Undecidability for arbitrary public announcement logic

TIM FRENCH AND HANS VAN DITMARSCH

ABSTRACT. *Arbitrary public announcement logic* ($APAL$) is an extension of multi-agent epistemic logic that allows agents' knowledge states to be updated by the public announcement of (possibly arbitrary) epistemic formulae. It has been shown to be more expressive than epistemic logic, and a sound and complete axiomatization has been given. Here we address the question of decidability. We present a proof that the satisfiability problem for *arbitrary public announcement logic* ($APAL$) is co-RE complete, via a tiling argument.

Keywords: Epistemic Logic, Public Announcement Logic, Decidability

1 Introduction

Arbitrary announcement logic ($APAL$) is an extension of multi-agent epistemic logics with *public announcements* and *arbitrary announcements*. A public announcement allows the information state of every agent to be updated by publicly informing them that some epistemic formula, ψ is true. An arbitrary announcement is added to the language to allow us to quantify over all possible announcements. This logic is described in detail in [1]. It is shown to be more expressive than normal epistemic logic and a sound and complete axiomatization is given. Furthermore, in [2] a tableau-calculus is presented to determine validity in $APAL$.

While the above results indicate that the set of validities for $APAL$ is recursively enumerable, the full decidability of the satisfiability problem has remained open. Here we show that the satisfiability problem for the logic is undecidable via a tiling argument. This is a surprising result since in [1] it is shown that $APAL$ is bisimulation invariant. Hence every $APAL$ formula is satisfied by a tree-like model, rather than the grid-like models typically required for tiling arguments (see [9] for a detailed analysis of undecidability in extensions of epistemic logic). The undecidability follows from the power of the arbitrary announcement operator. The arbitrary announcement operator, $\Diamond\phi$ expresses:

> *"there exists a true formula of epistemic logic, that when publicly announced establishes the truth of ϕ."*

Implicit in this statement is an existential quantification over *all* formulae of epistemic logic, and we show that this expressive power is sufficient to allow us to encode an undecidable tiling problem. This is not an entirely surprising result, despite the many other favorable properties of $APAL$. In [9] a detailed survey is presented of undecidable temporal and epistemic logics, and an analysis is presented of the properties leading to undecidability. The arbitrary announcement operator is transitive in nature and reminiscent of a temporal operator. However, most undecidable logics surveyed in [9] are not bisimulation invariant, indicating a certain uniqueness to this result. Another related result is the undecidability of iterated modal relativization [7]. This logic is shown to be highly undecidable (Σ^1_1-complete), again, by encoding a tiling problem. Other undecidable logics considered in [7] combine *common knowledge* with iterated relativization. 'Relativization' is another term to denote the structural restriction that constitutes the informative effect of an announcement. Iterated relativization is different from arbitrary announcement. The former means that one allows (arbitrary finite length) sequences of model restrictions for a given epistemic formula ('announcement'); note that after announcements of a modal formula, announcing that formula again may still be informative, as in the famous 'Muddy Children Problem' [4]. But the latter means model restriction for *any* epistemic formula. Now the iteration is implicit. It is there because the sequence of two epistemic announcements is again equivalent to an epistemic formula [1].

2 Syntax and semantics

The formulas of $APAL$, $\mathcal{L}_{apal}$ are inductively defined as

$$\phi ::= \ p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_a\phi \mid [\phi]\phi \mid \Box\phi$$

where a is taken from the set of agents A, and p is taken from the set of atomic propositions P. Let $\mathcal{L}_{el}$ be the set of formulas not containing any of the operators $[\phi]$ or $\Box$.

These formulas are interpreted over structures $M = (S, \sim, V)$ where S is a set of worlds, $\sim: A \longrightarrow \wp(S \times S)$ assigns a reflexive, transitive and symmetric accessibility relation, $\sim_a$ to each agent a, and $V : P \longrightarrow \wp(S)$ maps each proposition to the set of worlds where it is true.

Let $M = (S, \sim, V)$ and suppose that $s \in S$. The semantics of $APAL$ are given as:

$$\begin{array}{rll}
M, s \models p & \text{iff} & s \in V(p) \\
M, s \models \neg\phi & \text{iff} & M, s \not\models \phi \\
M, s \models \phi_1 \wedge \phi_2 & \text{iff} & M, s \models \phi_1 \text{ and } M, s \models \phi_2 \\
M, s \models K_a\phi & \text{iff} & \forall t \in S \text{ where } s \sim_a t,\ M, t \models \phi \\
M, s \models [\psi]\phi & \text{iff} & M, s \models \psi \Longrightarrow M^\psi, s \models \phi \\
M, s \models \Box\phi & \text{iff} & \forall \psi \in \mathcal{L}_{el},\ M, s \models [\psi]\phi
\end{array}$$

where $M^\psi = (S', \sim', V')$ is such that: $S' = \{s \in S \mid M, s \models \psi\}$; for all $a \in A$, $\sim'_a = \sim_a \cap (S' \times S')$; and for all $p \in P$, $V'(p) = V(p) \cap S'$. As usual we take $K_a\phi$ to mean *agent a knows* ϕ, and let $L_a\phi$ abbreviate $\neg K_a\neg\phi$ (*agent a considers* ϕ *possible*).

We say an $APAL$ formula ϕ is satisfiable if there exists some model $M = (S, \sim, V)$ and some world $s \in S$ such that $M, s \models \phi$, and if $M, s \models \phi$ for all model-world pairs, M, s, we say ϕ is valid.

Note that when defining the semantics of $\Box\phi$ we restrict the arbitrary announcements to range only over the epistemic formulas (i.e. those in $\mathcal{L}_{el}$). We reason for this is that we obviously cannot allow the arbitrary announcements to range over arbitrary announcements (i.e. formulae of the form $\Box\psi$) as the semantics would then be undefined. Further, we do not let the arbitrary announcements range over announcements (such as $[\psi]\alpha$ where $\psi,\ \alpha \in \mathcal{L}_{el}$) since such formulas are expressively equivalent to pure epistemic formulae (see [11] for a translation).

The formula $\Box\phi$ expresses the statement *"after publicly announcing any true formula of epistemic logic, ϕ must be true."* As we see in its formal semantics above, this statement implicitly quantifies over all true formulae of epistemic logic. For example, suppose ϕ were the formula $K_ap \rightarrow K_bp$. The formula $\Box\phi$ is true at some world where p is true, if and only if for every b-related world, u where p is not true, *for every epistemic formula* ψ, there is some a-related world, v, that agrees with u on the interpretation of ψ. (This is because otherwise the announcement of $p \vee \psi$ would be enough to make $K_ap \wedge \neg K_bp$ true). This is a strong property to be able to express. If two sets of worlds cannot be distinguished by any epistemic formula then they are, for the purposes the logic, identical. Given that the epistemic formulae can be arbitrarily large, using this notion of equivalence we are able to encode a grid-like property for finite grids of arbitrary size. This expressivity is exploited to encode an arbitrary tiling problem which is sufficient to show that the satisfiability problem for $APAL$ is co-RE complete.

3 Tilings and undecidability

We show the satisfiability problem is undecidable for $APAL$ by embedding a tiling problem that is known to be co-RE complete (i.e. equivalent to computing the membership of the complement of any recursively enumerable set).

The tiling problem is as follows:

DEFINITION 1. Let C be a finite set of *colours* and define a C-tile to be a four-tuple over C $\gamma = (\gamma^t, \gamma^r, \gamma^f, \gamma^\ell)$, where the elements are referred to as, respectively, *top, right, floor* and *left.* The tiling problem is, for any given finite set of C-tiles, Γ, determine if there is a function $\lambda : \omega \times \omega \longrightarrow \Gamma$ such that for all $(i, j) \in \omega \times \omega$:

1. $\lambda(i, j)^t = \lambda(i, j + 1)^f$
2. $\lambda(i, j)^r = \lambda(i + 1, j)^\ell$.

The tiling problem has been shown to be co-RE complete by Harel [6] (see [8] for an overview of the application of tiling problems to complexity for modal logics).

4 Encoding the tiling problem

To encode the tiling problem we seek to define a grid like structure in the model M. That is we define a formula *grid* whereby $M, s \models grid$ implies the structure of M is similar to $\omega \times \omega$. To do this we exploit one of the stronger properties of $APAL$: the ability to quantify over *all* epistemic formulas. This allows us to define an equivalence between the worlds in a model and modulo that equivalence, a grid like structure.

Such encodings are rarely elegant and this is no exception. We use the following atoms:

1. We label each world as either *white* (W) or *black* (B) with the understanding that B is an abbreviation for $\neg W$. We intend to label the model in chess-board pattern.
2. We use the set of agents a, b, c, d and t where we suppose that:
   - a and b describe some vertical successor relation (a goes from a black square to a white square, and b goes from a white square to a black square);
   - c and d describe some horizontal successor relation (c goes from a black square to a white square, and d goes from a white square to a black square);
   - the relation for the agent t includes the relations for the agents a, b, c and d.
3. For each tile $\gamma \in \Gamma$ we assign a proposition (also denoted γ) with the understanding that the tiles are mutually exclusive (i.e. $\gamma \rightarrow \bigwedge_{\delta \neq \gamma} \neg\delta$).

Such a structure is represented in Figure 1 (with the assumption that $\sim_t$ is a universal relation over all worlds).

Even though our accessibility relations are equivalence relations, in the multi-agent setting we can enforce directionality by composing equivalence relations for different agents (and grounding them by referring to truths in local or boundary conditions of our structure, such as the actual state, or the top-left state, or ...). More formally, even though $\sim_a$ and $\sim_d$ are equivalence relations, their composition $\sim_a \circ \sim_d$ is not symmetric: we may have that $x \sim_a y \sim_d z$, i.e., $x(\sim_a \circ \sim_d)z$, but not $z(\sim_a \circ \sim_d)x$. Although we were not inspired by this, it deserves mentioning that such emerging asymmetry in multi-agent conditions is used to great effect in the expressivity proofs in Chapter 8 of [11].

The encoding comes in three parts: Firstly, we would like to define t to contain the transitive closure of the other epistemic relations. Next, we define a grid like (or chessboard like) structure over the model. Finally we use this grid-like model to state that the given tiling exists.

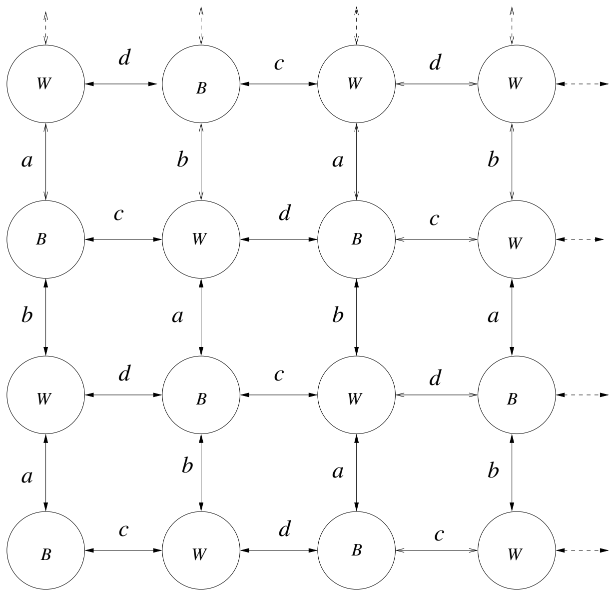

Figure 1. A grid-like model.

First note that these descriptions grossly over-simplify the actual construction. To properly execute these steps we would require a mechanism that allows us to define when one world is equivalent to another. This we do not have. However, the arbitrary announcement mechanism allows us to identify when two sets of worlds cannot be distinguished by any announcement. We will, for the moment ignore these considerations. We will use the term *equivalent* (rather than equal) to describe to worlds that are indistinguishable with respect to epistemic logic, and we will precisely define this notion in the subsequent sections.

4.1 Weak transitive closure

The following formula sets $\sim_t$ to include a weak transitive closure of $\sim_a$, $\sim_b$, $\sim_c$ and $\sim_d$. Particularly at every world w in the model, for each agent $x \in \{a,\ b,\ c,\ d\}$, if there is some world w' of a different colour to w where $w \sim_x w'$, then there is some world u where $w \sim_t u$ and u is equivalent to w'.

$$(1) \quad alt \quad = \quad K_t \begin{pmatrix} B \to (L_a W \wedge L_c W) \\ W \to (L_b B \wedge L_d B) \end{pmatrix}$$

$$(2) \quad T^* \quad = \quad K_t \Box \bigwedge \begin{pmatrix} K_t B \to (K_a B \wedge K_b B \wedge K_c B \wedge K_d B) \\ K_t W \to (K_a W \wedge K_b W \wedge K_c W \wedge K_d W) \end{pmatrix}.$$

The important part of this formula is the arbitrary announcement ($\Box$) in T^*. This states that no announcement can be made that informs agent t of the colour of the current square, without informing every other agent as well. To the contrary, suppose that this were not true. Specifically suppose the current world is *black*, and agent a considers some *white* world, u, possible where u was demonstrably different to every world t considers possible (say by formula χ_u). Then the public announcement $\chi_u \vee B$ would inform t that the current world is black, but not a.

4.2 Defining a grid

To define a grid-like structure we will require the following properties:

1. Every *black* world has an a-successor that is *white* and a c-successor that is *white*.
2. Every *white* world has a b-successor that is *black* and a d-successor that is *black*.
3. The current world is *black* and both b and d know this.
4. If the current world is *black*:
   - for every *white* world u that is a-reachable from the current world, every *black* world that is d-reachable from u is equivalent to some *black* world that is b-reachable from some *white* world that is c-reachable from the current world.
   - for every *white* world u that is c-reachable from the current world, every *black* world that is b-reachable from u is equivalent to some *black* world that is d-reachable from some *white* world that is a-reachable from the current world.
5. If the current world is *white*:
   - for every *black* world u that is b-reachable from the current world, every *white* world that is c-reachable from u is equivalent to some *white* world that is a-reachable from some *black* world that is d-reachable from the current world.
   - for every *black* world u that is d-reachable from the current world, every *white* world that is a-reachable from u is equivalent to some *white* world that is c-reachable from some *black* world that is b-reachable from the current world.

This is achieved with the following formulas:

$$st = B \wedge K_b B \wedge K_d B \tag{3}$$

$$C1 = B \rightarrow \Box\left((L_a(W \wedge L_d B)) \rightarrow (K_c(W \rightarrow L_b B))\right) \tag{4}$$

$$C2 = B \rightarrow \Box\left((L_c(W \wedge L_b B)) \rightarrow (K_a(W \rightarrow L_d B))\right) \tag{5}$$

$$C3 = W \rightarrow \Box\left((L_b(B \wedge L_c W)) \rightarrow (K_d(B \rightarrow L_a W))\right) \tag{6}$$

$$C4 = W \rightarrow \Box\left((L_d(B \wedge L_a W)) \rightarrow (K_b(B \rightarrow L_c W))\right) \tag{7}$$

Again, the arbitrary announcement is used to establish a notion of equivalence between two worlds. The formula *st* simply specifies the state of the initial world (the bottom, left hand corner of the grid which, to extend the chess-board analogy, is black). The other formulae $C1 - C4$ define a weak commutativity property (e.g. every *black* world that is $\sim_a\sim_d$ reachable from the current (*black*) world, is $\sim_c\sim_b$ reachable from the current world).

4.3 The existence of a tiling

Given the previous formulas are sufficient to set up the desired chessboard pattern, it is a simple matter to exploit it to assert the existence of a tiling. Suppose the set Γ is given as above. Let:

$$(8) \qquad blk \;=\; B \to \left(\bigwedge_{\gamma\in\Gamma} \left(\gamma \to \bigwedge \begin{bmatrix} K_a(W \to \bigvee_{\gamma^t=\delta^b} \delta)) \\ K_c(W \to \bigvee_{\gamma^r=\delta^\ell} \delta)) \end{bmatrix} \right) \right)$$

$$(9) \qquad wht \;=\; W \to \left(\bigwedge_{\gamma\in\Gamma} \left(\gamma \to \bigwedge \begin{bmatrix} K_b(B \to \bigvee_{\gamma^t=\delta^b} \delta)) \\ K_d(B \to \bigvee_{\gamma^r=\delta^\ell} \delta)) \end{bmatrix} \right) \right).$$

The interpretation of these formula is straightforward. Given a tile γ is true at the current state, we assert that the bottom of all successor vertical tiles [1] is the same colour as γ^t. In the case that the current state is *black* the successor vertical states are the a-reachable *white* states, and if the current state is *white* then all successor vertical states are the b-reachable *black* states. A similar characterization exists for the horizontal (*left-right*) correspondence.

Finally we can define the formula:

$$(10) \quad Tile_\Gamma = alt \wedge T^* \wedge st \wedge K_t(C1 \wedge C2 \wedge C3 \wedge C4 \wedge blk \wedge wht).$$

In the following section we show that the existence of a model for this formula is equivalent to the existence of a tiling of the ω-plane for γ.

5 Proof of correctness

In this section we show that the above formula, $Tile_\Gamma$, is satisfiable in $APAL$ if and only if the set of tiles Γ is able to tile the plane $\omega \times \omega$.

We first address the soundness of the construction of the formula $Tiles_\Gamma$.

LEMMA 2. *Given there is a Γ-tiling of the plane, $\lambda : \omega \times \omega \longrightarrow \Gamma$, we may define a model of $APAL$ that satisfies the formula $Tile_\Gamma$.*

Proof. This model is taken directly from the tiling see Figure 2 with the knowledge relation of t being the universal modality. That is we let our model be $M = (S, \sim, V)$ where:

- $S = \omega \times \omega$,

[1] Whilst the existence of multiple vertical successors is not very “grid-like” we will later show this is inconsequential.

- $V(B) = \{(i,j)|\ i+j \text{ is even}\}$ and $V(W) = \{(i,j)|\ i+j \text{ is odd}\}$,
- for each $\gamma \in \Gamma$, $V(\gamma) = \{(i,j)|\ \lambda(i,j) = \gamma\}$,
- $\sim_a$ is the transitive, reflexive and symmetric closure of the relation $\{((i,j),(i,j+1))|\ (i,j) \in V(B)\}$,
- $\sim_b$ is the transitive, reflexive and symmetric closure of the relation $\{((i,j),(i,j+1))|\ (i,j) \in V(W)\}$,
- $\sim_c$ is the transitive, reflexive and symmetric closure of the relation $\{((i,j),(i+1,j))|\ (i,j) \in V(B)\}$,
- $\sim_a$ is the transitive, reflexive and symmetric closure of the relation $\{((i,j),(i+1,j))|\ (i,j) \in V(W)\}$,
- $\sim_t = \{((i,j),(k,\ell))|\ i,j,k,\ell \in \omega\}$.

We now show that $M,(0,0) \models Tile_\Gamma$. For the parts of $Tile_\Gamma$ not containing arbitrary announcements this is straightforward. We can see that $M,(0,0) \models alt \wedge st$ by construction and as λ is a tiling it follows that $M,(0,0) \models K_t(blk \wedge wht)$. The remaining formulas T^* and $C1-C4$ involve arbitrary announcements. Let's first examine T^*. This is equivalent to, for all $i,j \in \omega$,

$$M,(i,j) \models \Box \bigwedge \begin{pmatrix} K_tB \rightarrow (K_aB \wedge K_bB \wedge K_cB \wedge K_dB) \\ K_tW \rightarrow (K_aW \wedge K_bW \wedge K_cW \wedge K_dW) \end{pmatrix}.$$

Suppose that $(i,j) \in V(B)$. If any announcement $[\phi]$ makes K_tB true, it must be that M^ϕ consists only of black worlds, and hence, $M^\phi,(i,j) \models (K_aB \wedge K_bB \wedge K_cB \wedge K_dB)$. A similar argument holds for $(i,j) \in V(W)$.

We will now show that $M,(0,0) \models K_t(C1)$, and the cases for $C2$-$C4$ can be shown similarly. Suppose that $(i,j) \in V(B)$. We must show that for all epistemic ψ where $M,(i,j) \models \psi$, $M^\psi,(i,j) \models L_a(W \wedge L_dB)) \rightarrow K_c(W \rightarrow L_bB)$. Since we are quantifying over all submodels M^ϕ corresponding to epistemic formula, it is sufficient to show that $M',(i,j) \models L_a(W \wedge L_dB)) \rightarrow K_c(W \rightarrow L_bB)$ where $M' = (S', \sim', V')$ is any submodel where $(i,j) \in S'$. In such a case, since $M',(i,j) \models L_a(W \wedge L_dB)$, it follows that $(i,j+1),(i+1,j+1) \in S'$. Also, $(i+1,j)$ is the only one white world c-related to (i,j). If $(i+1,j) \in S'$, then because $(i+1,j+1) \in S'$, we have $M',(i,j) \models K_c(W \rightarrow L_bB)$. If $(i+1,j) \notin S'$ then $M',(i,j) \models K_c(W \rightarrow L_bB)$ as $M',(i,j) \models K_c\neg W$. Thus for every sub-model, M' including (i,j) we have

$$M',(i,j) \models L_a(W \wedge L_dB)) \rightarrow K_c(W \rightarrow L_bB)$$

and thus for every epistemic announcement ψ where $M,(i,j) \models \psi$ we have $M^\psi,(i,j) \models L_a(W \wedge L_dB)) \rightarrow K_c(W \rightarrow L_bB)$. Therefore $M,(0,0) \models K_tC_1$. A similar argument can be applied for $C2$-$C4$ so it follows that given a Γ-tiling exists, we can show, $Tile_\Gamma$ is satisfiable. ■

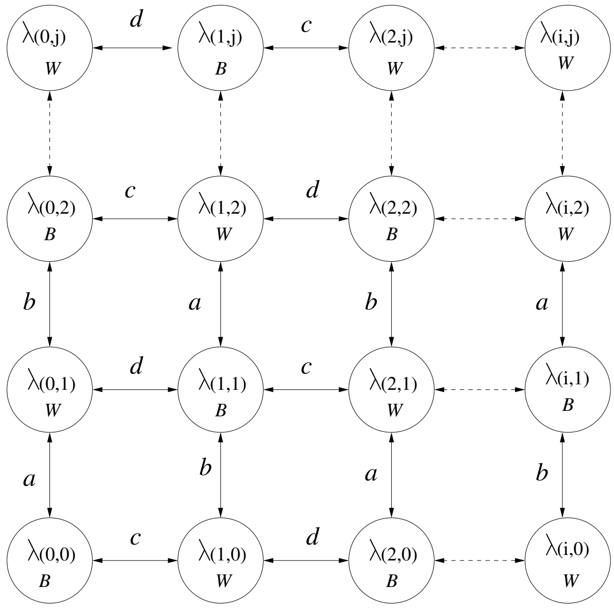

Figure 2. The conversion of a a tiling into a model.

For the completeness argument, we suppose that $M, s \models Tile_\Gamma$. From M we will show that, for each $n \in \omega$ we can construct a tiling, λ_n of an $n \times n$ grid. This is shown to be equivalent to tiling the full $\omega \times \omega$ plane in the following lemma.

LEMMA 3. *If Γ is able to tile an $n \times n$ grid for all $n \in \omega$, then Γ is able to tile the $\omega \times \omega$ plane.*

Proof. Let λ_n be the $n \times n$ tiling, and define λ^* as a tiling of the plane where $\lambda^*(0,0) = \gamma$ for some γ where for some infinite $N_{0,0} \subseteq \omega$, for all $n \in N_{0,0}$ $\lambda_n(0,0) = \gamma$. We then proceed by induction over $\omega \times \omega$ where $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 + j_1 < i_2 + j_2$ or $i_1 + j_1 = i_2 + j_2$ and $i_1 \leq i_2$. For $(i,j) > (0,0)$, we define $\lambda^*(i,j)$ such that

- if $i > 0$ then $\lambda^*(i,j) = \gamma$ where for some infinite $N_{i,j} \subseteq N_{i-1,j+1}$, for all $n \in N_{i,j}$ $\lambda_n(i,j) = \gamma$,

- otherwise (if $i = 0$) we let $\lambda^*(0,j) = \gamma$ where for some infinite $N_{0,j} \subseteq N_{j-1,0}$, for all $n \in N_{0,j}$ $\lambda_n(0,j) = \gamma$

It can be shown that such γ and $N_{i,j}$ can always be found (since there are infinitely many finite tilings and only finitely many tiles, the pigeon hole principle may be applied). Therefore such a λ^* may be defined by induction, (or indeed, Koenig's Lemma). ■

To proceed we require the following definition:

DEFINITION 4. Two worlds are 0-*Q-bisimilar*, iff they satisfy exactly the same set of propositional atoms taken from Q.
For all $n \in \omega$, two worlds, u and v in M, are *n-Q-bisimilar* (written $u \cong_n^Q v$) if and only if:

1. $u \cong_{n-1}^Q v$;
2. for every $x \in \{a, b, c, d, t\}$, for every world w where $u \sim_x w$, there is some world w' where $v \sim_x w'$ and $w \cong_{n-1}^Q w'$; and
3. for every $x \in \{a, b, c, d, t\}$, for every world w where $v \sim_x w$, there is some world w' where $u \sim_x w'$ and $w \cong_{n-1}^Q w'$.

We note for all n and Q, n-Q-bisimilarity is an equivalence relation.

LEMMA 5. *Suppose that the set of propositions, Q, is finite. Then for every n, there is a finite set of $\mathcal{L}_{el}$ formulas $\{\phi_1, \dots, \phi_m\}$ such that for every $u \in S$, there is some $i \leq m$ such that for all $v \in S$, $u \cong_n^Q v$ if and only if $M, v \models \phi_i$.*

Proof. This can be shown by induction. As a base case we take the set of formulas $\phi(Q') = \bigwedge_{x \in Q'} x \wedge \neg \bigvee_{x \in Q \setminus Q'} x$ for all $Q' \subseteq Q$. Clearly, for each $u \in S$, we can let $Q' = \{x | u \in V(x)\}$ and then for all $v \in S$, $M, v \models \phi(Q')$ if and only if $v \cong_0^Q u$.

For the inductive step, suppose that $\{\phi_1, \dots, \phi_m\}$ is a set of formulas such that for every u in S, there is some $i \leq m$ such that for all $v \in S$, $u \cong_n^Q v$ if and only if $M, v \models \phi_i$. For each $u \in S$ let the corresponding formula ϕ_i be denoted ϕ_u^n, and let

$$\phi_u^{n+1} = \phi_u^n \wedge \bigwedge_{x \in A} (succ_x^u \wedge nsucc_x^u)$$

where

$$\begin{aligned} succ_x^u &= \bigwedge \{L_x \phi_v^n | \ v \sim_x u\} \\ nsucc_x^u &= K_x \bigwedge \{\neg \phi_i | \ \forall v \sim_x u, \ M, v \not\models \phi_i\} \end{aligned}$$

for the set of agents, $A = \{a, b, c, d, t\}$.

Then for any $v \in S$ where $M, v \models \phi_u^{n+1}$ we have:

1. $v \cong_n^Q u$ since $M, v \models \phi_u^n$.
2. for every $x \in \{a, b, c, d, t\}$, for every world w where $u \sim_x w$, we have $succ_x^u \rightarrow L_x \phi_w^n$, so $M, v \models L_x \phi_w^n$, and thus there is some world w' where $v \sim_x w'$, $M, w' \models \phi_w^n$, so $w \cong_n^Q w'$.

3. for every $x \in \{a,b,c,d,t\}$, for every world w where $v \sim_x w$, there is some world w' where $u \sim_x w'$ and $w \cong_n^Q w'$. To see this, suppose for contradiction that there was some world w such that $v \sim_x w$ and for every world w' where $u \sim_x w'$ we have $w \not\cong_n^Q w'$. Therefore, we have $nsucc_x^u \to K_x \neg\phi_w^n$ so $M,v \models K_x\neg\phi_w^n$, contradicting $v \sim_x w$.

Therefore $v \cong_{n+1}^Q u$ as required. Conversely, if $v \cong_{n+1}^Q u$, then

1. $M,v \models \phi_u^n$ since $v \cong_n^Q u$.
2. for all $x \in A$, for every world w where $u \sim_x w$ there is some world w' where $v \sim_x w'$ and $w \cong_n^Q w'$ (hence $M,w' \models \phi_w^n$). Therefore $M,v \models succ_x^u$.
3. for every $x \in \{a,b,c,d,t\}$, for every world w where $v \sim_x w$, there is some world w' where $u \sim_x w'$ and $w \cong_n^Q w'$. By the induction hypothesis, for every w where $v \sim_x w$, we have $M,w \models \bigvee_{u \sim_x w'} \phi_{w'}^n$. For all w' where $u \sim_x w'$ we clearly have $\phi_{w'}^n \to \bigwedge\{\neg\phi_i | \ \forall v \sim_x u,\ M,v \not\models \phi_i\}$, so it follows that $M,v \models nsucc_x^u$.

Thus $M,v \models \phi_u^{n+1}$ completing the induction. ■

For the following proofs we define a new operator $\Box_n^Q$, to mean "for all public Q-announcements of depth n". The semantics are given as: $M,u \models \Box_n^Q\phi$ if and only if for all $\mathcal{L}_{el}$ formulae ψ with at most n nestings of knowledge operators and containing only the atoms Q, if $M,u \models \psi$, then $M^\psi, u \models \phi$.

LEMMA 6.

1. *For all n, for all Q, $\Box\phi \to \Box_n^Q\phi$ is a validity.*
2. *For any two worlds u, v where $u \cong_n^Q v$, for any L_{el} formula ϕ of depth at most n and containing only atoms from Q, $M,u \models \phi$ if and only if $M,v \models \phi$.*
3. *For any two worlds u, v where $u \cong_{n+m}^Q v$, for any L_{el} formula ϕ of depth at most n and containing atoms only from Q, $M,u \models \Box_m^Q\phi$ if and only if $M,v \models \Box_m^Q\phi$.*

Proof.

1. Obvious.
2. By induction. Clearly the statement holds for the case $n = 0$. Suppose the statement holds for n. Every L_{el} formula ϕ, of depth $n+1$, can be written as a Boolean combination of atoms and formulas $K_{x_i}\phi_i$ (for $i = 1,\ldots,m$) where ϕ_i is a formula of depth at most n. If $u \cong_{n+1}^Q v$, then for every $u' \sim_x u$ there is some $v' \sim_x v$ where $u' \cong_n^Q v'$, and vice-versa. By the induction hypothesis, $M,u' \models \phi_i$ for all $u' \sim_{x_i} u$ if and only if $M,v' \models \phi_i$ for all $v' \sim_{x_i} v$. It follows that $M,u \models \phi$ if and only if $M,v \models \phi$.

3. To prove this statement we extend the induction above with the case for m. In the case $m = 0$ it is effectively the second part of this lemma. Now suppose the statement holds for a given m (and for all n). Without loss of generality, suppose that $u \cong^Q_{n+m} v$ and $M, u \models \neg\Box^Q_m \phi$ where ϕ is of depth n. Therefore there is some announcement, ψ, of depth m that makes ϕ false. This is equivalent to $M, u \models \psi \wedge \neg\phi^\psi$ where ϕ^ψ is inductively defined by replacing subformulas $K_x(\alpha)$ with $K_x(\psi \to \alpha^\psi)$, and otherwise acting as the identity (see Proposition 4.22 of [11] for a formal description of this translation). Now $\psi \wedge \neg\phi^\psi$ is of depth $n + m$ so the result follows from the second part of this Lemma.

■

We will refer to a formula of depth n, containing only atoms from Q as an n-Q-formula. The above lemmas and definitions will be applied to show how the arbitrary announcements in the formula $Tile_\Gamma$ can allow us to establish that two worlds are n-Q-bisimilar for arbitrary n and arbitrary Q. To this end, for $n \in \omega$ and finite $Q \subset P$, we define the formula $Tile^{(n,Q)}_\Gamma$ to be the formula $Tile_\Gamma$ with every arbitrary announcement $\Box$ replaced by $\Box^Q_n$ (and likewise for subformulas such as $C1^{(n,Q)}$).

There are two types of public announcement in the formula $Tile_\Gamma$. The first appears in the sub-formula T^*, and the following lemma shows how this allows the knowledge relation of agent t to act as a weak kind of transitive closure for the other knowledge relations.

LEMMA 7. *Let $n \geq 1$, $x \in \{a, b, c, d\}$ and $Q \subset P$ be a finite set of propositional atoms including $\{B, W\} \cup \Gamma$. Suppose that $u \in V(B)$ (resp. $u \in V(W)$), $u \sim_x v$ for some $v \in V(W)$ (resp. $v \in V(B)$), $u \cong^Q_n w$, and $M, u \models (alt \wedge T^*)^{(n,Q)}$. Then there is some w' such that $w \sim_t w'$ and $w' \cong^Q_{n-1} v$.*

Proof. Suppose that $M, u \models (alt \wedge T^*)^{(n,Q)}$, $u \cong^Q_n w$, and $u \sim_x v$ where $u \in V(B)$ and $v \in V(W)$. From T^* we have $M, u \models \Box^Q_{n-1}(K_t B \to K_x B)$. As $u \cong^Q_n w$ and $n \geq 1$, by Lemma 6.3 we have $M, w \models \Box^Q_{n-1}(K_t B \to K_x B)$. Now, suppose for contradiction that for all $w' \sim_t w$ we have $w' \not\cong^Q_{n-1} v$. By Lemma 5 there is some formula ϕ^{n-1}_v such that $M, v \models \phi^{n-1}_v$, and for all $w' \sim_t w$ we have $M, w' \models \neg\phi^{n-1}_v$. Therefore we may make the public announcement $\psi = B \vee \phi^{n-1}_v$. Thus $M^\psi, w \models K_t B$, and since $M, w \models \Box^Q_{n-1}(K_t B \to K_x B)$, we have $M^\psi, w \models K_x B$. It follows that $M, w \models \psi \wedge K_x(\psi \to B)$. However, $M, u \models \psi \wedge \neg K_x(\psi \to B)$. Since $\psi = \phi^{n-1}_v \vee B$, ψ has depth $n - 1$ and thus $K_x(\psi \to B)$ has depth n. As $u \cong^Q_n w$, by Lemma 6.2, u and w agree on all formulas of depth n. This contradicts the inference that $M, u \models \neg K_x(\psi \to B)$ and $M, w \models K_x(\psi \to B)$. ■

The other occurrences of arbitrary announcements in the formula $Tile_\Gamma$ appear in the formulae $C1$–$C4$. These formulas use the arbitrary announcements to establish a weak type of commutativity property, which is essential

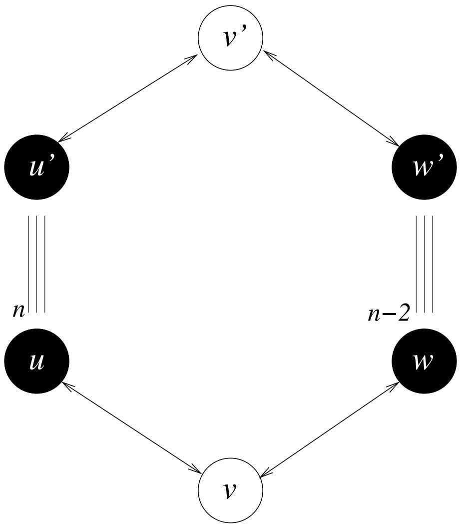

Figure 3. The construction of Lemma 8, inferring the worlds w and w' are $n-2$-bisimilar, given the worlds u and u' are n bisimilar.

in defining a grid. The following lemmas clarify this property and show that it is enforced by the formula $Tile_\Gamma$. The first lemma deals with the black worlds and the second lemma deals (symmetrically) with the white worlds.

LEMMA 8. *Suppose that $u \in V(B)$, $u \sim_a v$, for some $v \in V(W)$. Let $n \geq 2$ and suppose also that $u \cong_n^Q u'$ for some finite $Q \subset P$ including $\{B, W\} \cup \Gamma$, and $u' \sim_c v'$ for some $v' \in V(W)$. Given that $M, u \models (C1 \wedge C2)^{(n,Q)}$:*

1. *for all $w \in V(B)$ where $v \sim_d w$, if there is some $w' \in V(B)$ where $v' \sim_b w'$ then there is some such w' where either $w \cong_{n-2} w'$ or $w' \cong_{n-2}^Q u'$.*
2. *for all $w' \in V(B)$ where $v' \sim_b w'$, if there is some w where $v \sim_d w$, then there is some such w where either $w \cong_{n-2} w'$ or $w \cong_{n-2}^Q u$.*

Proof. We will show case 1, and case 2 can be shown similarly. So given the assumptions of the Lemma, let $w \in V(B)$ be such that $v \sim_d w$. Consider the announcement $\psi = W \vee \phi_u^{n-2} \vee \phi_w^{n-2}$. Since $M, u \models C1^{(n,Q)}$, $M^\psi, u \models L_a(W \wedge L_d B) \to K_c(W \to L_b B)$, and by Lemma 5, $M^\psi, u \models L_a(W \wedge L_d B)$. We may apply modus ponens and Lemma 6 to deduce, $M_{u'}^\psi \models K_c(W \to L_b B)$. Therefore $M_{v'}^\psi \models L_b B$, so there is some $w' \in V(B)$ where $v' \sim_b w'$, and $M, w' \models \psi$. As $w' \in V(B)$, we have either $M, w' \models \phi_u^{n-2}$ and thus $w' \cong_n^Q u'$, or $M, w' \models \phi_w^{n-2}$, and thus $w \cong_{n-2}^Q w'$, (by Lemma 5). This scenario is represented in Figure 3 ■

Notice that the lemma does not perfectly capture the notion of commutativity. Ideally we would like to have:
For all $u, w, u' \in V(B)$ for all $v \in V(W)$ where $u \sim_a v$, $v \sim_d w$ and $u \cong_n^Q u'$, there's some $v' \in V(W)$ and some $w' \in V(B)$ such that $u' \sim_c v'$, $v' \sim_b w'$ and $w' \cong_{n-2}^Q w$, (and vice-versa).
However, we must consider the additional possibility that there's some $v' \in V(W)$ and some $w' \in V(B)$ such that $u' \sim_c v'$, $v' \sim_b w'$ and $w' \cong_{n-2}^Q u'$. In such a case we would have, by the second part of the lemma, that there is some w where $v \sim_d w$ and either $w \cong_{n-2}^Q w'$ or $w \cong_{n-2}^Q u$. In either case, as $w' \cong_{n-2}^Q u'$ and $u \cong_n^Q u'$, we will have $w' \cong_{n-2}^Q u' \cong_{n-2}^Q u \cong_{n-2}^Q w$, which is sufficient for our purposes.

LEMMA 9. *Suppose that $u \in V(W)$, $u \sim_b v$, for some $v \in V(B)$. Suppose also that $u \cong_n^Q u'$ for some finite $Q \subset P$ including $\{B, W\} \cup \Gamma$, and $u' \sim_d v'$ for some $v' \in V(B)$. Given that $M, u \models (C3 \wedge C4)^{(n,Q)}$:*

1. *for all $w \in V(W)$ where $v \sim_c w$, if there is some $w' \in V(W)$ where $v' \sim_a w'$ then there is some such w' where either $w \cong_{n-2}^Q w'$ or $w' \cong_{n-2}^Q u'$.*
2. *for all $w' \in V(W)$ where $v' \sim_a w'$, if there is some w where $v \sim_c w$, then there is some such w where either $w \cong_{n-2}^Q w'$ or $w \cong_{n-2}^Q u$.*

Proof. The proof of this is symmetrical to the proof of Lemma 8 ■

These lemmas are sufficient to establish a finite grid structure, as depicted in Figure 4. The formulas *blk* and *wht* are then clearly sufficient to encode a finite tiling, so if $M, s \models Tile_\Gamma$ then by Lemma 3 a Γ tiling exists.

Recall

(11) $Tile_\Gamma = alt \wedge T^* \wedge st \wedge K_t(C1 \wedge C2 \wedge C3 \wedge C4 \wedge blk \wedge wht)$.

We give the following Lemma.

LEMMA 10. *Suppose that $M, s \models Tile_\Gamma$. Then for all $n \in \omega$, for $Q = \Gamma \cup \{B, W\}$ we may define a partial function $f : \{0, \ldots, n\}^2 \longrightarrow S$ such that:*

1. *$f(0,0) = s$;*
2. *if $f(i,j) \in V(B)$, then*
   (a) *if $i < n$, there is some $u \in V(W)$ where $f(i+1, j) \cong_{k(i,j)}^Q u$ and $f(i,j) \sim_c u$, and*
   (b) *if $j < n$, there is some $u \in V(W)$ where $f(i, j+1) \cong_{k(i,j)}^Q u$ and $f(i,j) \sim_a u$;*
   (c) *$M, f(i,j) \models blk$*

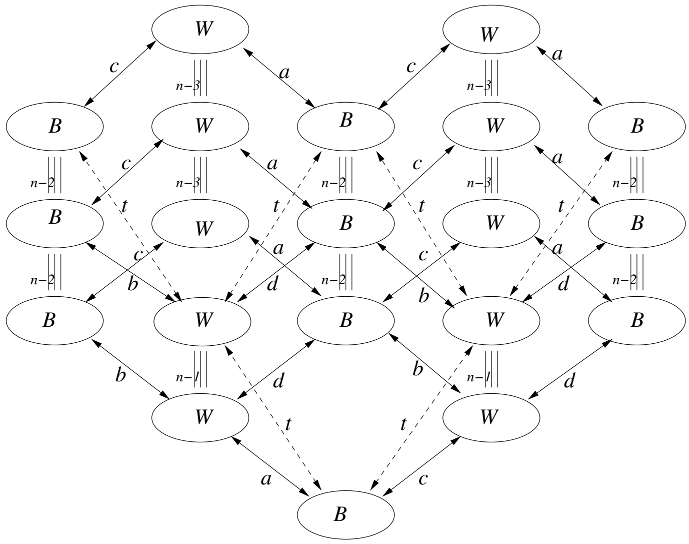

Figure 4. The construction of the finite grid.

3. if $f(i,j) \in V(W)$*, then*

(a) if $i < n$*, there is some* $u \in V(B)$ *where* $f(i+1,j) \cong^{Q}_{k(i,j)} u$ *and* $f(i,j) \sim_d u$*, and*

(b) if $j < n$*, there is some* $u \in V(B)$ *where* $f(i,j+1) \cong^{Q}_{k(i,j)} u$ *and* $f(i,j) \sim_b u$*;*

(c) $M, f(i,j) \models wht$

where $k(i,j) = 2n + 3 - (i + j)$*.*

We can show this by construction, applying Lemmas 7, 8 and 9. The function $k(i,j)$ is chosen such that for all $i, j \leq n$, $k(i,j) \geq 3$. This allows the preconditions of the Lemmas 7, 8 and 9 to be met for all $i, j \leq n$. The proof is illustrated in Figure 4. You can view this figure as a cube, cut in half diagonally up from a bottom corner. The base of the shape makes a finite grid. We construct a function, f, mapping $\{0, \dots, n\}^2$ to the states of the model, such that $f(i,j)$ is $k(i,j)$-Q-bisimilar to the corresponding world at the base of the grid. As we get further from the corner $k(i,j)$ decreases so this correspondence becomes progressively weaker. By the time $i + j > 2n$, $k(i,j) < 3$ so the preconditions for the necessary lemmas is not met. However, by this stage we have already defined an $n \times n$ grid.

Proof. We construct the a function, F, satisfying the stated properties by induction over $i + j$, where $i + j \leq 2n$. For the induction hypothesis we assume for all g, h where $g + h < i + j$, $f(g,h)$ is defined such that:

1. if $f(g,h) \in V(B)$, then

   (a) if $0 < g < n$, then $f(g-1,h) \in V(W)$ and for some $u \in V(B)$ we have $f(g,h) \cong^Q_{k(g,h)} u$ and $f(g-1,h) \sim_d u$, and

   (b) if $0 < h < n$, then $f(g,h-1) \in V(W)$ and for some $u \in V(B)$ we have $f(g,h) \cong^Q_{k(g,h)} u$ and $f(g,h-1) \sim_b u$;

2. if $f(g,h) \in V(W)$, then

   (a) if $0 < g < n$ then $f(g-1,h) \in V(B)$ and for some $u \in V(W)$ we have $f(g,h) \cong^Q_{k(g,h)} u$ and $f(g-1,h) \sim_c u$, and

   (b) if $0 < h < n$ then $f(g,h-1) \in V(B)$ and for some $u \in V(W)$ we have $f(g,h) \cong^Q_{k(g,h)} u$ and $f(g,h-1) \sim_a u$;

3. there is some u where $f(g,h) \cong^Q_{k(g,h)} u$ and

   (12) $M, u \models (alt \wedge T^* \wedge K_t(C1 \wedge C2 \wedge C3 \wedge C4 \wedge blk \wedge wht))^{(k(g,h),Q)}$.

For brevity let,

(13) $Hyp = alt \wedge T^* \wedge K_t(C1 \wedge C2 \wedge C3 \wedge C4 \wedge blk \wedge wht)$.

For the base case of this induction, suppose $M, s \models Tile_\Gamma$, and let $n \in \omega$. We define $f(0,0) = s$. Then

1. $s \cong_{k(0,0)} f_n(0,0)$,

2. $M, s \models alt \wedge T^*$,

3. $M, s \models C1 \wedge C2 \wedge C3 \wedge C4$,

so it clearly satisfies the inductive hypothesis.

For the induction, suppose that the inductive hypothesis holds for the pair i, j. There are three cases to consider, $i = 0$, $j = 0$ and $i, j \neq 0$.

1. if $i = 0$ we may assume $j \neq 0$ (since $f(0,0)$ is already defined). By the induction hypothesis, $f(i, j-1)$ is defined. We suppose, without loss of generality, that $f(i,j-1) \in V(B)$ and the case of $f(i,j-1) \in V(W)$ may be handled similarly. Also by the induction hypothesis, there is some $u \cong^Q_{k(i,j-1)} f(i,j-1)$, where $M, u \models Hyp^{(k(i,j-1),Q)}$. By Lemma 6 we have $M, f(i,j-1) \models L_a W$. Therefore there is some world $v \in V(W)$ where $f(i,j-1) \sim_a v$ and we let $f(i,j) = v$. By Lemma 7 there is some world $w \sim_t u$ such that $w \cong^Q_{k(i,j)} v$ and since $\models Hyp \to K_t Hyp$, $M, w \models Hyp^{(k(i,j),Q)}$ as required.

2. the case for $j = 0$ is symmetric to the case above.

3. if $i, j \neq 0$ suppose, without loss of generality, that $f(i,j) \in V(B)$ (and the case for $f(i-1,j-1) \in V(W)$ is handled similarly). By the induction hypothesis for some $u \cong^Q_{k(i-1,j-1)} f(i-1,j-1)$, we have $M, u \models Hyp^{(k(i-1,j-1),Q)}$, $u \sim_a v$ for some $v \cong^Q_{k(i-1,j)} f(i-1,j) \in V(W)$, and $u \sim_c v'$ for some $v' \cong_{k(i,j-1)} f(i,j-1)$. By Lemma 8, either:

   (a) there is some $w, w' \in V(B)$ such that $v \sim_d w$, $v' \sim_b w'$ and $w' \cong^Q_{k(i,j)} w$. In such a case we let $f(i,j) = w$; or

   (b) there is some $w' \in V(B)$ where $v' \sim_b w'$ and $w' \cong^Q_{k(i,j)} u$. In this case we let $f(i,j) = u$. By Lemma 8 we also have for all $w' \in V(B)$ where $v' \sim_b w'$ there is some $w \in V(B)$ where $v \sim_d w$ and $w \cong^Q_{k(i,j)} u$.

   Also by the induction hypothesis we have $M, v \models Hyp^{(f(i-1,j),Q)}$, so we may apply Lemma 7 to show that there is some world z such that $z \cong_{k(i,j)} f_n(i,j)$ and $v \sim_t z$. As the formula

   (14) $Hyp^{(f(i-1,j),Q)} \rightarrow K_t(Hyp^{(f(i-1,j),Q)})$

   is a tautology of epistemic logic and $M, v \models Hyp^{(f(i-1,j),Q)}$ we have $M, z \models (alt \wedge T^* \wedge K_t(C1 \wedge C2 \wedge C3 \wedge C4 \wedge blk \wedge wht))^{(k(i,j),Q)}$, for some $z \cong_{k(i,j)} f_n(i,j)$, as required.

Therefore, the induction hypothesis holds for the pair (n, n). Thus for all (i, j) where $i + j < 2n$ we have

1. if $f(i,j) \in V(W)$, then

   (a) $f(i+1,j) \in V(B)$ and for some $u \in V(W)$ we have $f(i,j) \cong^Q_{k(i,j)} u$ and $f(i+1,j) \sim_d u$, and

   (b) $f(i,j+1) \in V(B)$ and for some $u \in V(W)$ we have $f(i,j) \cong^Q_{k(i,j)} u$ and $f(i,j+1) \sim_b u$;

   (c) $M, f(i,j) \models wht$

2. if $f(i,j) \in V(B)$, then

   (a) $f(i+1,j) \in V(W)$ and for some $u \in V(B)$ we have $f(i,j) \cong^Q_{k(i,j)} u$ and $f(i+1,j) \sim_c u$, and

   (b) $f(i,j+1) \in V(W)$ and for some $u \in V(B)$ we have $f(i,j) \cong^Q_{k(i,j)} u$ and $f(i,j+1) \sim_a u$;

   (c) $M, f(i,j) \models blk$

so f satisfies the required properties. ■

Note that this lemma only defines an $n \times n$ grid, since as Lemmas 6, 8 and 9 are used the induction and Lemma 6 is only available when $k(i,j) > 2$, and Lemmas 8 and 9 are only available when $k(i,j) > 1$. However, n is chosen arbitrarily. Because $f(0,0) \cong_n^Q f(0,0)$ for all n, we can seed the induction with an arbitrarily large n. This allows us to apply Lemma 3 to define a tiling.

COROLLARY 11. *If $M, u \models Tile_\Gamma$ then a Γ-tiling exists*

Proof. If $M, u \models Tile_\Gamma$, then by Lemma 10, for all n we may define f such that for all i, j where $i + j < 2n$,

1. if $f(i,j) \in V(B)$ then
   (a) if $j < n$, $f(i,j) \sim_a u$ for some $u \cong^Q_{k(i,j+1)} f(i,j+1)$.
   (b) if $i < n$, $f(i,j) \sim_c u$ for some $u \cong^Q_{k(i,j+1)} f(i+1,j)$.
   (c) $M, f(i,j) \models blk$.
2. if $f(i,j) \in V(W)$ then
   (a) if $j < n$, $f(i,j) \sim_b u$ for some $u \cong^Q_{k(i,j+1)} f(i,j+1)$.
   (b) if $i < n$, $f(i,j) \sim_d u$ for some $u \cong^Q_{k(i,j+1)} f(i+1,j)$.
   (c) $M, f(i,j) \models wht$.

Recall the formulas:

$$(15) \qquad blk \quad = \quad B \to \left(\bigwedge_{\gamma\in\Gamma} \left(\gamma \to \bigwedge \left[\begin{array}{c} K_a(W \to \bigvee_{\gamma^t=\delta^b} \delta)) \\ K_c(W \to \bigvee_{\gamma^r=\delta^\ell} \delta)) \end{array} \right] \right) \right)$$

$$(16) \qquad wht \quad = \quad W \to \left(\bigwedge_{\gamma\in\Gamma} \left(\gamma \to \bigwedge \left[\begin{array}{c} K_b(B \to \bigvee_{\gamma^t=\delta^b} \delta)) \\ K_d(B \to \bigvee_{\gamma^r=\delta^\ell} \delta)) \end{array} \right] \right) \right).$$

Applying the semantics of epistemic logic:

1. if $f(i,j) \in V(B)$, and $M, f(i,j) \models \gamma$ then for some $u \cong^Q_{k(i,j+1)} f(i,j+1)$ we have $f(i,j) \sim_a u$ and $u \in V(W)$. Therefore $M, u \models \delta$ for some δ where $\gamma^t = \delta^b$. Thus $M, f(i+1,j) \models \delta$ for some δ where $\gamma^t = \delta^b$. Likewise for some $u \cong^Q_{k(i,j+1)} f(i+1,j)$ we have $f(i,j) \sim_c u$ and $u \in V(W)$, so $M, f(i,j+1) \models \delta$ for some δ where $\gamma^r = \delta^\ell$.

2. if $f(i,j) \in V(W)$, and $M, f(i,j) \models \gamma$ then for some $u \cong^Q_{k(i,j+1)} f(i,j+1)$ we have $f(i,j) \sim_a u$ and $u \in V(B)$. Therefore $M, u \models \delta$ for some δ where $\gamma^t = \delta^b$. Thus $M, f(i+1,j) \models \delta$ for some δ where $\gamma^t = \delta^b$. Likewise for some $u \cong^Q_{k(i,j+1)} f(i+1,j)$ we have $f(i,j) \sim_c u$ and $u \in V(B)$, so $M, f(i,j+1) \models \delta$ for some δ where $\gamma^r = \delta^\ell$.

Therefore f defines a tiling of an $n \times n$ grid with the tiles of Γ. Therefore the formula $Tile_\Gamma$ is satisfiable if and only if Γ can tile the $n \times n$ grid for arbitrary n. Applying Lemma 3 it follows a γ-tiling exists. ■

Thus the satisfiability problem $APAL$ is co-RE hard, as it is able to embed the tiling problem. As we know from [2] that the set of valid formulas for $APAL$ is recursively enumerable, it follows that the satisfiability problem is co-RE complete.

6 Future work

Up to this point arbitrary public announcement logic has shown some promise for practical reasoning applications: it has an axiomatization, a tableau-calculus, it is bisimulation invariant, naturally extends epistemic logic, and model checking is PSPACE-complete [1]. The notion of an arbitrary public announcement is also a natural concept (consider the plea, "*is there anything I can say to make you believe X*"). Therefore, a natural avenue of investigation is to consider whether we may be able to somehow expressively weaken $APAL$ to a decidable logic which also enjoys all these favorable properties.

One area of investigation may be to consider the set of formulae (or more abstractly, model-properties) that announcements may range over. From the encoding $Tile_\Gamma$ we have given, we note that the encoding of the tiling problem for Γ only requires five agents in the language (although it is conceivable a more complex encoding could do with less). Also, from the proof of correctness we have given, we note that the arbitrary announcements do not need to range over all epistemic formulae. The announcements only need to range over all formulae containing the atoms from $\Gamma \cup \{B, W\}$ (or the atoms appearing in the formula). However, the proof does require that the arbitrary announcements range over formulae of unbounded depth, so it may be of interest to consider restrictions where the arbitrary announcements ranged over announcements of a bounded depth (say, formulas of depth at most one).

We also note that generalizing the set of formulae that the arbitrary announcements range over would not affect this undecidability result. For example, if we allowed fixed-point operators to appear in the arbitrary announcements, then the proofs of Lemmas 7 and 8 would remain. However we may consider restrictions of the set of formulae. One natural restriction to consider would be to restrict arbitrary announcements to positive knowledge formulae (formulae where the knowledge operators K_x always appear in the scope of an even number of negations). For such a restricted set of formulae Lemma 5 would not hold so decidability may still be achievable.

An alternative approach is discussed [10]. Here it is suggested that informative events such as announcements, rather than simply being evaluated with respect to the given model, could add additional information (at an atomic level) into the model. This approach is motivated by the observation that while the $APAL$ is bisimulation invariant, it is not the case that two models bisimilar with respect to a subset of atoms, X, agree on all formulas that contain only the atoms X. The suggested *Future event logic* quantifies over refinements of a model, which includes all public announcements of epistemic formulae, but also other public or non-public informative events that can be described as action models [3]. It is shown to be decidable via a reduction to a bisimulation quantified logic [5]. Future event logic is interesting in its own right, and it aims to provide the sort of link between dynamic and temporal epistemics pointed out in [9]

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Hans van Ditmarsch

Computer Science, University of Otago, New Zealand & IRIT, France

`hans@cs.otago.ac.nz`

Tim French

School of Computer Science and Software Engineering,

The University of Western Australia, Australia

`tim@csse.uwa.edu.au`