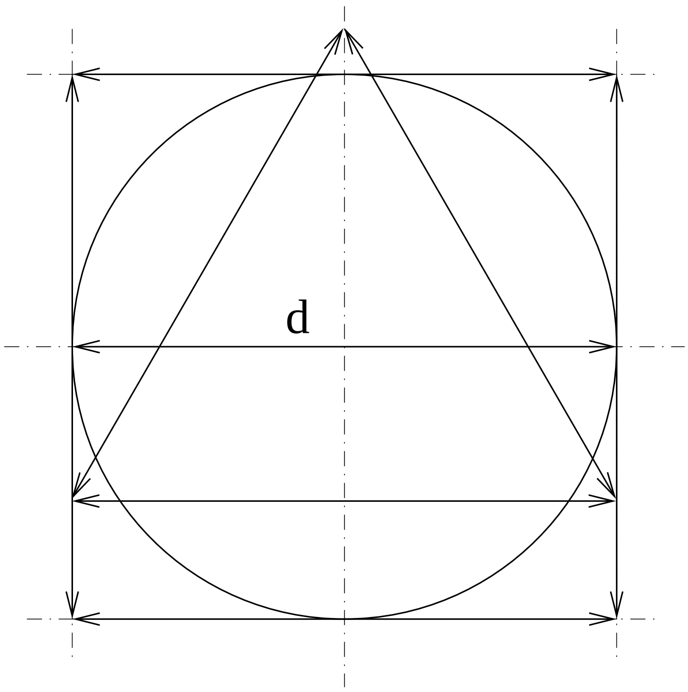

Figure 12.1: $3 < \pi < 4$

It is not possible to represent all numbers in all possible number systems with a finite number of symbols, so it is sometimes necessary to round this number to make it representable in the number system of choice. Let the set $\{0, \ldots, 9, .\}$ be the alphabet Σ with which we want to represent the number π, the ratio between the radius and the circumference of a circle on a plane by the rules of the decimal system:

Truncation

Setting the length of the radius to $r = \frac{1}{2}$ the circumference c of the circle with that radius is $c = 2r\pi = \pi$. That number is irrational and transcendental and can not be represented with that alphabet by concatenation of its elements, so it is not possible to describe both, the length of the circumference of a circle and its radius with that alphabet of symbols and the set of rules of the decimal system. In theory. *In praxi* such a high precision of measurement is not needed and not even possible in the real world. In the physical world every circle is a polygon with a finite number of sides.

Imagine a tourist center in the future that thinks it is needed to advertise the vast space of our universe to draw some tourists with fat purses but way smaller home universes. Some numbers are needed and instead of pulling these dimensions out of a very personal place—the one where the sun rarely shines—a young trainee is appointed to that job. Completely unexperienced in that number-pulling thing he walks to the next computer terminal.

Trainee:	Hi!
Computer:	LEAVE ME ALONE!
Trainee:	What the ...?
Computer:	SPEAK IN FULL SENTENCES, IDIOT!
Trainee:	I want ...
Computer:	WHO CARES?
Trainee:	Well, I ... I ...
Computer:	WHAT!?

As this seems to last a little bit longer, we will leave that fruitless attempts to get some computing time from a quite buggy AI and get our sliding rules out. The periodicals in the rack down in the ante room are a little bit older—"Galileo retracts!", is not really news, is it?—but sombody left last month's issue of "Astronomy Today" with a little table on the backside: "State of the Art in the Twentieth Century". The radius of the universe is given there at 39 billion lightyears. That is a bit off, but the footnote says it was a lower bound, whatever that might have meant at that time. That is, with $9\,460\,730\,472\,580\,800\, m$ in a lightyear and $\ell_P \approx 1.616252 \cdot 10^{-35}\, m$ the diameter of the universe d_U is

$$\begin{aligned} d_U = {} & 45\,657\,297\,058\,955\,063\,938\,049\,264\,594\,877\,531\,474\,052 \\ & 313\,624\,360\,557\,635\,814\,217\,\ell_P \end{aligned} \tag{12.22}$$

If we cannot measure less than one meter any result that is more precise is useless. For a circle with a diameter of one meter no fractional digit of π is needed, for a circle with a diameter of 10 meters two fractional digits are needed but also for $r = 5$ and one digit for $r = 2$, so to get maximal precision the number of fractional digits needed is the number of digits of the next decimal exponent.

$$n_d = \left\lceil \frac{\ln 2r}{\ln 10} \right\rceil + 1 \tag{12.23}$$

So a mere 63 digits of π are needed to calculate the circumference of the universe as exact as physically possible.

$$\begin{aligned}\pi_{63} = {} & 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\,502\,884\,197 \\ & 169\,399\,375\,105\,820\,974\,944\,592\end{aligned} \tag{12.24}$$

This gives the result

$$\begin{aligned}c_U = {} & 143\,436\,629\,023\,180\,101\,816\,772\,847\,242\,591\,648\,920 \\ & 495\,880\,549\,601\,544\,567\,220\,033\,\ell_P\end{aligned} \tag{12.25}$$

A 63-digit number will probably impress a lot of potential tourists and we should not be too mannerless and give the envelope—that with the calculation on its backside—to the sobbing picture of misery in the corner: the poor boy formerly known as the BEST APPRENTICE IN THE UNIVERSE; at least to his mother.

A decimal number x will be truncated n digits to result in the new number x'

$$x' = \begin{cases} x - \operatorname{rem}\left(\frac{x}{10^n}\right) & \text{for } x \in \mathbb{Z} \\ x - \operatorname{rem}\left(\frac{x}{\frac{1}{10^n}}\right) & \text{for } x \in \end{cases} \tag{12.26}$$

where $\operatorname{rem}(\frac{p}{q})$ is the remainder of the division $\frac{p}{q}$. Examples:

$$123000 = \operatorname{trunc}(123456, 3) \tag{12.27}$$

$$654.3 = \operatorname{trunc}(654.321, 2) \tag{12.28}$$

Rounding

After careful lecture of section 12.4.1 the attentive reader has most probably observed the problems of the calculations in that section. In the case of the universe the result with truncated π gives a polygon with a circumference that is obviously smaller than the circumference of the universe and it remains smaller, no matter which length ℓ_P has and after which digit π is truncated[1]. The tourist center is saved from a lawsuit for "Exaggerated Advertising" but that will not help us to resolve the problem of underestimating the real size of the universe—especially if we get paid for the meter.

"Meter" is a good keyword. For example the price of the gasoline the author's car needs to run its motor costs at the day of the writing 1.699 EUR per

[1]Actually, the circumference of the polygon is equal to the circumference of the universe if $\ell_P = 0$ and π is not truncated at all.

liter[2] but one EUR has only 100 cent in it[3]. Now the author, as forgetful as he is, tries to fill up a nearly full tank and the clock of the pump stops at $5.00\ \ell$, sold for the price of... how much?

Truncating 849.5 cent to 849 cent leaves the owner of the filling station with a little loss every time the price is not an integer. A lump sum, you might think but even lump sums add up.

The common solution is to round it—to the nearest integer in most cases but it can be extended to any number. Basically: for any pair of digits d_n, d_{n+1} of a number x there is a rule to change the value of d_n based on the value of d_{n+1} and x to be able to get rid of d_{n+1}.

Restricting the problem to points of the real line, the value of d_n can be increased or decreased resulting in an in- or decrease of the value of x, not necessarily in the same direction. The real line has two directions: towards ∞ and towards $-\infty$. Some rules for rounding add the directions "towards zero" and "away from zero", so we have 4 elements of direction and, in the case of decimal digits, 10 different values for d_{n+1} to build a rule from.

Symmetric Arithmetic The symmetric arithmetic rounding, also known as *round-half-up*, is the most common rounding—teached in primary school and used in accountance.

$$d'_n = \begin{cases} d_n + 1 & \text{if } d_{n+1} \geq 5 \\ d_n & \text{if } d_{n+1} < 5 \end{cases} \tag{12.29}$$

The value of x is ignored here, it acts on the absolute value of x.

$$\operatorname{rnd} 3.5 = 4$$
$$\operatorname{rnd} -3.5 = -4$$

Asymmetric Arithmetic Where the *symmetric* arithmetic rounding ignores the sign of the number to be rounded the *asymmetric* arithmetic rounding takes care of the sign.

$$d'_n = \begin{cases} d_n + 1 & \text{if } (\operatorname{sign}(x) d_{n+1}) \geq 5 \\ d_n & \text{if } (\operatorname{sign}(x) d_{n+1}) < 5 \end{cases} \tag{12.30}$$

Example:

$$\operatorname{rnd} 3.5 = 4$$
$$\operatorname{rnd} -3.5 = -3$$

This rounding is the method for ECMA-scripts `Math.round`.

[2]If you, dear reader, found this lines in the future and, after a longwinded search to find out what "EUR" was, starts to dream about how cheap gasoline was in that days: we were sure this price was offensively exaggerated, harmful to economy and pure and utter greed of the oil companies!

[3]Hence the name "cent"

Bankers method This method, known under many names is used when large sets of numbers have to be rounded like in statistics or, as the name implies, in financial institutes.

$$d'_n = \begin{cases} d_n + 1 & \begin{cases} \text{if } d_{n+1} \geq 6 \\ \text{if } d_{n+1} = 5 \wedge \text{even}(d_n) \end{cases} \\ d_n & \begin{cases} \text{if } d_{n+1} \leq 4 \\ \text{if } d_{n+1} = 5 \wedge \text{odd}(d_n) \end{cases} \end{cases} \tag{12.31}$$

Example:

$$\text{rnd}\, 3.5 = 3$$
$$\text{rnd}\, 4.5 = 5$$

Nearest Integer In this kind of rounding the value of the whole number x, or better: the sign of x is significant. It is a variation of the *Bankers Rounding* above and used in electronic computing. Here x' is the integer part of x and d_n is the digit at the place counting the tenths ($\frac{1}{10}$).

$$x' = \begin{cases} \text{sign}(x)(|x| + 1) & \begin{cases} \text{if } d_n = 5 \wedge \text{odd}(x) \\ \text{if } d_n > 5 \end{cases} \\ x' & \begin{cases} \text{if } d_n = 5 \wedge \text{even}(x) \\ \text{if } d_n < 5 \end{cases} \end{cases} \tag{12.32}$$

Example:

$$\text{rnd}\, 3.5 = 4$$
$$\text{rnd}\, 4.5 = 4$$

Towards Zero Rounding towards zero is the same as truncating: subtracting the fractional part of the absolute value of the number

$$\text{rnd}\, x = \lfloor x \rfloor \tag{12.33}$$

Away from Zero Rounding away from zero is a variation of truncating.

$$\text{rnd}\, x = \lceil x \rceil \tag{12.34}$$

Random Rounding Also known under the name "stochastic rounding".

$$d'_n = \begin{cases} d_n + 1 & \text{if } d_{n+1} > 5 \\ d_n & \text{if } d_{n+1} < 5 \\ d_n + f() & \text{if } d_{n+1} = 5 \end{cases} \tag{12.35}$$

Here $f()$ is a function that returns an element of the set $\{0, 1\}$ with a probability $\mathcal{P} = \frac{1}{2}$. As an example the out put of ten runs of rounding the number 3.5: 3, 3, 3, 4, 4, 3, 3, 4, 4, 4. The random number generator used timings of radiactive decay[4].

[4] A brick, a Geiger-Müller-counter and an atomic clock

12.4.2 Lucas Numbers

The Lucas numbers ([66]) are the same as the Fibonacci numbers with the exception that the sequence starts with 2 instead of 0.

12.5 $\mathbb{C}$

12.5.1 Discrete Fourier Transformation

12.6 Leftovers fitting nowhere else

Glossary

Affinely Extended Real Numbers It is the set $\mathbb{R} \cup \{-\infty, +\infty\}$, a 2-point compactification of the real line. The common notation seems to be $\overline{(\mathbb{R})}$. This set is not a field: for example $+\infty+(-\infty)$ and $-\infty+(+\infty)$ are not defined, so the inverse axiom of the field axiom does not hold here.

Blancmange Recipe by Paul Bocuse in [13]

250 g	almonds, peeled
2	bitter almonds
2/5 ℓ	fresh cream
100 g	sugar
1/2 tbl	vanilla sugar
7–8 leaves	gelatine
1/10 ℓ	thick cream (crème fraîche)
	some extra sugar

Soak the almonds[5] for at least 30 minutes in cold water, let them dripp off and pound them in a mortar until it is a fine paste. Add some drips of water after the first minutes of pounding and later the cream in small quantities[6]. Put the paste in a clean cloth and wring it out over a bowl. Solve the sugar, the vanilla sugar in that almond milk. Warm a small quantity of that milk up to solve the gelatine in it. Add that mix to the rest of the almond milk, let it cool off until the gelatine starts to set. Whip the rest of the cream[7] together with a bit of sugar and add. Fill into a mold or dish, cool of in the refrigerator (at least 2 hours), unmold and serve.

Blancmange Function The Blancmange-function is a continuous function but nowhere differentiable. It is also called the "Takagi fractal curve" by [110]

[5]if they are not peeled: cook about five minutes, quench and press them between thumb and forefinger to slip the peel of. It is quite a mess, the author suggests to buy peeled almonds, they are not very expensive.

[6]Yes, the whole process might last about 15–20 minutes!

[7] It is not in the recipe which of the two sorts of cream is meant to be whipped, but crème fraîche is nearly impossible to whip

because of [103]. The term Blancmange function seems more common, for example in [104].

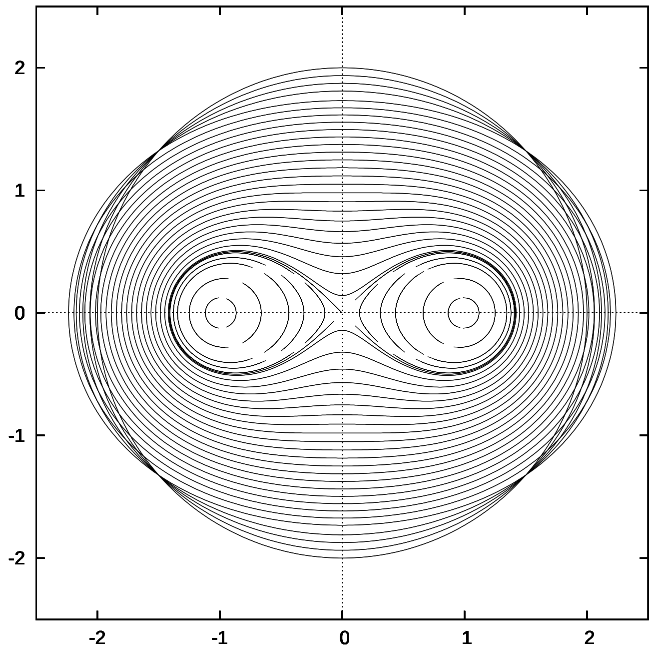

Figure 12.2: Cassini Ovales

Cassini Ovals The Cassini ovales are toric sections by a plane.The `Gnu_plot` script for figure 12.2 was[8]

```
set terminal postscript eps enhanced solid "Helvetica" 14
set output "cassini.eps";
set parametric;
set size ratio -1 1,1;
set xrange[-2.5:2.5];set yrange[-2.5:2.5];
set trange[-2*pi:2*pi];
set nokey;
set zeroaxis;
set samples 1000;
f(a,b,x) = a**2*cos(2*x)+ (sqrt(b**4 -a**2 * sin(2*x)**2));
g(a,b,x) = a**2*cos(2*x)- (sqrt(b**4 -a**2 * sin(2*x)**2));
```

[8]The gaps in parts of the curve are remnants of rounding errors caused by an insufficient number of samples.

```
h(a,b,x) = f(a,b,x);
y(s,a,b) = sin(s)*sqrt(h(a,b,s));
x(s,a,b) = cos(s)*sqrt(h(a,b,s));
set multiplot
plot x(t,1,0.5),y(t,1,0.5);
h(a,b,x) = g(a,b,x);
plot x(t,1,0.5),y(t,1,0.5);
plot x(t,1,0.75),y(t,1,0.75);
h(a,b,x) = f(a,b,x);
plot x(t,1,0.75),y(t,1,0.75);
plot x(t,1,0.9),y(t,1,0.9);
h(a,b,x) = g(a,b,x);
plot x(t,1,0.9),y(t,1,0.9);
plot x(t,1,0.95),y(t,1,0.95);
h(a,b,x) = f(a,b,x);
plot x(t,1,0.95),y(t,1,0.95);
plot x(t,1,0.99),y(t,1,0.99);
h(a,b,x) = g(a,b,x);
plot x(t,1,0.99),y(t,1,0.99);
h(a,b,x) = f(a,b,x);
plot x(t,1,1),y(t,1,1);
plot x(t,1,1.01),y(t,1,1.01);
plot x(t,1,1.05),y(t,1,1.05);
plot x(t,1,1.1),y(t,1,1.1);
plot x(t,1,1.15),y(t,1,1.15);
plot x(t,1,1.2),y(t,1,1.2);
plot x(t,1,1.25),y(t,1,1.25);
plot x(t,1,1.3),y(t,1,1.3);
plot x(t,1,1.35),y(t,1,1.35);
plot x(t,1,1.4),y(t,1,1.4);
plot x(t,1,1.45),y(t,1,1.45);
plot x(t,1,1.5),y(t,1,1.5);
plot x(t,1,1.55),y(t,1,1.55);
plot x(t,1,1.6),y(t,1,1.6);
plot x(t,1,1.65),y(t,1,1.65);
plot x(t,1,1.7),y(t,1,1.7);
plot x(t,1,1.75),y(t,1,1.75);
plot x(t,1,1.8),y(t,1,1.8);
plot x(t,1,1.85),y(t,1,1.85);
plot x(t,1,1.9),y(t,1,1.9);
plot x(t,1,1.95),y(t,1,1.95);
plot x(t,1,2),y(t,1,2);
plot x(t,.85,2),y(t,.85,2);
plot x(t,.7,2),y(t,.7,2);
plot x(t,.5,2),y(t,.5,2);
plot x(t,0,2),y(t,0,2);
set nomultiplot
```

Chaos Not defined. See for example in [118] for a list of some reasons, some trials to define it nevertheless, and a long list of references.

Continued Fraction A continued fraction is one form of a representation of a real number x with a sequence of integers. Two different notations are common:

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}} \tag{12.36}$$

$$= a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}\cdots \tag{12.37}$$

The notation used for example for the entries of [98][9] is a list of the variables a_n with the variables b_n set to one[10].

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \qquad \text{with } a_1, a_2, a_3, \ldots > 0 \tag{12.38}$$

For a randomly choosen example from [98]

$$[0; 8, 9, 1, 149083, 1, 1, 1, 4, 1, 1, 1, 3, 4, 1, 1, 1, 15, \ldots] \tag{12.39}$$

the continued fraction is

$$x = 0 + \cfrac{1}{8 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{149083 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}}}} \tag{12.40}$$

In a notation suitable for ECMA-script (just C&P)

$$0 + 1/(8 + 1/(9 + 1/(1 + 1/(149083 + 1/(1 + 1/(1 + 1/(1 + 1/(4 + 1/(1 + 1/(1 + 1/(1 + 1/(3 + 1/(4 + 1/(1 + 1/(1 + 1/(1 + 1/(15 + 1)))))))))))))))))) \tag{12.41}$$

[9] The notation most seen in today's literature is of the form $[a_0; a_1, a_2, \ldots, a_n]$

[10] The indexing starts not always at a_0 but from a_1 instead. That is not always made sufficiently clear in the accompanying text, so a bit of extra care should be taken in these cases.

Running gives 0.12345678910111, which is Champernowns constant to 13 digits precision. The real precision of the continued fraction 12.41 is 185 digits, the twenty digits a_{180}–a_{200} are 9596979**9**000102030405

Champernowns constant had been choosen as an example because it is extremely easy to check the correctness of the result and because the continued fraction expansion has a weird pattern[11] and is very hard to obtain.

Getting a continued fraction expansion is quite simple, at least in theory:

Example 12.6.1. Some warm-up: we are looking for the continued fraction expansion of $\sqrt{2}$. The rules are simple:

$$\rho_0 = \rho \tag{12.42}$$
$$a_n = \lfloor \rho_n \rfloor \tag{12.43}$$
$$\rho_{n+1} = \frac{1}{\rho_n - a_n} \tag{12.44}$$

$$a_0 = \left\lfloor \sqrt{2} \right\rfloor = 1 \tag{12.45}$$
$$\rho_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1 \tag{12.46}$$

$$a_1 = \left\lfloor \sqrt{2}+1 \right\rfloor = 2 \tag{12.47}$$
$$\rho_2 = \frac{1}{\sqrt{2}+1-2} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1 \tag{12.48}$$

$$a_2 = \left\lfloor \sqrt{2}+1 \right\rfloor = 2 \tag{12.49}$$

We can stop here because it is obvious that the partial quotients are always 2. So, with the overline marking the periodical sequence, the continued fraction expansion of $\sqrt{2}$ is $[1;\overline{2}]$.

[11] The next partial quotient a_{18} is $4.5754 * 10^{165}$

Computing the expansion with ten partial quotients (again fit for a simple C&P)

$$\begin{aligned} &1+1/(2+1/(2+1/(2+1/(2+1/(2+1/(2+ \\ &\quad 1/(2+1/(2+1/(2+1/(2))))))))) \\ &= \frac{8119}{5741} \end{aligned} \tag{12.50}$$

$$\approx 1.414213551 \tag{12.51}$$

$$\sqrt{2} \approx 1.414213562 \tag{12.52}$$

Ten partial quotients give seven decimal digits precision, not a very high converging speed but at least easy to compute.

The error is approximately the multiplicative inverse of the square of the last denominator. In this case

$$\frac{1}{5741^2} \approx 3.034 \times 10^{-08} \tag{12.53}$$

In the case of Champernowns constant

$$\frac{1}{490050000000^2} \approx 4.164 \times 10^{-24} \tag{12.54}$$

which is a bit off. Including partial quotient a_{18} gives an approxximation of 1.989×10^{-355}, which even more off. The arithmetic mean of the exponents is 189.5 which seems more passable in relation to the real precision of 185 decimal digits.

Dirichlet β-function The Dirichlet β-function is defined by the sum

$$\beta(x) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-x} \tag{12.55}$$

Equivalent equations include

$$\beta(x) = 2^{-x}\Phi\left(-1, x, \frac{1}{2}\right) \tag{12.56}$$

$$= \frac{1}{4^x}\left(\zeta\left(x, \frac{1}{4}\right) - \zeta\left(x, \frac{3}{4}\right)\right) \tag{12.57}$$

Where $\zeta(x, a)$ is Hurwitz' ζ-function and $\Phi(z, s, a)$ is the Lerch transcendent.

Extending the β-function to the complex plane can be done by

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{1}{2}\pi z \Gamma(z)\beta(z)\right) \tag{12.58}$$

where $\Gamma(z)$ is the Γ-function.

Field A field is unsuprisingly defined by its axioms

Name	Addition	Multiplication
Commutativity	$a + b = b + a$	$ab = ba$
Associativity	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Distributivity	$a(b + c) = ab + ac$	$(a + b)c = ac + bc$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = a = 1 \cdot a$
Inverse	$a + (-a) = 0 = (-a) + a$	$a\frac{1}{a} = 1 = \frac{1}{a}a$ with $a \neq 0$

Iverson Bracket The Iverson bracket ([48]) is a notation for a numerical equivalence of a Boolean result. Given a mathematical statement S, then the definition and notation is

$$[S] = \begin{cases} 0 & \text{if } S \text{ is false} \\ 1 & \text{if } S \text{ is true} \end{cases} \tag{12.59}$$

The notation with the brackets $[x]$ is sometimes used for the floor function and for rounding to the nearest integer, too. The former is rarely seen, the notation $\lfloor x \rfloor$ is probably the most common in the time of this writing. The latter used should be made clear by the accompanying text or, if that is not possible, with the named operator $\mathrm{nint}(x)$, which seems to be the most common abbreviation for that kind of rounding.

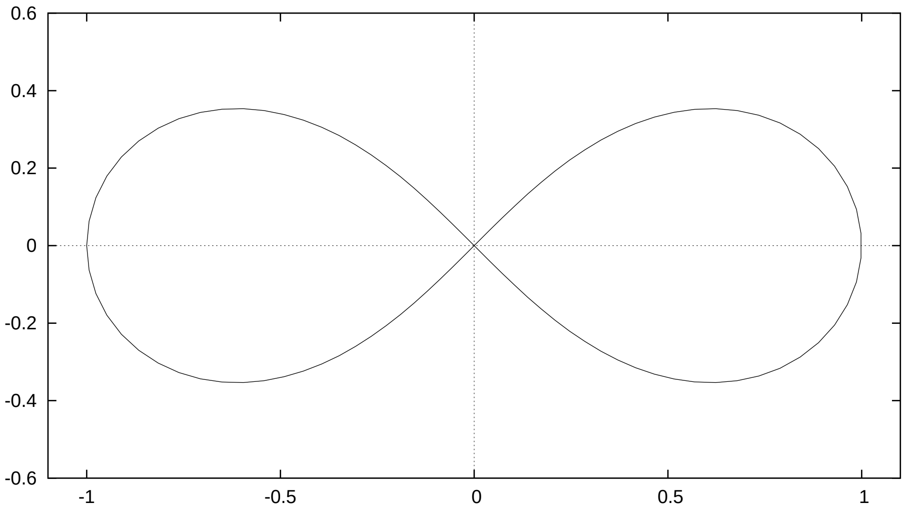

Figure 12.3: Lemniscate

Lemniskate The lemniscate or Bernoulli's lemniscate[12] is a special form of

[12] Bernoulli described that curve in [9]

Cassini ovals with the Cartesian equation

$$\left(x^2+y^2\right)=2c^2\left(x^2-y^2\right) \tag{12.60}$$

He called it *lemnicus* the Latin form of the greek $\lambda\eta\mu\nu\grave{\iota}\sigma\kappa o\varsigma$ with the meanings according to [65]

1. *woolen fillet,ribbon* by which chaplets were fastened
2. *ribbon* attached to bird's feet
3. surgical bandage
4. pledget

The Latin translation of $\lambda\eta\mu\nu\grave{\iota}\sigma\kappa o\varsigma$ would be *taenia* ([64]). Bernoulli's reasons to choose a latin transliteration of a greek word is unknown to the author.

With the parametric equations for a lemniscate with width a

$$x=\frac{a\cos t}{1+\sin^2 t} \tag{12.61}$$

$$y=\frac{a\sin t\cos t}{1+\sin^2 t} \tag{12.62}$$

a small `Gnuplot` script produced the figure 12.3

```
set terminal postscript eps enhanced solid "Helvetica"
    14
set output "lemniscate.eps";
set parametric;
set nokey;
set zeroaxis;
set size ratio -1 1,1
set trange[-pi:pi];
set xrange[-1.1:1.1];set yrange[-.6:.6];
x(s,a)=(a*cos(s))/(1+(sin(s)**2));
y(s,a)=(a*sin(s)*cos(s))/(1+(sin(s)**2));
plot x(t,1),y(t,1);
```

Logarithmic Spiral The logarithmic spiral has the polar equation

$$r=ae^{b\theta} \tag{12.63}$$

with r the distance from the origin and θ the angle from the x-axis. The two variables a and b are arbitray constants to describe form of the spiral. The parametric equations are

$$x=a\cos\theta e^{b\theta} \tag{12.64}$$

$$y=a\sin\theta e^{b\theta} \tag{12.65}$$

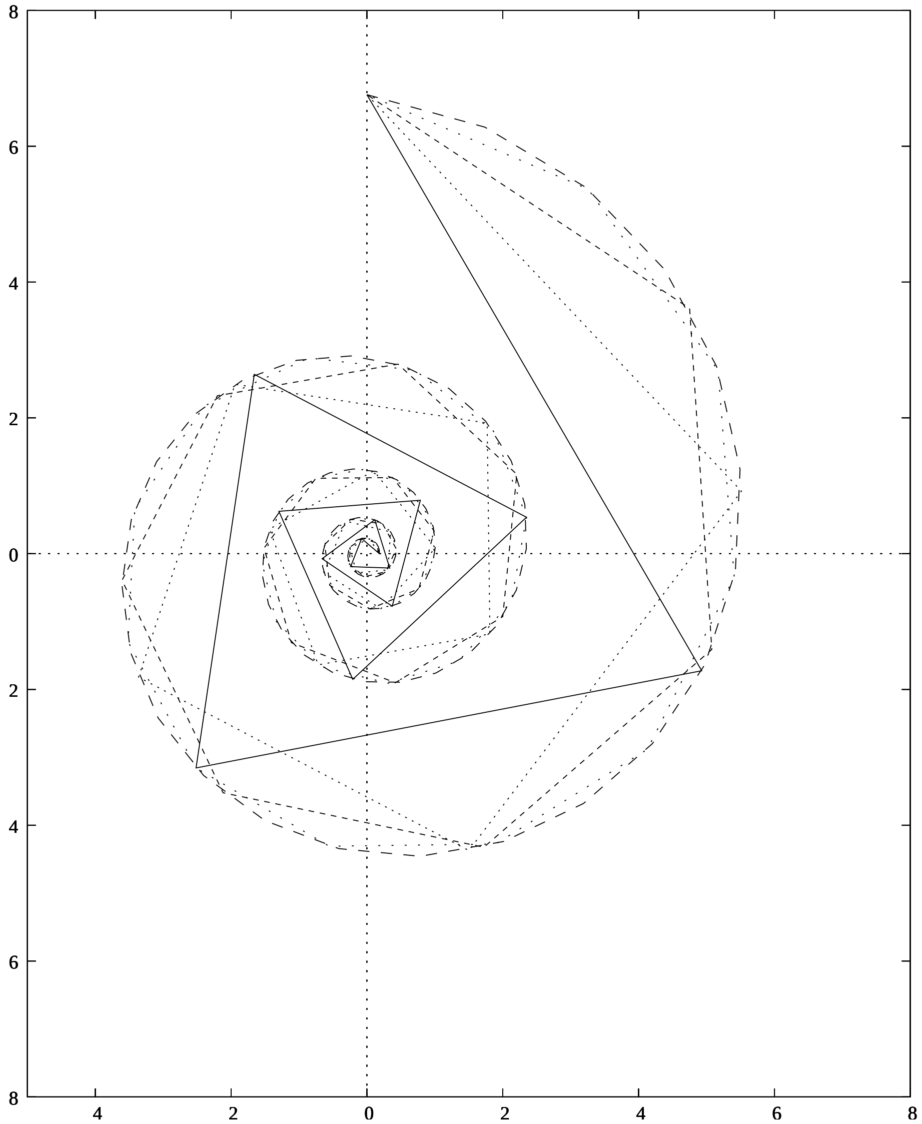

Figure 12.4: Logarithmic Spirals with increasing number of points calulated

The picture in figure 12.4 has been plotted with `Gnuplot` with the polar equation and $ae = \ln \pi$ resulting in the script:

```
set polar
set zeroaxis
set size ratio -1
set multiplot
plot [-4*pi:4.5*pi] [-5:8] [-8:8] log(pi)**t notitle
```

```
    with lines linetype 1
set samples 20
plot [-4*pi:4.5*pi] [-5:8] [-8:8] log(pi)**t notitle
    with lines linetype 2
set samples 30
plot [-4*pi:4.5*pi] [-5:8] [-8:8] log(pi)**t notitle
    with lines linetype 3
set samples 50
plot [-4*pi:4.5*pi] [-5:8] [-8:8] log(pi)**t notitle
    with lines linetype 4
set samples 100
plot [-4*pi:4.5*pi] [-5:8] [-8:8] log(pi)**t notitle
    with lines linetype 5
set nomultiplot
```

Möbius function The Möbius function [73]

$$\mu(n) \equiv \begin{cases} 0 & \text{if } n \text{ has one or more \textit{repeated} prime factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ \textit{distinct} primes} \end{cases} \tag{12.66}$$

If $\mu(n) \neq 0$ the number n is squarefree.

The symbol $\mu(x)$ is due to Mertens [72]

***q*-expansion** For any real number x such that $0 \leq x \leq \frac{q\lfloor q\rfloor}{q-1}$ exists the serial expansion

$$x = \sum_{n=0}^{\infty} a_n q^{-1} \tag{12.67}$$

with $n \geq 0$, $0 \leq a_n \leq \lfloor q \rfloor$ ($a_n \in \mathbf{S} \wedge \mathbb{N} \subseteq \mathbf{S}$) and can be found with the greedy algorithm ([2]).

Quotient Beneath the common interpretation as the ratio $x = \frac{p}{q}$ with $q \neq 0$ the term "quotation" might describe integer division, mostly notated with the symbol \ not to be confused with the sign used to annotate set subtraction.

$$p \setminus q = \left\lfloor \frac{p}{q} \right\rfloor \tag{12.68}$$

Squarefree A number is called *squarefree*[13] if its prime decomposition contains no repeated factors. By convention the number 1 is included in this club.

[13]The German *quadratfrei*, a verbatim translation, is also used

n	factors	squarefree
1	1	yes
2	2	yes
3	3	yes
4	$2 \cdot 2$	**no**
5	5	yes
6	$2 \cdot 3$	yes
7	7	yes
8	$2 \cdot 2 \cdot 2$	**no**
9	$3 \cdot 3$	**no**
10	$2 \cdot 5$	yes

The asymptotic density δ of such numbers n

$$\delta = \frac{1}{\zeta(2)} \qquad \text{for } n \in \mathbb{N} \tag{12.69}$$

Here, as elsewhere, $\zeta(n)$ is Riemann's ζ-function.

The equivalent for the integers on the complex plane, the Gaussian integers,

$$\delta = \frac{6}{\pi^2 K} \tag{12.70}$$

K is Catalan's constant

Stirling Number of the Second Kind A Stirling number of the second kind $S(m,k)$ describes the number of collections $\mathfrak{C}_j$ of same-sized non-empty subsets $\mathbf{L}_n$ of a power set $\mathcal{P}(S)$. More formaly:

$$\#\mathbf{S} > 0 \tag{12.71}$$
$$\mathfrak{C}_j \subseteq \{\mathcal{P}(S) \setminus \varnothing\} \tag{12.72}$$
$$\mathbf{L}_n \subseteq \mathfrak{C}_j \tag{12.73}$$
$$\mathfrak{C}_j = \{\{\mathbf{L}_0, \mathbf{L}_1, \ldots, \mathbf{L}_n\} : \#\mathbf{L}_n = \#\mathbf{L}_{n+1} \wedge \mathbf{L}_n \neq \varnothing\} \tag{12.74}$$
$$m = \#\mathbf{S} \tag{12.75}$$
$$k = \#\mathbf{L}_n \tag{12.76}$$
$$S(m,k) = \#\{\mathfrak{C}_0, \mathfrak{C}_1, \ldots, \mathfrak{C}_j\} \tag{12.77}$$

Some examples with the set $\mathbf{S} = \{2,3,4\}$:

$$\mathcal{P}(S) = \{\varnothing, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\}\} \tag{12.78}$$
$$S(3,1) = \#\{\{\{2\}, \{3\}, \{4\}\}\} = 1 \tag{12.79}$$
$$S(3,2) = \#\{\{2,3\}, \{2,4\}, \{3,4\}\} = 3 \tag{12.80}$$
$$S(3,3) = \#\{\{2,3,4\}\} = 1 \tag{12.81}$$

Thue-Morse Sequence The Thue-Morse sequence can can be easily generated with the substitution map

$$0 \to 01 \tag{12.82}$$
$$1 \to 10 \tag{12.83}$$

Interpreted as the concatenation of binary digits gives the Thue-Morse constant.

Wronskian The Wronskian, or Wronski-determinant, is defined for a set of functions differentiable over an interval I by

$$\mathrm{W}(f_1, f_2, \ldots, f_n)(x) \equiv \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \tag{12.84}$$

If the Wronskian is non-zero somewhere on the interval I the functions are said to be linearly independent on the interval.

Example:
Given the interval $\mathbb{R}$ and the three functions $f_1(x) = 1$, $f_2 = x$ and $f_3 = x^3$ which are luckily defined for every $x \in \mathbb{R}$ and are easily differentiable without the help of a CAS[14]. The Wronskian is in that case

$$\mathrm{W}(f_1, f_2, f_3)(x) \equiv \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} \tag{12.85}$$

The determinant of a 3×3 square matrix can be calculated without much headache by strictly following Sarrus' rule

$$\begin{aligned} &(a_{11}a_{22}a_{33}) \\ +&(a_{12}a_{23}a_{31}) \\ +&(a_{13}a_{21}a_{32}) \\ -&(a_{13}a_{22}a_{31}) \\ -&(a_{12}a_{21}a_{33}) \\ -&(a_{11}a_{23}a_{32}) \end{aligned} \tag{12.86}$$

Filling the blanks and calculating[15] the determinant comes out as $6x$ which is evidently non-zero on the given interval, the real line, so the functions are linearly independent.

[14]Just in case: the derivative of a constant is 0(zero), the derivative of a power is $cx^n = ncx^{n-1}$ (with c constant, the coeficient) which means for an exponent of 1(one): $x^1 = 1 \cdot x^{(1-1)} = 1 \cdot x^0 = 1 \cdot 1 = 1$

[15]Which has been left as an exercise to the reader, of course.

For another example consider the three functions $f_1(x) = 1$, $f_2(x) = x^2$ and $f_3(x) = 3 + 2x^2$ over the interval $\mathbb{R}$. The Wronskian is

$$\mathrm{W}(f_1, f_2, f_3)(x) \equiv \begin{vmatrix} 1 & x^2 & 3+2x^2 \\ 0 & 2x & 4x \\ 0 & 2 & 4 \end{vmatrix} = 0 \tag{12.87}$$

This result is a hint, that these functions might be linearly dependent, but it is not mandatory! We can see by further inspection that $f_3(x) = 3f_1(x) + 2f_2(x)$. That rule has it's continuation in the relation of the column vectors $c_3 = 3c_1 + 2c_2$ so these three functions are in fact linearly dependent.

It is not sufficient to assume that the functions are linearly independent from the value of the Wronski-determinant alone if that determinant is zero everywhere. A counter-example is easily constructed with the two functions $f_1(x) = x^2\,|x|$ and $f_2(x) = x^3$ over the interval $(-1, 1) \in \mathbb{R}$[16]. f_1 is differentiable at $x = 0$ and

$$f_1(x) = \begin{cases} -3x^2 & \text{if } x < 0 \\ 3x^2 & \text{if } x \geq 0 \end{cases} \tag{12.88}$$

The Wronskians are then

$$\mathrm{W}(f_1, f_2)(x) \equiv \begin{cases} \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0 & \text{if } x \geq 0 \\ \begin{vmatrix} -x^3 & x^3 \\ -3x^2 & 3x^2 \end{vmatrix} = 0 & \text{if } x < 0 \end{cases} \tag{12.89}$$

So for all $x \in (-1, 1)$ the Wronski-determinant is $\mathrm{W}(f_1, f_2) = 0$. Nevertheless it is obvious that these two functions are linearly independant on $(-1, 1)$, moreso on any interval $\mathbf{I} = (a, b)$ with $\{0\} \in \mathbf{I}$.

[16]Note that contrary to most textbooks the derivative of the function $f(x) = |x|$ does not exist. Only if it is restricted to the real line derivatives exist for all points except 0

Glossary

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Qualem commendes, etiam atque etiam aspice, ne mox
incutiant aliena tibi peccata pudorem.
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Copy from one, it's plagiarism; copy from two, it's research.
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Pooh!