On the Complexity of Modal Axiomatisations over Many-dimensional Structures

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Abstract

We show that all the complexities of a possible axiomatisation of $\mathbf{S5}^n$, the n-modal logic of products of n equivalence frames, are already present in any axiomatisation of $\mathbf{K}^n$. Then we show that if $3 \leq n < \omega$ then, for any set L of n-modal formulas between $\mathbf{K}^n$ and $\mathbf{S5}^n$, the class of all frames for L is not closed under ultraproducts and is therefore not elementary. So any modal axiomatisation for a Kripke complete logic in the interval between $\mathbf{K}^n$ and $\mathbf{S5}^n$ must contain modal formulas with no first-order correspondents. The proof is based on a construction of Hirsch and Hodkinson [15] showing that the class of strongly representable n-dimensional cylindric algebra atom structures is not closed under ultraproducts. We show that this construction can be carried through in a diagonal-free setting.

Keywords: many-dimensional modal logic, products of Kripke frames, ultraproducts

1 Introduction

As usual in any area of logic, when one considers the "logic" or "theory" of a class $\mathcal{C}$ of structures (the "intended models"), then there are always "non-intended", "non-standard" models of this "logic". These non-standard structures are often hard to describe. In this paper we discuss this problem in the setting of *n-modal logics*: propositional multi-modal logics having finitely many unary modal operators $\Diamond_0, \dots, \Diamond_{n-1}$ (and their duals $\Box_0, \dots, \Box_{n-1}$), where n is a non-zero natural number. Formulas of this language, using propositional variables from some fixed countably infinite set, are called *n-modal formulas*. Frames for n-modal logics — *n-frames* — are structures of the form $\mathfrak{F} = (W, T_i)_{i<n}$ where W is a non-empty set and each T_i is a binary relation on W, for $i < n$. *Validity* of a set Σ of n-modal formulas in an n-frame $\mathfrak{F}$ (in symbols: $\mathfrak{F} \models \Sigma$) is defined as usual. If $\mathfrak{F} \models \Sigma$ then we also say that $\mathfrak{F}$ is a *frame for* Σ. Given a class $\mathcal{C}$ of

n-frames, we denote by $\mathsf{Log}(\mathcal{C})$ the set of all n-modal formulas that are valid in every n-frame in $\mathcal{C}$.

Our "intended" structures are the following special n-frames. Given 1-frames $\mathfrak{F}_i = (W_i, R_i)$, $i < n$, their *product* is the n-frame

$$\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} = (W_0 \times \cdots \times W_{n-1}, \bar{R}_i)_{i<n},$$

where $W_0 \times \cdots \times W_{n-1}$ is the Cartesian product of the W_i and for all $\mathbf{u}, \mathbf{v} \in W_0 \times \cdots \times W_{n-1}$ and $i < n$,

$$\mathbf{u}\bar{R}_i\mathbf{v} \quad \text{iff} \quad u_i R_i v_i \text{ and } u_j = v_j \text{ for } j \neq i,\ j < n.$$

Such n-frames we call *n-dimensional product frames.* They have been introduced in [9,24] and have been extensively studied both in pure modal logic and in applications, see [8,21] and the references therein.

Two examples of classes of n-dimensional product frames are:

$$\mathcal{C}^n_{all} = \text{the class of all } n\text{-dimensional product frames},$$
$$\mathcal{C}^n_{equiv} = \text{the class of all } n\text{-dimensional products of equivalence frames}.$$

Let us also introduce notations for the n-modal logics they determine:

$$\mathbf{K}^n = \mathsf{Log}(\mathcal{C}^n_{all}),$$
$$\mathbf{S5}^n = \mathsf{Log}(\mathcal{C}^n_{equiv}).$$

It can be hard to describe an arbitrary n-frame for $\mathbf{K}^n$ or $\mathbf{S5}^n$. As is shown in [16], if $n \geq 3$ and L is any set of n-modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$, then it is undecidable whether a finite n-frame is a frame for L or not. (So no such logic L can be finitely axiomatisable.) Here we show that these non-standard n-frames are hard to "catch" in an other sense: They cannot be described in the first-order "frame language", that is, in the language having n binary predicate symbols and equality.

Theorem 1.1 *Let $3 \leq n < \omega$ and let L be any set of n-modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then the class of all frames for L is not closed under ultraproducts, and so is not elementary.*

Note that both $\mathbf{K}^2$ and $\mathbf{S5}^2$ are (finitely) axiomatisable by Sahlqvist-formulas (see [9,14]), so the respective classes of all their frames *are* elementary. Also note that Theorem 1.1 only says that the class of *all* frames for certain modal logics is not closed under ultraproducts. Such a logic can still be determined by some *smaller*, ultraproduct-closed class of n-frames. This is indeed the case for many, see Prop. 2.9 below. As is shown in [20], $\mathbf{K}^n$ is even determined by a class of n-frames that can be *finitely* axiomatised in the first-order frame language.

However, as a consequence of Theorem 1.1 we obtain the following quite discouraging result, as far as finding an explicit axiomatisation for the logics in question is concerned:

Corollary 1.2 *Let $3 \leq n < \omega$ and let L be any Kripke-complete n-modal logic such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then any axiomatisation for L must contain n-modal formulas with no first-order correspondents.*

We conjecture that, for canonical logics L in the interval between $\mathbf{K}^n$ and $\mathbf{S5}^n$, a combination of the techniques of the present paper with those of Hodkinson and Venema [17] might result in an even stronger statement: Any axiomatisation for such an L must contain infinitely many non-canonical n-modal formulas.

The structure of the paper is as follows. In Section 2 we give a general characterisation of arbitrary frames of multi-modal logics determined by frame-classes satisfying some closure conditions. Using this we show that if we could "deal" with non-standard n-frames for $\mathbf{K}^n$, then we could do that with arbitrary n-frames for $\mathbf{S5}^n$ as well. In particular, we show that $\mathbf{S5}^n$ is finitely axiomatisable over $\mathbf{K}^n$. Then in Sections 3 and 4 we prove Theorem 1.1. The proof is based on a construction of Hirsch and Hodkinson [15] showing that the class of strongly representable n-dimensional cylindric algebra atom structures is not closed under ultraproducts. We show that this construction can be carried through in a diagonal-free setting, and then apply the results of Section 2.

2 Non-standard frames for logics determined by classes of n-dimensional product frames

We begin with proving some general results on modal logics determined by special classes of relational structures of *any* signature. In what follows we use the words *frame* and *relational structure* as synonyms. (So the n-frames introduced in Section 1 are special frames.) We use without explicit reference standard notions and results from basic modal logic and universal algebra; such as *p-morphisms*, *generated subframes*, *Sahlqvist formulas* and *canonicity*, duality between relational structures and *Boolean algebras with operators* (*BAOs*), *homomorphisms*, *subalgebras*, *direct products*, *ultraproducts*, *varieties*, *subdirect embeddings* and *subdirectly irreducible* algebras. For notions and statements not defined or proved here, see [3,4,10,13].

If x is a point in a relational structure $\mathfrak{F}$ then we denote by $\mathfrak{F}^x$ the smallest generated subframe of $\mathfrak{F}$ containing x. We call $\mathfrak{F}^x$ a *point-generated subframe* of $\mathfrak{F}$. If $\mathfrak{F} = \mathfrak{F}^x$ for some x, then $\mathfrak{F}$ is called *rooted.* Apart from the usual operators $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ on classes of algebras (denoting homomorphic images, subalgebras, and isomorphic copies of direct products, respectively), we use the following operators on classes of frames of the same signature:

$$\mathbb{G}\mathsf{sf}\,\mathcal{C} = \text{isomorphic copies of generated subframes of frames in } \mathcal{C},$$

$$\mathbb{G}\mathsf{sf}_{\mathsf{p}}\,\mathcal{C} = \text{isomorphic copies of point-generated subframes of frames in } \mathcal{C}.$$

The (*full*) *complex algebra* of a frame $\mathfrak{F} = (W, R_i)_{i\in I}$ is denoted by $\mathfrak{Cm}\,\mathfrak{F}$. That is, $\mathfrak{Cm}\,\mathfrak{F} = (\mathcal{P}(W), \cap, -^W, f_i)_{i\in I}$, where $(\mathcal{P}(W), \cap, -^W)$ is the Boolean algebra of all subsets of W, and for each $k+1$-ary relation R_i, f_i is a k-ary function defined by taking, for every $X_1,\ldots,X_k \subseteq W$,

$$f_i(X_1,\ldots,X_k) = \{w \in W \,:\, R_i(w,x_1,\ldots,x_k) \text{ for some } x_1 \in X_1,\ldots,x_k \in X_k\}.$$

Given a class $\mathcal{C}$ of frames of the same signature, we denote by $\mathsf{Cm}\,\mathcal{C}$ the class of complex algebras of frames in $\mathcal{C}$. The starting point of the duality between Kripke complete modal logics and BAOs is the following well-known property. For any class $\mathcal{C}$ of frames, and for any frame $\mathfrak{F}$ of the same signature,

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \Longleftrightarrow \qquad \mathfrak{Cm}\,\mathfrak{F} \in \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}. \tag{1}$$

The following general result shows that if $\mathcal{C}$ satisfies some closure conditions, then $\mathbf{H}$ is not needed in generating the variety corresponding to $\mathsf{Log}(\mathcal{C})$:

Theorem 2.1 (Goldblatt [11]) *If $\mathcal{C}$ is a class of frames that is closed under ultraproducts, then $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}\,\mathcal{C}$ is a canonical variety.*

Let us have a closer look at the subdirectly irreducible algebras of these varieties.

Lemma 2.2 *For any class $\mathcal{C}$ of frames, the subdirectly irreducible members of $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}\,\mathcal{C}$ belong to $\mathbf{S}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}_{\mathsf{p}}\,\mathcal{C}$.*

Proof. Let $\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}\,\mathcal{C}$ and let $\mathfrak{A} \rightarrowtail \prod_{i\in I}\mathfrak{A}_i$ be a subdirect embedding, for some $\mathfrak{A}_i \in \mathbf{S}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}\,\mathcal{C}$, $i \in I$. If $\mathfrak{A}$ is subdirectly irreducible then there is an $i \in I$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_i$, and so $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm}\,\mathfrak{F}$ for some $\mathfrak{F} \in \mathbb{G}\mathsf{sf}\,\mathcal{C}$. Then for each point x in $\mathfrak{F}$, $\mathfrak{F}^x \in \mathbb{G}\mathsf{sf}_{\mathsf{p}}\,\mathbb{G}\mathsf{sf}\,\mathcal{C} \subseteq \mathbb{G}\mathsf{sf}_{\mathsf{p}}\,\mathcal{C}$. It is not hard to show (see e.g. [10, 3.3]) that $\mathfrak{Cm}\,\mathfrak{F} \rightarrowtail \prod_{x\in\mathfrak{F}}\mathfrak{Cm}\,\mathfrak{F}^x$ is a (subdirect) embedding. So there exist subalgebras $\mathfrak{B}_x$ of $\mathfrak{Cm}\,\mathfrak{F}^x$ such that $\mathfrak{A} \rightarrowtail \prod_{x\in\mathfrak{F}}\mathfrak{B}_x$ is a subdirect embedding as well. As $\mathfrak{A}$ is subdirectly irreducible, there is some x in $\mathfrak{F}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}_x$, and so $\mathfrak{A} \in \mathbf{S}\,\mathsf{Cm}\,\mathbb{G}\mathsf{sf}_{\mathsf{p}}\,\mathcal{C}$. $\square$

Now Theorem 2.1 and Lemma 2.2 imply the following characterisation of varieties generated by certain classes of complex algebras.

Theorem 2.3 *If $\mathcal{C}$ is a class of frames that is closed under ultraproducts and point-generated subframes, then $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C} = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ is a canonical variety.*

We can also have a 'dual' structural characterisation of subdirectly irreducible algebras of these varieties. Recall that an *ultrafilter* of a BAO $\mathfrak{A} = (A,\wedge,-,f_i)_{i\in I}$ is any subset μ of A such that, for all $a,b \in A$,

- if $a \in \mu$ and $a \wedge b = a$ then $b \in \mu$;
- if $a,b \in \mu$ then $a \wedge b \in \mu$;
- $a \in \mu$ iff $-a \notin \mu$.

Let $Uf(A)$ denote the set of all such ultrafilters. Given a BAO $\mathfrak{A} = (A,\wedge,-,f_i)_{i\in I}$, we denote by $\mathfrak{Uf}\,\mathfrak{A} = (Uf(A),R_i)_{i\in I}$ its *ultrafilter frame*, where for each k-ary function f_i,

R_i is the following $k+1$-ary relation: for any $\mu, \nu_1, \dots, \nu_k \in Uf(A)$,

$$R_i(\mu, \nu_1, \dots, \nu_k) \qquad \text{iff} \qquad \forall a_1 \in \nu_1, \dots, a_k \in \nu_k \; f_i(a_1, \dots, a_k) \in \mu.$$

The *ultrafilter extension* of a frame $\mathfrak{F}$ is $\mathfrak{Ue}\,\mathfrak{F} = \mathfrak{Uf}\,\mathfrak{Cm}\,\mathfrak{F}$.

Theorem 2.4 *Let $\mathcal{C}$ be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every subdirectly irreducible algebra $\mathfrak{A}$,*

$$\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C} \iff \mathfrak{A} \in \mathbf{S}\,\mathsf{Cm}\,\mathcal{C} \iff \mathfrak{Uf}\,\mathfrak{A} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Proof. $\Leftarrow$: By Jónsson and Tarski's [19] theorem, $\mathfrak{A}$ is embeddable into $\mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A}$. And by duality, $\mathfrak{Cm}\,\mathfrak{Uf}\,\mathfrak{A}$ is embeddable into $\mathfrak{Cm}\,\mathfrak{G} \in \mathsf{Cm}\,\mathcal{C}$.

$\Rightarrow$: If $\mathfrak{A} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}$ then there is a subdirect embedding $\mathfrak{A} \rightarrowtail \prod_{i \in I} \mathfrak{A}_i$, for some $\mathfrak{A}_i \in \mathbf{S}\,\mathsf{Cm}\,\mathcal{C}$, $i \in I$. As $\mathfrak{A}$ is subdirectly irreducible, there is an $i \in I$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_i$, that is, $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{Cm}\,\mathfrak{F}$ for some $\mathfrak{F} \in \mathcal{C}$. By duality, $\mathfrak{Uf}\,\mathfrak{A}$ is a p-morphic image of $\mathfrak{Ue}\,\mathfrak{F}$. As $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of an ultrapower of $\mathfrak{F}$ (see [7,1,2]) and $\mathcal{C}$ is closed under taking ultraproducts, the proof is completed. □

As a consequence, we obtain a characterisation of "non-standard" frames for certain logics of the form $\mathsf{Log}(\mathcal{C})$:

Corollary 2.5 *Let $\mathcal{C}$ be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every rooted frame $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \iff \qquad \mathfrak{Ue}\,\mathfrak{F} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Proof. By (1) and Theorem 2.3,

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \iff \qquad \mathfrak{Cm}\,\mathfrak{F} \in \mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}.$$

As the complex algebra of a rooted frame is subdirectly irreducible [10], the statement follows from Theorem 2.4. □

As the ultrafilter extension of a finite frame is isomorphic to the frame itself, we obtain:

Corollary 2.6 *Let $\mathcal{C}$ be a class of frames that is closed under ultraproducts and point-generated subframes. Then for every finite rooted frame $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}) \qquad \iff \qquad \mathfrak{F} \textit{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}.$$

Now we would like to apply these general results to various classes of n-dimensional product frames, whenever $0 < n < \omega$. To this end, observe that the product operation commutes with ultraproducts and point-generated subframes:

Claim 2.7 *Let U be an ultrafilter over some index set I, and let $\mathfrak{F}_k^i$ be a 1-frame, for $i \in I$, $k < n$. Then:*

$$\prod_{i\in I}(\mathfrak{F}_0^i \times \cdots \times \mathfrak{F}_{n-1}^i)/U \quad \text{is isomorphic to} \quad (\prod_{i\in I}\mathfrak{F}_0^i/U) \times \cdots \times (\prod_{i\in I}\mathfrak{F}_{n-1}^i/U).$$

Claim 2.8 *Let $\mathfrak{F} = \mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1}$ and $\mathbf{x}$ be a point in $\mathfrak{F}$. Then:*

$$\mathfrak{F}^{\mathbf{x}} = \mathfrak{F}_0^{x_0} \times \cdots \times \mathfrak{F}_{n-1}^{x_{n-1}}.$$

Given classes $\mathcal{C}_i$ of 1-frames, for $i < n$, let us define

$$\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1} = \{\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} : \mathfrak{F}_i \in \mathcal{C}_i,\ i < n\}.$$

As a consequence of Claims 2.7 and 2.8, we obtain:

Proposition 2.9 *If, for $i < n$, $\mathcal{C}_i$ is a class of 1-frames that is closed under ultraproducts and point-generated subframes, then the class $\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ of n-dimensional product frames is closed under ultraproducts and point-generated subframes.*

Now, by Theorem 2.3, (1) and Corollary 2.5, we have:

Theorem 2.10 *If, for $i < n$, $\mathcal{C}_i$ is a class of 1-frames that is closed under ultraproducts and point-generated subframes, then:*
(i) $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}) = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1})$ *is a canonical variety.*
(ii) $\mathsf{Log}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1})$ *is a canonical n-modal logic.*
(iii) *For every rooted n-frame $\mathfrak{F}$,*

$$\mathfrak{F} \models \mathsf{Log}(\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}) \iff \mathfrak{Ue}\,\mathfrak{F} \text{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}.$$

Remark 2.11 The condition of Theorem 2.10 clearly holds if each $\mathcal{C}_i$ is defined by a set of 1-modal formulas having first-order correspondents, such as the classes of all frames of well-known modal logics like **K**, **K4**, **K4.3**, **S4.3**, **S5**, $\mathsf{Log}\{(\mathbb{Q}, <)\}$.

In particular, the classes $\mathcal{C}^n_{all}$ and $\mathcal{C}^n_{equiv}$ introduced in Section 1 are examples of classes of the form $\mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ within the scope of Theorem 2.10. So, for every rooted n-frame $\mathfrak{F}$,

$$\mathfrak{F} \models \mathbf{K}^n \iff \mathfrak{Ue}\,\mathfrak{F} \text{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}^n_{all}, \tag{2}$$
$$\mathfrak{F} \models \mathbf{S5}^n \iff \mathfrak{Ue}\,\mathfrak{F} \text{ is a p-morphic image of some } \mathfrak{G} \in \mathcal{C}^n_{equiv}. \tag{3}$$

Also, $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{all}$ and $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{equiv}$ are canonical varieties. The latter is a variety well-known in algebraic logic: the variety of *n-dimensional representable diagonal-free cylindric algebras* [14].

The following lemma shows that any n-frame having n equivalence relations and being a p-morphic image of an arbitrary n-dimensional product frame is also a p-morphic image of a product of n equivalence frames.

Lemma 2.12 *Let $n > 0$ be an arbitrary natural number, and let $\mathfrak{F} = (W, T_i)_{i<n}$ be an n-frame such that every T_i is an equivalence relation, for $i < n$. Suppose that $f : \mathfrak{G}_0 \times \cdots \times \mathfrak{G}_{n-1} \to \mathfrak{F}$ is a surjective p-morphism, for some 1-frames $\mathfrak{G}_i = (U_i, R_i)$, $i < n$. Then there exist 1-frames $\mathfrak{G}_i^* = (U_i, R_i^*)$, $i < n$, such that*

- *each R_i^* is an equivalence relation extending R_i, and*
- *$f : \mathfrak{G}_0^* \times \cdots \times \mathfrak{G}_{n-1}^* \to \mathfrak{F}$ is still a surjective p-morphism.*

Proof. In order to obtain the 'equivalence-closure' R_i^* of each R_i, one can add the missing pairs step by step, like it is done for the $n = 2$ case in the proof of [8, Lemma 5.8]. The fact that now n is an arbitrary natural number does not make any difference. □

Remark 2.13 Note that a similar proof would prove a stronger statement. The property of each T_i being an equivalence relation can be replaced with any property of T_i that can be defined by a set of *universal Horn* formulas in the first-order language having a binary predicate symbol and equality (and there can be different such properties for different i).

As a consequence of Theorem 2.10 and Lemma 2.12 we obtain:

Theorem 2.14 *Let L be any canonical n-modal logic with $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then $\mathbf{S5}^n$ is finitely axiomatisable over L: $\mathbf{S5}^n$ is the smallest n-modal logic containing L and the $\mathbf{S5}$-axioms for $\Diamond_i$, $i < n$.*

Proof. One inclusion is clear, let us prove the other. The **S5**-axioms are well-known examples of Sahlqvist formulas, and their first-order correspondent is the property of being an equivalence relation. So, by Sahlqvist's completeness theorem, the smallest n-modal logic containing L and the **S5**-axioms for $\Diamond_i$, $i < n$ is canonical, and so Kripke complete. So it is enough to show that every rooted n-frame $\mathfrak{F}$ for this logic is a frame for $\mathbf{S5}^n$.

Take such an n-frame $\mathfrak{F}$. As $\mathfrak{F}$ is a frame for $\mathbf{K}^n = \mathsf{Log}(\mathcal{C}^n_{all})$, by (2), $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of some n-dimensional product frame $\mathfrak{G}$. As $\mathfrak{F}$ validates the canonical **S5**-axioms, they also hold in $\mathfrak{Ue}\,\mathfrak{F}$, and so all the relations in $\mathfrak{Ue}\,\mathfrak{F}$ are equivalence relations. Now by Lemma 2.12, $\mathfrak{Ue}\,\mathfrak{F}$ is a p-morphic image of some $\mathfrak{G}^* \in \mathcal{C}^n_{equiv}$, and so by (3), $\mathfrak{F}$ is a frame for $\mathbf{S5}^n = \mathsf{Log}(\mathcal{C}^n_{equiv})$. □

Remark 2.15 By Remarks 2.11 and 2.13 we can have similar statements for any $\mathsf{Log}(\mathcal{K})$ in place of $\mathbf{S5}^n$, whenever $\mathcal{K} = \mathcal{C}_0 \times \cdots \times \mathcal{C}_{n-1}$ for some classes $\mathcal{C}_i$ of 1-frames, each of which is definable by Sahlqvist formulas having universal Horn first-order correspondents.

Theorem 2.14 shows that any negative result on the equational axiomatisation of the variety on n-dimensional representable diagonal-free cylindric algebras (such as its non-finiteness [18], for $n \geq 3$) transfers not only to its logic counterpart $\mathbf{S5}^n$, but also to other many-dimensional modal logics like $\mathbf{K}^n$. In other words, this theorem also means that all the complexities of a possible axiomatisation of $\mathbf{S5}^n$ come from the many-dimensional nature of the product frames and are already present in an axiomatisation of $\mathbf{K}^n$. Though, by a general result of [9], $\mathbf{K}^n$ is known to be recursively enumerable, an axiomatisation of $\mathbf{K}^n$ should be quite complex, whenever $n \geq 3$: any such axiomatisation should contain modal formulas of arbitrary modal depth for each modality [20], and infinitely many propositional variables [22]. (At the moment we cannot use Theorem 2.14 to infer the latter, as it is not known whether $\mathbf{S5}^n$ can be axiomatised using finitely many variables, whenever $n \geq 3$.) As Theorem 1.1 above shows, it will be quite hard to find an explicit axiomatisation for $\mathbf{K}^n$, as any such must contain n-modal formulas having no first-order correspondents.

3 Frames constructed from graphs

This and the next section are devoted to the proof of Theorem 1.1. Throughout, we fix a natural number $n \geq 3$. We will use n as a notation for both this number and for the set $\{0, \dots, n-1\}$. In order to show Theorem 1.1, we will give n-frames $\mathfrak{G}_k$, for $k < \omega$, such that each $\mathfrak{G}_k$ is a frame for $\mathbf{S5}^n$, but any non-principal ultraproduct of the $\mathfrak{G}_k$s is not a frame for $\mathbf{K}^n$.

We will use a construction of Hirsch and Hodkinson [15], so let us introduce the necessary notions. To begin with, let us enrich n-frames by adding some unary relations. An *$n\delta$-frame* is a relational structure of the form $\mathfrak{F} = (W, T_i, E_{ij})_{i,j<n}$ where $(W, T_i)_{i<n}$ is an n-frame and $E_{ij} \subseteq W$ for all $i, j < n$. For any n-dimensional product frame $\mathfrak{F} = (W_0 \times \cdots \times W_{n-1}, \bar{R}_i)_{i<n}$, we define an $n\delta$-frame $\mathfrak{F}^\delta$ by taking

$$\mathfrak{F}^\delta = (W_0 \times \cdots \times W_{n-1}, \bar{R}_i, \delta_{ij})_{i,j<n},$$

where $\delta_{ij} = \{\mathbf{w} \in W_0 \times \cdots \times W_{n-1} : w_i = w_j\}$, for $i, j < n$. These δ_{ij}s are called *diagonal elements*. Now let

$$\mathcal{C}^{n\delta}_{cube} = \{(\underbrace{\mathfrak{F} \times \cdots \times \mathfrak{F}}_{n})^\delta : \mathfrak{F} = (U, U \times U) \text{ for some non-empty set } U\}.$$

Note that if $\mathfrak{F}^\delta \in \mathcal{C}^{n\delta}_{cube}$ then $\mathfrak{F} \in \mathcal{C}^n_{equiv}$. Using Claims 2.7 and 2.8, it is not hard to see that $\mathcal{C}^{n\delta}_{cube}$ is closed under ultraproducts and point-generated subframes. So, by Theorem 2.3, $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{cube}$ is a canonical variety, well-known in algebraic logic: the variety of *n-dimensional representable cylindric algebras* [14].

Next, we define special $n\delta$-frames with the help of graphs. By a *graph* we mean a pair (Γ, E), where Γ is non-empty set and E is an irreflexive and symmetric binary relation on Γ (the *edges*). We identify a graph with its underlying set Γ of *nodes*. Given a graph $\Gamma = (\Gamma, E)$, a set $X \subseteq \Gamma$ is called *independent*, if $(x, y) \notin E$ whenever $x, y \in X$. The *chromatic number* $\chi(\Gamma)$ of Γ is the smallest $k < \omega$ such that Γ can be partitioned into k independent sets, and ∞ is there is no such k. An *ultrafilter on* Γ is an ultrafilter of the Boolean algebra of all subsets of Γ. For any graph Γ and $n < \omega$, we define the graph $\Gamma \times n$ as n disjoint copies of Γ, with all possible edges between distinct copies being added. For notions not defined here and general information on graphs, see [5].

Given a graph Γ, Hirsch and Hodkinson [15] define an $n\delta$-frame

$$\mathfrak{F}_\Gamma = (H_\Gamma, \equiv_i, D_{ij})_{i,j<n}$$

as follows.

- H_Γ is the set of all pairs $(K, \sim)$, where $K : n \to \Gamma \times n$ is a partial map, and $\sim$ is an equivalence relation on n, satisfying one of the following properties:
  - Either: all distinct $i, j < n$ are not $\sim$-equivalent, $K(i)$ is defined for all $i < n$, and $\{K(0), \ldots, K(n-1)\}$ is not an independent set in $\Gamma \times n$.
  - Or: $\{i, j\}$ is a 2-element $\sim$-class, all other $\sim$-classes are singletons, $K(i)$ and $K(j)$ are both defined and $K(i) = K(j)$, and $K(k)$ is not defined for $k \neq i, j$.
  - Or: the number of $\sim$-classes is $\leq n - 2$ and $K = \emptyset$.
- For every $i < n$, $\equiv_i$ is a binary relation on H_Γ defined by

$$(K, \sim) \equiv_i (K', \sim') \quad \text{iff} \quad \begin{array}{l} \sim|_{n-\{i\}} = \sim'|_{n-\{i\}}, \text{ and} \\ \text{either both } K(i) \text{ and } K'(i) \text{ are undefined,} \\ \text{or both } K(i) \text{ and } K'(i) \text{ are defined and } K(i) = K'(i). \end{array}$$

- For all $i, j < n$, D_{ij} is the following subset of H_Γ:

$$D_{ij} = \{(K, \sim) \, : \, i \sim j\}.$$

The following two propositions are proved in [15]:

Proposition 3.1 [15, Prop.5.2]
If $\chi(\Gamma) = \infty$ then $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ is an n-dimensional representable cylindric algebra.

Proposition 3.2 [15, Prop.5.4]
If Γ is infinite and $\chi(\Gamma) < \omega$, then $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ is not an n-dimensional representable cylindric algebra.

Observe that $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ is a BAO of the form $(A, \wedge, -, c_i, d_{ij})_{i,j<n}$, where each c_i is a unary function on A and each d_{ij} is an element of A. If we forget about the d_{ij}s, we obtain what is called the *diagonal-free reduct* of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$. It should be clear that this diagonal-free reduct is in fact $\mathfrak{Cm}\,\mathfrak{F}_\Gamma^-$, where $\mathfrak{F}_\Gamma^-$ is the n-frame $(H_\Gamma, \equiv_i)_{i<n}$.

We would like to have the diagonal-free "analogues" of Propositions 3.1 and 3.2. On the one hand, it is straightforward to see that if $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ is an n-dimensional representable cylindric algebra, that is, it belongs to $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n\delta}_{cube}$, then its diagonal-free reduct $\mathfrak{Cm}\,\mathfrak{F}_\Gamma^-$ belongs to $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^{n}_{equiv}$. So by (1) and Prop. 3.1 we obtain:

Proposition 3.3 *If $\chi(\Gamma) = \infty$ then $\mathfrak{F}_\Gamma^-$ is a frame for $\mathbf{S5}^n$.*

On the other hand, having the analogue of Prop. 3.2 is not so easy. As is well-known in algebraic logic, there are $n\delta$-frames $\mathfrak{G}$ such that though $\mathfrak{Cm}\,\mathfrak{G}$ is *not* an n-dimensional representable cylindric algebra, yet its diagonal-free reduct $\mathfrak{Cm}\,\mathfrak{G}^-$ *is* an n-dimensional representable diagonal-free cylindric algebra [14]. We will show that if Γ is infinite and $\chi(\Gamma) < \infty$ then for $\mathfrak{G} = \mathfrak{F}_\Gamma$ this is not the case: $\mathfrak{Cm}\,\mathfrak{F}_\Gamma^-$ is not an n-dimensional representable diagonal-free cylindric algebra, and so $\mathfrak{F}_\Gamma^-$ is not a frame for $\mathbf{S5}^n$.

Let us begin with showing some further properties of $\mathfrak{F}_\Gamma$:

Claim 3.4 (i) *For every $i < n$, $\equiv_i$ is an equivalence relation, and $D_{ii} = H_\Gamma$.*
(ii) *For all $i, j < n$, $\equiv_i$ and $\equiv_j$ commute.*
(iii) *For all $i, j, k < n$, $i \neq j$, $k \neq i, j$ and for all $(K \sim) \in H_\Gamma$,*

$$(K,\sim) \in D_{ij} \quad \textit{iff} \quad \text{there is } (K',\sim') \in D_{ik} \cap D_{kj} \text{ such that } (K,\sim) \equiv_k (K',\sim').$$

(iv) *For all $i, j < n$, $i \neq j$, if $(K,\sim), (K',\sim') \in D_{ij}$ and $(K,\sim) \equiv_i (K',\sim')$, then $(K,\sim) = (K',\sim')$.*
(v) *$\mathfrak{F}_\Gamma$ is rooted.*

Proof. The proofs of items (i) and (ii) are tiresome at places, but straightforward.

(iii): Fix some $k \neq i, j$. First, let $(K,\sim) \in D_{ij}$. Then $i \sim j$ and $K(k)$ is not defined for $k \neq i, j$. Let $K' = \emptyset$ and $\sim'$ such that $\sim'|_{n-\{k\}} =\sim|_{n-\{k\}}$ and $k \sim' i \sim' j$. Then $(K',\sim') \in H_\Gamma$ as required. For the other direction, let $(K',\sim') \in D_{ik} \cap D_{kj}$ and $(K,\sim) \equiv_k (K',\sim')$. Then $i \sim' k \sim' j$ and $\sim'|_{n-\{k\}} =\sim|_{n-\{k\}}$, so $i \sim j$, thus $(K,\sim) \in D_{ij}$.

(iv): If $(K,\sim), (K',\sim') \in D_{ij}$ and $(K,\sim) \equiv_i (K',\sim')$, then $i \sim j$, $i \sim' j$ and $\sim|_{n-\{i\}} =\sim'|_{n-\{i\}}$. Therefor $\sim=\sim'$ follows. Then there are two cases: either all of $K(i)$, $K(j)$, $K'(i)$, $K'(j)$ are defined and equal, or none of them is defined. In either case, $K = K'$ follows.

(v): (cf. [15, proof of Lemma 5.1]) We show that $(\emptyset, n \times n) \in H_\Gamma$ is suitable as root. To this end, take any $(K,\sim) \in H_\Gamma$. For any $i < n$, define a partial function $K_i : n \to \Gamma \times n$ by taking

$$K_i(j) = \begin{cases} K(i), & \text{if } j = 0 \text{ or } j = i, \text{ and } K(i) \text{ is defined,} \\ \text{undefined,} & \text{else.} \end{cases}$$

Let $\sim_i$ be the unique equivalence relation such that $\sim_i|_{n-\{i\}} =\sim|_{n-\{i\}}$ and $i \sim_i 0$. Then $(K_i, \sim_i) \in H_\Gamma$ and $(K,\sim) \equiv_i (K_i, \sim_i)$. So we have

$$(K,\sim) \equiv_1 (K_1, \sim_1) \equiv_2 (K_{12}, \sim_{12}) \cdots \equiv_{n-1} (K_{12\ldots n-1}, \sim_{12\ldots n-1}).$$

As $n \geq 3$, we have $0 \sim_{12} 1 \sim_{12} 2$, so $K_{12} = \cdots = K_{12\ldots n-1} = \emptyset$. Also, $\sim_{12\ldots n-1}= n \times n$. Therefore, by item (i), $(\emptyset, n \times n)$ is a root of $\mathfrak{F}_\Gamma$. $\square$

Properties (i)–(iv) above form the definition of what is called in algebraic logic an *n-dimensional cylindric atom structure* (see [13, 2.7.40]). Complex algebras of these special $n\delta$-frames belong to the variety of *n-dimensional cylindric algebras*. The interested reader can find the definition of this class in e.g. [13]. Here we only use that, being a variety, the class of n-dimensional cylindric algebras is closed under subalgebras. So, in particular, by Claim 3.4 we have that

$$\text{any subalgebra of } \mathfrak{Cm}\,\mathfrak{F}_\Gamma \text{ is an } n\text{-dimensional cylindric algebra.} \tag{4}$$

An element a in an algebra $\mathfrak{A} = (A, \wedge, -, c_i, d_{ij})_{i,j<n}$ is called *$< n$-dimensional*, if there is some $i < n$ such that $c_i(a) = a$. We will use the following result:

Theorem 3.5 (Johnson [18], see also [12,14])
Let $\mathfrak{A}$ be an n-dimensional cylindric algebra that is generated by its $<n$-dimensional elements. If the diagonal-free reduct $\mathfrak{A}^-$ of $\mathfrak{A}$ is an n-dimensional representable diagonal-free cylindric algebra, then $\mathfrak{A}$ is an n-dimensional representable cylindric algebra.

In Section 4 below we will define a subalgebra $\mathfrak{A}_\Gamma$ of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ and show the following two statements:

Proposition 3.6 *$\mathfrak{A}_\Gamma$ is an n-dimensional cylindric algebra generated by its $<n$-dimensional elements.*

Proposition 3.7 (cf. [15, Prop.5.4])
If Γ is infinite and $\chi(\Gamma) < \omega$, then $\mathfrak{A}_\Gamma$ is not an n-dimensional representable cylindric algebra.

Now if Γ is infinite and $\chi(\Gamma) < \omega$ then, by Theorem 3.5, the diagonal-free reduct $\mathfrak{A}_\Gamma^-$ of $\mathfrak{A}_\Gamma$ is not an n-dimensional representable diagonal-free cylindric algebra, that is, it does not belong to $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{equiv}$. As $\mathfrak{A}_\Gamma^-$ is a subalgebra of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma^-$, it follows that $\mathfrak{Cm}\,\mathfrak{F}_\Gamma^-$ does not belong to $\mathbf{S}\,\mathbf{P}\,\mathsf{Cm}\,\mathcal{C}^n_{equiv}$ either. So, by Claim 3.4(v) and (3), $\mathfrak{Ue}\,\mathfrak{F}_\Gamma^-$ is not a p-morphic image of a product of n equivalence frames. On the other hand, by Claim 3.4(i), all the relations $\equiv_i$ in $\mathfrak{F}_\Gamma^-$ are equivalence relations, for $i < n$. Therefore, the n-frame $\mathfrak{F}_\Gamma^-$ validates the canonical **S5**-axioms, for all $i < n$, so they also hold in $\mathfrak{Ue}\,\mathfrak{F}_\Gamma^-$, meaning that all its relations are equivalence relations as well. So, by Lemma 2.12, $\mathfrak{Ue}\,\mathfrak{F}_\Gamma^-$ is not a p-morphic image of *any* product frame. So, by (2), we have the required analogue of Prop. 3.2:

Proposition 3.8 *If Γ is infinite and $\chi(\Gamma) < \omega$, then $\mathfrak{F}_\Gamma^-$ is not a frame for $\mathbf{K}^n$.*

Now we can complete the proof of Theorem 1.1 precisely as it is done in the proof of [15, Thm.6.1]: It is not hard to see that if U is a non-principal ultrafilter over some index set I, then

$$\prod_{i\in I}\mathfrak{F}_{\Gamma_i}^-/U \quad \text{is isomorphic to} \quad \mathfrak{F}^-_{\prod_{i\in I}\Gamma_i/U}\,. \tag{5}$$

So what is left is to have a sequence $(\Gamma_k)_{k<\omega}$ of graphs such that

- $\chi(\Gamma_k) = \infty$ for all $k < \omega$.
- If Γ is any non-principal ultraproduct of the Γ_k, then Γ is infinite and $\chi(\Gamma) < \omega$.

As is shown in [15], one can have such a sequence of graphs by using Erdős's famous theorem [6]. Now let L be any set of n-modal formulas such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$. Then, by Prop. 3.3, each $\mathfrak{F}_{\Gamma_k}^-$ is a frame for L. On the other hand, by (5) and Prop. 3.8, any non-principal ultraproduct of the $\mathfrak{F}_{\Gamma_k}^-$ is not a frame for $\mathbf{K}^n$, and so not a frame for L.

4 The algebra $\mathfrak{A}_\Gamma$

What is left is to define a subalgebra $\mathfrak{A}_\Gamma$ of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$, and prove Propositions 3.6 and 3.7 about it. We define $\mathfrak{A}_\Gamma$ using notions introduced in [15, Defs. 4.1, 4.4]. To this end, for

$i < n$, let

$$F_i \;=\; \bigcap_{j,k\neq i,\; j\neq k} (H_\Gamma - D_{jk}) \;=\; \{(K,\sim) \in H_\Gamma \,:\, K(i) \text{ is defined}\}.$$

Now, for any $X \subseteq \Gamma \times n$, put

$$X^{(i)} = \{(K,\sim) \in F_i \,:\, K(i) \in X\},$$

and let $\mathfrak{A}_\Gamma$ be the subalgebra of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ generated by the set

$$\{X^{(i)} \,:\, i < n,\, X \subseteq \Gamma \times n\}.$$

Proof of Prop. 3.6. By (4), $\mathfrak{A}_\Gamma$ is an n-dimensional cylindric algebra. Now take any $i < n$ and $X \subseteq \Gamma \times n$. Let $(K,\sim) \in X^{(i)}$ and $(K',\sim') \in H_\Gamma$ such that $(K,\sim) \equiv_i (K',\sim')$. Then both $K(i)$ and $K'(i)$ are defined, $(K',\sim') \in F_i$ and $K(i) = K'(i)$, so $(K',\sim') \in X^{(i)}$ as well. This shows that $c_i(X^{(i)}) = X^{(i)}$, so $X^{(i)}$ is $<n$-dimensional.

Proof of Prop. 3.7. We establish a connection between ultrafilters of $\mathfrak{A}_\Gamma$ and ultrafilters over $\Gamma \times n$, just like it is done in [15] between ultrafilters of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ and ultrafilters over $\Gamma \times n$.

For any $i < n$, let E_i denote the binary relation corresponding to c_i in the ultrafilter frame of $\mathfrak{A}_\Gamma$. For any $S \subseteq F_i$, put $S(i) = \{K(i) \,:\, (K,\sim) \in S\}$. For any $i < n$, and any ultrafilter μ of $\mathfrak{A}_\Gamma$, let

$$\mu(i) = \{S(i) \,:\, S \in \mu,\, S \subseteq F_i\}.$$

Claim 4.1 (analogue of [15, Lemma 4.6])
Let μ be an ultrafilter of $\mathfrak{A}_\Gamma$ such that $F_i \in \mu$ for some $i < n$. Then:
(i) *$\mu(i)$ is an ultrafilter on $\Gamma \times n$.*
(ii) *If $j < n$ and $D_{ij} \in \mu$, then $F_j \in \mu$ and $\mu(j) = \mu(i)$.*
(iii) *For any ultrafilter ν of $\mathfrak{A}_\Gamma$, we have $\mu E_i \nu$ iff $F_i \in \nu$ and $\mu(i) = \nu(i)$.*

Proof. (i): An arbitrary element of $\mu(i)$ is of the form $S(i)$ for some $S \in \mu$, $S \subseteq F_i$. Suppose that $S(i) \subseteq X \subseteq \Gamma \times n$. Then it is not hard to see that $S \subseteq S(i)^{(i)} \subseteq X^{(i)}$. As $X^{(i)}$ is an element of $\mathfrak{A}_\Gamma$ and μ is an ultrafilter of $\mathfrak{A}_\Gamma$, $X^{(i)} \in \mu$ follows. We also have $X^{(i)} \subseteq F_i$. So $X = X^{(i)}(i) \in \mu(i)$.

The proofs of the other two ultrafilter-properties, and of (ii) and (iii) are the same as those of the corresponding items in [15, Lemma 4.6]. $\Box$

Now we can complete the proof of Prop. 3.7 by following precisely the same steps as in the proof of [15, Prop.5.4]), using ultrafilters of $\mathfrak{A}_\Gamma$ in place of ultrafilters of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$. If $\chi(\Gamma) < \omega$, then also $\chi(\Gamma \times n) < \omega$. So $\Gamma \times n = I_0 \cup \cdots \cup I_{k-1}$ for some natural number k and independent sets I_j, for $j < k$. So, for every ultrafilter μ on $\Gamma \times n$, there is a unique $j < k$ such that $I_j \in \mu$. As Γ is infinite, so is H_Γ, and so is $\mathfrak{A}_\Gamma$.

Now suppose that $\mathfrak{A}_\Gamma$ is an n-dimensional representable cylindric algebra. As is shown in [15, Lemma 5.1], every subalgebra of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ is subdirectly irreducible, therefore so is $\mathfrak{A}_\Gamma$. Thus, by Theorem 2.4, $\mathfrak{Uf}\,\mathfrak{A}_\Gamma$ is a p-morphic image of some frame from $\mathcal{C}^{n\delta}_{cube}$, that is, there exist an infinite set U and a surjective function $h : U^n \to$ {ultrafilters of $\mathfrak{A}_\Gamma$} such that

(h1) for all $i < n$, $\mathbf{a}, \mathbf{b} \in U^n$, if $a_j = b_j$ for all $j < n$, $j \neq i$, then $h(\mathbf{a}) E_i h(\mathbf{b})$,

(h2) for all $i, j < n$, $\mathbf{a} \in U^n$, $a_i = a_j$ iff $D_{ij} \in h(\mathbf{a})$.

(We will not use the 'backward' condition w.r.t. E_i.) So if $\mathbf{a} \in U^n$ is such that all the a_i are different for $i < n$ then, by (h2) and Claim 4.1(i),

$$\big(h(\mathbf{a})(0), \ldots, h(\mathbf{a})(n-1)\big)$$

is an n-tuple of $(\Gamma \times n)$-ultrafilters. We show that for each $i < n$, $h(\mathbf{a})(i)$ depends only on the set $\{a_0, \ldots, a_{n-1}\} - \{a_i\}$. That is, such a function h determines of what is called in [15] a *patch system.*

Claim 4.2 *Let $i, j < n$ and $\mathbf{a}, \mathbf{b} \in U^n$ be such that*

- *$a_k \neq a_\ell$ whenever $k, \ell \neq i$, $k, \ell < n$,*
- *$b_k \neq b_\ell$ whenever $k, \ell \neq j$, $k, \ell < n$, and*
- *$\{a_k : k < n, k \neq i\} = \{b_k : k < n, k \neq j\}$.*

Then $h(\mathbf{a})(i) = h(\mathbf{b})(j)$.

Proof. This claim is claimed and proved in the proof of [15, Lemma 4.12(2)]. Using ultrafilters of $\mathfrak{A}_\Gamma$ instead of ultrafilters of $\mathfrak{Cm}\,\mathfrak{F}_\Gamma$ does not make any difference. □

As a consequence we obtain:

Claim 4.3 (cf. [15, Def. 4.11, Lemma 4.12(2)])
Given h as above, define a function

$$\partial h : \{n-1\text{-element subsets of } U\} \to \{\text{ultrafilters on } \Gamma \times n\}$$

by taking, for every n-element subset A of U an n-tuple $\mathbf{a} \in U^n$ such that $A = \{a_0, \ldots, a_{n-1}\} - \{a_i\}$ for some $i < n$ and putting

$$\partial h(A) = h(\mathbf{a})(i).$$

Then ∂h is well-defined.

Take the functions h and ∂h as defined above. As $\mathfrak{A}_\Gamma$ is infinite, the domain U^n of h should also be infinite. Choose an infinite sequence $a_0, a_1, \ldots$ of distinct elements from U, and define a function

$$f : \{n-1\text{-element subsets of } \omega\} \to k$$

by taking

$$f(\{i_1, \ldots, i_{n-1}\}) = j \quad \text{iff} \quad I_j \in h\big(\{a_{i_1}, \ldots, a_{i_{n-1}}\}\big).$$

By Ramsey's theorem [23], we may assume that the value of f is constant, say, c. Let $A = \{a_0, \ldots, a_{n-1}\}$ and $\mathbf{a} = (a_0, \ldots, a_{n-1})$. Then $I_c \in \partial h(A - \{a_i\}) = h(\mathbf{a})(i)$, for each

$i < n$. So for every $i < n$ there exists some $S_i \in h(\mathbf{a})$ such that $S_i \subseteq F_i$ and $S_i(i) = I_c$. As $h(\mathbf{a})$ is an ultrafilter of $\mathfrak{A}_\Gamma$ and $\bigcap_{i<n} S_i \in h(\mathbf{a})$, we have that $\bigcap_{i<n} S_i \neq \emptyset$. Take any $(K, \sim) \in \bigcap_{i<n} S_i$. Then on the one hand, $K(i)$ is defined for all $i < n$, so the set $\{K(0), \ldots, K(n-1)\}$ is not independent. (This argument is written in the proof of [15, Lemma 4.10].) On the other hand, as $S_i(i) = I_c$, we have $\{K(0), \ldots, K(n-1)\} \subseteq I_c$, so it is independent, a contradiction, completing the proof of Prop. 3.7.

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