REPEATED BINOMIAL COEFFICIENTS AND FIBONACCI NUMBERS

DAVID SINGMASTER
Polytechnic of the South Bank, London and Istituto Matematico, Pisa, Italy

ABSTRACT

In this note, I show that there are infinitely many solutions to the equation

$$\binom{n+1}{k+1} = \binom{n}{k+2},$$

given by $n = F_{2i+2}F_{2i+3} - 1$, $k = F_{2i}F_{2i+3} - 1$, where F_n is the n^{th} Fibonacci number, beginning with $F_0 = 0$. This gives infinitely many binomial coefficients occurring at least 6 times. The method and results of a computer search for repeated binomial coefficients, up to 2^{48}, will be given.

1. INTRODUCTION

In [6], I have conjectured that the number of times an integer can occur as a binomial coefficient is bounded. A computer search up to 2^{48} has revealed only the following seven nontrivial repetitions:

$$120 = \binom{16}{2} = \binom{10}{3}; \quad 210 = \binom{21}{2} = \binom{10}{4}; \quad 1540 = \binom{56}{2} = \binom{22}{3};$$

$$7140 = \binom{120}{2} = \binom{36}{3}; \quad 11628 = \binom{153}{2} = \binom{19}{5}; \quad 24310 = \binom{221}{2} = \binom{17}{8};$$

and

$$3003 = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}.$$

In [2], it has been shown that the only numbers which are both triangular, i.e., $= \binom{n}{2}$ for some n, and tetrahedral, i.e., $= \binom{n}{3}$ for some n, are 1, 10, 120, 1540 and 7140. The first two are trivial and the last three were also found by the computer, giving a check on the search procedure.

The coefficient 3003 occurs in the following striking pattern in Pascal's triangle:

$$\begin{array}{ccccc} 1001 & & 2002 & & 3003 \\ & 3003 & & 5005 & \\ & & 8008 & & \end{array}$$

I had noticed this pattern some years ago when I discovered that it is the only solution to

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = 1:2:3,$$

and that there is at most one solution to this relation when the right-hand side is replaced by $a:b:c$. Hence I was led to consider determining solutions when the right-hand side was $a:b:a+b$, or, equivalently and more simply, solutions of

$$\binom{n+1}{k+1} = \binom{n}{k+2}. \tag{1}$$

2. SOLUTION OF EQUATION (1)

From (1), we have $(n+1)(k+2) = (n-k)(n-k-1)$. Set $m = n+1$, $j = k+2$, thus obtaining $m^2 + (1-3j)m + j^2 - j = 0$. Solving for m gives

(2) $$m = [-1+3j \pm \sqrt{5j^2-2j+1}\,]/2 \quad .$$

For this to make sense, we must have that $5j^2 - 2j + 1$ is a perfect square, say v^2. We can rewrite this as

(3) $$(5j-1)^2 - 5v^2 = -4 .$$

Letting $u = 5j - 1$, $C = -4$, we have the Pell-like equation

(4) $$u^2 - 5v^2 = C .$$

This can be completely solved by standard techniques [5, section 58, p. 204 ff]. The basic solutions are:

$$9 \pm 4\sqrt{5} \text{ when } C = 1; \quad 2 \pm \sqrt{5} \text{ when } C = -1; \quad \text{and} \quad 1 \pm \sqrt{5} \text{ and } 4 \pm 2\sqrt{5} \text{ when } C = -4 .$$

The class of solutions determined by $4 + 2\sqrt{5}$ is the same as the class determined by $4 - 2\sqrt{5}$, i.e., the class is ambiguous, in the terminology of [5]. Hence all solutions are given by

$$u_i + v_i\sqrt{5} = (-1+\sqrt{5})(9+4\sqrt{5})^i, \quad u_i + v_i\sqrt{5} = (1+\sqrt{5})(9+4\sqrt{5})^i, \quad u_i + v_i\sqrt{5} = (4+2\sqrt{5})(9+4\sqrt{5})^i ,$$

and their conjugates and negatives.

Let $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ define the Fibonacci numbers and let $L_0 = 2$, $L_1 = 1$, $L_{n+1} = L_n + L_{n-1}$ define the Lucas numbers.

Lemma. $$(L_n + F_n\sqrt{5})(9+4\sqrt{5}) = L_{n+6} + F_{n+6}\sqrt{5} \ .$$

Proof. Let $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. By the Binet formulas, we have

$$F_n = (\alpha^n - \beta^n)/\sqrt{5}, \qquad L_n = \alpha^n + \beta^n ,$$

and so $L_n + F_n\sqrt{5} = 2\alpha^n$. Hence the lemma reduces to showing $\alpha^6 = 9 + 4\sqrt{5}$, which is readily done.

Since the basic solutions $u_0 + v_0\sqrt{5}$ given above are respectively

$$L_{-1} + F_{-1}\sqrt{5}, \qquad L_1 + F_1\sqrt{5} \qquad \text{and} \qquad L_3 + F_3\sqrt{5} ,$$

the general solution of (4) can be written as

(5) $$L_{2i-1} + F_{2i-1}\sqrt{5}, \qquad i = 0, 1, \cdots$$

and we may now ignore the conjugates and negatives.

To solve (3), we must have $5j - 1 = L_{2i-1}$. From the Binet formula, one may obtain $L_i \equiv 2\cdot 3^i \pmod 5$ and hence $L_i \equiv -1 \pmod 5$ if and only if $i \equiv 3 \pmod 4$. Recalling that $j = k + 2 \geq 2$, the solutions of (3) are thus

$$j = (L_{4i+3}+1)/5, \qquad v = F_{4i+3}, \qquad i = 1, 2, \cdots .$$

By standard manipulations, we obtain

(6) $$j = F_{2i}F_{2i+3} + 1, \quad k = F_{2i}F_{2i+3} - 1, \quad m = F_{2i+2}F_{2i+3} = (L_{4i+5}-1)/5, \quad n = F_{2i+2}F_{2i+3} - 1.$$

Finally, observe that

$$\binom{n}{k} : \binom{n}{k+1} = (k+1):(n-k) = F_{2i} : F_{2i+1} ,$$

hence

$$\binom{n}{k} : \binom{n}{k+1} : \binom{n}{k+2} = F_{2i} : F_{2i+1} : F_{2i+2} .$$

The case $i = 1$ gives $n = 14$, $k = 4$ and

$$\binom{15}{5} = \binom{14}{6} = 3003 .$$

The case $i = 2$ gives $n = 103$, $k = 38$, $k + 2 = 40$, and

$$\binom{104}{39} = \binom{103}{40} = 6121\ 81827\ 43304\ 70189\ 14314\ 82520 .$$

This number does not occur again as a binomial coefficient. The next values of (n,k) are (713, 271) and (4894, 1868).

Equation (1) has also been solved by Lind [4]. Hoggatt and Lind [3] have dealt with some related inequalities.

3. REMARKS

The coefficients

$$N = \binom{n+1}{k+1} = \binom{n}{k+1} = \binom{N}{1}$$

give us infinitely many binomial coefficients occurring at least six times. This has also been noted in [1, Theorem 3]. Since 3003 happens to be also a triangular number, one might hope that some more of these values might also be triangular. I first determined by calculation that

$$\binom{103}{40}$$

was not triangular and later I determined that it did not occur as any other binomial coefficient. These determinations are described below. I have not been able to discern any other patterns in the repetitions found.

One might try to extend the pattern of Eq. (1) and try to find

$$\binom{n}{k+4} = \binom{n+1}{k+3} = \binom{n+2}{k+2}.$$

This would require two solutions of (1) with consecutive values of n and inspection of (6) shows this is impossible.

The lemma is a special case of the general assertion that the solutions u_i, v_i of

$$u_i + v_i\sqrt{D} = (u_0 + v_0\sqrt{D})(a + b\sqrt{D})^i$$

both satisfy the same second-order recurrence relation:

$$u_{n+1} = 2au_n + (b^2D - a^2)u_{n-1}.$$

(In our particular case: $F_{n+6} = 18F_n - F_{n-6}$.) I do not see whether the fact that the three basic solutions happen to neatly fit together into a single linear recurrence is a happy accident or a general phenomenon. The converse problem of determining which pairs of recurrence relations give all solutions of a Pell-like equation seems interesting but I have not examined it.

4. THE COMPUTER SEARCH

Two separate computer searches were made. First an ALGOL program was used to search up to 2^{23} on the London Polytechnics' ICL 1905E. All the 4717 binomial coefficients $\binom{n}{k}$ with $k \geq 2$, $n \geq 2k$ and less than 2^{23} were formed by addition and stored in rows corresponding to the diagonals of Pascal's triangle. As each new coefficient was created, it was compared with the elements in the preceding rows. Since each row is in increasing order, a simple binary search was done in each preceding row and the process is quite quick. All the repeated values given in the Introduction were already determined in this search.

The second search was carried out using a FORTRAN program on the University of London Computer Centre's CDC 6600. Although the 6600 has a 60-bit word, it is difficult to use integers bigger than 2^{48} and overflow occurs with such integers. Consequently, I was only able to search up to 2^{48}. There are about 24×10^6 triangular numbers and about 12×10^4 tetrahedral numbers up to this limit. It is impractical to store all of these, so the program had to be modified. Fortunately, the results of [2], mentioned in the Introduction, implied that we did not have to compare these two sets. I wrote a subroutine to determine if an integer N was triangular or tetrahedral. This estimates the J such that $J(J+1)/2 = N$ by $J = [\sqrt{(2N)}] - 1$ and then computes the succeeding triangular numbers until they equal or exceed N. Two problems of overflow arose. Firstly: if N is large, the calculation of the first triangular number to be considered, i.e., $J(J+1)/2$, may cause an overflow when $J(J+1)$ is formed. This was resolved by examining J (mod 2) and computing either $(J/2)(J+1)$ or

$$J\left(\frac{J+1}{2}\right).$$

Secondly: if N is larger than the largest triangular number less than 2^{48}, the calculation of the successive triangular numbers will produce an overflow before the comparison with N reveals that we have gone far enough. This was resolved by testing the index of the triangular numbers to see if overflow was about to occur. The test for tetrahedral numbers was similar, but requires testing J (mod 6).

The search then proceeded much as before. All coefficients $\binom{n}{k}$ with $k \geqslant 4$ and $n \geqslant 2k$ and less than 2^{48} were formed by addition and stored in rows. As each coefficient was formed, the subroutine was used to see if it was triangular or tetrahedral and binary search was used to see if it occurred in a preceding row.

I was rather startled that the second search produced no new results. The results 210, 11628, 24310 and 3003 were refound, which gave me some confidence in the process. I reran the program with output of the searching steps and this indicated that the program works correctly. So I am reasonably sure of the results, although still startled. I hope someone can extend this to higher limits, say 2^{59} and see if there are more repetitions.

The calculation of $N = \binom{103}{40}$ and the computational determination that it was not triangular were also complicated by overflow, since $N > 2^{48}$. First I attempted to compute only the 103rd row of the Pascal triangle by use of

$$\binom{103}{k} = \frac{104-k}{k}\binom{103}{k-1},$$

using double precision real arithmetic. However, this showed inaccuracies in the units place, beginning with $k = 33$. I then computed the entire triangle up to the 103rd row (mod 10^{14}) by addition. I could then overlap the two results to get N. The double precision calculation had been accurate to 27 of the 29 places.

I applied the idea of the subroutine to determine if N were triangular. This required some adjustments. Since $2N$ is bigger than 2^{96}, one cannot truncate $\sqrt{(2N)}$ to an integer. Instead $\sqrt{(N/2)}$ was calculated, truncated to an integer, converted to a double precision real and then doubled. Then the process of the subroutine was carried out, working in double precision real form. N was found to lie about halfway between two consecutive triangular numbers. These results for N were independently checked by Cecil Kaplinsky using multiprecision arithmetic on an IBM 360.

In a personal letter, D. H. Lehmer pointed out that one could determine that N was not triangular by noting its residue (mod 13). Following up on this suggestion, I computed the Pascal triangle (mod p) for small primes. Since $\binom{n}{k}$ (mod p) is periodic as a function of n [7, Theorem 38; 8; 9], one can deduce that $N \neq \binom{n}{k}$ for various k's by examination of N (mod p) and the possible values of $\binom{n}{k}$ (mod p). For example, $N \equiv 4$ (mod 13), but $\binom{n}{k} \not\equiv 4$ (mod 13) for $k = 2, 4, 6, 7, 8, 9, 10, 11, 12$. Using the primes 13, 19, 29, 31, 37, 53, 59 and 61, one can exclude all possibilities for k, other than 39 and 40 and hence N occurs exactly six times.

On the basis of the computer search and the scarcity of solutions of (1), I am tempted to make the following:

CONJECTURE. No binomial coefficient is repeated more than 10 times. (Perhaps the right number is 8 or 12?)

REFERENCES

1. H. L. Abbott, P. Erdös and D. Hanson, "On the Number of Times an Integer Occurs as a Binomial Coefficient," *Amer. Math. Monthly*, 81 (1974), pp. 256–261.
2. E.T. Avanesov, "Solution of a Problem on Figurate Numbers," *Acta Arith.,* 12 (1966/67), pp. 409-420. (Russian). See the review: MR 35, No. 6619.
3. V. E. Hoggatt, Jr., and D. A. Lind, "The Heights of Fibonacci Polynomials and an Associated Function," *The Fibonacci Quarterly,* Vol. 5, No. 2 (April, 1967), pp. 141–152.
4. D. A. Lind, "The Quadratic Field $Q(\sqrt{5})$ and a Certain Diophantine Equation," *The Fibonacci Quarterly,* Vol. 6, No. 1 (February 1968), pp. 86–93.
5. T. Nagell, *Introduction to Number Theory,* 2nd ed., Chelsea Publ. Co., New York, 1964.
6. D. Singmaster, "How Often does an Integer Occur as a Binomial Coefficient?" *Amer. Math. Monthly,* 78 (1971), pp. 385–386.
7. D. Singmaster, "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers," to appear.
8. W.F. Trench, "On Periodicities of Certain Sequences of Residues," *Amer. Math. Monthly,* 67 (1960), pp. 652–656.
9. S. Zabek, "Sur la périodicite modulo m des suites de nombres $\binom{n}{k}$. *Ann. Univ. Mariae Curie-Sklodowska Sect. A,* 10 (1956), pp. 37–47.

The final preparation of this manuscript was supported by an Italian National Research Council research fellowship.

****