

# Sahlqvist Theorems for Precontact Logics

Philippe Balbiani<sup>1</sup>

*Institut de recherche en informatique de Toulouse  
CNRS — Université de Toulouse*

Stanislav Kikot<sup>2</sup>

*Department of Computer Science and Information Systems  
Birkbeck — University of London*

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## Abstract

Precontact logics are propositional modal logics that have been recently considered in order to obtain decidable fragments of the region-based theories of space introduced by De Laguna and Whitehead. We give the definition of Sahlqvist formulas to this region-based setting and we prove correspondence and canonicity results. Together, these results give rise to a completeness result for precontact logics that are axiomatized by Sahlqvist axioms.

*Keywords:* Sahlqvist theory; Precontact logics; Correspondence; Canonicity.

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## 1 Introduction

In modal logic, Sahlqvist formulas are modal formulas with remarkable properties [30,31]: the Sahlqvist correspondence theorem says that every Sahlqvist formula corresponds to a first-order definable class of frames; the Sahlqvist completeness theorem says that when Sahlqvist formulas are used as axioms in a normal logic, the logic is complete with respect to the elementary class of frames the axioms define. Roughly speaking, modal formulas in the Sahlqvist fragment are implications the antecedents of which do not contain occurrences of boxes taking scope over diamonds. As a result, their main characteristic consists in this: second-order quantifier elimination is complete for their standard translation in a second-order setting [10,17].

The Sahlqvist fragment does not contain all modal formulas corresponding to a first-order definable class of frames: there exists non-Sahlqvist formulas that correspond to first-order conditions. Moreover, it is undecidable, given a modal formula, to determine whether it has a first-order correspondent. As well, it is

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<sup>1</sup> Address: Institut de recherche en informatique de Toulouse, CNRS — Université de Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 9, FRANCE; balbiani@irit.fr.

<sup>2</sup> Address: Department of Computer Science and Information Systems, Birkbeck — University of London, Malet Street, London WC1E 7HX, UK; kikot@dcs.bbk.ac.uk.

undecidable, given a first-order sentence, to determine whether it has a modal correspondent. For more on this, see [8,9].

There is quite a lot of literature around Sahlqvist theorem, which roughly can be divided into the following groups. A first group concerns the study of algorithms performing second-order quantifier elimination, see [32] for a recent account of this area. A second group deals with generalizations of Sahlqvist theorem within the classical syntax and semantics of modal logic [21,34,35]. A third group deals with generalization of Sahlqvist theorem to stronger or weaker variants of modal language: hybrid logics [7], distributive modal logic [16], polyadic modal languages [19,20], relevant modal logics [33], modal fixed point logics [5]. This paper certainly belongs to this group.

Recently, in order to obtain decidable fragments of the region-based theories of space introduced by De Laguna [28] and Whitehead [38], propositional languages with topological semantics have been considered [15,25,36,37]. The main tools in the completeness proofs of the associated logics are the representation theorems for precontact algebras and adjacency spaces presented in [11,12,13,14]. At first sight, the modal nature of the logics in question is not patently visible. Nevertheless, almost all known tools and techniques in modal logic — e.g. the method of canonical models and the filtration method — can be transferred to them with slight modifications for obtaining the above-mentioned completeness proofs [1,2,3,4].

Hence, a natural question is to ask whether a Sahlqvist-like theory — i.e. a theory that identifies a set of formulas that correspond to first-order definable classes of frames and that define logics complete with respect to the elementary classes of frames they correspond to — can be elaborated on the setting of the region-based propositional modal logics of space (RBPMLS). With the object of answering this question, we give the definition of Sahlqvist formulas to this RBPMLS setting and we prove correspondence and canonicity results. Together, these results give rise to a completeness result for RBPMLS that are axiomatized by Sahlqvist axioms. Note that the translation  $C(a, b) = \diamond_u(b \wedge \diamond a)$  conservatively embeds RBPMLS into the basic modal language extended with universal modality, hence the correspondence part for RBPMLS follows from the classical Sahlqvist theorem. However, the completeness part for RBPMLS is genuinely new, since it provides an inference in a relatively weak calculus for every RBPMLS formula, which follows semantically from the initial 'Sahlqvist' RBPMLS formula. We assume the reader is at home with tools and techniques in modal logic. For more on this, see [6,27].

## 2 Syntax

The language is defined using a countable set  $BV$  of Boolean variables (with typical members denoted by  $p, q$ , etc). We inductively define the set  $t(BV)$  of terms (with typical members denoted by  $a, b$ , etc) as follows:

- $a ::= p \mid 0 \mid -a \mid (a \cup b)$ .

The other Boolean constructs for RBPMLS terms are defined as usual:  $1$  for  $-0$  and  $(a \cap b)$  for  $-(-a \cup -b)$ . A term  $a$  is positive iff  $a$  is built up from Boolean variables using only  $1$ ,  $\cup$  and  $\cap$ . We inductively define the set  $f(BV)$  of formulas (with typical members denoted by  $\phi$ ,  $\psi$ , etc) as follows:

- $\phi ::= a \equiv b \mid C(a, b) \mid \perp \mid \neg\phi \mid (\phi \vee \psi)$ .

The other Boolean constructs for RBPMLS formulas are defined as usual:  $\top$  for  $\neg\perp$ ,  $(\phi \wedge \psi)$  for  $\neg(\neg\phi \vee \neg\psi)$ ,  $(\phi \rightarrow \psi)$  for  $(\neg\phi \vee \psi)$  and  $(\phi \leftrightarrow \psi)$  for  $(\neg(\phi \vee \psi) \vee \neg(\neg\phi \vee \neg\psi))$ . We obtain the formulas  $a \not\equiv b$  and  $\bar{C}(a, b)$  as abbreviations:

- $a \not\equiv b ::= \neg a \equiv b$ ,
- $\bar{C}(a, b) ::= \neg C(a, b)$ .

If a formula  $\phi$  is built up from  $a \not\equiv 0$  and  $C(a, b)$  (where  $a$  and  $b$  are positive terms) using only  $\top$ ,  $\vee$  and  $\wedge$  then we say that  $\phi$  is negation-free. A formula  $\phi$  is positive iff  $\phi$  is built up from  $a \not\equiv 0$ ,  $\neg a \equiv 0$ ,  $C(a, b)$  and  $\bar{C}(-a, -b)$  (where  $a$  and  $b$  are positive terms) using only  $\top$ ,  $\vee$  and  $\wedge$ . The notion of subterm and the notion of subformula are standard. We adopt the standard rules for omission of the parentheses. If a formula  $\phi$  is an implication  $\psi \rightarrow \chi$  in which  $\psi$  is negation-free and  $\chi$  is positive then we say that  $\phi$  is a Sahlqvist formula. Let us consider the following 8 formulas:

- (i)  $\top \rightarrow C(1, 1)$ ,
- (ii)  $p \not\equiv 0 \rightarrow C(p, 1)$ ,
- (iii)  $p \not\equiv 0 \rightarrow C(p, p)$ ,
- (iv)  $C(p, q) \rightarrow C(q, p)$ ,
- (v)  $C(p, q) \rightarrow C(p, r) \vee C(-r, q)$ ,
- (vi)  $p \not\equiv 0 \wedge \neg p \not\equiv 0 \rightarrow C(p, -p)$ ,
- (vii)  $(p \cup q) \equiv 1 \wedge (p \cap q) \equiv 0 \rightarrow C(p, p) \vee C(q, q)$ ,
- (viii)  $(p \cap -q) \not\equiv 0 \rightarrow C(p, -q) \vee C(q, -q)$ .

Obviously, the first 4 formulas are Sahlqvist formulas whereas the last 4 formulas are not Sahlqvist formulas.

### 3 Kripke-type semantics

RBPMLS have 3 kinds of semantics:

- an algebraic semantics based on some classes of abstract contact algebras of regions,
- a topological semantics based on concrete contact algebras of regions over some classes of topological spaces,
- a Kripke-type semantics based on some classes of Kripke frames regarded as adjacency spaces.

The main tools in the equivalence of these 3 kinds of semantics are the representation theorems for precontact algebras and adjacency spaces presented

in [11,12,13,14]. In this paper, seeing that we want to elaborate a Sahlqvist-like theory on the setting of RBPMLS, we concentrate attention to the Kripke-type semantics. A Kripke frame is an ordered pair  $\mathcal{F} = (W, R)$  where  $W$  is a non-empty set of possible worlds and  $R$  is a binary relation on  $W$ . A valuation based on  $\mathcal{F}$  is a function  $V$  assigning to each Boolean variable  $p$  a subset  $V(p)$  of  $W$ . As usual,  $V$  induces a homomorphism  $(\cdot)^V$  assigning to each term  $a$  a subset  $(a)^V$  of  $W$  as follows:

- $(p)^V = V(p)$ ,
- $(0)^V = \emptyset$ ,
- $(-a)^V = W \setminus (a)^V$ ,
- $(a \cup b)^V = (a)^V \cup (b)^V$ .

We shall say that  $V$  is smaller than a valuation  $V'$  based on  $\mathcal{F}$ , in symbols  $V \leq V'$ , iff for all Boolean variables  $p$ ,  $V(p) \subseteq V'(p)$ .

**Lemma 3.1** *Let  $V, V'$  be valuations based on  $\mathcal{F}$  such that  $V \leq V'$ . For all positive terms  $a$ ,  $(a)^V \subseteq (a)^{V'}$ .*

**Proof.** The proof is done by induction on  $a$ . □

A Kripke model is an ordered triple  $\mathcal{M} = (W, R, V)$  where  $\mathcal{F} = (W, R)$  is a frame and  $V$  is a valuation based on  $\mathcal{F}$ . The satisfiability of a formula  $\phi$  in  $\mathcal{M}$ , in symbols  $\mathcal{M} \models \phi$ , is defined as follows:

- $\mathcal{M} \models a \equiv b$  iff  $(a)^V = (b)^V$ ,
- $\mathcal{M} \models C(a, b)$  iff there exists  $x, y \in W$  such that  $xRy$ ,  $x \in (a)^V$  and  $y \in (b)^V$ ,
- $\mathcal{M} \not\models \perp$ ,
- $\mathcal{M} \models \neg\phi$  iff  $\mathcal{M} \not\models \phi$ ,
- $\mathcal{M} \models \phi \vee \psi$  iff  $\mathcal{M} \models \phi$  or  $\mathcal{M} \models \psi$ .

As a result,  $\mathcal{M} \models a \not\equiv b$  iff  $(a)^V \neq (b)^V$  and  $\mathcal{M} \models \bar{C}(a, b)$  iff for all  $x, y \in W$ , if  $xRy$  then  $x \notin (a)^V$  or  $y \notin (b)^V$ .

**Lemma 3.2** *Let  $V, V'$  be valuations based on  $\mathcal{F}$  such that  $V \leq V'$ .*

- (i) *For all negation-free formulas  $\phi$ , if  $(\mathcal{F}, V) \models \phi$  then  $(\mathcal{F}, V') \models \phi$ .*
- (ii) *For all positive formulas  $\phi$ , if  $(\mathcal{F}, V) \models \phi$  then  $(\mathcal{F}, V') \models \phi$ .*

**Proof.** Both items follow by induction on  $\phi$ , using Lemma 3.1. □

Let  $\mathcal{F}$  be a frame. A formula  $\phi$  is valid in  $\mathcal{F}$ , in symbols  $\mathcal{F} \models \phi$ , iff for all models  $\mathcal{M}$  based on  $\mathcal{F}$ ,  $\mathcal{M} \models \phi$ .

## 4 Standard translation into a first-order language

In the above Kripke-type semantics, satisfaction is a binary relation between models and formulas whereas in the semantics for the basic modal language, satisfaction is a ternary relation between models, possible worlds and formulas. Such a difference is illustrated by the following translation of our language into a first-order language. Let  $\mathcal{L}^1(BV)$  be the first-order language with equality

which has the unary predicates  $P_0, P_1$ , etc corresponding to the Boolean variables  $p_0, p_1$ , etc in  $BV$  and the binary predicate  $R_C$  corresponding to the modal operator  $C$ . If  $u$  is a first-order variable and  $a$  is a term then the first-order formula  $ST(u, a)$  is inductively defined as follows:

- $ST(u, p_n) = P_n(u)$ ,
- $ST(u, 0) = \perp$ ,
- $ST(u, \neg a) = \neg ST(u, a)$ ,
- $ST(u, a \cup b) = ST(u, a) \vee ST(u, b)$ .

If  $\phi$  is a formula then the first-order sentence  $ST(\phi)$  is inductively defined as follows:

- $ST(a \equiv b) = \forall u (ST(u, a) \leftrightarrow ST(u, b))$ ,
- $ST(C(a, b)) = \exists u \exists v (R_C(u, v) \wedge ST(u, a) \wedge ST(v, b))$ ,
- $ST(\perp) = \perp$ ,
- $ST(\neg \phi) = \neg ST(\phi)$ ,
- $ST(\phi \vee \psi) = ST(\phi) \vee ST(\psi)$ .

**Proposition 4.1** *Let  $\mathcal{M} = (W, R, V)$  be a model.*

- (i) *For all terms  $a$  and for all  $x \in W$ ,  $x \in (a)^V$  iff  $\mathcal{M} \models ST(u, a)[x]$ .*
- (ii) *For all formulas  $\phi$ ,  $\mathcal{M} \models \phi$  iff  $\mathcal{M} \models ST(\phi)$ .*

**Proof.** The first item follows by induction on  $a$  and the second one follows by induction on  $\phi$ , using the first item.  $\square$

**Proposition 4.2** *Let  $\mathcal{F} = (W, R)$  be a frame. For all formulas  $\phi$ ,  $\mathcal{F} \models \phi$  iff  $\mathcal{F} \models ST(\phi)$ .*

**Proof.** By Proposition 4.1.  $\square$

Obviously,  $ST(\phi)$  belongs to the 2-variable fragment of  $\mathcal{L}^1(BV)$  for each formula  $\phi$ . Since the satisfiability problem for the 2-variable fragment of any first-order language with equality is decidable in nondeterministic exponential time [22,29], then the embedding of our language into  $\mathcal{L}^1(BV)$  considered in Proposition 4.2 implies that if  $\mathcal{C}$  is a class of frames definable by a first-order sentence with at most 2 variables then the satisfiability problem in models based on  $\mathcal{C}$ -frames for RBPMLS formulas is decidable in nondeterministic exponential time.

## 5 Correspondence theorem

We shall say that a formula  $\phi$  and a first-order sentence  $\alpha$  of the first-order language  $\mathcal{L}^1(\emptyset)$  with equality which has the binary predicate  $R_C$  corresponding to the modal operator  $C$  are frame correspondents iff for all frames  $\mathcal{F} = (W, R)$ ,  $\mathcal{F} \models \phi$  iff  $\mathcal{F} \models \alpha$ .

**Theorem 5.1** *Let  $\phi$  be a Sahlqvist formula. There exists a first-order sentence  $\alpha$  of the first-order language  $\mathcal{L}^1(\emptyset)$  such that  $\phi$  and  $\alpha$  are frame correspondents.*

Moreover,  $\alpha$  is effectively computable from  $\phi$ .

**Proof.** Since  $\phi$  is a Sahlqvist formula, then  $\phi$  is an implication  $\psi \rightarrow \chi$  in which  $\psi$  is negation-free and  $\chi$  is positive. Without loss of generality, we may assume that  $\psi$  is equal either to  $\top$  or to a disjunction  $\psi_1 \vee \dots \vee \psi_n$  of  $\top$ -free  $\vee$ -free negation-free formulas. Consider a frame  $\mathcal{F} = (W, R)$ . Let  $p_1, \dots, p_N$  be an enumeration of the Boolean variables occurring in  $\chi$  and  $P_1, \dots, P_N$  be the corresponding unary predicates. We need to consider the following 2 cases.

**Case “ $\psi$  is equal to  $\top$ ”.** The following properties are equivalent:

- (i)  $\mathcal{F} \models \phi$ ,
- (ii) for all valuations  $V$  based on  $\mathcal{F}$ ,  $(\mathcal{F}, V) \models \phi$ ,
- (iii) for all valuations  $V$  based on  $\mathcal{F}$ ,  $(\mathcal{F}, V) \models \chi$ .

Let  $V_{min}$  be the empty valuation. Since  $\chi$  is positive, then by the second item of Lemma 3.2, (iii) is equivalent to the following property:

- (iv)  $(\mathcal{F}, V_{min}) \models \chi$ .

By Proposition 4.1, (iv) is equivalent to the following property:

- (v)  $(\mathcal{F}, V_{min}) \models ST(\chi)$ .

Since  $V_{min}$  is definable in  $\mathcal{L}^1(\emptyset)$  by  $P_m(\cdot) ::= \perp$  for each  $m \in \{1, \dots, N\}$ , then (v) is equivalent to the following property:

- (vi)  $\mathcal{F} \models \theta(ST(\chi))$ ,  $\theta$  being a substitution that replaces  $P_m(\cdot)$  by  $\perp$  for each  $m \in \{1, \dots, N\}$ .

As a result, one may take  $\alpha$  to be  $\theta(ST(\chi))$ .

**Case “ $\psi$  is equal to a disjunction  $\psi_1 \vee \dots \vee \psi_n$  of  $\top$ -free  $\vee$ -free negation-free formulas”.** Let  $i \in \{1, \dots, n\}$ . Without loss of generality, we may assume that  $\psi_i$  is equal to a conjunction of the form  $a_{i,1} \neq 0 \wedge \dots \wedge a_{i,k_i} \neq 0 \wedge C(b_{i,l_i}, c_{i,l_i})$  where all  $a_{i,\star}$  are equal to an intersection of Boolean variables and all  $b_{i,\star}, c_{i,\star}$  are equal either to 1 or to an intersection of Boolean variables. The following properties are equivalent:

- (i)  $\mathcal{F} \models \psi_i \rightarrow \chi$ ,
- (ii) for all valuations  $V$  based on  $\mathcal{F}$ ,  $(\mathcal{F}, V) \models \psi_i \rightarrow \chi$ ,
- (iii) for all valuations  $V$  based on  $\mathcal{F}$ , if  $(\mathcal{F}, V) \models \psi_i$  then  $(\mathcal{F}, V) \models \chi$ .

For all  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$ , if  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  then for all  $x_1, \dots, x_{k_i} \in W$ , let  $V_{i,min}$  be a valuation such that for all  $m \in \{1, \dots, N\}$ ,  $V_{i,min}(p_m) = \{x_j: 1 \leq j \leq k_i \text{ and } p_m \text{ occurs in } a_{i,j}\} \cup \{y_j: 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } b_{i,j}\} \cup \{z_j: 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } c_{i,j}\}$ . By Proposition 4.1, the following properties are equivalent:

- (iv)  $(\mathcal{F}, V) \models \psi_i$ ,
- (v)  $(\mathcal{F}, V) \models ST(\psi_i)$ ,
- (vi)  $(\mathcal{F}, V) \models \exists u_1 ST(u_1, a_{i,1}) \wedge \dots \wedge \exists u_{k_i} ST(u_{k_i}, a_{i,k_i}) \wedge \exists v_1 \exists w_1 (R_C(v_1, w_1) \wedge ST(v_1, b_{i,1}) \wedge ST(w_1, c_{i,1})) \wedge \dots \wedge \exists v_{l_i} \exists w_{l_i} (R_C(v_{l_i}, w_{l_i}) \wedge ST(v_{l_i}, b_{i,l_i}) \wedge$

$ST(w_{l_i}, c_{i,l_i}),$

- (vii) there exists  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$  such that  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  and there exists  $x_1, \dots, x_{k_i} \in W$  such that  $(\mathcal{F}, V) \models ST(u_1, a_{i,1}) \wedge \dots \wedge ST(u_{k_i}, a_{i,k_i}) \wedge (ST(v_1, b_{i,1}) \wedge ST(w_1, c_{i,1})) \wedge \dots \wedge (ST(v_{l_i}, b_{i,l_i}) \wedge ST(w_{l_i}, c_{i,l_i})) [u_1 := x_1, \dots, u_{k_i} := x_{k_i}, v_1 := y_1, w_1 := z_1, \dots, v_{l_i} := y_{l_i}, w_{l_i} := z_{l_i}]$ ,
- (viii) there exists  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$  such that  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  and there exists  $x_1, \dots, x_{k_i} \in W$  such that  $V_{i,min} \leq V$ .

Since  $\chi$  is positive, then by the second item of Lemma 3.2, (iii) is equivalent to the following property:

- (ix) for all  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$ , if  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  then for all  $x_1, \dots, x_{k_i} \in W$ ,  $(\mathcal{F}, V_{i,min}) \models \chi$ .

By Proposition 4.1, (ix) is equivalent to the following property:

- (x) for all  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$ , if  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  then for all  $x_1, \dots, x_{k_i} \in W$ ,  $(\mathcal{F}, V_{i,min}) \models ST(\chi)$ .

Since  $V_{i,min}$  is definable in  $\mathcal{L}^1(\emptyset)$  by  $P_m(\cdot) ::= \bigvee \{ \cdot = u_j : 1 \leq j \leq k_i \text{ and } p_m \text{ occurs in } a_{i,j} \} \vee \bigvee \{ \cdot = v_j : 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } b_{i,j} \} \vee \bigvee \{ \cdot = w_j : 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } c_{i,j} \}$  for each  $m \in \{1, \dots, N\}$ , then (x) is equivalent to the following properties:

- (xi) for all  $y_1, z_1, \dots, y_{l_i}, z_{l_i} \in W$ , if  $y_1 R z_1, \dots, y_{l_i} R z_{l_i}$  then for all  $x_1, \dots, x_{k_i} \in W$ ,  $\mathcal{F} \models \theta_i(ST(\chi))$ ,  $\theta_i$  being a substitution that replaces  $P_m(\cdot)$  by  $\bigvee \{ \cdot = u_j : 1 \leq j \leq k_i \text{ and } p_m \text{ occurs in } a_{i,j} \} \vee \bigvee \{ \cdot = v_j : 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } b_{i,j} \} \vee \bigvee \{ \cdot = w_j : 1 \leq j \leq l_i \text{ and } p_m \text{ occurs in } c_{i,j} \}$  for each  $m \in \{1, \dots, N\}$ ,
- (xii)  $\mathcal{F} \models \forall u_1 \dots \forall u_{k_i} \forall v_1 \forall w_1 \dots \forall v_{l_i} \forall w_{l_i} (R_C(v_1, w_1) \wedge \dots \wedge R_C(v_{l_i}, w_{l_i}) \rightarrow \theta_i(ST(\chi)))$ .

As a result, one may take  $\alpha$  to be the conjunction of all  $\forall u_1 \dots \forall u_{k_i} \forall v_1 \forall w_1 \dots \forall v_{l_i} \forall w_{l_i} (R_C(v_1, w_1) \wedge \dots \wedge R_C(v_{l_i}, w_{l_i}) \rightarrow \theta_i(ST(\chi)))$  for each  $1 \leq i \leq n$ .  $\square$

By way of examples, we determine the first-order sentences corresponding to the 4 Sahlqvist formulas considered at the end of Section 2.

(i) Concerning the formula  $\top \rightarrow C(1, 1)$ , its frame correspondent is the first-order sentence  $\exists u' \exists v' (R_C(u', v') \wedge ST(u', 1) \wedge ST(v', 1))$ . It is equivalent to  $\exists u' \exists v' R_C(u', v')$ .

(ii) As for the formula  $p \not\equiv 0 \rightarrow C(p, 1)$ , its frame correspondent is the first-order sentence  $\forall u \theta(\exists u' \exists v' (R_C(u', v') \wedge ST(u', p) \wedge ST(v', 1)))$  where  $\theta(P(\cdot))$  is  $\cdot = u$ . It is equivalent to  $\forall u \exists v' R_C(u, v')$ .

(iii) Concerning the formula  $p \not\equiv 0 \rightarrow C(p, p)$ , its frame correspondent is the first-order sentence  $\forall u \theta(\exists u' \exists v' (R_C(u', v') \wedge ST(u', p) \wedge ST(v', p)))$  where  $\theta(P(\cdot))$  is  $\cdot = u$ . It is equivalent to  $\forall u R_C(u, u)$ .

(iv) As for the formula  $C(p, q) \rightarrow C(q, p)$ , its frame correspondent is the first-order sentence  $\forall v \forall w (R_C(v, w) \rightarrow \theta(\exists u' \exists v' (R_C(u', v') \wedge ST(u', q) \wedge$

$ST(v', p)))$  where  $\theta(P(\cdot))$  is  $\cdot = v$  and  $\theta(Q(\cdot))$  is  $\cdot = w$ . It is equivalent to  $\forall v \forall w (R_C(v, w) \rightarrow R_C(w, v))$ .

## 6 Logics

We shall say that a set  $L$  of formulas is a logic iff

- $L$  is closed under the rule of modus ponens,
- $L$  is closed under the rule of uniform substitution,
- $L$  contains all instances of tautologies of the classical propositional logic,
- $L$  contains all instances of axioms for non-degenerate Boolean algebras in terms of  $\equiv$ ,
- $L$  contains all instances of the following 3 formulas:
  - $C(a, b) \rightarrow a \neq 0 \wedge b \neq 0$ ,
  - $C(a \cup b, c) \leftrightarrow C(a, c) \vee C(b, c)$ ,
  - $C(a, b \cup c) \leftrightarrow C(a, b) \vee C(a, c)$ .

We will use  $L, M$ , etc, for logics. Obviously, the set of all logics is a partially ordered set with respect to set inclusion. Since the intersection of any collection of logics is again a logic, then there exists a least logic, denoted  $L_{min}$ . Note that the greatest logic is the set of all formulas. Of course, a logic  $L$  is the set of all formulas iff  $\perp \in L$ . A logic  $L$  will be defined to be consistent iff  $\perp \notin L$ . We now come to an important convention of notation:

until the end of this paper,  $L$  will denote a consistent logic.

For all formulas  $\phi$ , let  $L + \phi$  be the least logic containing  $L$  and  $\phi$ .

## 7 Theories

We shall say that a set  $\Gamma$  of formulas is an  $L$ -theory iff

- $\Gamma$  is closed under the rule of modus ponens,
- $\Gamma$  contains  $L$ .

We will use  $\Gamma, \Delta$ , etc, for  $L$ -theories. Let us be clear that the set of all  $L$ -theories is a partially ordered set with respect to set inclusion. The least  $L$ -theory is  $L$  and the greatest  $L$ -theory is the set of all formulas. Of course, an  $L$ -theory  $\Gamma$  is the set of all formulas iff  $\perp \in \Gamma$ . An  $L$ -theory  $\Gamma$  will be defined to be consistent iff  $\perp \notin \Gamma$ . Since each intersection of  $L$ -theories is an  $L$ -theory, then there exists a least  $L$ -theory, denoted  $\Gamma \oplus \phi$ , containing a given  $L$ -theory  $\Gamma$  and a given formula  $\phi$ : namely,  $\Gamma \oplus \phi = \{\psi: \phi \rightarrow \psi \in \Gamma\}$ . Obviously, if  $\neg\phi \notin \Gamma$  then  $\Gamma \oplus \phi$  is consistent. We shall say that an  $L$ -theory  $\Gamma$  is maximal iff for all formulas  $\phi$ ,  $\phi \in \Gamma$  or  $\neg\phi \in \Gamma$ . In Lemma 7.2 below, the expression “maximal consistent set of terms” refers to the notions of maximality and consistency in Boolean logic which can be found in most elementary logic texts.

**Lemma 7.1** *Let  $\Gamma$  be a consistent  $L$ -theory. There exists a maximal consistent  $L$ -theory  $\Delta$  such that  $\Gamma \subseteq \Delta$ .*

**Proof.** This is the Lindenbaum’s lemma, a standard result. □



**Lemma 7.2** *Let  $\Gamma$  be a maximal consistent  $L$ -theory.*

- (i) *For all terms  $a$ , if  $a \not\equiv 0 \in \Gamma$  then there exists a maximal consistent set  $x$  of terms such that  $a \in x$  and for all terms  $a'$ , if  $a' \in x$  then  $a' \not\equiv 0 \in \Gamma$ .*
- (ii) *For all terms  $a, b$ , if  $C(a, b) \in \Gamma$  then there exists maximal consistent sets  $x, y$  of terms such that  $a \in x$ ,  $b \in y$  and for all terms  $a', b'$ , if  $a' \in x$  and  $b' \in y$  then  $C(a', b') \in \Gamma$ .*

**Proof.** See [4]. □

## 8 Canonical model

Let  $\Gamma$  be a maximal consistent  $L$ -theory. The canonical model for  $\Gamma$  is the ordered triple  $\mathcal{M}_\Gamma = (W_\Gamma, R_\Gamma, V_\Gamma)$  where:

- $W_\Gamma$  is the set of all maximal consistent sets  $x$  of terms such that for all terms  $a$ , if  $a \in x$  then  $a \not\equiv 0 \in \Gamma$ ,
- $R_\Gamma$  is the binary relation on  $W_\Gamma$  such that  $xR_\Gamma y$  iff for all terms  $a, b$ , if  $a \in x$  and  $b \in y$  then  $C(a, b) \in \Gamma$ ,
- $V_\Gamma$  is the function assigning to each Boolean variable  $p$  the subset  $V_\Gamma(p)$  of  $W_\Gamma$  such that  $x \in V_\Gamma(p)$  iff  $p \in x$ .

Lemma 8.1 below plays for our Kripke-type semantics the role usually played by the truth lemma in the semantics for the basic modal language.

**Lemma 8.1** (i) *For all terms  $a$ ,  $x \in (a)^{V_\Gamma}$  iff  $a \in x$ .*

(ii) *For all formulas  $\phi$ ,  $(W_\Gamma, R_\Gamma, V_\Gamma) \models \phi$  iff  $\phi \in \Gamma$ .*

**Proof.** The first item follows by induction on  $a$  and the second one follows by induction on  $\phi$ , using Lemma 7.2 and the first item. □

## 9 Finite valuations and admissible valuations

Let  $\Gamma$  be a maximal consistent  $L$ -theory. The pair  $\mathcal{F}_\Gamma = (W_\Gamma, R_\Gamma)$  is called the canonical frame for  $\Gamma$ .  $V_\Gamma$  is called the canonical valuation for  $\Gamma$ . We shall say that a valuation  $V$  based on  $\mathcal{F}_\Gamma$  is finite iff for all Boolean variables  $p$ ,  $V(p)$  is a finite subset of  $W_\Gamma$ . A valuation  $V$  based on  $\mathcal{F}_\Gamma$  is said to be admissible iff for all Boolean variables  $p$ , there exists a term  $a$  such that  $V(p) = (a)^{V_\Gamma}$ . For all valuations  $V$  based on  $\mathcal{F}_\Gamma$ , let  $\text{adm}(V)$  be the set of all admissible valuations  $V'$  based on  $\mathcal{F}_\Gamma$  such that  $V \leq V'$ .

**Lemma 9.1** *Let  $V$  be an admissible valuation based on  $\mathcal{F}_\Gamma$ . For all  $\phi \in L$ ,  $(\mathcal{F}_\Gamma, V) \models \phi$ .*

**Proof.** Let  $\phi \in L$ . Let  $p_1, \dots, p_n$  be an enumeration of the Boolean variables occurring in  $\phi$ . Since  $V$  is admissible, then there exists terms  $a_1, \dots, a_n$  such that  $V(p_1) = (a_1)^{V_\Gamma}, \dots, V(p_n) = (a_n)^{V_\Gamma}$ . Obviously, for all terms  $b(p_1, \dots, p_n)$  and for all formulas  $\psi(p_1, \dots, p_n)$ :

- $(b(p_1, \dots, p_n))^V = (b(a_1, \dots, a_n))^{V_\Gamma}$ ,
- $(\mathcal{F}_\Gamma, V) \models \psi(p_1, \dots, p_n)$  iff  $(\mathcal{F}_\Gamma, V_\Gamma) \models \psi(a_1, \dots, a_n)$ .

The first item follows by induction on  $b$  and the second one follows by induction on  $\psi$ , using the first item. Since  $\phi(p_1, \dots, p_n) \in L$ , then  $\phi(a_1, \dots, a_n) \in L$ . Hence,  $\phi(a_1, \dots, a_n) \in \Gamma$ . By the second item of Lemma 8.1,  $(\mathcal{F}_\Gamma, V_\Gamma) \models \phi(a_1, \dots, a_n)$ . By the second item above,  $(\mathcal{F}_\Gamma, V) \models \phi(p_1, \dots, p_n)$ .  $\square$

**Lemma 9.2** *Let  $A \subseteq W_\Gamma$ . If  $A$  is finite then  $A \supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ .*

**Proof.** Suppose  $A$  is finite. Hence, there exists a nonnegative integer  $n$  such that  $\text{Card}(A) = n$ . We need to consider the following 3 cases.

**Case “ $n = 0$ ”.** Hence,  $A$  is empty. Thus,  $0$  is a term such that  $A \subseteq (0)^{V_\Gamma}$ . Since  $(0)^{V_\Gamma} = \emptyset$ , then  $\bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\} = \emptyset$ . Therefore,  $A \supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ .

**Case “ $n = 1$ ”.** Hence, there exists  $x \in W_\Gamma$  such that  $A = \{x\}$ . By the first item of Lemma 8.1, for all terms  $a$ ,  $x \in (a)^{V_\Gamma}$  iff  $a \in x$ . Thus, the following sets are equal:

- $\bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ ,
- $\bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } x \in (a)^{V_\Gamma}\}$ ,
- $\bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } a \in x\}$ .

Obviously,  $x$  is the only element in  $\bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } a \in x\}$ . Therefore,  $A \supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ .

**Case “ $n \geq 2$ ”.** Hence, there exists  $x_1, \dots, x_n \in W_\Gamma$  such that  $A = \{x_1, \dots, x_n\}$ . For all  $i = 1 \dots n$ , by the second case,  $\{x_i\} \supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } \{x_i\} \subseteq (a)^{V_\Gamma}\}$ . If  $A \not\supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$  then there exists  $x \in W_\Gamma$  such that  $x \notin A$  and  $x \in \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ . Since  $x \notin A$ , then for all  $i = 1 \dots n$ ,  $x \neq x_i$  and there exists a term  $a_i$  such that  $x_i \in (a_i)^{V_\Gamma}$  and  $x \notin (a_i)^{V_\Gamma}$ . Thus,  $x \notin (a_1 \cup \dots \cup a_n)^{V_\Gamma}$ . Since for all  $i = 1 \dots n$ ,  $x_i \in (a_i)^{V_\Gamma}$ , then  $A \subseteq (a_1 \cup \dots \cup a_n)^{V_\Gamma}$ . Since  $x \in \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } A \subseteq (a)^{V_\Gamma}\}$ , then  $x \in (a_1 \cup \dots \cup a_n)^{V_\Gamma}$ : a contradiction.  $\square$

**Lemma 9.3** *Let  $V$  be a valuation based on  $\mathcal{F}_\Gamma$ . If  $V$  is finite then  $V \geq \bigcap \{V' : V' \in \text{adm}(V)\}$ .*

**Proof.** Suppose  $V$  is finite. If  $V \not\geq \bigcap \{V' : V' \in \text{adm}(V)\}$  then there exists a Boolean variable  $p$  such that  $V(p) \not\supseteq \bigcap \{V'(p) : V' \in \text{adm}(V)\}$ . Hence, there exists  $x \in W_\Gamma$  such that  $x \notin V(p)$  and  $x \in \bigcap \{V'(p) : V' \in \text{adm}(V)\}$ . Since  $V$  is finite, then by Lemma 9.2,  $V(p) \supseteq \bigcap \{(a)^{V_\Gamma} : a \text{ is a term such that } V(p) \subseteq (a)^{V_\Gamma}\}$ . Since  $x \notin V(p)$ , then there exists a term  $a$  such that  $V(p) \subseteq (a)^{V_\Gamma}$  and  $x \notin (a)^{V_\Gamma}$ . Let  $V'$  be the valuation based on  $\mathcal{F}_\Gamma$  such that  $V'(p) = (a)^{V_\Gamma}$  and  $V'(q) = W_\Gamma$  for every Boolean variable  $q$  distinct from  $p$ . Obviously,  $V' \in \text{adm}(V)$ . Since  $x \in \bigcap \{V'(p) : V' \in \text{adm}(V)\}$ , then  $x \in V'(p)$ . Since  $V'(p) = (a)^{V_\Gamma}$ , then  $x \in (a)^{V_\Gamma}$ : a contradiction.  $\square$

**Lemma 9.4** *Let  $V$  be a valuation based on  $\mathcal{F}_\Gamma$ . If  $V$  is finite then for all positive terms  $a$ ,  $(a)^V \supseteq \bigcap \{(a)^{V'} : V' \in \text{adm}(V)\}$ .*

**Proof.** Suppose  $V$  is finite and let  $a$  be a positive term. The proof is done by induction on  $a$ .

**Case “ $a = p$ ”.** Since  $V$  is finite, then by Lemma 9.3,  $V(p) \supseteq \bigcap \{V'(p) : V' \in \text{adm}(V)\}$ . Hence,  $(a)^V \supseteq \bigcap \{(a)^{V'} : V' \in \text{adm}(V)\}$ .

**Case “ $a = 1$ ”.** Left to the reader.

**Case “ $a = b \cap c$ ” where  $b$  and  $c$  are positive terms.** Left to the reader.

**Case “ $a = b \cup c$ ” where  $b$  and  $c$  are positive terms.** By induction hypothesis,  $(b)^V \supseteq \bigcap \{(b)^{V'} : V' \in \text{adm}(V)\}$  and  $(c)^V \supseteq \bigcap \{(c)^{V'} : V' \in \text{adm}(V)\}$ . Hence, it suffices to demonstrate that  $\bigcap \{(b)^{V'} : V' \in \text{adm}(V)\} \cup \bigcap \{(c)^{V'} : V' \in \text{adm}(V)\} \supseteq \bigcap \{(b)^{V'} \cup (c)^{V'} : V' \in \text{adm}(V)\}$ . Let  $x \in \bigcap \{(b)^{V'} \cup (c)^{V'} : V' \in \text{adm}(V)\}$ . If  $x \notin \bigcap \{(b)^{V'} : V' \in \text{adm}(V)\} \cup \bigcap \{(c)^{V'} : V' \in \text{adm}(V)\}$  then  $x \notin \bigcap \{(b)^{V'} : V' \in \text{adm}(V)\}$  and  $x \notin \bigcap \{(c)^{V'} : V' \in \text{adm}(V)\}$ . Thus, there exists  $V'_b \in \text{adm}(V)$  such that  $x \notin (b)^{V'_b}$  and there exists  $V'_c \in \text{adm}(V)$  such that  $x \notin (c)^{V'_c}$ . Let  $V' = V'_b \cap V'_c$ . Obviously,  $V' \in \text{adm}(V)$ ,  $V' \leq V'_b$  and  $V' \leq V'_c$ . Since  $b$  and  $c$  are positive terms, then by Lemma 3.1,  $(b)^{V'} \subseteq (b)^{V'_b}$  and  $(c)^{V'} \subseteq (c)^{V'_c}$ . Since  $V' \in \text{adm}(V)$  and  $x \in \bigcap \{(b)^{V'} \cup (c)^{V'} : V' \in \text{adm}(V)\}$ , then  $x \in (b)^{V'}$  or  $x \in (c)^{V'}$ . Since  $(b)^{V'} \subseteq (b)^{V'_b}$  and  $(c)^{V'} \subseteq (c)^{V'_c}$ , then  $x \in (b)^{V'_b}$  or  $x \in (c)^{V'_c}$ : a contradiction.  $\square$

**Lemma 9.5** *Let  $V$  be a valuation based on  $\mathcal{F}_\Gamma$ . If  $V$  is finite then for all positive terms  $a, b$ ,  $((a)^V \times W_\Gamma) \cup (W_\Gamma \times (b)^V) \supseteq \bigcap \{((a)^{V'} \times W_\Gamma) \cup (W_\Gamma \times (b)^{V'}) : V' \in \text{adm}(V)\}$ .*

**Proof.** Suppose  $V$  is finite and let  $a$  and  $b$  be positive terms. If  $((a)^V \times W_\Gamma) \cup (W_\Gamma \times (b)^V) \not\supseteq \bigcap \{((a)^{V'} \times W_\Gamma) \cup (W_\Gamma \times (b)^{V'}) : V' \in \text{adm}(V)\}$  then there exists  $x, y \in W_\Gamma$  such that  $x \notin (a)^V$ ,  $y \notin (b)^V$  and  $x \in (a)^{V'}$  or  $y \in (b)^{V'}$  for each  $V' \in \text{adm}(V)$ . Let  $\cong$  be the binary relation on  $\text{adm}(V)$  defined as follows:  $V' \cong V''$  iff for all Boolean variables  $p$  occurring in  $a$ ,  $V'(p) = V''(p)$ . Obviously,  $\cong$  is an equivalence relation on  $\text{adm}(V)$ . Moreover,  $\text{adm}(V)_{|\cong}$ , the quotient set of  $\text{adm}(V)$  modulo  $\cong$ , is countable. Hence, there exists an  $\omega$ -sequence  $(|V'_n|)_{n \in \mathbb{N}}$  of equivalence classes modulo  $\cong$  enumerating  $\text{adm}(V)_{|\cong}$ . Let  $(|V''_n|)_{n \in \mathbb{N}}$  be the  $\omega$ -sequence of equivalence classes modulo  $\cong$  defined as follows: if  $n = 0$  then  $|V''_0| = |V'_0|$  else  $|V''_n| = |V''_{n-1} \cap V'_n|$ . Since  $x \in (a)^{V'}$  or  $y \in (b)^{V'}$  for each  $V' \in \text{adm}(V)$ , then  $x \in (a)^{V''_n}$  or  $y \in (b)^{V''_n}$  for each  $n \in \mathbb{N}$ . Since  $V$  is finite and  $a$  and  $b$  are positive terms, then by construction of  $(|V''_n|)_{n \in \mathbb{N}}$  and by Lemma 9.4,  $(a)^V \supseteq \bigcap \{(a)^{V''_n} : n \in \mathbb{N}\}$  and  $(b)^V \supseteq \bigcap \{(b)^{V''_n} : n \in \mathbb{N}\}$ . Since  $x \in (a)^{V''_n}$  or  $y \in (b)^{V''_n}$  for each  $n \in \mathbb{N}$ , then by construction of  $(|V''_n|)_{n \in \mathbb{N}}$  and by Lemma 3.1,  $x \in (a)^{V''_n}$  for each  $n \in \mathbb{N}$  or  $y \in (b)^{V''_n}$  for each  $n \in \mathbb{N}$ . Since  $(a)^V \supseteq \bigcap \{(a)^{V''_n} : n \in \mathbb{N}\}$  and  $(b)^V \supseteq \bigcap \{(b)^{V''_n} : n \in \mathbb{N}\}$ , then  $x \in (a)^V$  or  $y \in (b)^V$ : a contradiction.  $\square$

**Lemma 9.6** *Let  $V$  be a finite valuation based on  $\mathcal{F}_\Gamma$ . Let  $a$  be a positive term such that for all  $V' \in \text{adm}(V)$ ,  $(a)^{V'} \neq \emptyset$ . Then  $(a)^V \neq \emptyset$ .*

**Proof.** Let  $\cong$ ,  $(|V'_n|)_{n \in \mathbb{N}}$  and  $(|V''_n|)_{n \in \mathbb{N}}$  be defined as in the proof of Lemma 9.5. Since  $(a)^{V'} \neq \emptyset$  for each  $V' \in \text{adm}(V)$ , then  $(a)^{V''_n} \neq \emptyset$  for each  $n \in \mathbb{N}$ . Since  $V$  is finite and  $a$  is a positive term, then by construction of  $(|V''_n|)_{n \in \mathbb{N}}$  and by Lemma 9.4,  $(a)^V \supseteq \bigcap \{(a)^{V''_n} : n \in \mathbb{N}\}$ . Since  $V''_n$  is admissible for each  $n \in \mathbb{N}$ , then there exists a term  $a_n$  such that  $(a)^{V''_n} = (a_n)^{V''_n}$ .

for each  $n \in \mathbb{N}$ . Remark that  $(a_0)^{V_\Gamma} \supseteq (a_1)^{V_\Gamma} \supseteq \dots$ . Since  $(a)^{V''} \neq \emptyset$  for each  $n \in \mathbb{N}$ , then  $(a_n)^{V_\Gamma} \neq \emptyset$  for each  $n \in \mathbb{N}$ . Since  $(a_0)^{V_\Gamma} \supseteq (a_1)^{V_\Gamma} \supseteq \dots$ , then there exists  $x \in W_\Gamma$  such that  $a_n \in x$  for each  $n \in \mathbb{N}$ . Thus,  $\bigcap\{(a_n)^{V_\Gamma} : n \in \mathbb{N}\} \neq \emptyset$ . Therefore,  $\bigcap\{(a)^{V''} : n \in \mathbb{N}\} \neq \emptyset$ . Since  $(a)^V \supseteq \bigcap\{(a)^{V''} : n \in \mathbb{N}\}$ , then  $(a)^V \neq \emptyset$ .  $\square$

**Lemma 9.7** *Let  $V$  be a finite valuation based on  $\mathcal{F}_\Gamma$ . Let  $a$  and  $b$  be positive terms such that for all  $V' \in \text{adm}(V)$ , there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^{V'}$  and  $y \in (b)^{V'}$ . Then there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^V$  and  $y \in (b)^V$ .*

**Proof.** Let  $\cong, (| V'_n |)_{n \in \mathbb{N}}$  and  $(| V''_n |)_{n \in \mathbb{N}}$  be defined as in the proof of Lemma 9.5. Since there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^{V'}$  and  $y \in (b)^{V'}$  for each  $V' \in \text{adm}(V)$ , then there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^{V''_n}$  and  $y \in (b)^{V''_n}$  for each  $n \in \mathbb{N}$ . Since  $V$  is finite and  $a$  and  $b$  are positive terms, then by construction of  $(| V''_n |)_{n \in \mathbb{N}}$  and by Lemma 9.4,  $(a)^V \supseteq \bigcap\{(a)^{V''_n} : n \in \mathbb{N}\}$  and  $(b)^V \supseteq \bigcap\{(b)^{V''_n} : n \in \mathbb{N}\}$ . Since  $V''_n$  is admissible for each  $n \in \mathbb{N}$ , then there exists terms  $a_n, b_n$  such that  $(a)^{V''_n} = (a_n)^{V_\Gamma}$  and  $(b)^{V''_n} = (b_n)^{V_\Gamma}$  for each  $n \in \mathbb{N}$ . Remark that  $(a_0)^{V_\Gamma} \supseteq (a_1)^{V_\Gamma} \supseteq \dots$  and  $(b_0)^{V_\Gamma} \supseteq (b_1)^{V_\Gamma} \supseteq \dots$ . Since there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^{V''_n}$  and  $y \in (b)^{V''_n}$  for each  $n \in \mathbb{N}$ , then there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a_n)^{V_\Gamma}$  and  $y \in (b_n)^{V_\Gamma}$  for each  $n \in \mathbb{N}$ . Since  $(a_0)^{V_\Gamma} \supseteq (a_1)^{V_\Gamma} \supseteq \dots$  and  $(b_0)^{V_\Gamma} \supseteq (b_1)^{V_\Gamma} \supseteq \dots$ , then there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $a_n \in x$  and  $b_n \in y$  for each  $n \in \mathbb{N}$ . Thus,  $x \in \bigcap\{(a_n)^{V_\Gamma} : n \in \mathbb{N}\}$  and  $y \in \bigcap\{(b_n)^{V_\Gamma} : n \in \mathbb{N}\}$ . Therefore,  $x \in \bigcap\{(a)^{V''_n} : n \in \mathbb{N}\}$  and  $y \in \bigcap\{(b)^{V''_n} : n \in \mathbb{N}\}$ . Since  $(a)^V \supseteq \bigcap\{(a)^{V''_n} : n \in \mathbb{N}\}$  and  $(b)^V \supseteq \bigcap\{(b)^{V''_n} : n \in \mathbb{N}\}$ , then  $x \in (a)^V$  and  $y \in (b)^V$ .  $\square$

**Lemma 9.8** *Let  $V$  be a valuation based on  $\mathcal{F}_\Gamma$ . Let  $\phi$  be a negation-free formula such that  $(\mathcal{F}_\Gamma, V) \models \phi$ . Then there exists a finite valuation  $V_0$  based on  $\mathcal{F}_\Gamma$  such that  $V_0 \leq V$  and  $(\mathcal{F}_\Gamma, V_0) \models \phi$ .*

**Proof.** Without loss of generality, we may assume that  $\phi$  is equal either to  $\top$  or to a disjunction  $\phi_1 \vee \dots \vee \phi_n$  of  $\top$ -free  $\vee$ -free negation-free formulas. In the former case, let  $V_0$  be the empty valuation. In the latter case, there exists  $i \in \{1, \dots, n\}$  such that  $(\mathcal{F}_\Gamma, V) \models \phi_i$ . Since we may also assume that  $\phi_i$  is equal to a conjunction of the form  $a_1 \neq 0 \wedge \dots \wedge a_k \neq 0 \wedge C(b_1, c_1) \wedge \dots \wedge C(b_l, c_l)$  where all  $a_*$  are equal to an intersection of Boolean variables and all  $b_*, c_*$  are equal either to 1 or to an intersection of Boolean variables, then there exists  $x_1, \dots, x_k, y_1, z_1, \dots, y_l, z_l \in W_\Gamma$  such that  $x_1 \in (a_1)^V, \dots, x_k \in (a_k)^V$ ,  $y_1 \in (b_1)^V, z_1 \in (c_1)^V$  and  $y_1 R_\Gamma z_1, \dots, y_l R_\Gamma z_l$ . Let  $V_0$  be the finite valuation based on  $\mathcal{F}_\Gamma$  such that for all Boolean variables  $p$ ,  $V_0(p) = V(p) \cap (\{x_1, \dots, x_k\} \cup \{y_1, z_1, \dots, y_l, z_l\})$ . Obviously,  $V_0 \leq V$  and  $(\mathcal{F}_\Gamma, V_0) \models \phi$ .  $\square$

**Lemma 9.9** *Let  $V$  be a finite valuation based on  $\mathcal{F}_\Gamma$ . Let  $\phi$  be a positive formula such that for all  $V' \in \text{adm}(V)$ ,  $(\mathcal{F}_\Gamma, V') \models \phi$ . Then  $(\mathcal{F}_\Gamma, V) \models \phi$ .*

**Proof.** The proof is done by induction on  $\phi$ .

**Case “ $\phi = a \neq 0$ ”.** Since  $(\mathcal{F}_\Gamma, V') \models \phi$  for each  $V' \in \text{adm}(V)$ , then  $(a)^{V'} \neq \emptyset$  for each  $V' \in \text{adm}(V)$ . Since  $a$  is a positive term, then by Lemma 9.6,  $(a)^V \neq \emptyset$ . Hence,  $(\mathcal{F}_\Gamma, V) \models \phi$ .

**Case “ $\phi = -a \equiv 0$ ”.** Since  $(\mathcal{F}_\Gamma, V') \models \phi$  for each  $V' \in \text{adm}(V)$ , then  $(a)^{V'} = W_\Gamma$  for each  $V' \in \text{adm}(V)$ . Since  $V$  is finite and  $a$  is a positive term, then by Lemma 9.4,  $(a)^V = W_\Gamma$ . Hence,  $(\mathcal{F}_\Gamma, V) \models \phi$ .

**Case “ $\phi = C(a, b)$ ”.** Since  $(\mathcal{F}_\Gamma, V') \models \phi$  for each  $V' \in \text{adm}(V)$ , then there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^{V'}$  and  $y \in (b)^{V'}$  for each  $V' \in \text{adm}(V)$ . Since  $a$  and  $b$  are positive terms, then by Lemma 9.7, there exists  $x, y \in W_\Gamma$  such that  $xR_\Gamma y$ ,  $x \in (a)^V$  and  $y \in (b)^V$ . Hence,  $(\mathcal{F}_\Gamma, V) \models \phi$ .

**Case “ $\phi = \bar{C}(-a, -b)$ ”.** Since  $(\mathcal{F}_\Gamma, V') \models \phi$  for each  $V' \in \text{adm}(V)$ , then for all  $x, y \in W_\Gamma$ , if  $xR_\Gamma y$  then  $x \in (a)^{V'}$  or  $y \in (b)^{V'}$  for each  $V' \in \text{adm}(V)$ . Since  $V$  is finite and  $a$  and  $b$  are positive terms, then by Lemma 9.5, for all  $x, y \in W_\Gamma$ , if  $xR_\Gamma y$  then  $x \in (a)^V$  or  $y \in (b)^V$ . Hence,  $(\mathcal{F}_\Gamma, V) \models \phi$ .

**Case “ $\phi = \top$ ”.** Left to the reader.

**Case “ $\phi = \psi \vee \chi$ ” where  $\psi$  and  $\chi$  are positive formulas.** Let  $\cong, (| V'_n |)_{n \in \mathbb{N}}$  and  $(| V''_n |)_{n \in \mathbb{N}}$  be defined as in the proof of Lemma 9.5. Since  $(\mathcal{F}_\Gamma, V') \models \phi$  for each  $V' \in \text{adm}(V)$ , then  $(\mathcal{F}_\Gamma, V''_n) \models \phi$  for each  $n \in \mathbb{N}$ . Since  $\psi$  and  $\chi$  are positive, then by construction of  $(| V''_n |)_{n \in \mathbb{N}}$  and by the second item of Lemma 3.2,  $(\mathcal{F}_\Gamma, V''_n) \models \psi$  for each  $n \in \mathbb{N}$  or  $(\mathcal{F}_\Gamma, V''_n) \models \chi$  for each  $n \in \mathbb{N}$ . Since  $\psi$  and  $\chi$  are positive, then by construction of  $(| V''_n |)_{n \in \mathbb{N}}$  and by the second item of Lemma 3.2,  $(\mathcal{F}_\Gamma, V') \models \psi$  for each  $V' \in \text{adm}(V)$  or  $(\mathcal{F}_\Gamma, V') \models \chi$  for each  $V' \in \text{adm}(V)$ . Hence, by induction hypothesis,  $(\mathcal{F}_\Gamma, V) \models \phi$ .

**Case “ $\phi = \psi \wedge \chi$ ” where  $\psi$  and  $\chi$  are positive formulas.** Left to the reader.  $\square$

## 10 Completeness theorem

We shall say that  $L$  is canonical iff for all maximal consistent  $L$ -theories  $\Gamma$ ,  $\mathcal{F}_\Gamma \models L$ .

**Theorem 10.1** *Let  $\phi$  be a Sahlqvist formula. If  $L$  is canonical then  $L + \phi$  is canonical.*

**Proof.** Suppose  $L$  is canonical. If  $L + \phi$  is not canonical then there exists a maximal consistent  $L + \phi$ -theory  $\Gamma$  such that  $\mathcal{F}_\Gamma \not\models L + \phi$ . Hence,  $\Gamma$  is a maximal consistent  $L$ -theory such that  $\mathcal{F}_\Gamma \not\models L$  or  $\mathcal{F}_\Gamma \not\models \phi$ . Since  $L$  is canonical, then  $\mathcal{F}_\Gamma \models L$ . Since  $\mathcal{F}_\Gamma \not\models L$  or  $\mathcal{F}_\Gamma \not\models \phi$ , then  $\mathcal{F}_\Gamma \not\models \phi$ . Thus, there exists a valuation  $V$  based on  $\mathcal{F}_\Gamma$  such that  $(\mathcal{F}_\Gamma, V) \not\models \phi$ . Since  $\phi$  is a Sahlqvist formula, then  $\phi$  is an implication  $\psi \rightarrow \chi$  in which  $\psi$  is negation-free and  $\chi$  is positive. Since  $(\mathcal{F}_\Gamma, V) \not\models \phi$ , then  $(\mathcal{F}_\Gamma, V) \models \psi$  and  $(\mathcal{F}_\Gamma, V) \not\models \chi$ . Since  $\psi$  is negation-free, then by Lemma 9.8, there exists a finite valuation  $V_0$  based on  $\mathcal{F}_\Gamma$  such that  $V_0 \leq V$  and  $(\mathcal{F}_\Gamma, V_0) \models \psi$ . Since  $\chi$  is positive and  $(\mathcal{F}_\Gamma, V) \not\models \chi$ , then by the second item of Lemma 3.2,  $(\mathcal{F}_\Gamma, V_0) \not\models \chi$ . Since  $V_0$  is finite and  $\chi$  is positive, then by Lemma 9.9,  $(\mathcal{F}_\Gamma, V') \not\models \chi$  for some  $V' \in \text{adm}(V_0)$ . Since  $\psi$  is negation-free and  $(\mathcal{F}_\Gamma, V_0) \models \psi$ , then by Lemma 3.2,  $(\mathcal{F}_\Gamma, V') \models \psi$ . Since  $V'$  is admissible and  $\phi \in L + \phi$ , then by Lemma 9.1,  $(\mathcal{F}_\Gamma, V') \models \phi$ . Since  $(\mathcal{F}_\Gamma, V') \models \psi$ , then

$(\mathcal{F}_\Gamma, V') \models \chi$ : a contradiction. □

As a result,

**Theorem 10.2** *Let  $\phi$  be a Sahlqvist formula and  $\alpha$  be the first-order sentence of the first-order language  $\mathcal{L}^1(\emptyset)$  that corresponds to it by Theorem 5.1. For all formulas  $\psi$ ,  $\psi \in L_{min} + \phi$  iff for all frames  $\mathcal{F} = (W, R)$ , if  $\mathcal{F} \models \alpha$  then  $\mathcal{F} \models \psi$ .*

**Proof.** Firstly, let us prove the direction from left to right. Let  $\psi$  be a formula. If  $\psi \in L_{min} + \phi$  then let  $\mathcal{F} = (W, R)$  be a frame such that  $\mathcal{F} \models \alpha$ . Hence, by Theorem 5.1,  $\mathcal{F} \models \psi$ . Since  $\psi \in L_{min} + \phi$ , then there exists a proof of  $\psi$  in the axiomatic system based on the rules and the instances of formulas considered at the beginning of Section 6 and  $\phi$ . By induction on the length of this proof, one can show that  $\mathcal{F} \models \psi$ .

Secondly, let us prove the direction from right to left. Let  $\psi$  be a formula. If  $\psi \notin L_{min} + \phi$  then  $(L_{min} + \phi) \oplus \neg\psi$  is a consistent  $(L_{min} + \phi)$ -theory. Hence, by Lemma 7.1, there exists a maximal consistent  $(L_{min} + \phi)$ -theory  $\Gamma$  such that  $(L_{min} + \phi) \oplus \neg\psi \subseteq \Gamma$ . Now, it suffices to demonstrate that  $\mathcal{F}_\Gamma \models \alpha$  and  $\mathcal{F}_\Gamma \not\models \psi$ . Since  $\phi$  is a Sahlqvist formula, then by Theorem 10.1,  $L_{min} + \phi$  is canonical. Thus,  $\mathcal{F}_\Gamma \models \phi$ . Therefore, by Theorem 5.1,  $\mathcal{F}_\Gamma \models \alpha$ . Since  $\neg\psi \in \Gamma$ , then  $\psi \notin \Gamma$ . Consequently, by Lemma 8.1,  $\mathcal{M}_\Gamma \not\models \psi$ . Hence,  $\mathcal{F}_\Gamma \not\models \psi$ . □

## 11 Conclusion

As we already said, the last 4 formulas considered at the end of Section 2 are not Sahlqvist formulas. However, there could be the possibility of finding 4 Sahlqvist formulas corresponding to them. At this point, it might be useful to remark that the first of these 4 formulas, namely  $C(p, q) \rightarrow C(p, r) \vee C(-r, q)$ , corresponds to a first-order property whereas the last 3 of them, namely  $p \neq 0 \wedge -p \neq 0 \rightarrow C(p, -p)$ ,  $(p \cup q) \equiv 1 \wedge (p \cap q) \equiv 0 \rightarrow C(p, p) \vee C(q, q)$  and  $(p \cap -q) \neq 0 \rightarrow C(p, -q) \vee C(q, -q)$ , corresponds to second-order properties. For more on this, see [3,4]. Hence, a first question presents itself: the decidability of determining whether a given RBPMLS formula is equivalent to a Sahlqvist RBPMLS formula.

Important problems are the so-called algorithmic problems in correspondence theory: given a RBPMLS formula, determine whether it has a first-order correspondent; given a first-order sentence, determine whether it has a RBPMLS correspondent. In modal logic, such problems have been proved to be undecidable by Chagrova in 1989. For more on this, see [8,9]. Chagrova's proof of the undecidability of modal definability of first-order sentences can be almost reproduced word for word in the setting of RBPMLS (Tinko Tinchev, personal communication, Sofia (Bulgaria), February 24, 2012). Hence, a second question presents itself: the decidability of determining whether a given RBPMLS formula corresponds to a first-order sentence.

The undecidability of RBPMLS definability of first-order sentences shows that any sufficient condition for RBPMLS definability is very interesting by itself. In modal logic, Kracht formulas are the first-order counterparts of Sahlqvist formulas [26]. By means of an algorithm constructing a Sahlqvist formula from

a given Kracht formula, one can axiomatize validity in such and such elementary class of frames determined by Kracht formulas. Recently, Kracht theorem has been extended to the class of generalized Sahlqvist formulas introduced by Goranko and Vakarelov [20]. For more on this, see [23,24]. Hence, a third question presents itself: the definition of Kracht formulas in the setting of RBPMLS.

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