CHOW RINGS OF MATROIDS AND ATOMISTIC LATTICES

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Abstract. After Feichner and Yuzvinsky introduced the Chow ring associated to ranked atomistic lattices in 2003, little study of them was made before Adiprisito, Huh, and Katz used them to resolve the long-standing Heron-Rota-Walsh conjecture, proving along the way that the Chow rings of geometric lattices satisfy versions of Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations. Here, we seek to remedy the lack of basic knowledge about the Chow rings of atomic lattices by providing some general techniques for computing their Hilbert series, by making detailed study a few fundamental examples, and by providing a number of interesting conjectures based on our observations. Using the incidence algebra, we give a compact formula for the Hilbert series of Chow rings associated to both ranked atomic lattices and products of them. In a special case, we define a generalization of the Hilbert series and give a formula for the Hilbert series of the product in terms of differential operators.

In addition to general techniques, we study in detail the Chow rings associated to the lattices of flats of uniform and linear matroids. We show that the Hilbert series of uniform and linear matroids take forms of combinatorial interest; in particular, the Hilbert series of the linear matroid associated to an n-dimensional vector space over a finite field $\mathbb{F}_q$ can be described in terms of the q-Eulerian polynomial defined by Shareshian and Wachs in [SW10], and the Hilbert series of the Chow ring of a uniform matroid can be described in terms of elementary statistics on $\mathfrak{S}_n$. We also compute the Charney-Davis quantities of the rings, which come out to linear combinations of the secant numbers in the uniform case, and to a linear combination of the q-secant numbers of Foata and Han in the case of the linear matroids. Finally, we assert that Poincaré duality holds for the Chow rings a a slightly more general class of ranked atomistic lattices than those studied by Adiprasito, Huh, and Katz, and make a conjecture about the class of ranked atomic lattices for which Poincaré duality holds.

Contents

1. Introduction

Since Stanley's 1975 proof of the upper bound conjecture for simplicial spheres via the Stanley-Reisner ring, the study of graded rings associated to combinatorial objects has yielded many deep insights into combinatorics (and vice versa). An important example of this pattern is found in Stanley's use of the Chow ring of a toric variety and the geometric hard Lefschetz theorem to establish (one direction of) the famed g-theorem, which characterizes the numbers of faces of a simplicial polytope.

More recently, Feichtner and Yuzvinsky defined another useful graded ring, the *Chow ring* of an atomistic lattice, and provided a Gröbner basis for the ring in [FY04]. The power of the construction of [FY04] was demonstrated by Adiprasito, Huh, and Katz, who applied a slight variation of it to the lattice of flats of a matroid in order to resolve the following long-standing conjectures of Heron, Rota, and Walsh

Conjecture 1.1 (Resolved by [AHK15])**.** *Let M be a matroid of rank $r+1$ with characteristic polynomial $\chi_M(\lambda)$. Let w_k be the absolute value of the coefficient of λ^{r-k+1} in $\chi_M(\lambda)$. Then the sequence $(w_k)_k$ is log-concave.*

Conjecture 1.2 (Resolved by [AHK15])**.** *If M is as above, and f_k is the number of independent sets of cardinality k then the sequence $(f_k)_k$ is log-concave.*

In their paper, [AHK15] also use techniques inspired by Peter McMullen's combinatorial proof of the g-theorem [McM93] to prove that Chow rings arising from geometric lattices satisfy Poincaré duality and versions of the hard Lefschetz theorem and the Hodge-Riemann relations.

In the course of proving Conjectures 1.1 and 1.2, Adiprasito, Huh, and Katz show that Chow rings associated to geometric lattices (that is, to the lattice of flats of a matroid) satisfy versions of Poincaré duality, the hard Lefschetz theorem, and the Minkowski-Riemann relations. However, little more is known about basic properties of the Chow rings of geometric lattices, let along about those of more general lattices. Accordingly, our work here is primarily concerned with one of the most basic invariants of a Chow ring: its Hilbert series.

1.1. **Organization.** This report is organized as follows. In the remainder of this section, we summarize some of our main results; full definitions of all the objects involved are given in Section 2. In Section 3, we discuss a mild generalization of the Poincaré duality theorem of [AHK15]. Next, in Section 4, we provide a few methods for calculating the Hilbert series of a Chow ring. Section 5 contains explicit determinations of the Hilbert series of some example of special interest, and in Section 6 we present conjectures and ideas for further work.

1.2. **Summary of main results.** The results in this section are numbered according to their numbers in the body of the paper proper. We define a set of lattices that we call *nice* lattices, and state the following:

Theorem 1.3. *The Chow ring of a nice lattice exhibits Poincaré duality.*

Next, we provide three results allowing one to more easily calculate the Hilbert series of a Chow ring of a graded, atomic lattice L. Our first result in this direction uses the incidence algebra $(\mathbb{Q}[t])[L]$ of the ranked atomistic lattice L with coefficients in $\mathbb{Q}[t]$ to simplify a formula of [FY04]. Define the elements $\zeta_L, \eta_L, \gamma_L$ by

$$\begin{aligned}
\zeta_L(x,y) &= 1 \\
\alpha_L(x,y) &= \operatorname{rank} y - \operatorname{rank} x \\
\eta_L(x,y) &= \begin{cases} \frac{t - t^{\operatorname{rank} y - \operatorname{rank} x - 1}}{1-t} & \operatorname{rank} y - \operatorname{rank} x - 1 \geq 1 \\ 0 & \operatorname{rank} y - \operatorname{rank} x < 1 \end{cases} \\
\gamma_L &= \zeta_L (1 - \eta_L)^{-1}.
\end{aligned}$$

Proposition 1.4. *For $\zeta_L, \eta_L, \gamma_L$ as above, we have*

$$\begin{aligned}
\alpha_L &= (1-t)(1-\eta_L) + t\zeta_L \\
\eta_L &= \frac{1}{t-1}(\alpha_L - t\zeta_L) \\
\gamma_L(x,y) &= H(A([x,y]),t)
\end{aligned}$$

where $[x,y] = \{z \in L : x \leq z \leq y\}$.

The incidence algebra also allows us to derive a formula for the Hilbert series of the Chow ring of a product of lattices $L \times K$ in terms of the Hilbert series of the Chow rings of L and K. In particular, with the notation above, we show

Proposition 1.5.

$$\gamma_{L\times K} = (\gamma_L \otimes \gamma_K)(1 - t(1-\gamma_L) \otimes (t - \gamma_K))^{-1}$$

Next, we make use of the results of [AHK15] in order to derive a recurrence relation for the Hilbert series of the Chow ring of a sufficiently symmetric lattice L (examples include boolean lattices, partition lattices, and lattices of subspaces) in terms of the Hilbert series of intervals of the form $[z, \top]$. For such a lattice of rank $r+1$, there are elements $z_2, \ldots, z_r \in L$ such that

Proposition 1.6.

$$H(L,t) = [r+1]_t + t\sum_{i=2}^{r} |L_i|\,[i-1]_t\, H([z_i, \top], t).$$

Finally, we use differential operators to compute the Hilbert series of the Chow rings of some special products.

After presenting general methods for computing the Hilbert series, we make use of them to establish properties of Chow rings associated to (the lattices of flats of) two matroids of special importance: uniform matroids and linear matroids over finite fields. The Hilbert series and Charney-Davis quantities of these Chow rings have interpretations in term of elementary and Mahonian statistics on permutations.

Corollary 1.7 (Later Corollary 5.6). *The Hilbert function of the Chow ring of the uniform matroid of rank r on n elements $A(U_{n,r})$ is given explicitly by*

$$\dim A(U_{n,r})_k = \#\left\{\sigma \in \mathfrak{S}_n \,:\, \operatorname{exc}(\sigma) = k\right\} - \sum_{i=r}^{n-1} \#\left\{\sigma \in E_{n,n-i} \,:\, \operatorname{exc}(\sigma) = i - k\right\}$$

where $E_{n,k}$ is the set of permutations in $\mathfrak{S}_n$ with at least k fixed points and $\operatorname{exc}(\sigma)$ *is the number of excedances of σ; that is, the number of $i \in [n]$ such that $\sigma(i) > i$.*

Theorem 1.8 (Later Theorem 5.8). *For even r, the Charney-Davis quantity for the uniform matroid, $U_{n,r}$ of rank r on $[n]$ is 0. For odd r, the Charney-Davis quantity for $U_{n,r}$ is*

$$\sum_{k=0}^{\frac{r-1}{2}} \binom{n}{2k} E_{2k}$$

where $E_{2\ell}$ is the ℓth secant number.

Let $M_r(\mathbb{F}_q^n)$ be the matroid of subspaces of $\mathbb{F}_q^n$ with rank at most r. The Hilbert series of the Chow ring of the lattice of flats of $M_r(\mathbb{F}_q^n)$ is a q-analogue of the Hilbert series of the corresponding uniform matroid. The full rank case is particularly pleasing; its Hilbert series is the q-Eulerian polynomial defined by Shaeshian and Wachs in [SW10] and [SW07].

Corollary 1.9 (Later Corollary 5.26). *The Hilbert series of $A\big(M_r(\mathbb{F}_q^n)\big)$ is given by*

$$H\big(A(M_r(\mathbb{F}_q^n)), t\big) = \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{maj}(\sigma) - \operatorname{exc}(\sigma)} t^{\operatorname{exc}(\sigma)} - \sum_{j=0}^{n-1} \sum_{\sigma \in E_{n,n-r}} q^{\operatorname{maj}(\sigma) - \operatorname{exc}(\sigma)} t^{r - \operatorname{exc}(\sigma)}$$

The Charney-Davis quantities also have a nice form.

Theorem 1.10 (Later Theorem 5.28). *The Charney-Davis quantity of the chow ring $A\big(M_r(\mathbb{F}_q^n)\big)$ is*

$$1 + [n]_q! \sum_{a=1}^{k} \frac{(-1)^a}{[n-2a]_q!} \Delta_a$$

for Δ_a the determinant

$$\Delta_a = \det \begin{pmatrix} \frac{1}{[2]_q!} & 1 & 0 & \cdots & 0 \\ \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \frac{1}{[2a-8]_q!} & \cdots & 1 \\ \frac{1}{[2a-2]_q!} & \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \cdots & \frac{1}{[2]_q!} \end{pmatrix}.$$

The Charney-Davis quantities of $M_r(\mathbb{F}_q^n)$ can be expressed as a linear combination of the q-secant numbers defined by Foata and Han in [FH10]. As corollaries of the theorems above, we give new proofs of recurrences for the Eulerian and q-Eulerian numbers, provide another interpretation of the q-secant and q-tangent numbers, and give an alternate proof of unimodality and symmetry of the Eulerian and q-Eulerian numbers.

After concluding our study of uniform and linear matroids, we offer a pair of conjectures. The first is our best guess as to the family of lattices whose Chow rings exhibit Poincaré duality, and the second relates the Hilbert series of uniform matroids to the h-vector of the order complex of a truncated boolean poset.

2. Definitions and Background

In this section, we will cover most of the combinatorics and commutative algebra background used later in the paper. Readers with a background in commutative algebra and combinatorics may wish to skim the next few sections to pick up our notation, and then skip to Section 2.6, where we define the Chow ring associated to an atomic lattice.

2.1. **Posets.** In this section, we review some facts about partially ordered sets that will be relevant in our later writing. In particular, we define atomistic lattices and the incidence algebra. For more background, see [Sta12].

Definition 2.1. A *partially ordered set* or *poset* is a pair $P = (E, \leq)$, where E is a set and $\leq$ is a *partial order relation* on E satisfying

(1) If $a \in E$, then $a \leq a$.
(2) If $a, b \in E$ and both $a \leq b$ and $b \leq a$ then $a = b$.
(3) If $a, b, c \in E$ and if both $a \leq b$ and $b \leq c$, then $a \leq c$.

We call an element $p \in P$ *maximal* if there does not exist any $q \neq p \in P$ such that $q \geq p$. Likewise, we all p *minimal* if there is no $q \neq p$ such that $q \leq p$. If P has a unique maximal (resp. unique minimal) element, we denote it by $\top$ (resp. $\perp$). We call P *bounded* if it has both a unique minimal and unique maximal element. An element $p \in P$ is said to *cover* a different element $q \in P$ if $p \geq q$ and there is no $p' \in P$ such that $p \geq p' \geq q$. When this is the case, we will write $p \succ q$.

Two elements $p, q \in P$ are *comparable* if either $p \leq q$ or $q \leq p$; otherwise, they are *incomparable*. A subset $S \subseteq P$ is *totally ordered* if every pair of elements of S are comparable. A *chain* of P is a totally ordered subset of P. An *antichain* is a subset in which every pair of elements is incomparable. For $S \subseteq P$, the *order ideal* $P_{\leq S}$ defined by S is the set of all elements $q \in P$ such that there exists $s \in S$ with $s \geq q$. Likewise, the *order filter* $P_{\geq S}$ is the set of all elements $q \in P$ such that there exists $s \in S$ with $q \geq s$.

Example 2.2. Subsets of the real numbers form a poset under their usual ordering. Since the real numbers are totally ordered, every pair of elements in comparable, so every subset of the reals forms a chain. An order ideal of the real numbers is simply a ray $(-\infty, r]$, while an order filter is of the form $[r, \infty)$ for some $r \in \mathbb{R}$.

Example 2.3. The subsets of $[n]$ form a poset under inclusion. We call this poset the *boolean algebra* or *boolean poset* of rank n, and denote it by B_n. In B_3, $\{\{1,2\},\{2,3\}\}$ form an antichain because neither element contains the other. An example of a chain is $\{\varnothing, \{1\}, \{1,2\}\}$.

The order ideal generated by $\{\{1,2\},\{2,3\}$ is $\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$. The order filter generated by the same two elements is $\{\{1,2\},\{2,3\},\{1,2,3\}\}$.

Definition 2.4. A *lattice* is a poset L such that for all $p, q \in L$,

(1) p and q have a unique least upper bound $p \wedge q \in L$, called the *join* of p and q.
(2) p and q have a unique greatest lower bound $p \vee q \in L$ called the *meet* of p and q.

Every finite lattice L has $\top$ and $\bot$. The *atoms* of a finite lattice are the element $a \in L$ such that $a \succ \bot$. If every element of L can be written as join of atoms, then we call L *atomistic.*

Example 2.5. B_n is an atomistic lattice. The meet of two elements is their intersection, the join of two elements is their union, and the atoms are the 1-element subsets of $[n]$.

Example 2.6. A *set partition* of $[n]$ is a collection of disjoint sets $\Pi = \{S_1, \ldots, S_k\}$ such that $\bigcup_{i\in[k]} S_i = [n]$. We say a partition $\Pi' = \{T_1, \ldots, T_\ell\}$ *refines* Π if for all $i \in [\ell]$, there exists $j \in [k]$ such that $T_i \subseteq S_j$. The collection Π_n of set partitions of $[n]$ forms a lattice when ordered by refinement.

2.1.1. *The Incidence Algebra of a Poset.* The incidence algebra of a poset is a commutative algebra associated to the poset. It can be used to concisely express many quantities related to the structure of the poset, such as the number of chains of various lengths. Many important functions, such as the Möbius function of number theory, can be thought of as elements of the incidence algebra of particular posets. Throughout this section, P will be a finite poset.

Definition 2.7. Let R be a commutative ring. The **incidence algebra** of P over R, denoted $R[P]$, is the free R-module on the set $\{(a,b) \in P \times P : a \leq b\}$. Multiplication in the incidence algebra is defined by

$$(a,b)(c,d) = \begin{cases} (a,d) & b = c \\ 0 & b \neq c \end{cases}$$

and extends R-linearly to other elements.

When defining or referring to an element of an incidence algebra, we sometimes write $f(x,y)$ to refer to the coefficient of (x,y) in $f \in R[P]$. We will use the following well-known facts about the incidence algebra of a poset:

Proposition 2.8. *Let P be a poset and $f \in R[P]$. Then f is invertible in $R[P]$ if and only if $f(x,x)$ is invertible in R for all $x \in P$.*

Proposition 2.9. *Let P be a poset and $f \in R[P]$. If $f(x,x) = 0$ for all $x \in P$, then f is nilpotent of order k, where k is the maximum length of a chain in P. In particular, $1 - f$ is invertible and $(1-f)^{-1} = \sum_{i=0}^{k} f^i$.*

2.2. **Matroids.** Matroids are combinatorial structures that capture many seemingly different notions of independence, such as linear independence, acyclicness of graphs, and algebraic independence. One can axiomatize matroids in many equivalent ways; a few of the more common ones are described below. Throughout the following definitions, we will let E be a finite set, called the *ground set.*

Definition 2.10 (Independent sets)**.** A *matroid* is a pair $M = (E, \mathcal{I})$ where $\mathcal{I} \subseteq 2^E$ is the collection of *independent sets* of M and satisfies

(1) The empty set is in $\mathcal{I}$.
(2) If $I \in \mathcal{I}$, then $2^I \subseteq \mathcal{I}$.
(3) If $I, J \in \mathcal{I}$ and $\#I > \#J$, then there exists $i \in I$ such that $J \cup \{i\} \in \mathcal{I}$.

The *rank* of a subset S of a matroid, denoted $\mathrm{rank}(S)$, is the cardinality of the largest independent set it contains. In particular, the rank of a matroid is the size of its largest independent set. The set of maximal independent sets are called *bases*, and provide another way to define matroids.

Definition 2.11 (Bases)**.** A *matroid* is a pair $M = (E, \mathcal{B})$ where $\mathcal{B} \subseteq 2^E$ is the collection of bases of M and satisfies

(1) $\mathcal{B}$ is nonempty
(2) If $A, B \in \mathcal{B}$ are distinct and $a \in A - B$, then there exists $b \in B - A$ such that $(A - \{a\}) \cup \{b\} \in \mathcal{B}$.

Example 2.12. Any vector space V over finite field can be turned into a matroid $M(V)$ by taking the vector space as the ground set and the independent sets to be the linearly independent subsets of the vector space. The bases of the vector space are also the bases of the matroid. More generally, for an n-dimensional vector space V over the finite field $\mathbb{F}_q$ and for $0 \le r \le n$, let $M_r(V)$ denote the matroid with ground set $E = V$ and independent sets collections of at most r independent vectors in E. Then $M(V) = M_n(V)$.

Example 2.13. The *uniform matroid of rank r on* $[n]$ is defined by independent sets as

$$U_{n,r} \coloneqq \big([n], \big\{S \in 2^{[n]} \, : \, \#S \le r\big\}\big)$$

Example 2.14. Let $G = (V, E)$ be a graph and let $\mathcal{T} \subseteq 2^E$ be the set of acyclic subgraphs of G. The *graphic matroid of* G is defined by independent sets as $M(G) \coloneqq (E, \mathcal{T})$. A case of special interest is when $G = K_n$, the complete graph on n vertices. We call $M(K_n)$ the *complete graphic matroid of rank n.*

One might wish to "take the span" of a subset of a matroid. To do this, we define a *closure operator*

$$\begin{aligned} \mathrm{cl} : 2^M &\to 2^M \\ S &\mapsto \{m \in M \, : \, \mathrm{rank}(S \cup \{m\}) = \mathrm{rank}(S)\} \end{aligned}$$

Definition 2.15. A subset $S \subseteq M$ such that $\mathrm{cl}(S) = S$ is called a *flat* of M. The set of flats, ordered by inclusion, is the *lattice of flats* of M, denoted $\mathcal{L}(M)$. The lattices of flats of finite matroids are precisely lattices that are both atomistic and *semimodular*, a term that will not be defined or directly used here. See definition 3.9 in chapter 3 of [Sta+04] for further details.

Example 2.16. The lattice of flats of $U_{n,n}$ is B_n.

Example 2.17. The lattice of flats of $M(K_n)$ is Π_n.

Example 2.18. The lattice of flats of $M_r(\mathbb{F}_q^n)$ is collection of subspaces of $\mathbb{F}_q^n$ of dimension at most r, ordered by inclusion, together with the top dimensional subspace $\top = \mathbb{F}_q^n$.

Some common operations on matroids include contractions and restrictions.

Definition 2.19. Given a matroid M on E and an element $e \in E$, the *restriction matroid* $M \backslash e$ is the matroid on $E - \{e\}$ whose independent sets are given by

$$\mathcal{I}(M \backslash e) = \{I \in \mathcal{I}(M) \,:\, e \notin I\}$$

For $Z \subseteq E$ with $E \setminus Z = \{a_1, \ldots, a_k\}$, define the restriction

$$M^Z = (\cdots((M \backslash a_1) \backslash a_2) \cdots \backslash a_k).$$

Example 2.20. Let $r \leq m \leq n$. The restriction of $U_{n,r}$ with respect to $Z = [m]$ is $(U_{n,r})^Z = U_{m,r}$.

Example 2.21. The restriction of a graphic matroid, $M(G)$, about an edge e is the matroid of the graph G' obtained from G by removing e.

Definition 2.22. Given a matroid M on E and an element $e \in E$, the *contraction matroid* M/e is the matroid on $E - \{e\}$ whose independent sets are given by

$$\mathcal{I}(M/e) = \{E \subseteq E - \{e\} \,:\, E \cup \{e\} \in \mathcal{I}(M)\}.$$

For $Z = \{a_1, \ldots, a_k\}$, set

$$M_Z = (\cdots((M/a_1)/a_2) \cdots /a_k).$$

Example 2.23. The contraction of the uniform matroid $U_{n,r}$ at a set $Z = [k]$ is the uniform matroid $(U_{n,r})_Z = U_{n-k,r-k}$

Example 2.24. The contraction of a graphic matroid $M(G)$ about an edge e is the matroid of the contraction graph $G' = G/e$.

2.3. **q-analogs.** In general, a q-analog of an identity is an expression in terms of a variable q that specializes to the original identity when one sets $q = 1$. Many statistics relating to binomial coefficients and statistics on $\mathfrak{S}_n$ have natural q-analogues which allow one to refine many combinatorial theorems and identities. In particular, q-analogues often appear as invariants associated to vector spaces over finite fields.

Definition 2.25. For any $n \in \mathbb{N}$, we define the *q-analog of n* to be

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

and the *q-analog of $n!$* to be

$$[n]!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q.$$

Definition 2.26. For any natural numbers $0 \leq k \leq n$, we define the *q-binomial coefficient* to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}.$$

Proposition 2.27. *The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q.*

When q is a prime power, the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ computes the number of k-dimensional subspaces of $\mathbb{F}_q^n$. Also note that when $q = 1$, the above definitions specialize to the usual factorial and binomial coefficient.

2.4. **Simplicial Complexes.** Simplicial complexes are geometric and combinatorial structures strongly related to both posets and commutative algebra. Here, we give the minimal set of definitions needed for our own purposes.

Definition 2.28. A *simplicial complex* Δ on a vertex set $V(\Delta)$ is a subset of $2^{V(\Delta)}$ such that if $S \in \Delta$, then $2^S \subseteq \Delta$.

We call the elements of Δ the *faces* of Δ. The *face poset* of Δ, denoted $P(\Delta)$ is the poset whose ground set is Δ under the inclusion order. The maximal elements of P that are proper subsets of $V(\Delta)$ are the *facets* of Δ. The dimension of a face $F \in \Delta$ is $\dim F := \#F - 1$, and $\dim \Delta := \max\{\dim F\}_{F\in\Delta}$. A simplicial complex is *pure* if all of its facets have the same dimension.

Example 2.29. The *d-simplex* is $\Delta_d := 2^{[d+1]}$. Δ_d can be realized geometrically as the convex hull of a basis of $\mathbb{R}^{d+1}$; the dimension of this geometric realization is d.

Definition 2.30. The *order complex* of a poset P is

$$\Delta(P) := \{S \subseteq P \,:\, S \text{ is a chain}\}$$

Example 2.31. See Figure 1.

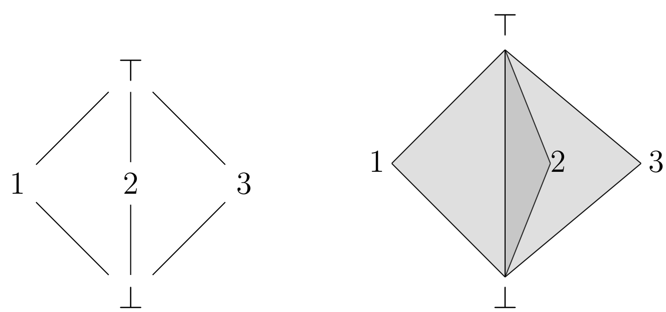

FIGURE 1. A poset P (left) and its order complex $\Delta(P)$ (right). Each edge of the order complex corresponds to a chain of length two in P, and each two dimensional face corresponds to one of the maximal chains of length three.

2.4.1. *The Stanley-Reisner ring.* The *Stanley-Reisner ring* of a simplicial complex provides a bridge between simplicial complexes and commutative algebra that has proved enlightening for both fields. For more information, see [Sta96].

Definition 2.32. Let k be a field. The *Stanley-Reisner ring* or *face ring* of a simplicial complex Δ is $k[\Delta] := k[x_v : v \in V(\Delta)]/I_\Delta$ where

$$I_\Delta := (x_{v_1}x_{v_2}\cdots x_{v_k} \,:\, \{v_1, \ldots, v_k\} \notin \Delta)$$

Proposition 2.33. *If Δ is a simplicial complex of dimension n, then the Stanley-Reisner ring has Krull dimension $n+1$.*

2.5. **Graded Rings and Modules.** Our main objects of study, Chow rings, belong to a class of rings called *graded rings*.

Definition 2.34. A *graded ring* or $\mathbb{N}$-*graded ring* R is a ring together with a decomposition of abelian groups

$$R = \bigoplus_{n \geq 0} R_n$$

such that for any $v \in R_n$ and $w \in R_m$, the product vw is in R_{mn}. The direct summand R_n in this decomposition is called the *nth homogeneous component* of R. An element $r \in R_n$ is called *homogeneous of degree* n or said to be of *degree* n, written $\deg r = n$.

One of the most important invariants of a graded ring is its *Hilbert series*, which records the free ranks of the abelian groups R_n, $n \geq 0$, in a formal power series.

Definition 2.35. The *Hilbert series* of a graded ring R is the series

$$H(R,t) \coloneqq \sum_{n \geq 0} \dim_{\mathbb{Z}} R_n t^n \in \mathbb{N}[\![t]\!]$$

The coefficients of the power series are given by the *Hilbert function* $h(R,n) \coloneqq \dim_{\mathbb{Z}} R_n$.

The Hilbert series of some rings, including those that we will study, are *symmetric*, meaning that there exists a $d \geq 0$ such that $h(R,n) = 0$ for $n > d$, $h(R,d) \neq 0$, and $h(R,n) = h(R,d-n)$ for all $0 \leq n \leq d$. The information in a symmetric Hilbert series (and more generally, in a symmetric polynomial) can be reformatted in different and sometimes more enlightening ways, one of which is the γ-*vector*.

Definition 2.36. Let $h(t)$ be a symmetric polynomial in t of degree d. The γ-*vector* of h is the unique vector $\gamma = (\gamma_0, \ldots, \gamma_{\lceil d/2 \rceil - 1})$ such that

$$h(t) = \sum_{i=0}^{\lceil d/2 \rceil - 1} \gamma_i t^i (t+1)^{d-2i}$$

Example 2.37. If $h(t) = t^3 + 5t^2 + 5t + 1$, then $\gamma = (1,2)$. Indeed,

$$t^3 + 5t^2 + 5t + 1 = (t+1)^3 + 2t(t+1)$$

A weaker invariant than the Hilbert series is the *Charney-Davis quantity* of R, defined to be the value $H(R,-1)$. The Charney-Davis quantity was introduced in [CD95] and is related to a conjecture of Charney and Davis for posets associated to flag simplicial complexes. See [RW05] for a more recent framework towards approaching questions stemming from Charney and Davis' original conjecture.

2.6. **Chow Rings.** We now define the main object of study. Following the conventions of Fiechner-Yuzvinsky, we define the Chow ring for a general atomistic lattice.

Definition 2.38 (From [FY04]). Let L be an atomistic lattice with atoms $a_1, \ldots, a_k$. The *Chow ring* of L is defined to be

$$A(L) = \mathbb{Z}[\{x_p : p \in L, p \neq \bot, \top\}]/(I + J)$$

where I and J are the ideals

$$I = (x_p x_q \,:\, p \text{ and } q \text{ are incomparable})$$
$$J = \left(\sum_{q \geq a_i} x_q \,:\, 1 \leq i \leq k\right).$$

Chow rings of atomistic lattices are graded by degree in the usual way.

If L is the lattice of flats of a matroid M, then we denote the Chow ring $A(\mathcal{L}(M))$ by $A(M)$. If we speak of the Hilbert series, γ-vector, or Charney-Davis quantity of a matroid M, then we are refering to that of its Chow ring $A(M)$.

Theorem 2.39 ([FY04] Corollary 2)**.** *Let L be a finite, ranked, atomistic lattice. Then the Hilbert series of $A(L)$ is*

$$H(A(L), t) = 1 + \sum_{\perp = x_0 < x_1 < \cdots < x_m} \prod_{i=1}^{m} \frac{t(1 - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1} - 1})}{1 - t}.$$

where the sum is taken over all chains $\perp = x_0 < x_1 < \cdots < x_m$ of L.

2.6.1. *Chow Rings of Matroids.* Chow rings that arise from geometric lattices (that is, from the lattices of flats of finite matroids) are known to satisfy many nice properties. The main one that we will use is the fact that Chow rings of matroids satisfy a version of Poincaré duality; for more information, see [AHK15]. For mild generalizations of these results, see Section 3.

Let M be a matroid of rank $r + 1$. By Proposition 6.2 of [AHK15], $A^q(M) = 0$ for $q > r$, and by Corollary 6.11 of [AHK15], $A^r(M) \cong \mathbb{Z}$. Combining these facts with the following theorem yields that the Hilbert series of $A(M)$ is symmetric of degree r.

Theorem 2.40 ([AHK15] Theorem 6.19)**.** *For any nonnegative integer $q \leq r$, the multiplication map*

$$A^q(M) \times A^{r-q}(M) \to A^r(M)$$

defines an isomorphism between groups

$$A^{r-q}(M) \cong \operatorname{Hom}_{\mathbb{Z}}(A^q(M), A^r(M))$$

Observe that Theorem 2.40 above implies that the Charney-Davisquantity of a matroid M is 0 if M is of even rank, and is $\dim_{\mathbb{Z}} A^{(\operatorname{rank} M - 1)/2}(M)$ if $\operatorname{rank} M$ is odd.

2.7. **Eulerian Polynomials, Tangent/Secant Numbers and Statistics on $\mathfrak{S}_n$.** Let $\mathfrak{S}_n$ denote the symmetric group on n letters. At various times in this report, we will refer to the following statistics on permutations.

Definition 2.41. Let $\omega \in \mathfrak{S}_n$ be a permutation. Then, define the statistics

$$\operatorname{inv}(\sigma) = \#\left\{(i, j) \,:\, \sigma(i) > \sigma(j)\right\}$$
$$\operatorname{des}(\sigma) = \#\left\{i \in [n-1] \,:\, \sigma(i+1) < \sigma(i)\right\}$$
$$\operatorname{exc}(\omega) = \#\left\{i \in [n] \,:\, \omega(i) > i\right\}$$
$$\operatorname{maj}(\omega) = \sum_{i,\, \omega(i) < \omega(i+1)} i$$

The Eulerian polynomial $A_n(t)$ is the polynomial

$$A_n(t) = \sum_{\omega \in \mathfrak{S}_n} t^{\mathrm{exc}(\omega)}$$

These polynomials are classical and have many interesting applications; see [Pet15] for further exposition. We note here that the polynomials $A_n(t)$ satisfy the following identities

Proposition 2.42. *([Pet15], thm 1.4)* $A_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(t)(t+1)^k.$

Proposition 2.43. *([Pet15], Quadratic polynomial recurrence, thm 1.5)*

$$A_n(t) = A_{n-1}(t) + t \sum_{i=0}^{n-2} \binom{n-1}{i} A_i(t) A_{n-1-i}(t).$$

Proposition 2.44. *([Pet15], thm 1.6) The exponential generating function of the $A_n(t)$ is*

$$\sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{z(t-1)}}.$$

We also note that the h-polynomial of the n-dimensional permutohedron is given by $A_n(t)$. The coefficient of t^k in $A_n(t)$ can be expressed as,

$$A(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} = \# \left\{ \sigma \in \mathfrak{S}_n \, : \, \mathrm{exc}(\sigma) = k \right\}.$$

These quantities are referred to as the *Eulerian numbers.*

Definition 2.45. The *n-th tangent/secant number* E_n is the n-th coefficient in the exponential generating function

$$\tanh(x) + \operatorname{sech}(x) = \sum_{n \geq 0} E_n \frac{x^n}{n!}$$

Remark 2.46. In the literature, the numbers E_{2n} are often referred to as the Euler numbers. To avoid confusion with the Eulerian numbers, we will refrain from using this language.

Since $\tanh(x)$, resp. $\operatorname{sech}(x)$, is odd, resp. even, it follows that

$$\tanh(x) = \sum_{n \geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \text{ and } \operatorname{sech}(x) = \sum_{n \geq 0} E_{2n} \frac{x^{2n}}{(2n)!}$$

Moreover, for all n, $(-1)^n E_{2n} > 0$ and $(-1)^n E_{2n+1} > 0$. Also,

$$\tan(x) = \sum_{n \geq 0} (-1)^n E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \text{ and } \sec(x) = \sum_{n \geq 0} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}.$$

(Hence, we refer to E_n as a tangent/secant number.) The following is a well-known combinatorial description of E_n.

Proposition 2.47. *([Pet15], exercise 4.2) The number $|E_n| = (-1)^n E_n$ counts the number of down-up permutations in $\mathfrak{S}_n$. That is, $|E_n|$ is the number of permutations $\omega \in \mathfrak{S}_n$ such that*

$$\omega(1) > \omega(2) < \omega(3) > \cdots$$

In the sequel, we will also make use of the following standard recurrence for the secant numbers.

Proposition 2.48 ([Sun05])**.** *For all n, $E_{2n} = -\sum_{k=0}^{n-1} \binom{2n}{2k} E_{2k}$.*

3. Generalizations of Poincaré Duality to nice Atomic Lattices

Here, we give some extensions of the results of [AHK15], including Poincaré duality, to an expanded class of lattices. We will define the class of lattices shortly. With such lattices, it is possible to make definitions so that the original proofs of [AHK15] essentially just go through, so we defer most definitions, proofs, and intermediate results to Appendix A because they will not be used directly.

Definition 3.1. Let L be a ranked atomistic lattice with atoms $E = \{\alpha_1, \ldots, \alpha_k\}$. For any $x, y \in L$ with $x < y$, let $d(x, y)$ be the minimum number d of atoms $\alpha_{i_1}, \ldots, \alpha_{i_d}$ such that

$$y = x \wedge \bigwedge_{j=1}^{d} \alpha_{i_j}.$$

Say that L is *nice* if $d(x, y) = \operatorname{rank}(y) - \operatorname{rank}(x)$ for all $x, y \in L$.

Having defined the lattices under consideration, we give the definitions and results regarding these lattices that we will make direct use of.

Let L be a nice lattice with atoms E, and for $I \subseteq E$, let $\operatorname{cl}_L(I) := \bigwedge_{Z \in I} Z$. Next, let

$$\mathcal{I}(L) := \left\{ I \in 2^E \;:\; \#I = \operatorname{rank}(\operatorname{cl}_L(I)) \right\}.$$

We can now define the Chow ring of a lattice with respect to a filter.

Definition 3.2. Let L be a lattice with set of atoms E and let $\mathscr{P}$ be an order filter on L. As in [AHK15], let $S_{E \cup \mathscr{P}} := \mathbb{Z}[x_i, x_F \mid i \in E, F \in \mathscr{P}]$, and define $A(L, \mathscr{P}) := S_{E \cup \mathscr{P}}/(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4)$ where

$$\mathcal{J}_1 = \left(x_{F_1} x_{F_2} \colon F_1, F_2 \text{ incomparable in } L \right)$$

$$\mathcal{J}_2 = \left(x_i x_F \colon F \in \mathscr{P}, i \in E \setminus F \right)$$

$$\mathcal{J}_3 = \left(\prod_{i \in I} x_i \colon I \in \mathcal{I}(L),\, \operatorname{cl}_L(I) \in \mathscr{P} \cup \{E\} \right)$$

$$\mathcal{J}_4 = \left(\left(x_i + \sum_{i \in F} x_F \right) - \left(x_j + \sum_{j \in F} x_F \right) \colon i \neq j \in E \right)$$

With the above definitions, the machinery of [AHK15] in the proof of Poincaré duality largely runs without change. In particular, we have the following results. Let $\mathscr{P}_-$ be a filter in L, and let $\mathscr{P}_+ = \mathscr{P}_- \cup \{Z\}$ where Z is a maximal element of $L \setminus \mathscr{P}_-$.

Proposition 3.3 (c.f. [AHK15] Proposition 6.6)**.** *The pullback homomorphism*

$$\Phi_Z \colon A(L, \mathscr{P}_-) \to A(L, \mathscr{P}_+)$$

defined by taking $x_F \mapsto x_F$ and

$$x_i \mapsto \begin{cases} x_i + x_F & \text{if } i \in Z \\ x_i & \text{if } i \notin Z \end{cases}$$

is a well-defined graded ring homomorphism. We will use the notation Φ_Z^q to denote the group homomorphism induced by Φ_Z on the degree-q component of $A(L, \mathscr{P}_-)$.

Proposition 3.4 (c.f. [AHK15] Proposition 6.8)**.** *For $p, q > 0$ integers, there are group homomorphisms*

$$\Psi^{p,q}\colon A^{q-p}([Z,\top]) \to A^q(L, \mathscr{P}_+) \quad \text{and} \quad \Gamma_Z^{p,q}\colon A^{q-p}([\bot, Z]) \to A^q(L)$$

sending $x_{\mathcal{F}} \mapsto x_Z^p x_{\mathcal{F}}$.

Theorem 3.5 (c.f. [AHK15] Theorem 6.18)**.** *For any $q > 0$,*

$$\Phi_Z^q \oplus \bigoplus_{p=1}^{\operatorname{rank}(Z)-1} \Psi_Z^{p,q}\colon \left(A^q(L, \mathscr{P}_-) \oplus \bigoplus_{p=1}^{\operatorname{rk}(Z)-1} A^{q-p}([Z,\top]) \right) \to A^q(L, \mathscr{P}_+)$$

is an isomorphism of groups.

Theorem 3.6 (c.f. [AHK15] Theorem 6.19)**.** *For $q \geq r$, the multiplication map*

$$A^q(L, \mathscr{P}) \times A^{r-q}(L, \mathscr{P}) \to A^r(L, \mathscr{P})$$

defines an isomorphism

$$A^{r-q}(L, \mathscr{P}) \simeq \operatorname{Hom}_{\mathbb{Z}}(A^q(L, \mathscr{P}), A^r(M, \mathscr{P}))$$

Remark 3.7. Theorem 3.6 implies that the Hilbert series of $A(L)$ for L a nice ranked atomistic lattice is a symmetric polynomial. In particular, this allows one to consider its associated γ-vector and can be suggestive of further structure. (i.e. Whether natural isomorphisms can be written between symmetric graded components via multiplication by combinatorial analogues of ample elements.)

4. Methods for calculating Hilbert series of Chow rings

4.1. **Incidence algebra.** Let L be a ranked lattice and consider the incidence algebra $(\mathbb{Q}(t))[L]$ of L with coefficients in the ring $\mathbb{Q}(t)$ of rational functions. Define distinguished elements in $(\mathbb{Q}(t))[L]$ as

$$\begin{aligned} \zeta_L(x,y) &= 1 \\ \alpha_L(x,y) &= t^{\operatorname{rank} y - \operatorname{rank} x} \\ \eta_L(x,y) &= \begin{cases} \frac{t - t^{\operatorname{rank} y - \operatorname{rank} x}}{1-t} & \operatorname{rank} y - \operatorname{rank} x - 1 \geq 1 \\ 0 & \operatorname{rank} y - \operatorname{rank} x - 1 < 1 \end{cases} \\ \gamma_L &= (1-\eta_L)^{-1}\zeta_L. \end{aligned}$$

To describe the relations of these elements, we have the following proposition.

Proposition 4.1. *Let L be a ranked atomistic lattice. We have*

$$\begin{aligned}1-\eta_L &= \frac{1}{t-1}(\alpha_L - t\zeta_L)\\ \alpha_L &= (1-t)(1-\eta_L)+t\zeta_L\\ \gamma_L(x,y) &= H(A([x,y]),t)\end{aligned}$$

where $[x,y]=\{z\in L : x\le z\le y\}$.

Proof. The first identity follows straightforwardly from the second. To verify the second, let $x\le y$. If $x=y$ then

$$\begin{aligned}(1-t)(1(x,x)-\eta_L(x,x))+t\zeta_L(x,x) &= (1-t)(1-0)+t\\ &= 1\\ &= \alpha_L(x,x).\end{aligned}$$

Similarly, if $\operatorname{rank} y = \operatorname{rank} x + 1$ we have

$$\begin{aligned}(1-t)(1(x,y)-\eta_L(x,y))+t\zeta_L(x,y) &= (1-t)(0-0)+t\\ &= t\\ &= \alpha_L(x,y)\end{aligned}$$

and if $\operatorname{rank} y - \operatorname{rank} x \ge 2$ then

$$\begin{aligned}(1-t)(1(x,y)-\eta_L(x,y))+t\zeta_L(x,y) &= -(1-t)\frac{t-t^{\operatorname{rank} y-\operatorname{rank} x}}{1-t}+t\\ &= t^{\operatorname{rank} y-\operatorname{rank} x}\\ &= \alpha_L(x,y).\end{aligned}$$

For the last identity, we have by Theorem 2.39 that

$$\begin{aligned}H(A([x,y]),t) &= 1+\sum_{\perp=x_0<x_1<\cdots<x_m}\left(\prod_{i=1}^{m}\eta_L(x_i,x_{i+1})\right)\zeta_L(x_m,y)\\ &= \eta_L^0(x,y)\zeta_L(x,y)+\sum_{m=0}^{\infty}\eta_L^m(x,y)\zeta_L(x,y)\\ &= (1-\eta_L)^{-1}(x,y)\zeta_L(x,y)\end{aligned}$$

with the last line using the fact that η_L is nilpotent, since L is finite. □

In particular, the above proposition shows that $\gamma_L(\perp,\top)=H(L,t)$. It can be shown that

$$\gamma_L(\perp,\top)=\sum_{x\in L}(1-\eta_L)^{-1}(\perp,x)$$

and it can further be shown that $(1-\eta_L)^{-1}(\perp,x)$ can be computed using $O(N)$ operations in the ring $\mathbb{Q}[t]$, where N is the number of elements in L. Thus $H(A(L),t)$ can be computed using $O(N^2)$ operations in $\mathbb{Q}[t]$. Since all polynomials involved will have degree at most $\operatorname{rank} L$, this gives an algorithm for computing $H(A(L),t)$ which uses $O((\operatorname{rank} L)N^2)$ arithmetic operations. (Note that N may not be polynomial in $\operatorname{rank} L$.)

4.1.1. *Application to products.* Let L and K be two ranked lattices. Recall that for any ring R we have the R-algebra isomorphism

$$\begin{aligned} R[L] \otimes R[K] &\longrightarrow R[L \times K] \\ (a_1, b_1) \otimes (a_2, b_2) &\longmapsto ((a_1, a_2), (b_1, b_2)) \end{aligned}$$

and the existence of this isomorphism does not depend on L or K being ranked or being lattices.

Proposition 4.2. *We have*

$$\gamma_{L\times K} = (\gamma_L \otimes \gamma_K)(1 - t(1-\gamma_L) \otimes (1-\gamma_K))^{-1}.$$

Proof. We have

$$\begin{aligned} 1 - \eta_{L\times K} &= \frac{1}{1-t}(\alpha_{L\times K} - t\zeta_{L\times K}) \\ &= (1-t)(1-\eta_L) \otimes (1-\eta_K) + t(\zeta_L \otimes (1-\eta_K) + (1-\eta_L) \otimes \zeta_K) - t\zeta_L \otimes \zeta_K \\ &= (1-\eta_L) \otimes (1-\eta_K) - t(1-\eta_L-\zeta_L) \otimes (1-\eta_K-\zeta_K) \\ &= ((1-\eta_L) \otimes (1-\eta_K))(1-t(1-\gamma_L) \otimes (1-\gamma_K)) \end{aligned}$$

so

$$\begin{aligned} (1-\eta_{L\times K})^{-1} &= (1-t(1-\gamma_L) \otimes (1-\gamma_K))^{-1}((1-\eta_L)^{-1} \otimes (1-\eta_L)^{-1}) \\ \gamma_{L\times K} &= (1-t(1-\gamma_L) \otimes (1-\gamma_K))^{-1}(\gamma_L \otimes \gamma_K) \end{aligned}$$

which can be rewritten as

$$\gamma_{L\times K} = \sum_{k=0}^{\min(\operatorname{rank} L, \operatorname{rank} K)} t^k((1-\gamma_L)^k\gamma_L) \otimes ((1-\gamma_K)^k\gamma_K)$$

because $(1-\gamma_L)$ is nilpotent of order $\operatorname{rank} L$ for any finite ranked lattice L. $\square$

Example 4.3. When $K = \{0 \leq 1\}$, we have that $(1-\gamma_K)^2 = 0$ and $\gamma_K(1-\gamma_K) = (1-\gamma_K)\gamma_K = 1-\gamma_K$. Thus

$$\begin{aligned} \gamma_{L\times K} &= \gamma_L \otimes \gamma_K + t((1-\gamma_L)\gamma_L) \otimes ((1-\gamma_K)\gamma_K) \\ &= \gamma_L \otimes \gamma_K + t\gamma_L \otimes 1_K - t\gamma_L^2 \otimes 1_K - t\gamma_L \otimes \gamma_K + t\gamma_L^2 \otimes \gamma_K \end{aligned}$$

which gives

$$H(A(L_1 \times L_2), t) = (1-t)H(A(L_1), t) + t\gamma_1^2(\bot, \top).$$

In the case that $L_1 = B_n$ is the Boolean algebra on n elements, we have

$$H(A(B_{n+1}), t) = (1-t)H(A(B_n), t) + t\sum_{k=0}^{n}\binom{n}{k}H(A(B_k), t)H(A(B_{n-k}), t)$$

which gives the well-known quadratic recurrence for the Eulerian polynomials. By taking $K = B_k$, it is possible to get k-th order recurrences. However, these are not obviously equivalent to the k-th order recurrences given by repeatedly applying the quadratic recurrence.

4.2. **Applications of results of Adiprasito-Huh-Katz to lattices.**

Let M be a matroid of rank $r+1$ on a ground set E. For $z \in E$, let M_z be the contraction of M at z, and recall that $\mathcal{L}(M_z) \cong [\{z\}, \top] \subseteq \mathcal{L}(M)$. This fact, combined with the results of [AHK15] can be applied to find formulas for the Hilbert series of Chow rings of certain matroids in terms of their contractions. More generally we can find a formula for the Hilbert series of any graded lattice L of rank $r+1$ with the property that $[z, \top] \cong [z', \top]$ for all $z, z' \in L$ with $\operatorname{rank}(z) = \operatorname{rank}(z')$. In the following, we assume that L has this property.

Let $\mathscr{P}_-$ be an order filter of L, and let $\mathscr{P}_+ = \mathscr{P}_- \cup \{z\}$ for some maximal $z \in L \setminus \mathscr{P}_-$. Next, recall from Propositions 3.3 and 3.4 the pullback homomorphism in degree $q > 0$,

$$\Phi_z^q : A^q(L, \mathscr{P}_-) \to A^q(L, \mathscr{P}_+)$$

and the Gysin homomorphism

$$\Psi_z^{p,q} : A^{q-p}([z, \top]) \to A^q(L, \mathscr{P}_+).$$

By Theorem 3.5, $\Phi_z^q \oplus \bigoplus_{i=1}^{\operatorname{rank}(z)-1} \Psi_z^{p,q}$ is an isomorphism. Hence, if $\mathscr{P}_i$ is the order filter of L obtained by removing from L all elements of rank less than or equal to i and $(z_1, \ldots, z_r)$ is a sequence of elements of L with $\operatorname{rank}(z_i) = i$ for all i, then for all $q > 0$, we have isomorphisms

$$A^q(L) = A^q(L, \mathscr{P}_0) \cong A^q(L, \mathscr{P}_1) \cong A^q(L, \mathscr{P}_2) \oplus \left(A^{q-1}([z_2, \top])\right)^{\oplus |L_2|} \cong$$
$$\cdots \cong A^q(L, \mathscr{P}_r) \oplus \left(\bigoplus_{i=2}^{r} \left(\bigoplus_{p=1}^{i-1} A^{q-p}([z_i, \top]) \right)^{\oplus |L_i|} \right).$$

Since $A^q(L, \mathscr{P}_r) \cong \mathbb{Z}$, it follows that

$$\dim_{\mathbb{Z}} A^q(L) = 1 + \sum_{i=2}^{r} |L_i| \sum_{p=1}^{i-1} \dim_{\mathbb{Z}} A^{q-p}([z_i, \top]).$$

This recurrence for the dimension of a homogeneous component can be lifted to a recurrence for the Hilbert series of $A(L)$ in the following manner. For a fixed $0 \leq k \leq r-1$, let $(z_1, \ldots, z_r)$ be a sequence of elements of L with $\operatorname{rank}(z_i) = i$ for all i. Then

$$\begin{aligned}
H(L, t) &= \sum_{q=0}^{r} \dim_{\mathbb{Z}} A^q(L)\, t^q \\
&= \sum_{q=0}^{r} \left(1 + \sum_{i=2}^{r} |L_i| \sum_{p=1}^{i-1} \dim_{\mathbb{Z}} A^{q-p}([z_i, \top]) \right) t^q \\
&= [r+1]_t + \sum_{i=2}^{r} |L_i| \sum_{p=1}^{i-1} \sum_{q=0}^{r} \dim_{\mathbb{Z}} A^{q-p}([z_i, \top])\, t^q
\end{aligned}$$

Since $\dim_{\mathbb{Z}} A^{q-p}([z_i, \top]) = 0$ when $q - p < 0$ by convention, the innermost sum above really only runs from $q = p$ to $q = r$. Making this change and setting $k = q - p$, we can rewrite the above as

$$[r+1]_t + \sum_{i=2}^{r} |L_i| \sum_{p=1}^{i-1} t^p \sum_{k=0}^{r-p} \dim_{\mathbb{Z}} A^k([z_i, \top])\, t^k.$$

Now, observe that $\operatorname{rank}([z_i, \top]) = r + 1 - i$ and that $p \le i - 1$ implies $r - p \ge r - i + 1$. Hence, $\sum_{k=0}^{r-p} \dim_{\mathbb{Z}} A^k([z_i, \top])t^k = H([z_i, \top], t)$ for every p and i, so we obtain the following expression for $H(L, t)$.

Proposition 4.4. *If L is a ranked atomistic lattice such that $[z, \top] \cong [z', \top]$ for all $z, z' \in L$ with $\operatorname{rank}(z) = \operatorname{rank}(z')$, and if $(z_1, \dots, z_r)$ be a sequence of elements of L with $\operatorname{rank}(z_i) = i$ for all i, then*

$$H(L, t) = [r+1]_t + t \sum_{i=2}^{r} |L_i| \, [i-1]_t \, H([z_i, \top], t).$$

We will now state the Hilbert series that one gets by applying 4.4 to several matroids of special interest.

Uniform matroids. Each upper interval of $\mathcal{L}(U_{n,r+1})$ is the lattice of flats of a uniform matroid on a smaller ground set and of lower rank. Hence

$$H(U_{n,r+1}, t) = [r+1]_t + t \sum_{i=2}^{r} \binom{n}{i} [i-1]_t \, H(U_{n-i,r+1-i}, t).$$

Subspaces of vector spaces over finite fields. The formula for vector spaces over finite fields is a q-analog of the one for the uniform matroid.

$$H\left(A\big(M_{r+1}(\mathbb{F}_q^n)\big), t\right) = [r+1]_t + t \sum_{i=2}^{r} [i-1]_t \begin{bmatrix} n \\ i \end{bmatrix}_q H\left(A\big(M_{r+1-i}(\mathbb{F}_q^n)\big), t\right)$$

Complete graphic matroids. If $S(n, m)$ is the Stirling number of the second kind, then

$$H\left(A\big(M(K_{n+1})\big), t\right) = [n]_t + t \sum_{i=2}^{n-1} S(n+1, i) \, [i-1]_t \, h\left(A\big(M(K_{n+1-i})\big), t\right).$$

4.3. **Computing Hilbert series of $A(L \times B_1)$ using differential operators.** While the incidence algebra provides a general and concise way to write the Hilbert series of a product, it is slightly unsatisfying in that, given the Hilbert series of both lattice in the product, one must still make reference to posets to write the Hilbert series of the product. Here, we present a very simple case, in which one can write the Hilbert series of a product without reference to any information except for the Hilbert series' of the two lattices being multiplied together. Let L be a ranked atomistic lattice. For a chain $C = \{x_0 < x_1 < \cdots < x_m\}$ in L and $0 < i \le m$, let

$$m_{C,i} = \frac{t - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1}}}{1 - t}.$$

Let s be a new variable, and in analogy to Theorem 2.39, write

$$H(A(L), t, s) := 1 + \sum_{C = \{\bot = x_0 < x_1 < \cdots < x_m\}} \prod_{i=1}^{m} \left((1 + m_{C,i}) \, e^s - 1 \right).$$

Observe that $H(A(L), t, 0) = H(A(L), t)$. Finally, let $\partial_s : \mathbb{Q}[s, t] \to \mathbb{Q}[s, t]$ be the formal derivative operator with respect to s.

Proposition 4.5.

$$H\big(A(L \times B_2), t\big) = (1 + t\,\partial_s) H\big(A(L), t, s\big)\big|_{s=0}$$

Proof. Strict chains in $L \times B_2$ that begin at $\perp$ take three different forms. If $C = \{\perp = x_0 < x_1 < \cdots < x_m\}$ is a chain in L, then a chain in $L \times B_2$ must look like one of the following:

$$(\perp, 0) < (x_1, 0) < \cdots < (x_m, 0)$$
$$(\perp, 0) < (x_1, 0) < \cdots < (x_i, 0) < (x_{i+1}, 1) < \cdots < (x_m, 1)$$
$$(\perp, 0) < (x_1, 0) < \cdots < (x_i, 0) < (x_i, 1) < (x_{i+1}, 1) < \cdots < (x_m, 1) \tag{1}$$

Since $\operatorname{rank}(x_i, 0) = \operatorname{rank}(x_i, 1) - 1$, chains like the one in (1) contribute 0 to the sum in the statement of Theorem 2.39. Hence, by Theorem 2.39,

$$H(A(L \times B_2), t) = 1 + \sum_{C=\{\perp = x_0 < x_1 < \cdots < x_m\}} \left(\prod_{i=1}^{m} m_{C,i} + \sum_{k=1}^{m} t(1 + m_{C,i}) \prod_{\substack{i=1 \\ i \neq k}}^{m} m_{C,i} \right) \tag{2}$$

By the product rule for the derivative, if $f(t) \in \mathbb{Q}[t] \subseteq \mathbb{Q}[s,t]$, then

$$\partial_s\big((1 + f(t)) - 1\big) = (1 + f(t))e^s.$$

Hence, equation (2) is equal to

$$H(A(L), t, 0) + t\,\partial_s H(A(L), t, s)\big|_{s=0} = (1 + t\,\partial_s) H(A(L), t, s)\big|_{s=0}. \qquad \square$$

Similarly, suppose we instead define

$$H(A(L), t, s) = 1 + \sum_{C=\{\perp = x_0 < x_1 < \cdots < x_m\}} \prod_{i=1}^{m} \frac{te^s - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1}} e^{ts}}{1 - t}.$$

Then we have

Proposition 4.6.

$$H(A(L \times B_1), t, s) = (1 + \partial_s) H(A(L), t, s).$$

Proof. Similarly to the proof of Proposition 4.5, define

$$m_{C,i} = \frac{te^s - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1}} e^{ts}}{1 - t}$$

for a chain $C = \{x_0 < x_1 < \cdots < x_m\}$. A similar casework argument on the structure of chains in $L \times B_1$ gives

$$\begin{aligned} H(A(L \times B_1), t, s) &= 1 + \sum_{C=\{\perp = x_0 < x_1 < \cdots < x_m\}} \left(\prod_{i=1}^{m} m_{C,i} + \sum_{k=1}^{m} t(e^s + m_{C,k}) \prod_{\substack{i=1 \\ i \neq k}}^{m} m_{C,i} \right) \\ &= (1 + \partial_s) H(A(L), t, s) \end{aligned}$$

using the fact that

$$\partial_s \frac{te^s - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1}} e^{ts}}{1-t} = \frac{te^s - t^{(\operatorname{rank} x_i - \operatorname{rank} x_{i-1})+1} e^{ts}}{1-t}$$
$$= t\left(e^s + \frac{te^s - t^{\operatorname{rank} x_i - \operatorname{rank} x_{i-1}} e^{ts}}{1-t}\right).$$

□

5. Explicit determination of Hilbert series and Charney-Davis quantities

5.1. **The Uniform Matroid.** Recall the uniform matroid of rank r and dimension n is the matroid $U_{n,r}$ on ground set $E = [n]$ with independent sets all subsets of $[n]$ of size at most r. $U_{n,r}$ has lattice of flats

$$\mathcal{L}(U_{n,r}) = (B_n)_{[r]} = \{S \in B_n \,:\, \#S < r \text{ or } S = \top\}.$$

That is, $U_{n,r}$ has lattice of flats $\mathcal{L}(U_{n,r})$ equal to the collection of all subsets of $[n]$ of size at most r, ordered under inclusion, together with a unique top element $\top = [n]$.

For $\sigma \in \mathfrak{S}_n$, let

$$\operatorname{fix}(\sigma) = \{i \in [n] \,:\, \sigma(i) = i\}.$$

Also let

$$\mathcal{D}_n = \{\sigma \in \mathfrak{S}_n \,:\, \operatorname{fix}(\sigma) = \varnothing\}$$

be the set of derangements of n and put

$$E_{n,k} = \{\sigma \in \mathfrak{S}_n \,:\, \sigma \text{ has at least } k \text{ fixed points}\}.$$

5.1.1. *Hilbert Series of $A(U_{n,n})$.* Recall the Stanley-Reisner ring over $\mathbb{Z}$ of $\mathcal{L}(U_{n,n})$ is $S := \mathbb{Z}[x_F]_{F\in\mathcal{L}(M)}/I$, where

$$I := (x_F x_G \,:\, F, G \in \mathcal{L}(M),\ F \not\leq G,\ G \not\leq F).$$

Moreover, since $\mathcal{L}(U_{n,n})$ has dimension n, S has Krull dimension $n+1$. For each $i \in [n]$, let $\gamma_i := \sum_{F\ni i} x_F$ and $\alpha_i := \gamma_i - x_\top$ be in S.

Lemma 5.1. *$\{\alpha_i - \alpha_n \,:\, i \in [n-1]\}$ is algebraically independent*

Proof. First, observe that

$$\{\alpha_i - \alpha_n \,:\, i \in [n-1]\} = \{\gamma_i - \gamma_n \,:\, i \in [n-1]\}.$$

Furthermore, if $\{\gamma_i \,:\, i \in [n]\}$ is algebraically independent, then so is $\{\gamma_i - \gamma_n \,:\, i \in [n-1]\}$, since any polynomial in $\{\gamma_i - \gamma_n \,:\, i \in [n-1]\}$ can be expanded out into a polynomial in $\{\gamma_i \,:\, i \in [n]\}$.

Hence, we have reduced to showing that $\{\gamma_i \,:\, i \in [n]\}$ is algebraically independent. Let

$$p(y_1, \ldots, y_n) := \sum_{\substack{\mathbf{i}=(i_1,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0} \\ \sum_j i_j \leq m}} a_{\mathbf{i}}\, y_1^{i_1} \cdots y_n^{i_n} \in R[y_1, \ldots, y_n]$$

be a polynomial of total degree m and let $\operatorname{NZ}(\mathbf{i}) := \{j \in [n] \,:\, i_j \neq 0\}$ be the set of indices of nonzero entries of $\mathbf{i}$. Suppose that $p(\gamma_1, \ldots, \gamma_n) = 0$. If this is the case, we will show that $a_{\mathbf{i}} = 0$ for all $\mathbf{i}$ by induction on $\#\operatorname{NZ}(\mathbf{i})$.

First, if $\#\operatorname{NZ}(\mathbf{i}) = 0$ then $a_{\mathbf{i}} = 0$ because none of the γ_i's have constant terms. Now, assume that $a_{\mathbf{i}} = 0$ for all $\mathbf{i}$ with $\mathbf{i} < k$, and let $a_{\mathbf{i}}$ be the coefficient of a term of p with

$\mathrm{NZ}(\mathbf{i}) = \{j_1 < \ldots < j_k\}$. When multiplied out, the term $\gamma_1^{i_1} \cdots \gamma_n^{i_n}$ of $p(\gamma_1, \ldots, \gamma_n)$ has a monomial $\prod_{\ell \in [k]} x_{F_\ell}^{i_{j_\ell}}$ where $\{F_1 < F_2 < \ldots < F_k\} \subseteq \mathcal{L}(U_{n,n})$ is an increasing chain, with $F_\ell \ni j_\ell$ for each ℓ. No other term of $p(\gamma_1, \ldots, \gamma_n)$ has such a monomial, so $a_{\mathbf{i}} = 0$. Therefore, $p(y_1, \ldots, y_n) = 0$ and $\{\gamma_i : i \in [n]\}$ is algebraically independent. $\square$

Theorem 5.2. *The dimension of the* i*th graded component of* $A(U_{n,n}) = \bigoplus_{i=0}^{n-1} A^i$ *is* $\dim_{\mathbb{Z}} A^i = A(n,i)$*, the* (n,i)*th Eulerian number.*

Proof. By inspection of the defining relations, it is evident that the Chow ring $A(U_{n,n})$ is equal to S/J, where

$$J := (x_\top, x_\perp) + (\alpha_i - \alpha_n \,:\, i \in [n-1]) \subseteq S.$$

Since neither $x_\top$ nor $x_\perp$ appear in any element of $\{\alpha_i - \alpha_n \,:\, i \in [n-1]\}$, J is generated by a maximal set of algebraically independent elements. Therefore, ([BGS82], Corollary 4.5) gives a monomial basis for S whose m-th graded component is indexed by permutations in $\mathfrak{S}_n$ with m excedences. Hence, we conclude that $\dim_{\mathbb{Z}} A(U_{n,n})_m = A(n,m)$. $\square$

5.1.2. *Hilbert Series of* $A\big(U_{n,r}\big)$ *for* $r < n$. For any $r \leq n$, there is a surjective graded map of rings $\pi_{n,r} \colon A(U_{n,r+1}) \to A(U_{n,r})$ acting on variables x_F by

$$x_F \mapsto \begin{cases} x_F & \text{if } \operatorname{rank}(F) < r \\ 0 & \text{if } \operatorname{rank}(F) = r \end{cases}$$

Let $K_{n,r}$ be the kernel of this ring map.

Theorem 5.3. *The Hilbert function of* $K_{n,r}$ *is given by the formula*

$$\dim_{\mathbb{Z}}(K_{n,r})_k = \#\left\{\sigma \in E_{n,n-r} \,:\, \operatorname{exc}(\sigma) = r - k\right\}$$

To prove theorem 5.3, we will use the monomial basis of [FY04]. Let the variable $x_\top$ denote the element $\sum_{F \ni i} x_F \in A(U_{n,r+1})$ for any fixed $i \in E$. From the linear relations defining the chow ring, $x_\top$ is uniquely defined in $A(U_{n,r+1})$. From [FY04], set of monomials of the form

$$x_{F_1}^{\alpha_1} \cdots x_{F_\ell}^{\alpha_\ell},$$

where $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_\ell$ is a flag of nonempty flats in $\mathcal{L}(U_{n,r})$ and for $r_i = \operatorname{rank}(F_i)$, we have the inequalities $1 \leq \alpha_i \leq r_i - r_{i-1} - 1$, forms a $\mathbb{Z}$-basis for the ring $A(U_{n,r})$. Moreover, under $\pi_{n,r}$ each such basis element, b, of $A(U_{n,r+1})$ is either carried to the corresponding basis element of $A(U_{n,r})$ or is sent to 0, the latter case holding if and only if $b = x_\top^a x_{F_1}^{\alpha_1} \cdots x_{F_\ell}^{\alpha_\ell}$ for $a \geq 0$, F_i and α_i as above, and $\operatorname{rank}(F_1) = r - a$.

To prove theorem 3, we consider the following refinement.

Lemma 5.4. *Consider the numbers*

$$\gamma_{r,k,a}^n := \#\left\{ x_\top^a x_{F_1}^{\alpha_1} \cdots x_{F_\ell}^{\alpha_\ell} \,:\, {\scriptstyle F_\top \supsetneq F_1 \supsetneq \cdots \supsetneq F_\ell \text{ is a flag of nonempty flats in } \mathcal{L}(U_{n,r+1}) \atop \scriptstyle a + \sum \alpha_j = k,\ 1 \leq \alpha_i \leq r_i - r_{i+1} - 1, \text{ and } \operatorname{rank}(F_1) = r - a} \right\}$$

and

$$\Gamma_{r,k,a}^n := \#\left\{\sigma \in \mathfrak{S}_n \,:\, \#\operatorname{fix}(\sigma) = n - r + a \text{ and } \operatorname{exc}(\sigma) = r - k\right\}$$

Then, $\gamma_{r,k,a}^n = \Gamma_{r,k,a}^n$.

Proof. We immediately make the following reductions. Partitioning by the choice of a flag $F_1 \in \mathcal{L}(U_{n,r+1})_{r-a}$,

$$\gamma^n_{r,k,a} = \binom{n}{r-a} \cdot \#\left\{x_\top^{\alpha_1} x_{F_2}^{\alpha_2} \cdots x_{F_\ell}^{\alpha_\ell} \ : \ \substack{F_\top \supsetneq F_2 \supsetneq \cdots \supsetneq F_\ell \text{ is a flag of nonempty flats in } \mathcal{L}(U_{r-a,r-a}) \\ \sum \alpha_j = k-a \text{ and } 1 \leq \alpha_i \leq r_i - r_{i+1} - 1}\right\},$$

while, partitioning on the choice of fixed points,

$$\Gamma^n_{r,k,a} = \binom{n}{r-a} \cdot \#\left\{\sigma \in \mathfrak{S}_{r-a} \ : \ \#\mathrm{fix}(\sigma) = 0 \text{ and } \mathrm{exc}(\sigma) = r-k\right\}$$

Hence, it suffices to show equality of the latter two sets. In particular, set

$$\Theta_{n,k} = \left\{x_\top^{\alpha_1} x_{F_2}^{\alpha_2} \cdots x_{F_\ell}^{\alpha_\ell} \ : \ \substack{F_\top \supsetneq F_2 \supsetneq \cdots \supsetneq F_\ell \text{ is a flag of nonempty flats in } \mathcal{L}(U_{n,n}) \\ \sum \alpha_j = k \text{ and } 1 \leq \alpha_i \leq r_i - r_{i-1} - 1}\right\}$$

$$\mathcal{D}_{n,k} = \{\sigma \in \mathcal{D}_n \ : \ \mathrm{exc}(\sigma) = n-k\}$$

We will show that $\#\Theta_{n,k} = \#\mathcal{D}_{n,k}$ for all n, k.

We proceed by induction on k. The base case $k = 0$ is trivial as both sets are empty. Assume $k \geq 1$. Set

$$S_{n,k} = \left\{x_{F_1}^{\alpha_1} x_{F_2}^{\alpha_2} \cdots x_{F_\ell}^{\alpha_\ell} \ : \ \substack{F_1 \supsetneq F_2 \supsetneq \cdots \supsetneq F_\ell \text{ is a flag of nonempty flats in } \mathcal{L}(U_{n,n}) \\ \sum \alpha_j = k \text{ and } 1 \leq \alpha_i \leq r_i - r_{i+1} - 1}\right\}$$

and

$$S_{n,k,a} = \left\{x_\top^{a} x_{F_1}^{\alpha_1} \cdots x_{F_\ell}^{\alpha_\ell} \ : \ \substack{F_\top \supsetneq F_1 \supsetneq \cdots \supsetneq F_\ell \text{ is a flag of nonempty flats in } \mathcal{L}(U_{n,n}) \\ a + \sum \alpha_j = k,\ 1 \leq \alpha_i \leq r_i - r_{i+1} - 1, \text{ and } \mathrm{rank}(F_1) = n-a}\right\}$$

Consider the map of sets $\varphi_{n,k} \colon \Theta_{n,k} \to S_{n,k-1}$ taking $x_\top^{\alpha_1} x_{F_2}^{\alpha_2} \cdots x_{F_\ell}^{\alpha_\ell} \mapsto x_\top^{\alpha_1 - 1} x_{F_2}^{\alpha_2} \cdots x_{F_\ell}^{\alpha_\ell}$. Then, $\varphi_{n,k}$ is injective and $S_{n,k-1}$ decomposes into a disjoint union

$$S_{n,k-1} = \mathrm{Image}(\varphi_{n,k}) \sqcup \coprod_{a \geq 1} S_{n,k-1,a}. \tag{3}$$

Considering the choice of second highest element,

$$\#S_{n,k-1,a} = \binom{n}{n-a} \#\Theta_{n-a,k-a-1}.$$

Also note that $\#S_{n,k}$ computes the Eulerian numbers from 5.2. Then, using the decomposition 3 and applying the induction hypothesis,

$$\begin{aligned}
\#\Theta_{n,k} &= \#S_{n,k-1} - \sum_{a \geq 1} \#S_{n,k-1,a} \\
&= \#\{\sigma \in \mathfrak{S}_n \ : \ \mathrm{exc}(\sigma) = n-k\} - \sum_{a \geq 1} \binom{n}{n-a} \#\{\sigma \in \mathcal{D}_{n-a} \ : \ \mathrm{exc}(\sigma) = n-k\}
\end{aligned}$$

Then, using that $\binom{n}{n-a} \#\{\sigma \in \mathcal{D}_{n-a} \ : \ \mathrm{exc}(\sigma) = n-k\} = \#\{\sigma \in \mathfrak{S}_n \ : \ \#\mathrm{fix}(\sigma) = a \text{ and } \mathrm{exc}(\sigma) = n-k\}$ for all $a \leq n$, we can reduce the above to

$$\begin{aligned}
\#\Theta_{n,k} &= \#\{\sigma \in \mathfrak{S}_n \ : \ \mathrm{exc}(\sigma) = n-k\} - \sum_{a \geq 1} \#\{\sigma \in \mathfrak{S}_n \ : \ \#\mathrm{fix}(\sigma) = a \text{ and } \mathrm{exc}(\sigma) = n-k\} \\
&= \#\{\sigma \in \mathfrak{S}_n \ : \ \mathrm{exc}(\sigma) = n-k\} - \#\{\sigma \in E_{n,1} \ : \ \mathrm{exc}(\sigma) = n-k\} \\
&= \#\{\sigma \in \mathcal{D}_n \ : \ \mathrm{exc}(\sigma) = n-k\} = \#\mathcal{D}_{n,k}
\end{aligned}$$

Hence, $\#\Theta_{n,k} = \#\mathcal{D}_{n,k}$ by induction and the lemma is proven. $\square$

Remark 5.5. Theorem 5.3 follows from the lemma above since the difference in dimensions $\dim_{\mathbb{Z}} A(U_{n,r+1})_k - A(U_{n,r})_k$ can be summed to

$$\sum_{a\geq 0}\gamma^n_{r,k,a} = \sum_{a\geq 0}\Gamma^n_{r,k,a} = \#\{\sigma\in\mathfrak{S}_n \,:\, \#\mathrm{fix}(\sigma)\geq n-r \text{ and } \mathrm{exc}(\sigma)=r-k\}$$
$$= \#\{\sigma\in E_{n,n-r} \,:\, \mathrm{exc}(\sigma)=r-k\}.$$

Corollary 5.6. *The Hilbert function of $A(U_{n,r})$ is given explicitly by*

$$\dim_{\mathbb{Z}} A(U_{n,r})_k = \#\{\sigma\in\mathfrak{S}_n \,:\, \mathrm{exc}(\sigma)=k\} - \sum_{i=r}^{n-1}\#\{\sigma\in E_{n,n-i} \,:\, \mathrm{exc}(\sigma)=i-k\}$$

In particular, when $r=n-1$, the Hilbert function of $A(U_{n,n-1})$ is

$$\dim_{\mathbb{Z}}(A(U_{n,r}))_k = \#\{\sigma\in\mathcal{D}_n \,:\, \mathrm{exc}(\sigma)=k+1\} \tag{4}$$

Proof. Theorem 3 implies the series of equalities

$$\begin{aligned}\dim_{\mathbb{Z}}(A(U_{n,r}))_k &= \dim_{\mathbb{Z}}(A(U_{n,r+1})) - \#\{\sigma\in E_{n,n-r} \,:\, \mathrm{exc}(\sigma)=r-k\}\\ &= \cdots = \dim_{\mathbb{Z}}(A(U_{n,n})) - \sum_{i=r}^{n-1}\#\{\sigma\in E_{n,n-i} \,:\, \mathrm{exc}(\sigma)=i-k\}\\ &= \#\{\sigma\in\mathfrak{S}_n \,:\, \mathrm{exc}(\sigma)=k\} - \sum_{i=r}^{n-1}\#\{\sigma\in E_{n,n-i} \,:\, \mathrm{exc}(\sigma)=i-k\}\end{aligned}$$

where the latter equality follows from the characterization of the Hilbert series of $A(U_{n,n})$ given by theorem 1.

In the case $r=n-1$, this becomes

$$\#\{\sigma\in\mathfrak{S}_n \,:\, \mathrm{exc}(\sigma)=k\} - \#\{\sigma\in E_{n,1} \,:\, \mathrm{exc}(\sigma)=n-1-k\}$$

By Poincaré duality for $A(U_{n,n})$,

$$\#\{\sigma\in\mathfrak{S}_n \,:\, \mathrm{exc}(\sigma)=k\} = \#\{\sigma\in\mathfrak{S}_n \,:\, \mathrm{exc}(\sigma)=n-1-k\}$$

So 4 follows from the equality of sets

$$\begin{aligned}&\{\sigma\in\mathfrak{S}_n : \mathrm{exc}(\sigma)=n-1-k\}\\ &\quad= \{\sigma\in E_{n,1} \,:\, \mathrm{exc}(\sigma)=n-1-k\}\sqcup\{\sigma\in\mathcal{D}_n \,:\, \mathrm{exc}(\sigma)=n-1-k\}.\end{aligned}$$ ◻

Corollary 5.7. *The total dimension of the chow ring $A(U_{n,n-r})$ is given by*

$$\dim_{\mathbb{Z}} A(U_{n,n-r}) = \#\mathcal{D}_n - \sum_{k=1}^{r-1}\#E_{n,k}$$

for all $r\geq 1$. Equivalently,

$$\frac{\dim_{\mathbb{Z}} A(U_{n,n-r})}{n!} = E[1-\min(\#\mathrm{fix}(\sigma),r)]$$

for all $r\geq 0$, where σ is chosen uniformly from $\mathfrak{S}_n$.

Proof. The total dimension of $K_{n,r}$ can be computed as

$$\sum_{k\geq 0} \#\left\{\sigma \in E_{n,n-r} \,:\, \mathrm{exc}(\sigma) = r-k\right\} = \#E_{n,n-r}.$$

Hence, the corollary follows by descending induction on r. ☐

5.1.3. *Charney-Davis Quantities for $A\big(U_{n,r}\big)$.* We now compute the Charney-Davis quantities of $A\big(U_{n,r}\big)$ in terms of the secant numbers E_{2k}. We show that in the full rank case, the formula for the Charney-Davis quantity as a linear combination of the secant numbers can be interpreted as a recurrence for the tangent numbers.

Theorem 5.8. *For even r, the Charney-Davis quantity for the uniform matroid, $U_{n,r}$ of rank r on $[n]$ is 0. For odd r, the Charney-Davis quantity for $U_{n,r}$ is*

$$\sum_{k=0}^{\frac{r-1}{2}} \binom{n}{2k} E_{2k}$$

where $E_{2\ell}$ is the ℓth secant number.

Proof. When r is even, the Charney-Davis quantity being zero is an immediate consequence of Poincaré duality. From here on, assume that r is odd. We proceed by induction on ℓ for $r = 2\ell+1$. If $\ell = 0$, then $A(M) = \mathbb{Z}$ and the theorem follows trivially.

Let $\ell > 0$. By [FY04], Hilbert series of $A(U_{n,r})$ is given by

$$H(A(U_{n,r}),t) = 1 + \sum_{\mathbf{r}} \prod_{i=1}^{k(r)} \left\{ \left[1 + \frac{t(1-t^{r-r_{k(r)}-1})}{1-t}\right] \frac{t(1-t^{r_i - r_{i-1}-1})}{1-t} \right\} + \frac{t(1-t^{\ell-1})}{1-t}$$

where the sum is over all subsets $\mathbf{r} = (0 = r_0 < r_1 < \cdots < r_{k(r)} < r)$. Then, evaluating at $t = -1$ gives

$$1 + \sum_{\substack{\mathbf{r},\, r_k < r \\ \forall i, r_i - r_{i-1} \text{ is even}}} (-1)^{k(r)} \prod_{i=1}^{k(r)} \binom{n - r_{i-1}}{r_i - r_{i-1}}$$

Then, this quantity can be expressed as a sum

$$\left\{ 1 + \sum_{\substack{\mathbf{r},\, r_k < r-2 \\ \forall i, r_i - r_{i-1} \text{ is even}}} (-1)^{k(r)} \prod_{i=1}^{k(r)} \binom{n - r_{i-1}}{r_i - r_{i-1}} \right\} + \left\{ \sum_{\substack{\mathbf{r},\, r_k = r-1 \\ \forall i, r_i - r_{i-1} \text{ is even}}} (-1)^{k(r)} \prod_{i=1}^{k(r)} \binom{n - r_{i-1}}{r_i - r_{i-1}} \right\}$$

This former term is exactly the Charney-Davis quantity for $U_{r-2,n}$, so by induction, it suffices to show that the latter summand is equal to $E_{r-1}\binom{n}{r-1}$. Let this term by denoted by T_ℓ.

Reindexing and then summing over r such that $r_{k-1} = 2a$, we get the recursion

$$\begin{aligned}
T_r &= -\sum_{\substack{\mathbf{r},\, r_k < r-2 \\ \forall i, r_i - r_{i-1} \text{ is even}}} (-1)^{k(r)} \binom{n - r_k}{r - r_k - 1} \prod_{i=1}^{k(r)} \binom{n - r_{i-1}}{r_i - r_{i-1}} \\
&= -\sum_{a=0}^{\ell-1} \binom{n-2a}{r-2a-1} \sum_{\substack{\mathbf{r},\, r_k = 2a \\ \forall i, r_i - r_{i-1} \text{ is even}}} (-1)^{k(r)} \prod_{i=1}^{k(r)} \binom{n - r_{i-1}}{r_i - r_{i-1}} \\
&= -\sum_{a=0}^{\ell-1} \binom{n-2a}{r-2a-1} T_a
\end{aligned}$$

Then, by induction, we conclude

$$\begin{aligned}
T_r &= -\sum_{a=0}^{\ell-1} \binom{n-2a}{r-2a-1} \binom{n}{2a} E_{2a} = -\sum_{a=0}^{\ell-1} \binom{n}{r-1} \binom{r-1}{2a} E_{2a} \\
&= \binom{n}{r-1} \left(-\sum_{a=0}^{\ell-1} \binom{r-1}{2a} E_{2a} \right) = \binom{n}{r-1} E_{r-1}
\end{aligned}$$

where the last equality follows from the recursion $E_{2\ell} = -\sum_{k=0}^{\ell-1} \binom{2\ell}{2k} E_{2k}$ (see [Pet15], pg 88, ex 4.2). The theorem is therefore proved. □

Remark 5.9. When $r = n$ is odd, the Charney-Davis quantity given above is equal to the tangent number E_n via the following proposition.

Proposition 5.10. *For any $n \geq 0$, $E_{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} E_{2k}$.*

Proof. Consider the series expansion of $\tanh(x)$. We have

$$\begin{aligned}
\tanh(x) = \sinh(x)\operatorname{sech}(x) &= \left(\sum_{k\geq 0} \frac{x^{2k+1}}{(2k+1)!} \right) \left(\sum_{k \geq 0} E_{2k} \frac{x^{2k}}{(2k)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{2n+1}{2k} E_{2k} \right) \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Since also $\tanh(x) = \sum_{n\geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$, equating coefficients gives the relation

$$\sum_{k=0}^{n} \binom{2n+1}{2k} E_{2k} = E_{2n+1}$$

as desired. □

5.1.4. *γ-Vectors of $A(U_{n,n})$ and $A(U_{n,n-1})$.* By inspection, $H\big(A(U_{n,n}), t\big)$ is the h-vector of the permutohedron of dimension n. Hence, [PRW06] gives the following characterization of γ-polynomial of $A(U_{n,n})$

Proposition 5.11. *([PRW06], Thm 11.1) The γ-polynomial of $A(U_{n,n})$ is of the form*

$$\sum_{w\in\tilde{\mathfrak{S}}_n} t^{\operatorname{des}(w)}$$

where $\tilde{\mathfrak{S}}_n$ denotes the set of permutations in $\mathfrak{S}_n$ which do not contain any final descents or double descents.

Since $H\big(A(U_{n,n-1}),t\big) = \sum_{w\in\mathcal{D}_n} t^{\operatorname{des}(w)}$ is the local h-vector of the barycentric subdivision of the permutohedron, Athanasiadis' survey [Ath16] gives the analogous interpretation of the γ-vector of $H\big(A(U_{n,n-1}),t\big)$.

Proposition 5.12. *The γ-vector of $A(U_{n,n-1})$ is given by $\gamma = (\gamma_{n,i})$ where $\gamma_{n,i}$ denotes the number of permutations in $\mathfrak{S}_n$ with i descending runs and no descending run of size one.*

5.2. **Matroid of subspaces of vector spaces over finite fields.** Let V be an n-dimensional vector space over the finite field $\mathbb{F}_q$. Recall, for $0 \le r \le n$, $M_r(V)$ denotes the matroid with ground set $E = V$ and independent sets collections of at most r independent vectors in E. We also put $M(V) \coloneqq M_n(V)$. For the purposes of computing the Chow ring $A\big(M_r(V)\big)$, it suffices to consider the simplification-the matroid with ground set $\mathbb{P}V$ whose independent sets consist of all subspaces in $\mathbb{P}V$ of size at most r.

The lattice of flats of $M_r(\mathbb{F}_q^n)$ is given by the collection of subspaces of $\mathbb{F}_q^n$ of dimension at most r ordered by inclusion together with the top dimensional subspace $\mathbb{F}_q^n$.

The Gröbner basis in [FY04] gives a formula for the Hilbert series of $A\big(M(\mathbb{F}_q^n)\big)$,

$$H\Big(A\big(M_r(\mathbb{F}_q^n)\big),t\Big) = 1 + \sum_{\mathbf{r}} \prod_{i=1}^{k(r)} \frac{t(1-t^{r_i-r_{i-1}-1})}{1-t} \begin{bmatrix} n-r_{i-1} \\ r_i - r_{i-1}\end{bmatrix}_q$$

where the sum is over all tuples $\mathbf{r} = (0 = r_0 < r_1 < \cdots < r_{k(r)} \le r)$. In particular, this specializes to the formula for $h\Big(A(U_{n,r}),t\Big)$ when $q = 1$. Consequently, $M_r(\mathbb{F}_q^n)$ is best seen as a q-analogue of $A(U_{n,r})$.

5.2.1. *Hilbert Series of $A\big(M(\mathbb{F}_q^n)\big)$.* We show that the the Hilbert series of $A\big(M(\mathbb{F}_q^n)\big)$ agrees with the definition of Shareshian and Wachs in [SW10] and whose generating function is studied in [SW07]. To characterize the Hilbert series of $A\big(M(\mathbb{F}_q^n)\big)$, we first compute its q-exponential generating function, and we use this result to give a combinatorial interpretation.

Proposition 5.13. *Define $h_0 \coloneqq 1$. The q-exponential generating function of $h_n(t) \coloneqq H\Big(A\big(M(\mathbb{F}_q^n)\big),t\Big)$ is given by*

$$F(t,x) = \sum_{n\ge 0} h_n(t)\frac{x^n}{[n]_q!} = \frac{(t-1)e_q(t)}{te_q(t) - e_q(tx)}$$

where e_q denotes the q-exponential function $e_q(x) \coloneqq \sum_{n\ge 0} \frac{x^n}{[n]_q!}$.

Proof. From the recurrence 4.2, we have the relation

$$h_n = 1 + t\sum_{i=1}^{n-1} [i-1]_t \begin{bmatrix} n \\ i \end{bmatrix}_q h_{n-i}$$

Then, the generating function $F(t,x)$ satisfies

$$\begin{aligned} F(t,x) &= 1 + \sum_{n\geq 1} \frac{x^n}{[n]_q!} + t\sum_{n\geq 1}\left([i-1]_t \begin{bmatrix} n \\ i \end{bmatrix}_q h_{n-i}\right)\frac{x^n}{[n]_q!} \\ &= e_q(x) + t\sum_{n\geq 1}\sum_{i=1}^{n}\left([i-1]_t \frac{x^i}{[i]_q!}\right)\left(h_{n-i}\frac{x^{n-i}}{[n-i]_q!}\right) \\ &= e_q(x) + tF(t,x)G(t,x) \end{aligned}$$

for $G(t,x) = \sum_{i\geq 1}[i-1]_t \frac{x^i}{[i]_q!}$. We can evaluate G by

$$G(t,x) = \frac{1}{t-1}\sum_{i\geq 1}(t^{i-1}-1)\frac{x^i}{[i]_q!} = \frac{1}{t-1}\left(\frac{e_q(tx)}{t} - e_q(x)\right) = \frac{1}{t^2-t}\Big(e_q(tx) - te_q(x)\Big)$$

Substituting into the equation above and solving for F, we get

$$F(t,x) = \frac{e_q(x)}{1 - \frac{1}{t-1}\Big(e_q(tx) - te_q(x)\Big)} = \frac{(t-1)e_q(x)}{te_q(x) - e_q(tx)} \qquad \square$$

From [SW07], this generating function uniquely classifies the polynomials $h_n(t)$ as the q-Eulerian polynomial $A_n(q,t)$ defined as follows.

Definition 5.14. The *q-Eulerian polynomial* $A_n(q,t)$ is the polynomial

$$A_n(q,t) := A_n^{\mathrm{maj},\mathrm{exc}}(q, tq^{-1}) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}t^{\mathrm{exc}(\sigma)}$$

Also set the q-Eulerian number $\begin{bmatrix} n \\ j \end{bmatrix}_q$ to be

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j}} q^{\mathrm{maj}(\sigma)-j}$$

Clearly,

$$A_n(q,t) = \sum_{j=1}^{n-1}\begin{bmatrix} n \\ j \end{bmatrix}_q t^j$$

Theorem 5.15. *For all n and all prime powers q,*

$$h\Big(A\big(M(\mathbb{F}_q^n)\big), t\Big) = A_n(q,t).$$

Next, we prove a recurrence for $A_n(q,t)$. We need the following lemmas.

Lemma 5.16. *For any positive q and $k \geq 0$,*

$$\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}}\begin{bmatrix} k \\ i \end{bmatrix}_q = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{else} \end{cases}$$

Proof. When $k = 0$, the lemma is clear. By the q-binomial theorem (see, for example, [Sta97], ch 3, exercise 119), for $k > 0$,

$$\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q x^{k-i} = \prod_{i=0}^{n-1}(x - q^i).$$

Setting $x = 1$ gives the desired identity. $\square$

Lemma 5.17. *For all $n \geq 0$,*

$$t^n = \sum_{k=0}^{n} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=1}^{k}(t - q^i)$$

Proof. We can expand the expression

$$\prod_{i=1}^{k}(t - q^i) = \sum_{i=0}^{k}(-1)^i \begin{bmatrix} k \\ i \end{bmatrix}_q q^{\frac{i(i-1)}{2}} t^{n-i}$$

and collect powers of q to see the righthand side of the desired identity is

$$\begin{aligned} \sum_{k=0}^{n} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=1}^{k} &= \sum_{0 \leq i \leq k \leq n} (-1)^i q^{n-k} q^{\binom{i+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q t^{n-i} \\ &= \sum_{\ell=0}^{n} \left((-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ \ell + i \end{bmatrix} \begin{bmatrix} \ell + i \\ i \end{bmatrix} \right) q^{n-\ell} t^{\ell} \\ &= \sum_{\ell=0}^{n} \left((-1)^i q^{\binom{i}{2}} \begin{bmatrix} n - \ell \\ i \end{bmatrix} \right) \begin{bmatrix} n \\ \ell \end{bmatrix} q^{n-\ell} t^{\ell} \end{aligned}$$

Then, applying lemma 5.16 to the latter expression gives the result. $\square$

Remark 5.18. Observe that equation 5.17 is a q-analogue of the binomial expansion

$$t^n = \big((t-1)+1\big)^n = \sum_{k=0}^{n} \binom{n}{k} (t-1)^k$$

Proposition 5.19. *Let $H_n(t) = H(A(M(\mathbb{F}_q^n)), t)$ denote the Hilbert series of $A(M(\mathbb{F}_q^n))$, and let $(a; q)_n \coloneqq (1-a)(1-aq)\cdots(1-aq^{n-1})$ be the Pochhammer symbol. Then, h_n satisfies the recurrence*

$$h_n(t) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q h_k(t) \prod_{i=1}^{n-1-k} (t - q^i) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q t^{n-1-k} \cdot h_k(t) \cdot (q/t, q)_{n-1-k} \tag{5}$$

Proof. Let $g_n(t)$ denote the sequence satisfying $g_0 = 1$ and the recurrence 5. Then, the q-exponential generating function $G(t, x)$ of $g_n(t)$ satisfies

$$\begin{aligned} G(t, x) = \sum_{n \geq 0} g_n(t) \frac{x^n}{[n]_q!} &= 1 + \sum_{n \geq 1} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q g_k(t) \prod_{i=1}^{n-1-k} (t - q^i) \frac{x^n}{[n]_q!} \\ &= 1 + G(t, x) \sum_{n \geq 1} \frac{x^n}{[n]_q!} \prod_{i=1}^{n-1} (t - q^i) \end{aligned}$$

By proposition 5.13, the q-exponential generating function of h_n is

$$F(t,x) = \frac{(t-1)e_q(t)}{te_q(t) - e_q(tx)}.$$

Hence, proposition 5.19 above is equivalent to showing

$$\sum_{n\geq 1} \frac{x^n}{[n]_q!}\prod_{i=1}^{n-1}(t-q^i) = \frac{e_q(tx)-e_q(x)}{(t-1)e_q(x)}$$

Clearly both sides of the above have zero constant term. Therefore, taking the Jackson q derivative, we are reduced to showing

$$\sum_{n\geq 0} \frac{x^n}{[n]_q!}\prod_{i=1}^{n}(t-q^i) = D_q\left[\frac{e_q(tx)-e_q(x)}{(t-1)e_q(x)}\right] = \frac{e_q(tx)}{e_q(qx)}$$

Rearranging the above gives

$$e_q(tx) = e_q(qx)\sum_{n\geq 0}\frac{x^n}{[n]_q!}\prod_{i=1}^{n}(t-q^i) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} q^{n-k}\prod_{i=1}^{k}(t-q^i)\begin{bmatrix} n \\ k\end{bmatrix}_q\right)\frac{x^n}{[n]_q!}$$

Therefore, the conjecture again reduces to proving the identity

$$t^n = \sum_{k=0}^{n} q^{n-k}\begin{bmatrix} n \\ k\end{bmatrix}_q \prod_{i=1}^{k}(t-q^i) \tag{6}$$

But this is precisely the statement of lemma 5.17. We conclude that $h_n(t)$ satisfies the desired recurrence. □

Remark 5.20. When $q = 1$, proposition 5.19 becomes the well-known recurrence for the Eulerian polynomials,

$$A_n(t) = \sum_{k=0}^{n-1}\binom{n}{k}A_k(t)(t-1)^{n-1-k}$$

To the authors' knowledge, the recurrence in proposition 5.19 does not yet appear in the literature.

Corollary 5.21.

$$\begin{bmatrix} n \\ k\end{bmatrix}_q = \#\left\{x_{V_1}^{\alpha_1}\dots x_{V_\ell}^{\alpha_\ell} \;:\; \substack{V_1\subsetneq\cdots\subsetneq V_\ell \text{ are subspaces of } \mathbb{F}_q^n \\ 1\leq \alpha_i\leq \dim_{\mathbb{Z}} V_i - \dim_{\mathbb{Z}} V_{i-1}-1,\ \sum_i \alpha_i = k}\right\}$$

Proof. Both quantities count $\dim_{\mathbb{Z}} A(M(\mathbb{F}_q^n))_k$ □

5.2.2. *Hilbert Series of $A(M_r(\mathbb{F}_q^n))$ for $r < n$.* We study the Hilbert series of $A\big(M_r(\mathbb{F}_q^n)\big)$ by descending induction on rank. As in the case of the uniform matroids, there are graded, surjective ring homomorphisms

$$\pi_{n,r,q}\colon A(M_{r+1}\big(\mathbb{F}_q^n)\big) \to A\big(M_r(\mathbb{F}_q^n)\big)$$

defined by taking variables $x_V \in A(M_{r+1}\big(\mathbb{F}_q^n)\big)$ to zero if $\dim_{\mathbb{Z}}(V) = r+1$ and to the corresponding variable $x_V \in A\big(M_r(\mathbb{F}_q^n)\big)$ otherwise. The Hilbert functions of the kernels $K_{n,r,q} = \ker(\pi_{n,r,q})$ are q-analogues of the numbers

$$\#\left\{\sigma\in E_{n,n-r} \,:\, \operatorname{exc}(\sigma) = r-k\right\}$$

In particular, we can express

$$\dim_{\mathbb{Z}}(K_{n,r,q})_k = \sum_{i=0}^{r} \begin{bmatrix} n \\ i \end{bmatrix}_q D_{i,r-k,q} = \sum_{i=0}^{r} \begin{bmatrix} n \\ r-i \end{bmatrix}_q D_{r-i,k-i,q}$$

where $D_{n,k,q}$ is a q-analogue of the number

$$\# \{\sigma \in \mathcal{D}_n \,:\, \operatorname{exc}(\sigma) = r-k\}$$

More explicitly, define

$$T_{n,k,q} = \left\{ x_\top^{\alpha_0} x_{V_1}^{\alpha_1} \cdots x_{V_\ell}^{\alpha_\ell} \,:\, \substack{\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_\ell \text{ are subspaces of } \mathbb{F}_q^n \text{ of rank } \leq r \\ \sum \alpha_j = k,\ 1 \leq \alpha_i \leq \dim_{\mathbb{Z}}(V_i) - \dim_{\mathbb{Z}}(V_{i+1}) - 1} \right\}$$

and set $D_{n,k,q} \coloneqq \# T_{n,k,q}$. Then, from the Gröbner basis of [FY04], note that

$$\dim_{\mathbb{Z}}(K_{n,r,q})_k = \# \left\{ x_\top^{i} x_{V_1}^{\alpha_1} \cdots x_{V_\ell}^{\alpha_\ell} \,:\, \substack{\mathbb{F}_q^n = V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_\ell \text{ are subspaces of } \mathbb{F}_q^n \text{ of rank } \leq r \\ i + \sum \alpha_j = k,\ 1 \leq \alpha_i \leq \dim_{\mathbb{Z}}(V_i) - \dim_{\mathbb{Z}}(V_{i+1}) - 1,\ \dim_{\mathbb{Z}}(V_1) = r-i} \right\}$$

So summing over possible values of i gives

$$\dim_{\mathbb{Z}}(K_{n,r,q})_k = \sum_{i=0}^{r} \begin{bmatrix} n \\ r-i \end{bmatrix}_q D_{r-i,k-i,q} \tag{7}$$

We will now give a combinatorial description of $D_{n,k,q}$. To do so, we establish some notation. For $\sigma \in \mathfrak{S}_A$ for $A = \{a_1 < \cdots a_k\}$ an ordered set, let the *reduction* of σ be the permutation $\overline{\sigma}$ in $\mathfrak{S}_k$ such that $\sigma(a_i) = a_{\overline{\sigma}(i)}$. For $\sigma \in \mathfrak{S}_n$, its *derangment part* $\operatorname{dp}(\sigma)$ is the reduction of σ along its nonfixed points.

The following lemma of Wachs will be essential.

Lemma 5.22. *([Wac89] Corollary 3) For all $\gamma \in \mathcal{D}_b$ and $n \geq b$,*

$$\sum_{\substack{\operatorname{dp}(\sigma)=\gamma \\ \sigma \in \mathfrak{S}_n}} q^{\operatorname{maj}(\sigma)} = q^{\operatorname{maj}(\gamma)} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Corollary 5.23. *For any integers $n, q, k \geq 0$,*

$$\sum_{\substack{\sigma \in \mathcal{D}_{n-i} \\ \operatorname{exc}(\sigma)=k}} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)} \begin{bmatrix} n \\ n-i \end{bmatrix}_q = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{exc}(\sigma)=k \\ \#\operatorname{fix}(\sigma)=i}} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}$$

Proof. From Lemma 5.22, we have the identity

$$\sum_{\substack{\gamma \in \mathcal{D}_{n-i} \\ \operatorname{exc}(\gamma)=k}} q^{\operatorname{maj}(\gamma)-\operatorname{exc}(\gamma)} \begin{bmatrix} n \\ n-i \end{bmatrix}_q = \sum_{\substack{\gamma \in \mathcal{D}_{n-i} \\ \operatorname{exc}(\gamma)=k}} q^{-\operatorname{exc}(\gamma)} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{dp}(\sigma)=\gamma}} q^{\operatorname{maj}(\sigma)} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \operatorname{exc}(\sigma)=k \\ \#\operatorname{fix}(\sigma)=i}} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}. \qquad \square$$

Theorem 5.24. *For $D_{n,k,q}$ as above,*

$$D_{n,k,q} = \sum_{\substack{\sigma \in \mathcal{D}_n \\ \operatorname{exc}(\sigma)=n-k}} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}$$

Proof. We proceed by induction on k. For $k=0$, the result is vacuous. For $k>0$, set

$$S_{\alpha_0}=\left\{x_\top^{\alpha_0}x_{V_1}^{\alpha_1}\cdots x_{V_\ell}^{\alpha_\ell}\ :\ \substack{\mathbb{F}_q^n=V_0\supsetneq V_1\supsetneq\cdots\supsetneq V_\ell \text{ are subspaces of } \mathbb{F}_q^n,\ \sum\alpha_j=k-1\\ 1\le\alpha_i\le\dim_{\mathbb{Z}}(V_i)-\dim_{\mathbb{Z}}(V_{i+1})-1,\quad \dim_{\mathbb{Z}}(V_1)=n-\alpha_0-1}\right\}$$

$$S=\left\{x_{V_1}^{\alpha_1}\cdots x_{V_\ell}^{\alpha_\ell}\ :\ \substack{V_1\supsetneq\cdots\supsetneq V_\ell \text{ are subspaces of } \mathbb{F}_q^n\\ \sum\alpha_j=k-1,\quad 1\le\alpha_i\le\dim_{\mathbb{Z}}(V_i)-\dim_{\mathbb{Z}}(V_{i+1})-1}\right\}$$

Then, the map on monomials taking $x_\top^{\alpha_0}x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}\mapsto x_\top^{\alpha_0-1}x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}$ gives an injective map

$$\varphi\colon T_{n,k,q}\to S.$$

Moreover, S is the disjoint union $S=\operatorname{Im}(\varphi)\sqcup\coprod_{a\ge 0}S_a$. Considering the choice of the second largest flag,

$$\#S_a=\begin{bmatrix} n\\ n-a-1\end{bmatrix}_q D_{n-a-1,k-a-1,q}$$

While from Corollary 5.21,

$$\#S=\begin{bmatrix} n\\ k-1\end{bmatrix}_q=\begin{bmatrix} n\\ n-k\end{bmatrix}_q$$

where the latter equality follows from Poincaré duality for $A\big(M(\mathbb{F}_q^n)\big)$. Therefore, by induction,

$$\begin{aligned} D_{n,k,q}=\#T_{n,k,q}=\#S-\sum_{a\ge 0}\#S_a&=\begin{bmatrix} n\\ n-k\end{bmatrix}_q-\sum_{b\ge 1}\begin{bmatrix} n\\ n-b\end{bmatrix}_q D_{n-b,k-b,q}\\ &=\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \operatorname{exc}(\sigma)=n-k}}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}-\sum_{b\ge 1}\sum_{\substack{\gamma\in\mathcal{D}_{n-b}\\ \operatorname{exc}(\gamma)=n-k}}q^{\operatorname{maj}(\gamma)-\operatorname{exc}(\gamma)}\begin{bmatrix} n\\ n-b\end{bmatrix}_q \qquad (8)\end{aligned}$$

Then applying Corollary 5.23, the righthand side of equation 8 can be expanded as

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \operatorname{exc}(\sigma)=n-k}}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}-\sum_{b\ge 1}\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \operatorname{exc}(\sigma)=n-k\\ \#\operatorname{fix}(\sigma)=b}}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}=\sum_{\substack{\sigma\in\mathcal{D}_n\\ \operatorname{exc}(\sigma)=n-k}}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}$$

This completes the induction, and the theorem is proven. □

Recall $E_{n,k}$ denotes the set $\{\sigma\in\mathfrak{S}_n\ :\ \#\operatorname{fix}(\sigma)\ge k\}$.

Corollary 5.25. *The Hilbert series of the kernel $K_{n,r,q}=\ker(\pi_{n,r,q})$ is given by*

$$H(K_{n,r,q},t)=\sum_{\sigma\in E_{n,n-r}}t^{r-\operatorname{exc}(\sigma)}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}$$

In particular, its Hilbert function is

$$\dim_{\mathbb{Z}}(K_{n,r,q})_k=\sum_{\substack{\sigma\in E_{n,n-r,q}\\ \operatorname{exc}(\sigma)=r-k}}q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)} \qquad (9)$$

Proof. Applying theorem 5.24 and corollary 5.23 to equation 7 gives

$$\begin{aligned}
\dim_{\mathbb{Z}}(K_{n,r,q})_k &= \sum_{i=0}^{r} \begin{bmatrix} n \\ r-i \end{bmatrix}_q D_{r-i,k-i,q} = \sum_{i=0}^{r} \sum_{\substack{\sigma\in\mathcal{D}_{r-i} \\ \mathrm{exc}(\sigma)=r-k}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} \\
&= \sum_{i=0}^{r} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \#\mathrm{fix}(\sigma)=n-r+i \\ \mathrm{exc}(\sigma)=r-k}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} \\
&= \sum_{\substack{\sigma\in E_{n,n-r} \\ \mathrm{exc}(\sigma)=r-k}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}.
\end{aligned}$$

□

Corollary 5.26. *The Hilbert series of $A\big(M_r(\mathbb{F}_q^n)\big)$ is given by*

$$H\big(A(M_r(\mathbb{F}_q^n)), t\big) = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} - \sum_{j=0}^{n-1} \sum_{\sigma\in E_{n,n-r}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{r-\mathrm{exc}(\sigma)} \tag{10}$$

In particular, if $r=n-1$, the Hilbert series of $A\big(M_{n-1}(\mathbb{F}_q^n)\big)$ is

$$h\Big(A\big(M_r(\mathbb{F}_q^n)\big), t\Big) = \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)-1}$$

Proof. Equation 10 follows from a direct substitution of 9 into the formula

$$H\big(A(M_r(\mathbb{F}_q^n), t\big) = H\big(A(M_{r+1}(\mathbb{F}_q^n)), t\big) + H\big(K_{n,r,q}, t\big) = \cdots = H\big(A(M(\mathbb{F}_q^n)), t\big) + \sum_{j=r}^{n-1} H\big(K_{n,j,q}, t\big)$$

Recall that $E_{n,r} = \{\sigma \in \mathfrak{S}_n : \#\mathrm{fix}(\sigma) \geq r\}$. For the case $r = n-1$, Corollary 5.23 implies that the coefficient of q^k in 10 can be simplified as follows.

$$\begin{aligned}
&\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=k}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} - \sum_{i=0}^{n-1} \sum_{\substack{\sigma\in\mathcal{D}_{n-i-1} \\ \mathrm{exc}(\sigma)=n-k-1}} \begin{bmatrix} n \\ n-i-1 \end{bmatrix}_q q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} \\
&\qquad = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=n-k-1}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} - \sum_{i=0}^{n-1} \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=n-k-1 \\ \#\mathrm{fix}(\sigma)=i+1}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} \\
&\qquad = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=n-k-1}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} - \sum_{\substack{\sigma\in E_{n,1} \\ \mathrm{exc}(\sigma)=n-k-1}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} \\
&\qquad = \sum_{\substack{\sigma\in\mathcal{D}_n \\ \mathrm{exc}(\sigma)=n-k-1}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}.
\end{aligned}$$

Then,

$$H\Big(A\big(M_r(\mathbb{F}_q^n)\big), t\Big) = \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{n-1-\mathrm{exc}(\sigma)} = \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)-1}$$

where the last equality follows from Poincaré duality of $A(M_{n-1}(\mathbb{F}_q^n))$. □

Remark 5.27. Note that the characterization of the Hilbert series of $A(M_r(\mathbb{F}_q^n))$ for $r = n-1, n$ together with the results of [AHK15] give an alternate proof of the unimodality and symmetry of the polynomials

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}t^{\mathrm{exc}(\sigma)} \text{ and } \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}t^{\mathrm{exc}(\sigma)-1}.$$

It should be noted that in [SW10], Shareshian and Wachs prove more general statements; namely, the coefficients of the above polynomials are q-unimodal and, in fact, q-γ-nonnegative. That is, the differences of consecutive coefficients is not only positive when evaluated at a particular value of q, but it also lies in $\mathbb{N}[q]$ as a polynomial in q, and moreover, its γ-vector has coordinates in $\mathbb{N}[q]$ (see subsection 5.2.4 below).

5.2.3. *Charney-Davis Quantities of $A(M(\mathbb{F}_q^n))$.* The Charney-Davis quantities can be computed much the same as in the uniform case.

Theorem 5.28. *The Charney-Davis quantity of the chow ring $A\big(M_r(\mathbb{F}_q^n)\big)$ is*

$$1+[n]_q!\sum_{a=1}^{k}\frac{(-1)^a}{[n-2a]_q!}\Delta_a$$

for Δ_a the determinant

$$\Delta_a = \det\begin{pmatrix} \frac{1}{[2]_q!} & 1 & 0 & \cdots & 0 \\ \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \frac{1}{[2a-8]_q!} & \cdots & 1 \\ \frac{1}{[2a-2]_q!} & \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \cdots & \frac{1}{[2]_q!} \end{pmatrix}.$$

Proof. As in the uniform case, we proceed by induction on the rank. Let $CD(n,r) = h\Big(A\big(M_r(\mathbb{F}_q^n)\big),-1\Big)$. The formula for the Hilbert series from [FY04] is

$$CD(n,r) = 1+\sum_{\substack{\mathbf{r},\, r_k<r\\ \forall i, r_i-r_{i-1}\text{ is even}}}(-1)^{k(r)}\prod_{i=1}^{k(r)}\begin{bmatrix} n-r_{i-1}\\ r_i-r_{i-1}\end{bmatrix}_q.$$

As in the uniform case, we get a decomposition of the above as

$$\left\{1+\sum_{\substack{\mathbf{r},\, r_k<r-2\\ \forall i, r_i-r_{i-1}\text{ is even}}}(-1)^{k(r)}\prod_{i=1}^{k(r)}\begin{bmatrix} n-r_{i-1}\\ r_i-r_{i-1}\end{bmatrix}_q\right\}+\left\{\sum_{\substack{\mathbf{r},\, r_k=r-1\\ \forall i, r_i-r_{i-1}\text{ is even}}}(-1)^{k(r)}\prod_{i=1}^{k(r)}\begin{bmatrix} n-r_{i-1}\\ r_i-r_{i-1}\end{bmatrix}_q\right\}$$

where the former term is $CD(n, r-2)$ and the latter we denote by $T_{n,q}(r-1)$. Then, partitioning on the rank of the second largest element in the flag, one obtains the recurrence

$$T_{n,q}(2a) = -\sum_{b=1}^{a-1}\begin{bmatrix} n-2b\\ 2a-2b\end{bmatrix}_q T_{n,q}(2b) \text{ with initial condition } T_{n,q}(2)=\begin{bmatrix} n\\ 2\end{bmatrix}_q$$

Solving this system with Cramer's rule gives

$$T_{n,q}(2a) = (-1)^a \begin{bmatrix} n \\ 2 \end{bmatrix}_q \det \begin{pmatrix} \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q & 1 & 0 & \cdots & 0 \\ \begin{bmatrix} n-2 \\ 4 \end{bmatrix}_q & \begin{bmatrix} n-4 \\ 2 \end{bmatrix}_q & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} n-2 \\ 2a-4 \end{bmatrix}_q & \begin{bmatrix} n-4 \\ 2a-6 \end{bmatrix}_q & \begin{bmatrix} n-6 \\ 2a-8 \end{bmatrix}_q & \cdots & 1 \\ \begin{bmatrix} n-2 \\ 2a-2 \end{bmatrix}_q & \begin{bmatrix} n-4 \\ 2a-4 \end{bmatrix}_q & \begin{bmatrix} n-6 \\ 2a-6 \end{bmatrix}_q & \cdots & \begin{bmatrix} n-2a+2 \\ 2 \end{bmatrix}_q \end{pmatrix} \tag{11}$$

Rewriting the determinant in 11 by pulling out common factors in the numerator, resp. denominators, of each column, resp. row, gives

$$\begin{aligned} T_{n,q}(2a) &= (-1)^a \frac{[n]_q!}{[n-2a]_q!} \det \begin{pmatrix} \frac{1}{[2]_q!} & 1 & 0 & \cdots & 0 \\ \frac{1}{[4]_q!} & \frac{1}{[2]_q!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \frac{1}{[2a-8]_q!} & \cdots & 1 \\ \frac{1}{[2a-2]_q!} & \frac{1}{[2a-4]_q!} & \frac{1}{[2a-6]_q!} & \cdots & \frac{1}{[2]_q!} \end{pmatrix} \\ &= (-1)^a \frac{[n]_q!}{[n-2a]_q!} \Delta_a \end{aligned}$$

Then, the Charney-Davis quantities for $r = 2k+1$ odd are

$$\begin{aligned} CD(n,r) &= CD(n,r-2) + T_{n,q}(2k) = \cdots = CD(n,1) + \sum_{a=1}^{k} T_{n,q}(2a) \\ &= 1 + [n]_q! \sum_{a=1}^{k} \frac{(-1)^a}{[n-2a]_q!} \Delta_a \end{aligned}$$

□

Remark 5.29. Note that when $q = 1$, $(-1)^a \Delta_a|_{q=1} = \frac{1}{(2a)!} E_{2a}$, where E_{2a} denotes the $(2a)$th secant number. Hence, when $q = 1$,

$$\left. \frac{(-1)^a [n]_q!}{[n-2a]_q!} \Delta_a \right|_{q=1} = \binom{n}{2a} E_{2a},$$

and the formula agrees with the Charney-Davis quantity of $A(U_{n,r})$.

In analogy with the uniform case, the q-exponential generating functions of the Charney-Davis quantities, and the corresponding q-Euler numbers can be computed. Let

$$\sinh_q(t) = \sum_{n \geq 0} \frac{t^{2n+1}}{[2n+1]_q!} \quad \text{and} \quad \cosh_q(t) = \sum_{n \geq 0} \frac{t^{2n}}{[2n]_q!}$$

Set $\operatorname{sech}_q(t) = 1/\cosh_q(t)$ and $\tanh_q(t) = \sinh_q(t)/\cosh_q(t)$.

Put $E_{2n}(q) := (-1)^n [2n]_q! \Delta_n$ and

$$E_{2n+1}(q) := CD(2n+1, 2n+1) = 1 + [2n+1]_q! \sum_{a=1}^{n} \frac{(-1)^a}{[2n-2a+1]_q!} \Delta_a.$$

Proposition 5.30. *The following identities of q-exponential generating functions hold:*

$$\operatorname{sech}_q(t) = \sum_{n\geq 0} E_{2n,q}\frac{t^{2n}}{[2n]_q!}$$

$$\tanh_q(t) = \sum_{n\geq 0} E_{2n+1,q}\frac{t^{2n+1}}{[2n+1]_q!}.$$

Proof. First, consider the generating function

$$F(t) = \sum_{n\geq 0} E_{2n,q}\frac{t^{2n}}{[2n]_q!}.$$

Observe that by expanding by minors in the first column, Δ_n satisfies the recurrence

$$\Delta_n = \sum_{k=1}^{n}\frac{(-1)^{n+1}}{[2n]_q!}\Delta_{n-k}$$

Then,

$$\begin{aligned} F(t) &= \sum_{n\geq 0}(-1)^n t^{2n}\Delta_n = 1 + \sum_{n\geq 1}(-1)^n t^{2n}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{[2k]_q!}\Delta_{n-k} \\ &= 1 + \sum_{r\geq 0}\sum_{k\geq 1}(-1)^{r+1}\Delta_r\frac{t^{2(r+k)}}{[2k]_q!} \\ &= 1 + \left(\sum_{k\geq 1}\frac{t^{2k}}{[2k]_q!}\right)\left(\sum_{r\geq 0}(-1)^{r+1}\Delta_r t^{2r}\right) = 1 - (\cosh_q(t) - 1)F(t) \end{aligned}$$

Therefore, solving for $F(t)$ gives

$$F(t) = 1/\cosh_q(t) = \operatorname{sech}_q(t)$$

Now consider the latter generating function. It follows that

$$\begin{aligned} \sum_{n\geq 0} E_{2n+1,q}\frac{t^{2n+1}}{[2n+1]_q!} &= \left(\sum_{n\geq 0}\frac{t^{2n+1}}{[2n+1]_q!}\right)\left(\sum_{k\geq 0}(-1)^k\Delta_k t^{2k}\right) \\ &= \sinh_q(t)/\cosh_q(t) = \tanh_q(t) \end{aligned}$$

□

In analogy with the classical tangent/secant numbers, we have the following recurrences:

Proposition 5.31. *For any integer $n \geq 0$,*

$$E_{2n,q} = -\sum_{k=0}^{n-1}\begin{bmatrix}2n\\2k\end{bmatrix}_q E_{2k,q} \tag{12}$$

$$E_{2n+1}, q = \sum_{k=0}^{n}\begin{bmatrix}2n+1\\2k\end{bmatrix}_q E_{2k,q} \tag{13}$$

Proof. Consider equation 12. Expanding the product $1 = \operatorname{sech}_q(t)\cosh_q(t)$ gives

$$\begin{aligned}\operatorname{sech}_q(t)\cosh_q(t) &= \left(\sum_{n\geq 0} E_{2n,q}\frac{t^{2n}}{[2n]_q!}\right)\left(\sum_{n\geq 0}\frac{t^{2n}}{[2n]_q!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} E_{2n,q}\begin{bmatrix}2n\\2k\end{bmatrix}_q\right)\frac{t^{2n}}{[2n]_q!}\end{aligned}$$

Hence, equating coefficients gives 12. Now, consider equation 13. We expand the product

$$\begin{aligned}\tanh_q(t) &= \operatorname{sech}_q(t)\sinh_q(t)\\ &= \left(\sum_{n\geq 0} E_{2n,q}\frac{t^{2n}}{[2n]_q!}\right)\left(\sum_{n\geq 0}\frac{t^{2n}}{[2n]_q!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} E_{2k,q}\begin{bmatrix}2n+1\\2k\end{bmatrix}_q\right)\frac{t^{2n+1}}{[2n+1]_q!}\end{aligned}$$

Hence, equating coefficients gives equation 13. □

Remark 5.32. The generating functions in proposition 5.30 imply that the numbers $E_{n,q}$ agree with the q-secant and q-tangent numbers introduced in [FH10]. In particular, we also have

$$E_{n,q} = \sum_{\sigma\in\mathfrak{I}_n} q^{\operatorname{exc}(\sigma)}$$

where $\mathfrak{I}_n$ denotes the number of alternating permutations of size n. The determinantal formula given above and the recurrences of proposition 5.31 do not yet appear in the literature to the authors' knowledge.

5.2.4. *γ-polynomials for $M(\mathbb{F}_q^n)$ and $M_{n-1}(\mathbb{F}_q^n)$.* For $\sigma\in\mathfrak{S}_n$, say i is a double descent of σ if $\sigma(i) > \sigma(i+1) > \sigma(i+2)$. We say that σ has a final descent if $\sigma(n-1) > \sigma(n)$ and an initial descent if $\sigma(1) > \sigma(2)$.

The γ-vectors of $M_r(\mathbb{F}_q^n)$ were considered in [SW17] for $r = n-1, n$. In particular, the following results were presented.

Theorem 5.33 ([SW17], thm 4.4)**.** *Let*

$$\Gamma_{n,k} = \left\{\sigma\in\mathfrak{S}_n \;:\; {\scriptstyle \sigma \text{ has no double descents, no} \atop \text{final descent, and } \operatorname{des}(\sigma)=k}\right\}.$$

The γ-vector of $H\big(A(M(\mathbb{F}_q^n)),t\big) = \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)}t^{\operatorname{exc}(\sigma)}$ is $\gamma = (\gamma_{n,k}(q))$ for

$$\gamma_{n,k}(q) = \sum_{\sigma\in\Gamma_{n,k}} q^{\operatorname{inv}(\sigma)}$$

Theorem 5.34 ([SW17], thm 6.1)**.** *Let*

$$\Gamma^0_{n,k} = \left\{\sigma\in\mathfrak{S}_n \;:\; {\scriptstyle \sigma \text{ has no double descents, no initial} \atop \text{descent, no final descent, and } \operatorname{des}(\sigma)=k}\right\}.$$

The γ-vector of $H\big(A(M_{n-1}(\mathbb{F}_q^n)),t\big) = \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$ is $\gamma = (\gamma^0_{n,k}(q))$, where

$$\gamma^0_{n,k}(q) = \sum_{\sigma\in\Gamma^0_{n,k}} q^{\mathrm{inv}(\sigma)}$$

6. Conjectures and future work

6.1. **Relationship between order complexes and Chow rings.** Let $\Delta(P)$ be the order complex of a poset P and let $h(\Delta(P),t)$ be its h polynomial.

Proposition 6.1. *For all $n \geq 1$,*

$$h\big(\Delta(\mathcal{L}(U_{n,n})),t\big) = H\big(A(U_{n,n}),t\big)$$

(See [Pet15] thm 9.1 or `https://oeis.org/A008292`.) For the corresponding statement for the uniform matroids $U_{n,r}$ with $r < n$ has small counterexamples, but can be modified as follows.

Conjecture 6.2. *For $r < n$, we have*

$$h\big(\Delta(\mathcal{L}(U_{n,r})),t\big) = t^2 \sum_{i=1}^{r} \binom{n-i-1}{r-i} H(A(U_{n,i}),t).$$

Since it is relatively simple to compute the f-vector of $\Delta(\mathcal{L}(U_{n,r}))$, this also gives a formula for $H(A(U_{n,i+1}),t)$.

Remark 6.3. The shift by t^2 is explained by the fact that the poset we are dealing with has a unique maximum and unique minimum element. For a simplicial complex S, let CS denote the simplicial complex obtained by taking a cone over S. Now, let P be a finite poset with unique maximal element $\top$ and unique minimal element $\bot$. Then $\Delta(P) = CC\Delta(P - \{\top,\bot\})$, since every element of P is comparable to both $\top$ and $\bot$. When we cone a simplicial complex, we multiply its f-polynomial by $t+1$, so its h-polynomial is multiplied by t; here, we cone twice, so we multiply the h-polynomial by t^2.

Remark 6.4. Conjecture 6.2 is equivalent to the equality $F_n(t,u) = H_n(t,u+1)$ for the polynomials

$$F_n(t,u) = \sum_{r=0}^{n-2} h(\Delta(\mathcal{L}(U_{n,r+1} \setminus \{\top,\bot\})),t)u^{n-2-r}$$
$$H_n(t,u) = \sum_{r=0}^{n-2} H(A(U_{n,r+1}),t)u^{n-2-r}$$

Remark 6.5. Conjecture 6.2 above can be interpreted combinatorially as follows. Let

$$h(\Delta(\mathcal{L}(U_{n,r})),t) = h_0 + h_1 t + \cdots + h_{r-1}t^{r-1}.$$

Expanding the right hand side and using Corollary 5.6 gives

$$\begin{aligned}
h_k &= \sum_{i=k+1}^{r} \binom{n-i-1}{r-i} \dim_{\mathbb{Z}} A(U_{n,i})_k \\
&= A_{n,k}\left[\sum_{i=k+1}^{r} \binom{n-i-1}{r-i}\right] - \sum_{j=k+1}^{n-1}\left[\sum_{i=k+1}^{j} \binom{n-i-1}{r-i}\right] \cdot \#\{\sigma \in E_{n,n-j} \mid \operatorname{exc}(\sigma) = j-k\}
\end{aligned}$$

To compute the f vector of $\Delta(\mathcal{L}(U_{n,r}))$, we need the following notions from [BGS82].

Definition 6.6. Let P be a graded poset. A $\mathbb{Z}$-valued labeling λ of the covering relations $x \prec y$ in P is called an *EL-labeling* if for every interval $[x, y]$, there is a unique chain $a_{[x,y]} \colon x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ such that $\lambda(x_0 \prec x_1) \leq \cdots \lambda(x_{k-1} \prec x_k)$ and for every other chain $b \colon x = y_0 \prec y_1 \prec \cdots \prec y_k = y$,

$$\lambda(b) = \Big(\lambda(y_0 \prec y_1), \ldots, \lambda(y_{k-1} \prec y_k)\Big) >_L \Big(\lambda(x_0 \prec x_1), \ldots, \lambda(x_{k-1} \prec x_k)\Big)\lambda(a_{[x,y]})$$

under the lexicographic ordering.

A graded poset P is called *EL-shellable* is it admits an EL-labeling.

Definition 6.7. Let P be a rank $d+1$ EL-shellable bounded ranked poset with EL labeling λ. For any maximal chain $m \colon \bot = x_0 \prec \cdots \prec x_{d+1} = \top$, the *Descent set* of M is the set

$$\mathcal{D}(m) = \{i \in [d] \,:\, \lambda(x_{i-1} \prec x_i) > \lambda(x_i \prec x_{i+1})\}\,.$$

Let P be a graded poset of rank $d+1$ with unique minimal and maximal elements $\bot, \top$. For $S \subseteq [d]$, set

$$P_S = \{x \in P \,:\, x = \bot,\, x = \top, \text{ or } \operatorname{rank}(x) \in S\}$$

Then, for any such $S \subseteq [d]$, let $\alpha(S)$ be the number of maximal chains in P_S and set

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T)$$

By inclusion-exclusion, this is equivalent to

$$\alpha(S) = \sum_{T \subseteq S} \beta(T)$$

Theorem 6.8. *([BGS82], thm 2.2) For $S \subseteq [d]$, $\beta(S)$ equals the number of maximal chains m in P with $\mathcal{D}(m) = S$.*

In our case, $\mathcal{L}(U_{n,r})$ is EL-shellable with shelling given by letting $\lambda(S \prec S \cup \{x\}) = x$ for $\operatorname{rank}(S) < r$ and $\lambda(S < \top) = \min\{x \in [d] \,:\, x \notin S\}$. Let $\operatorname{des}(S)$ denote the number of maximal chains m with $\mathcal{D}(m) = S$. Since $\mathcal{L}(U_{n,r})$ is EL-shellable, the f vector of its order complex can be reduced to counting maximal chains:

$$\operatorname{des}(S) = \begin{cases} \displaystyle\sum_{k=1}^{r-1} \#\left\{m \subseteq \mathcal{L}(U_{r-1,r-1}) \,:\, {\scriptstyle m \text{ is maximal} \atop \scriptstyle \operatorname{des(m)}=S-\{r\},\, m_{r-1}=k}\right\} \left[\binom{n}{r-1} - \binom{n}{r-k-1}\right] & \text{if } r \in S \\ \displaystyle\sum_{k=1}^{r-1} \#\left\{m \subseteq \mathcal{L}(U_{r-1,r-1}) \,:\, {\scriptstyle m \text{ is maximal} \atop \scriptstyle \operatorname{des(m)}=S,\, m_{r-1}=k}\right\} \binom{n}{r-k-1} & \text{if } r \notin S \end{cases}$$

Therefore,

$$
\begin{aligned}
f_i &= \sum_{\substack{S\subseteq[d]\\ \#S=i}} \alpha(S) = \sum_{\substack{S\subseteq[d]\\ \#S=i}} \sum_{T\subseteq S} \beta(S) \\
&= \sum_{\substack{\#S=i\\ r\in S}} \sum_{r\in T\subseteq S} \sum_{k=1}^{r-1} \binom{n}{r-1} \#\left\{m \subseteq \mathcal{L}(U_{r-1,r-1}) \,:\, {}^{\;m \text{ is maximal}}_{\text{des(m)}=T-\{r\},\, m_{r-1}=k}\right\} \\
&\quad + \sum_{\substack{\#S=i\\ r\notin S}} \sum_{T\subseteq S} \sum_{k=1}^{r-1} \binom{n}{r-k-1} \#\left\{m \subseteq \mathcal{L}(U_{r-1,r-1}) \,:\, {}^{\;m \text{ is maximal}}_{\text{des(m)}=T,\, m_{r-1}=k}\right\} \\
&= \sum_{j=0}^{i} \binom{r-j-1}{i-j} \sum_{k=1}^{r-1} \binom{n}{r-k-1} \#\{\sigma \in \mathfrak{S}_{r-1} \,:\, \mathrm{exc}(\sigma)=j,\ \sigma(r-1)=k\} \\
&\quad + \binom{n}{r-1} \sum_{j=0}^{i-1} \binom{r-j-1}{i-j-1} A_{r-1,j}
\end{aligned}
$$

This gives a combinatorial formula equivalent to the stated conjecture, and suggests a possible avenue for verification.

6.2. **Other posets.** One can define Chow rings entirely in terms of atomic lattices, so it is natural to ask whether the Chow rings of atomic lattices not arising from matroids might have nice structure. We offer some suggestions on atomic lattices to consider.

Face lattices of polytopes: Experimentally, all of the Hilbert series of face lattices of polytopes have been symmetric. The experiments have included cubes, cyclic polytopes, cross polytopes, and some flow polytopes. Note that while some of the application of results from [AHK15] apply to face lattices of pure simplicial complexes adjoined with a top element, these do not necessarily apply to face lattices of polytopes.

Convex closure: Given a set of points $X \subseteq \mathbb{R}^n$, one can define a lattice whose elements are $C \cap X$ where $C \subseteq \mathbb{R}^n$ is convex, and the subsets are ordered by inclusion. Experimentally, all Hilbert series of these lattices have been symmetric.

Operations on matroids and lattices: One could consider how the Chow rings and corresponding Hilbert series of products of lattices, star products of ranked lattices, and direct sums of matroids can be expressed in terms of the Hilbert series of the smaller Chow rings. We have already considered the case of products to some extent, but further work should be done.

Appendix A. Poincaré duality for ranked atomistic lattices

Here, we provide some details to support the assertions made in Section 3. We provide minimal discussion, only including places where proofs or statements must be modified from the original ones presented in Adiprisito, Huh, and Katz's work. To compare these to the originals, see Section 6 of [AHK15].

Lemma A.1. *For a nice lattice L with $Z \in L$ and α an atom of L,*

$$\operatorname{rank}(Z \wedge \alpha) \leq \operatorname{rank}(L) + 1.$$

Proof. Suppose for some element $Z \in L$ and atom $\alpha \in L$, $\operatorname{rank}(Z \wedge \alpha) > \operatorname{rank}(L) + 1$. Then,

$$d(Z, Z \wedge \alpha) = 1 \text{ while } \operatorname{rank}(Z \wedge \alpha) - \operatorname{rank}(Z) > 1,$$

contradicting the assumption that L is nice. □

Lemma A.2. *Suppose that $F \in L$ has $r = \operatorname{rank}(F)$, and say $I \subseteq F$ has $\operatorname{cl}_L(I) = F$. Then, there exists $J \subseteq I$ such that $\operatorname{cl}_L(J) = F$ and $J \in \mathcal{I}(L)$, i.e. $\#J = r$.*

Proof. Since $\operatorname{cl}_L(I) = F$, we can write $F = \bigwedge_{i=1}^{N} \alpha_i$ for $I = \{\alpha_1, \ldots, \alpha_n\}$. Then, by Lemma A.1, taking joins with an atom α_i either increases the rank by one or leads to equality. Hence, for every n, we can remove α_i if $\operatorname{rank}(\bigwedge_{i=1}^{n-1} \alpha_i) = \operatorname{rank}(\bigwedge_{i=1}^{n} \alpha_i)$. Doing so must leave exactly r distinct indices i_j such that $F = \bigwedge_{j=1}^{r} \alpha_{i_j}$. Setting $J = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ gives the desired set. □

Let L be a finite, ranked, atomistic lattice with E the set of L's atoms. Consider the rings $A(L, \mathscr{P})$ for $\mathscr{P}$ an order filter in L as in Section 3. Observe that when $\mathscr{P} = \varnothing$, the relations in $\mathcal{J}_4$ imply $x_i = x_j$ for $i, j \in E$. Hence $A(M, \varnothing) = \mathbb{Z}[x]/(x^{r+1})$. Also observe that, when $\mathscr{P} = \mathscr{P}(L) = L - \{\perp, \top\}$, the relations in $\mathcal{J}_3$ together with the observation that $\{i\} \in \mathcal{I}(L)$ imply $x_i = 0 \in A(M, \mathscr{P}(M))$ for all i. Hence, $A(L, \mathscr{P}(M)) = A(L)$.

Proposition A.3 (compare with [AHK15], prop. 6.2)**.** *If $I \subseteq E$ and $F \in \mathscr{P}$, then*

(1) *If I has cardinality at least the rank of F, then*

$$\left(\prod_{i \in I} x_i\right) x_F = 0 \in A(L, \mathscr{P})$$

(2) *If I has cardinality at least $r + 1$, then*

$$\prod_{i \in I} x_i = 0 \in A(L, \mathscr{P})$$

Proof. Same as in [AHK15] except for the base case of induction. Namely, in the case $\operatorname{rank}(I) = \operatorname{rank}(F)$, we can assume (by relations in $\mathcal{J}_2$) that $I \subseteq F$. In this case, lemma A.2 implies the existence of a subset J of I of cardinality and rank exactly $\operatorname{rank}(F)$, i.e. $\operatorname{cl}_L(J) = F$ and $J \in \mathcal{I}(L)$. Hence, both products are zero in the base case. □

Proposition A.4 (compare with [AHK15], prop. 6.6)**.** *The pullback homomorphism*

$$\Phi_Z \colon A(L, \mathscr{P}_-) \to A(L, \mathscr{P}_+)$$

taking $x_F \mapsto x_F$ and

$$x_i \mapsto \begin{cases} x_i + x_F & \text{if } i \in Z \\ x_i & \text{if } i \notin Z \end{cases}$$

is well-defined.

Proof. It suffices to show that for $\phi_Z\colon S_{E\cup\mathscr{P}_-} \to S_{E\cup\mathscr{P}_+}$ the corresponding map of polynomial rings, we have $\phi_Z(\mathcal{J}_3) \subseteq \mathcal{J}_2 + \mathcal{J}_3$.

If $I \in \mathcal{I}(L)$ has $\mathrm{cl}_L(I) \in \mathscr{P}_- \cup \{E\}$, then

$$\phi_Z\left(\prod_{i\in I} x_i\right) = \prod_{i\in I\setminus Z} x_i \prod_{i\in I\cap Z}(x_i + x_Z)$$

Then, in the polynomial ring $S_{E\cup\mathscr{P}_+}$, we have $\prod_{i\in I} x_i \in \mathcal{J}_3$. Moreover, because Z is minimal, $I \setminus Z \neq \varnothing$ so $\left(\prod_{i\in I\setminus Z} x_i\right) x_Z \in \mathcal{J}_2$. The rest of the proof is identical to [AHK15], prop. 6.6. ☐

Adiprisito, Huh, and Katz go on to prove a string of lemmas, each holding verbatim with references to M replaced by a nice atomistic lattice L, and references to bases of M replaced by maximal sets (under inclusion) in $\mathcal{I}(L)$. These lemmas are used to prove first that Φ_Z is an isomorphism in rank 1 and $\Phi_Z^q \oplus \bigoplus_{p=1}^{\mathrm{rank}(Z)-1} \Phi_Z^{p,q}$ is a surjective group homomorphism (see Propositions 3.3 and 3.4 for definitions of these maps). Finally, an inductive proof is given of the main theorems, stated here in Section 3, Theorems 3.5 and 3.6. As these latter steps and their proofs for matroids do not make reference to the relations in $\mathcal{J}_3$, they go through unchanged for more the more general class of finite nice atomistic lattices.

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References

[AHK15] K. Adiprasito, J. Huh, and E. Katz. “Hodge Theory for Combinatorial Geometries”. In: *ArXiv e-prints* (Nov. 2015). arXiv: `1511.02888 [math.CO]`.

[Ath16] Christos A Athanasiadis. “A survey of subdivisions and local h-vectors”. In: *The Mathematical Legacy of Richard P. Stanley* (2016), pp. 39–52.

[Bak17] M. Baker. “Hodge theory in combinatorics”. In: *ArXiv e-prints* (May 2017). arXiv: `1705.07960 [math.CO]`.

[BGS82] Anders Björner, Adriano M Garsia, and Richard P Stanley. “An introduction to Cohen-Macaulay partially ordered sets”. In: *Ordered sets*. Springer, 1982, pp. 583–615.

[CD95] Ruth Charney and Michael Davis. “The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold”. In: *Pacific Journal of Mathematics* 171.1 (1995), pp. 117–137.

[DE10] E. Deutsch and S. Elizalde. “The largest and the smallest fixed points of permutations”. In: *European Journal of Combinatorics* 31 (July 2010), pp. 1404–1409. DOI: `https://doi.org/10.1016/j.ejc.2009.12.002`. eprint: `math/09042792`.

[FH10] Dominique Foata and Guo-Niu Han. “The q-tangent and q-secant numbers via basic Eulerian polynomials”. In: *Proceedings of the American Mathematical Society* 138.2 (2010), pp. 385–393.

[FY04] E. M. Feichtner and S. Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". In: *Inventiones Mathematicae* 155 (Mar. 2004), pp. 515–536. DOI: 10.1007/s00222-003-0327-2. eprint: math/0305142.

[McM93] Peter McMullen. "On simple polytopes". In: *Inventiones mathematicae* 113.1 (Dec. 1993), pp. 419–444. ISSN: 1432-1297. DOI: 10.1007/BF01244313. URL: https://doi.org/10.1007/BF01244313.

[Pet15] T Kyle Petersen. *Eulerian numbers*. Birkhäuser, 2015.

[PRW06] Alexander Postnikov, Victor Reiner, and Lauren Williams. "Faces of generalized permutohedra". In: *Doc. Math* 13.207-273 (Sept. 2006), p. 51. DOI: https://doi.org/10.1016/j.ejc.2009.12.002. eprint: math/0609184.

[RW05] Victor Reiner and Volkmar Welker. "On the Charney–Davis and Neggers–Stanley Conjectures". In: *Journal of Combinatorial Theory, Series A* 109.2 (2005), pp. 247–280.

[Sta+04] Richard P Stanley et al. "An introduction to hyperplane arrangements". In: *Geometric combinatorics* 13 (2004), pp. 389–496.

[Sta12] Richard P. Stanley. *Enumerative combinatorics. Volume 1.* Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626. ISBN: 978-1-107-60262-5.

[Sta96] Richard P. Stanley. *Combinatorics and commutative algebra*. Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996, pp. x+164. ISBN: 0-8176-3836-9.

[Sta97] Richard P Stanley. *Enumerative Combinatorics. Vol. 1, vol. 49 of Cambridge Studies in Advanced Mathematics*. 1997.

[Sun05] Zhi-Wei Sun. "On Euler numbers modulo powers of two". In: *Journal of Number Theory* 115.2 (2005), pp. 371–380.

[SW07] John Shareshian and Michelle Wachs. "q-Eulerian polynomials: excedance number and major index". In: *Electronic Research Announcements of the American Mathematical Society* 13.4 (2007), pp. 33–45.

[SW10] John Shareshian and Michelle L Wachs. "Eulerian quasisymmetric functions". In: *Advances in Mathematics* 225.6 (2010), pp. 2921–2966.

[SW17] John Shareshian and Michelle L Wachs. "Gamma-positivity of variations of Eulerian polynomials". In: *arXiv preprint arXiv:1702.06666* (2017).

[Wac89] Michelle L Wachs. "On q-derangement numbers". In: *Proceedings of the American Mathematical Society* 106.1 (1989), pp. 273–278.

[WWZ17] J. D. Wiltshire-Gordon, A. Woo, and M. Zajaczkowska. "Specht Polytopes and Specht Matroids". In: *ArXiv e-prints* (Jan. 2017). arXiv: 1701.05277 [math.CO].