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The Asymptotic Formulas Related to Exponents in Factoring Integers

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Presented by P. Kenderov

For any positive integer $n>1$, say $n=q_1^{\alpha_1}q_2^{\alpha_2}\dots q_t^{\alpha_t}$, define $h(n)=\min(\alpha_1, \alpha_2,\dots, \alpha_t)$ and $H(n)=\max(\alpha_1, \alpha_2,\dots, \alpha_t)$. We take $h(1)=H(1)=1$. In this paper, making only use of elementary argument, we prove that $\sum_{i=1}^{n} h(i)=n+n^{1/2}\zeta(3/2)/\zeta(3)+n^{1/3}(\zeta(2/3)/\zeta(2)+c_0)+o(n^{1/3})$. Finally we give the formula of sum of $H(n)$.

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P. Erdös suggested that $\sum_{i=1}^{n} h(i)=n+c\sqrt{n}+o(\sqrt{n})$. His conjecture was proved by Ivan Niven [1]. In [1], he proved that

$$\sum_{i=1}^{n} h(i)=n+\sqrt{n}\,\zeta(3/2)/\zeta(3)+o(\sqrt{n})$$

and

$$\lim_{n\to\infty} 1/n \sum_{i=1}^{n} H(i)=1+\sum_{k=2}^{\infty}(1-\zeta(k)^{-1}),$$

where $\zeta(k)$ is the Riemann-zeta-function.

In this paper, making only use of elementary argument we prove the following theorem.

Theorem. $\sum_{i=1}^{n} h(i)=n+n^{1/2}\zeta(3/2)/\zeta(3)+n^{1/3}(\zeta(2/3)/\zeta(2)+c_0)+o(n^{1/3})$, *where* $c_0=\prod_p(1+p^{-4/3}+p^{-5/3})$.

Finally we shall give the formula of sum of $H(n)$.

Proof of Theorem. Define $S_2=\{k^2\,|\,k\in N\}$, $S_3=\{k^3\,|\,k\in N\}$ and $S_3(n)=\sum_{\substack{m\leq n\\ m\in S_3}} 1$, and define $T_k=\{m\,|\,h(m)\geqq k \text{ and } m\in N\}$ and $T_k(n)=\sum_{\substack{m\leq n\\ m\in T_k}} 1$.

It is clear that

(1) $$n+T_2(n)+T_3(n)\leqq \sum_{i=1}^{n} h(i)\leqq n+T_2(n)+T_3(n)+T_4(n)\log_2 n.$$

In this paper, p_i is the i-th prime.

From (1), we know that

$$T_3(n)=\Sigma S_3(n/(p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r})),$$

where the sum is over the 3^r terms with $\beta_i=0, 4$ or 5 and r chosen so that $p_{r+1}>n$.

By $S_3(n)=[\sqrt[3]{n}]<\sqrt[3]{n}$, we have

$$T_3(n)\leqq\Sigma(n/(p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r}))^{1/3}\leqq n^{1/3}\prod_{i=1}^{r}(1+p_i^{-4/3}+p_i^{-5/3})$$

and

(2) $$\overline{\lim_{n\to\infty}}\, T_3(n)/n^{1/3}\leqq c_0.$$

In the other hand, let $\{n_j\}$ be any sequence of integers with $32<n_1<n_2<n_3\dots$ and $\lim_{j\to\infty} n_j=\infty$. Let r_j be the largest positive integer such that $n_j\geqq(p_1p_2\dots p_{r_j})^5$, we have $r_j\to\infty$ when $n_j\to\infty$.

We have

$$T_3(n_j)\geqq\Sigma S_3(n_j/(p_1^{\beta_1}p_2^{\beta_2}\dots p_{r_j}^{\beta_{r_j}}),$$

where the sum is over the 3^{r_j} terms with $\beta_i=0$, 4, or 5.

By $S_3(n)>\sqrt[3]{n}-1$, we get

$$T_3(n_j)>\Sigma(n_j/(p_1^{\beta_1}p_2^{\beta_2}\dots p_{r_j}^{\beta_{r_j}}))^{1/3}-3^{r_j}$$

and

$$T_3(n_j)/n_j^{1/3}>\prod_{i=1}^{r_i}(1+p_i^{-4/3}+p_i^{-5/3})-3^{r_j}/n_j^{1/3}.$$

Note that

$$3^{r_j}/n_j^{1/3}<3^{r_j}/(p_1p_2\dots p_{r_j})<3/2(3/5)^{r_j-2}.$$

Hence we get

(3) $$\underline{\lim_{n\to\infty}}\, T_3(n)/n^{1/3}\geqq c_0.$$

By (2) and (3) we get

(4) $$T_3(n)=c_0n^{1/3}+o(n^{1/3}).$$

From [1], we know that any element m of T_2 not in S_2 can be written uniquely in the form

$$m=k^2q_1q_2\dots q_t,\ (q_1q_2\dots q_t)|k,\ t\geqq1,$$

where $q_1, q_2,\dots, q_t$ are distinct primes. Hence we have

$$(5)\qquad T_2(n)=\sum_{\substack{m\leqq n\\ m\in T_2}}1=\sum_{m^2r^3\leqq n}|\mu(r)|=\sum_{m^2r^3\leqq n}\sum_{k^2|r}\mu(k)=\sum_{m^2r^3\leqq n}\sum_{hk^2=r}\mu(k)$$

$$=\sum_{m^2h^3k^6\leqq n}\mu(k)=\sum_{k\leqq n^{1/6}}\mu(k)\sum_{m^2h^3\leqq n/k^6}1,$$

where m, r are positive integers and $\mu(n)$ is the Möbius function.

By an elementary argument, E. Landau [2] proved that if d_1 and d_2 are fixed positive integers and $d_1\neq d_2$, then

$$(6)\qquad \sum_{m^{d_1}n^{d_2}\leqq x}1=\zeta(d_2/d_1)x^{1/d_1}+\zeta(d_1/d_2)x^{1/d_2}+(x^{1/(d_1+d_2)}).$$

By (5) and (6), we get

$$T_2(n)=\sum_{k\leqq n^{1/6}}\mu(k)(\zeta(3/2)(n/k^6)^{1/2}+\zeta(2/3)(n/k^6)^{1/3}+O((n/k^6)^{1/5})$$

$$(7)\qquad =n^{1/2}\zeta(3/2)/\zeta(3)+n^{1/3}\zeta(2/3)/\zeta(2)+O(n^{1/2}\sum_{k>n^{1/6}}1/k^3)+O(n^{1/3}\sum_{k>n^{1/6}}1/k^2)$$

$$+O(n^{1/5})=n^{1/2}\zeta(3/2)/\zeta(3)+n^{1/3}\zeta(2/3)/\zeta(2)+O(n^{1/5}).$$

By (1), (4) and (7), and since $T_k(n)=O(n^{1/k})$ for any integer $k\geqq 3$ (see [1], p. 358), this completes the proof.

We discuss the formula of sum of $H(n)$ below.

Let $Q_k(x)$ denote the number of $n\leqq x$ such that n is a k-power free integer. From [1], we know that when $k\geqq 2$

$$\sum_{i=1}^{n}H(i)=[\log_2 n]n-\sum_{k=2}^{[\log_2 n]}Q_k(n)$$

and

$$|Q_k(n)-n\zeta(k)^{-1}|<3n^{1/k}.$$

Hence we get

$$\sum_{i=1}^{n}H(i)=[\log_2 n]n-n\sum_{k=2}^{[\log_2 n]}\zeta(k)^{-1}+O(n^{1/2}\log n)$$

$$(8)\qquad =\Big(1+\sum_{k=2}^{[\log_2 n]}\{1-\zeta(k)^{-1}\}\Big)n+O(n^{1/2}\log n)$$

$$=\Big(1+\sum_{k=2}^{\infty}\{1-\zeta(k)^{-1}\}\Big)n+O(n\sum_{k>[\log_2 n]}\{1-\zeta(k)^{-1}\})+(n^{1/2}\log n).$$

We have

$$\sum_{k>[\log_2 n]}\{1-\zeta(k)^{-1}\}=-\sum_{k>[\log_2 n]}\sum_{h=2}^{\infty}\mu(h)/h^k=-\sum_{h=2}^{\infty}\mu(h)\sum_{k>[\log_2 n]}1/h^k$$

$$(9)\quad =-\sum_{h=2}^{\infty}\mu(h).h^{-([\log_2 n]+1)}/(1-h^{-1})=O(2^{1-[\log_2 n]}\sum_{h=2}^{\infty}h^{-1}(h-1)^{-1})=O(1/n).$$

By (8) and (9), we obtain

$$\sum_{i=1}^{n} H(i) = c_0' n + O(n^{1/2} \log n),$$

where $c_0' = 1 + \sum_{k=2}^{\infty} \{1 - \zeta(k)^{-1}\}$, and the series $\sum_{k=2}^{\infty} \{1 - \zeta(k)^{-1}\}$ converges to 0.7 approximately.

From the above proof, it is easy to see that making use of the result $Q_2(n) = n/\zeta(2) + O(n^{1/2} \exp\{-c_1(\log n)^{3/5}(\log\log n)^{-1/5})$ (see [3]) we can obtain $\sum_{i=1}^{n} H(i) = c_0' n + O(n^{1/2} \exp\{-c_2(\log n)^{3/5}(\log\log n)^{-1/5})$.

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