ZERO-ONE SEQUENCES AND FIBONACCI NUMBERS

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1. INTRODUCTION

It is well known that the number of zero-one sequences of length n:

(1.1) $$(a_1, a_2, \cdots, a_n) \qquad (a_i = 0 \text{ or } 1)$$

with consecutive ones forbidden is equal to the Fibonacci number F_{n+2}. Moreover the number of such sequences with $a_n = a_1 = 1$ also forbidden is equal to the Lucas number L_n. This suggests the following two problems.

1. Let $n_{00}, n_{01}, n_{10}, n_{11}$ be non-negative integers such that

$$n_{00} + n_{01} + n_{10} + n_{11} = n - 1.$$

We seek the number of sequences (1.1) with exactly n_{00} occurrences of 00, n_{01} occurrences of 01, n_{10} occurrences of 10 and n_{11} occurrences of 11.

2. Let $n_{00}, n_{01}, n_{10}, n_{11}$ be non-negative integers such that

$$n_{00} + n_{01} + n_{10} + n_{11} = n.$$

We again seek the number of sequences (1.1) with n_{ij} occurrences of ij, but now $a_n a_1$ is counted as a consecutive pair.

Let $a(n_{00}, n_{01}, n_{10}, n_{11})$ denote the number of solutions of Problem 1 and $b(n_{00}, n_{01}, n_{10}, n_{11})$ denote the number of solutions of Problem 2. Put

$$f_n = f_n(x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{n_{ij}=0}^{\infty} a(n_{00}, n_{01}, n_{10}, n_{11}) x_{00}^{n_{00}} x_{01}^{n_{01}} x_{10}^{n_{10}} x_{11}^{n_{11}},$$

$$g_n = g_n(x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{n_{ij}=0}^{\infty} b(n_{00}, n_{01}, n_{10}, n_{11}) x_{00}^{n_{00}} x_{01}^{n_{01}} x_{10}^{n_{10}} x_{11}^{n_{11}}.$$

It is convenient to take

$$f_0 = g_0 = 0, \qquad f_1 = g_1 = 2.$$

Put

$$F(u) = \sum_{n=0}^{\infty} f_n u^n, \qquad G(u) = \sum_{n=0}^{\infty} g_n u^n.$$

We show that

(1.2) $$F(u) = \frac{2u + (x_{01} + x_{10} - x_{00} - x_{11})u^2}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2}$$

and

(1.3) $$2 + G(u) = \frac{2 - (x_{00} + x_{11})u}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2}.$$

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The special case

(1.4) $$x_{00} = x_{01} = x_{10} = 1, \qquad x_{11} = x$$

is of some interest. In this case (1.2) and (1.3) reduce to

(1.5) $$1 + F(u) = \frac{1 + (1-x)u}{1 - (1+x)u - (1-x)u^2}$$

(1.6) $$2 + G(u) = \frac{2 - (1+x)u}{1 - (1+x)u - (1-x)u^2},$$

respectively. These generating functions evidently contain the enumeration of zero-one sequences with a given number of occurrences of 11.

For $x = 0$, (1.5) and (1.6) reduce to the generating functions for F_{n+2} and L_n, respectively. Thus it is natural to put

$$1 + F(u) = \sum_{n=0}^{\infty} f_{n+2}(x)u^n, \qquad f_n(x) = \sum_k F_{n,k}x^k,$$

$$2 + G(u) = \sum_{n=0}^{\infty} g_n(x)u^n, \qquad g_n(x) = \sum_k L_{n,k}x^k .$$

We find that $f_n(x)$, $g_n(x)$ both satisfy

$$v_{n+2} = (1+x)v_{n+1} + (1-x)v_n ,$$

which implies

$$F_{n+2,k} = F_{n+1,k} + F_{n,k} + F_{n+1,k-1} - F_{n,k-1}$$

and similarly for $L_{n,k}$. Moreover there is the striking relation

$$g_n(x) = f_{n+3}(x) - 2f_{n+2}(x) + 2f_{n+1}(x) \qquad (n \geqslant 0).$$

2. PROBLEM 1

In order to enumerate the number of sequences of Problem 1 it is convenient to define

(2.1) $$a^i_{rs}(n_{00}, n_{01}, n_{10}, n_{11}) \qquad (i = 0, 1)$$

as the number of zero-one sequences with r zeros, s ones, n_{jk} occurrences of jk and ending with i, where

$$n_{00} + n_{01} + n_{10} + n_{11} = r + s - 1.$$

Put

(2.2) $$f_i(r,s) = f_i(r,s|x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{r,s} a^i_{rs}(n_{00}, n_{01}, n_{10}, n_{11})x_{00}^{n_{00}}x_{01}^{n_{01}}x_{10}^{n_{10}}x_{11}^{n_{11}} .$$

It is convenient to take

(2.3) $$\begin{cases} f_0(0,0) = 0, & f_0(1,0) = 1, \quad f_0(0,1) = 1 \\ f_1(0,0) = 0, & f_1(1,0) = 0, \quad f_1(0,1) = 1 . \end{cases}$$

Deleting the final element in a given sequence, we obtain the following recurrences:

(2.4) $$\begin{cases} f_0(r,s) = x_{00}f_0(r-1, s) + x_{10}f_1(r-1, s) \\ f_1(r,s) = x_{01}f_0(r, s-1) + x_{11}f_1(r, s-1) \end{cases} \quad (r+s > 1).$$

Put

(2.5) $$F_i = F_i(u,v) = \sum_{r,s=0}^{\infty} f_i(r,s)u^rv^s \qquad (i = 0, 1).$$

Then by the first of (2.4)

$$F_0(u,v) = uf_0(1,0) + vf_0(0,1) + x_{00}u \sum_{r+s\geqslant 2} u^{r-1}v^s f_0(r-1,s) + x_{10}v \sum_{r+s\geqslant 2} u^{r-1}v^s f_1(r-1,s),$$

so that

(2.6) $$F_0(u,v) = u + x_{00}uF_0(u,v) + x_{10}uF_1(u,v).$$

Similarly

(2.7) $$F_1(u,v) = v + x_{01}vF_0(u,v) + x_{11}vF_1(u,v).$$

This pair of formulas can be written compactly in matrix form:

(2.8) $$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + M \begin{pmatrix} F_0 \\ F_1 \end{pmatrix},$$

where

(2.9) $$M = \begin{pmatrix} x_{00}u & x_{10}u \\ x_{01}v & x_{11}v \end{pmatrix}.$$

It follows at once from (2.8) that

$$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = (I-M)^{-1} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since

$$(I-M)^{-1} = \frac{1}{D}\begin{pmatrix} 1-x_{11}v & x_{10}u \\ x_{01}v & 1-x_{00}u \end{pmatrix},$$

where

(2.10) $$D = \det M = 1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv,$$

we get

(2.11) $$\begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} u + (x_{10} - x_{11})uv \\ v + (x_{01} - x_{00})uv \end{pmatrix}.$$

Hence

(2.12) $$F(u,v) = F_0(u,v) + F_1(u,v) = \frac{u + v + (x_{01} + x_{10} - x_{00} - x_{11})uv}{1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv}.$$

This furnishes a generating function for the enumeration of sequences with a given number of zeros and a given number of ones and n_{ij} occurrences of ij.

Finally, taking $u = v$, we get the desired solution of Problem 1.

(2.13) $$F(u) = F(u,u) = \frac{2u + (x_{01} + x_{10} - x_{00} - x_{11})u^2}{1 - (x_{00} + x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2}.$$

Explicit formulas for

$$f(r,s) = f_0(r,s) + f_1(r,s)$$

can be obtained from (2.12). The extreme right member is equal to

$$\frac{u(1-x_{11}v) + v(1-x_{00}u) - (x_{01}+x_{11})uv}{(1-x_{00}u)(1-x_{11}v) - x_{01}x_{10}uv} = \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})^k u^{k+1} v^k}{(1-x_{00}u)^{k+1}(1-x_{11}v)^k}$$

$$+ \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})x^k u^k v^{k+1}}{(1-x_{00}u)^k(1-x_{11}v)^{k+1}} - (x_{01}+x_{10}) \sum_{k=0}^{\infty} \frac{(x_{01}x_{10})^k u^{k+1} v^{k+1}}{(1-x_{00}u)^{k+1}(1-x_{11}v)^{k+1}}.$$

Expanding, we get after some manipulation

(2.14) $$f(r,s) = \sum_{k>0} \binom{r-1}{k}\binom{s-1}{k-1}(x_{01}x_{10})^k x_{00}^{r-k-1} x_{11}^{s-k} + \sum_{k>0} \binom{r-1}{k-1}\binom{s-1}{k}(x_{01}x_{10})^k x_{00}^{r-k} x_{11}^{s-k-1}$$

$$- (x_{01}+x_{10}) \sum_{k>0} \binom{r-1}{k-1}\binom{s-1}{k-1}(x_{01}x_{10})^k x_{00}^{r-k-1} x_{11}^{s-k-1} \quad (r > 0,\ s > 0,\ r+s > 2).$$

3. SPECIAL CASES OF PROBLEM 1

If we take

(3.1) $$x_{00} = x_{01} = x_{10} = 1, \quad x_{11} = x,$$

(2.3) reduces to

(3.2) $$1 + F(u) = \frac{1 + (1-x)u}{1 - (1+x)u - (1-x)u^2}.$$

For $x = 0$ the right-hand side becomes

$$\frac{1+u}{1-u-u^2} = \sum_{n=0}^{\infty} F_{n+2}u^n$$

as anticipated. We now define $F_{n,j}$ by means of

(3.3) $$\frac{1 + (1-x)u}{1 - (1+x)u - (1-x)u^2} = \sum_{n=0}^{\infty} f_{n+2}(x)u^n,$$

where

(3.4) $$f_n(x) = \sum_{j \geq 0} F_{n,j}x^j.$$

It follows from (3.3) that $f_n(x)$ satisfies

(3.5) $$f_{n+2}(x) = (1+x)f_{n+1}(x) + (1-x)f_n(x) \quad (n \geq 2)$$

together with $f_2(x) = 1$, $f_3(x) = 2$; if we take $f_1(x) = 1$, then (3.5) holds for $n \geq 1$. From (3.5) we get the recurrence

(3.6) $$F_{n+2,k} = F_{n+1,k} + F_{n+1,k-1} + F_{n,k} - F_{n,k-1} \quad (n \geq 1).$$

The following table is now easily computed.

$F_{n,k}$:

n \ k	0	1	2	3	4	5	6	7
1	1							
2	1							
3	2							
4	3	1						
5	5	2	1					
6	8	5	2	1				
7	13	10	6	2	1			
8	21	20	13	7	2	1		
9	34	38	29	16	8	2	1	
10	55	71	60	39	19	9	2	1

Note that

(3.7) $$f_n(1) = \sum_{j \geq 0} F_{n,j} = 2^{n-2} \quad (n \geq 2).$$

This follows at once by taking $x = 1$ in (3.3). If we take $x = -1$ we get

$$\sum_{n=0}^{\infty} f_{n+2}(-1)u^n = \frac{1+2u}{1-2u^2},$$

which yields

(3.8) $$f_{2n}(-1) = 2^{n-1}, \qquad f_{2n+1}(-1) = 2^n \qquad (n \geq 1).$$

The table suggests

(3.9)
$$\begin{cases} F_{n,n-3} = 1 & (n > 3) \\ F_{n,n-4} = 2 & (n > 4) \\ F_{n,n-5} = n-1 & (n \geq 5) \end{cases}$$

Since

$$\frac{1+(1-x)u}{1-(1+x)u-(1-x)u^2} = \frac{1}{1-u-u^2} + \sum_{k=1}^{\infty} \frac{u^{k+1}(1-u)^{k-1}x^k}{(1-u-u^2)^{k+1}},$$

we have also

(3.10)
$$\sum_{n=k+3}^{\infty} F_{n,k}u^n = \frac{u^{k+1}(1-u)^{k-1}}{(1-u-u^2)^{k+1}} \qquad (k \geq 1).$$

Replacing x by x/u in (3.3) we get

(3.11)
$$\frac{1-x+u}{1-x-(1-x)u-u^2} = \sum_{n=0}^{\infty} u^n \sum_{k=0}^{\infty} F_{n+k+2,k}x^k,$$

which furnishes a generating function for diagonals, namely

(3.12)
$$D_n(x) \equiv \sum_{k=0}^{\infty} F_{n+k+2,k}x^k = \sum_{2s \leq n+1} \binom{n-s+1}{s}(1-x)^{-s}.$$

For example

$$D_0(x) = 1, \quad D_1(x) = 1 + \frac{1}{1-x}, \quad D_2(x) = 1 + \frac{2}{1-x}, \quad D_3(x) = 1 + \frac{3}{1-x} + \frac{1}{(1-x)^2},$$

in agreement with (3.9). Also,

$$D_4(x) = 1 + \frac{4}{1-x} + \frac{3}{(1-x)^2}, \quad D_5(x) = 1 + \frac{5}{1-x} + \frac{6}{(1-x)^2} + \frac{1}{(1-x)^3}, \text{ etc.}$$

The special case

(3.10)
$$x_{00} = x_{10} = x_{11} = 1, \quad x_{01} = x$$

is considerably simpler than (3.1). Using (3.10), (2.13) reduces to

(3.11)
$$1 + F(u) = \frac{1}{1-2u+(1-x)u^2}.$$

Since

$$\frac{1}{1-2u+(1-x)u^2} = \frac{1}{(1-u)^2 - xu^2} = \sum_{k=0}^{\infty} \frac{x^k u^{2k}}{(1-u)^{2k+2}} = \sum_{k=0}^{\infty} x^k u^{2k} \sum_{j=0}^{\infty} \binom{2k+k+1}{j} u^j$$

$$= \sum_{n=0}^{\infty} u^n \sum_{2k \leq n} \binom{n+1}{2k+1} x^k,$$

so that (3.11) becomes

(3.12)
$$1 + F(u) = \sum_{n=0}^{\infty} u^n \sum_{2k \leq n} \binom{n+1}{2k+1} x^k.$$

It follows from (3.12) that the number of sequences of length n with k occurrences of 01 is equal to the binomial coefficient $\binom{n+1}{2k+1}$. It is not difficult to give a direct combinatorial proof of this result.

4. PROBLEM 2

Let

(4.1)
$$a_{rs}^{ij}(n_{00}, n_{01}, n_{10}, n_{11}) \qquad (i,j = 0)$$

denote the number of sequences with r zeros and s ones, where $r+s = n_{00}+n_{01}+n_{10}+n_{11}+1$, with n_{hk} occurrences of hk, beginning with i and ending with j. Also put

$$f_{ij}(r,s) = f_{ij}(r,s\,|\,x_{00},x_{01},x_{10},x_{11}) = \sum_{n_{hk}=0}^{\infty} a_{rs}^{ij}(n_{00},n_{01},n_{10},n_{11})x_{00}^{n_{00}}x_{01}^{n_{01}}x_{10}^{n_{10}}x_{11}^{n_{11}}, \tag{4.2}$$

$$F_{ij} = F_{ij}(u,v) = \sum_{r,s=0}^{\infty} f_{ij}(r,s)u^r v^s . \tag{4.3}$$

Exactly as in § 2, we have

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} + M \begin{pmatrix} F_{00} & F_{01} \\ F_{01} & F_{11} \end{pmatrix}, \tag{4.4}$$

where M is defined in (2.9). Thus

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = (I-M)^{-1}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} .$$

It follows that

$$\begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix} = \frac{1}{D}\begin{pmatrix} u - x_{11}uv & x_{10}uv \\ x_{01}uv & v - x_{00}uv \end{pmatrix}, \tag{4.5}$$

where as before

$$D = 1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv. \tag{4.6}$$

For Problem 2 we require

$$G(u,v) = x_{00}F_{00} + x_{10}F_{01} + x_{01}F_{10} + x_{11}F_{11} . \tag{4.7}$$

Hence, by (4.5) and (4.6),

$$G(u,v) = \frac{x_{00}u + x_{11}v - 2(x_{00}x_{11} - x_{01}x_{10})uv}{1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv} .$$

It is convenient to replace this by

$$2 + G(u,v) = \frac{2 - x_{00}u - x_{11}v}{1 - x_{00}u - x_{11}v + (x_{00}x_{11} - x_{01}x_{10})uv} . \tag{4.8}$$

In particular, for $u = v$, (4.8) becomes

$$2 + g(u,u) = \frac{2 - (x_{00}+x_{11})u}{1 - (x_{00}+x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2} . \tag{4.9}$$

Thus (4.9) furnishes a generating function for Problem 2.

If we put

$$2 + G(u,u) = \sum_{n=0}^{\infty} g_n u^n, \qquad F(u) = \sum_{n=0}^{\infty} f_n u^n ,$$

where, by (2.13)

$$F(u) = \frac{2u + (x_{01}+x_{10}-x_{00}-x_{11})u^2}{1 - (x_{00}+x_{11})u + (x_{00}x_{11} - x_{01}x_{10})u^2} ,$$

then it is clear that

$$(2 - (x_{00}+x_{11})u) \sum_{0}^{\infty} f_n u^n = (2u + (x_{01}+x_{10}-x_{00}-x_{11})u^2) \sum_{0}^{\infty} g_n u^n .$$

Comparison of coefficients gives

$$f_n - (x_{00}+x_{11})f_{n-1} = 2g_{n-1} + (x_{01}-x_{10}-x_{00}-x_{11})g_{n-2} . \tag{4.10}$$

5. SPECIAL CASES OF PROBLEM 2

We take

(5.1) $$x_{00} = x_{01} = x_{10} = 1, \qquad x_{11} = x.$$

Then (4.9) reduces to

(5.2) $$2 + G(u,u) = \frac{2-(1+x)u}{1-(1+x)u-(1-x)u^2} \quad .$$

For $x = 0$ the right side of (5.2) becomes

$$\frac{2-u}{1-u-u^2} = \sum_0^\infty L_n u^n$$

as was expected. We now define $L_{n,j}$ by means of

(5.3) $$\frac{2-(1+x)u}{1-(1+x)u-(1-x)u^2} = \sum_{n=0}^\infty g_n(x)u^n ,$$

where

(5.4) $$g_n(x) = \sum_{j \geqslant 0} L_{n,j}x^j \ .$$

It follows from (5.3) that $g_n(x)$ satisfies

(5.5) $$g_{n+2}(x) = (1+x)g_{n+1}(x) + (1-x)g_n(x) \quad (n \geqslant 0)$$

together with $g_0(x) = 2, g_1(x) = 1 + x$. It is also clear that $L_{n,k}$ satisfies the recurrence

(5.6) $$L_{n+2,k} = L_{n+1,k} + L_{n+1,k-1} + L_{n,k} - L_{n,k-1} \quad (n \geqslant 0)$$

which is of course the same as (3.6).

The following table is easily computed.

$L_{n,k}$:

n \ k	0	1	2	3	4	5	6	7	8	9	10
0	2										
1	1	1									
2	3	0	1								
3	4	3	0	1							
4	7	4	4	0	1						
5	11	10	5	5	0	1					
6	18	18	15	6	6	0	1				
7	29	35	28	21	7	7	0	1			
8	47	64	60	40	28	8	8	0	1		
9	76	117	117	93	54	36	9	9	0	1	
10	123	210	230	190	135	70	45	10	10	0	1

It is easily proved by means of (5.3) and (5.4) that

(5.7) $$g_n(1) = \sum_{k=0}^n L_{n,k} = 2^n \qquad (n \geqslant 1),$$

(5.8) $$g_{2n}(-1) = 2^{n+1}, \qquad g_{2n+1}(-1) = 0 \qquad (n \geqslant 0).$$

The table suggests that $L_{nn} = 1$,

$$\begin{cases} L_{n,n-1} = 0 & (n > 1), \\ L_{n,n-2} = n & (n > 2), \\ L_{n,n-3} = n & (n > 3). \end{cases} \tag{5.9}$$

These results are easily proved by induction using (5.6).

Comparison of (5.3) with (3.3) gives

$$g_n(x) + (1-x)g_{n-1}(x) = 2f_{n+2}(x) - (1+x)f_{n+1}(x). \tag{5.10}$$

In view of (3.5), this implies

$$g_n(x) + (1-x)g_{n-1}(x) = f_{n+2}(x) + (1-x)f_n(x) \qquad (n \geqslant 1). \tag{5.11}$$

In particular (5.11) contains the familiar relation $L_{n+1} = F_{n+2} + F_n$. It would be of interest to express $g_n(x)$ in terms of $f_k(x)$.

We find that

$$g_0(x) = f_3(x), \quad g_1(x) = f_4(x) - f_3(x), \quad g_2(x) = f_5(x) - 2f_4(x) + 2f_3(x),$$

$$g_3(x) = f_6(x) - 2f_5(x) + 2f_4(x), \quad g_4(x) = f_7(x) - 2f_6(x) + 2f_5(x), \quad g_5(x) = f_8(x) - 2f_7(x) + 2f_6(x),$$

$$g_6(x) = f_9(x) - 2f_8(x) + 2f_7(x), \quad g_7(x) = f_{10}(x) - 2f_9(x) + 2f_8(x).$$

This suggests that

$$g_n(x) = f_{n+3}(x) - 2f_{n+2}(x) + 2f_{n+1}(x) \qquad (n = 0, 1, 2, \cdots). \tag{5.12}$$

To prove (5.12) we make use of the identity

$$u(2-(1+x)u) = (1-2u+2u^2)(1+(1-x)u) - (1-2u)(1-(1+x)u-(1-x)u^2).$$

Dividing both sides by $D = 1-(1+x)u-(1-x)u^2$, this becomes

$$u\,\frac{2-(1+x)u}{D} = (1-2u+2u^2)\,\frac{1+(1-x)u}{D} - 1 + 2u.$$

Hence, by (3.3) and (5.3),

$$u\sum_{n=0}^{\infty} g_n(x)u^n = (1-2u+2u^2)\sum_{n=0}^{\infty} f_{n+2}(x)u^n - 1 + 2u.$$

Comparing coefficients of u^n, we get

$$g_{n-1}(x) = f_{n+2}(x) - 2f_{n+1}(x) + 2f_n(x) \qquad (n \geqslant 1),$$

which is equivalent to (5.12).

From (5.12) we get

$$L_{n,k} = F_{n+3,k} - 2F_{n+2,k} + 2F_{n+1,k} \qquad (k = 0, 1, 2, \cdots). \tag{5.13}$$

Note that, for $k = 0$, (5.13) reduces to the familiar

$$L_n = F_{n+3} - 2F_{n+2} + 2F_{n+1} = -F_{n+2} + 3F_{n+1} = 2F_{n+1} - F_n = F_{n+1} + F_{n-1}.$$

Finally, replacing x by x/u in (5.3), we get

$$\frac{2-x-u}{1-x-(1-x)u-u^2} = \sum_{n=0}^{\infty} u^n \sum_{k=0}^{\infty} L_{n+k,k}x^k.$$

This yields

$$\sum_{k=0}^{\infty} L_{n+k,k}x^k = \frac{3-2x}{1-x}\sum_{2s\leqslant n} \frac{1}{(1-x)^s} - \sum_{2s\leqslant n+1} \binom{n-s+1}{s}\frac{1}{(1-x)^s}. \tag{5.14}$$

For example

$$\sum_{k=0}^{\infty} L_{k+1,k}x^k = \frac{3-2x}{1-x} - \left(1 + \frac{1}{1-x}\right) = 1,$$

which is correct.

REFERENCE

1. L. Carlitz, "Zero-One Sequences and Fibonacci Numbers of Higher Order," *The Fibonacci Quarterly,* Vol. 12 (1974), pp. 1–10.

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THE UNIFIED NUMBER IDENTITY

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The identity illustrated below shows a relation connecting all of the most important constants and numbers in mathematics.

$$e^{i\pi}\left(2\beta+\sum_{n=0}^{\infty}(-1)^n(\sqrt{5}F_{n+1}-L_{n+1})\right)+\alpha\sum_{n=0}^{\infty}\frac{(-1)^nE_{2n}\sum_{k=1}^{\infty}(1/k)^{2n}}{B_n(10)^{2n}}+1=0.$$

In the usual notation the above identity has the following constants and numbers:

CONSTANTS

$$0,\ 1,\ -1,\ 2,\ \sqrt{5},\ i=\sqrt{-1},\ e,\ \pi,\ \alpha=\frac{1+\sqrt{5}}{2},\ \beta=\frac{1-\sqrt{5}}{2},\ 10.$$

NUMBERS

Notation		Explanation
n	$n = 0, 1, \cdots$	denotes zero and the set of positive integers.
$1/k$	$k = 1, 2, \cdots$	is the collection of fractions of the form $1/k$.
F_{n+1}	$n = 0, 1, \cdots$	denotes the $(n+1)^{th}$ Fibonacci number.
L_{n+1}	$n = 0, 1, \cdots$	" " " Lucas number.
B_n	$n = 0, 1, \cdots$	" " n^{th} Bernoulli number.
E_{2n}	$n = 0, 1, \cdots$	" " $2n^{th}$ even Euler number.

The author of this note wishes to point out that since the letter n denotes zero and the set of positive integers, then it must denote most of the conceivable numbers defined by mathematicians so far. Let us name some of these numbers. Prime, Fermat, Guy Moebius, Perfect, Pythagorean, Random, Triangular, Amicable, Automorphic, Palindromic, and the list goes on and on $\cdots$.

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