

Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
<http://www.mathbalkanica.info>

or contact:

Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

The Asymptotic Formulas Related to Exponents in Factoring Integers

Cao Hui-Zhong

Presented by P. Kenderov

For any positive integer $n > 1$, say $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r}$, define $h(n) = \min(\alpha_1, \alpha_2, \dots, \alpha_r)$ and $H(n) = \max(\alpha_1, \alpha_2, \dots, \alpha_r)$. We take $h(1) = H(1) = 1$. In this paper, making only use of elementary argument, we prove that $\sum_{i=1}^n h(i) = n + n^{1/2} \zeta(3/2) / \zeta(3) + n^{1/3} (\zeta(2/3) / \zeta(2) + c_0) + o(n^{1/3})$. Finally we give the formula of sum of $H(n)$.

For any positive integer $n > 1$, say $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r}$, define $h(n) = \min(\alpha_1, \alpha_2, \dots, \alpha_r)$ and $H(n) = \max(\alpha_1, \alpha_2, \dots, \alpha_r)$. We take $h(1) = H(1) = 1$.

P. Erdős suggested that $\sum_{i=1}^n h(i) = n + c\sqrt{n} + o(\sqrt{n})$. His conjecture was proved by Ivan Niven [1]. In [1], he proved that

$$\sum_{i=1}^n h(i) = n + \sqrt{n} \zeta(3/2) / \zeta(3) + o(\sqrt{n})$$

and

$$\lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n H(i) = 1 + \sum_{k=2}^{\infty} (1 - \zeta(k)^{-1}),$$

where $\zeta(k)$ is the Riemann-zeta-function.

In this paper, making only use of elementary argument we prove the following theorem.

Theorem. $\sum_{i=1}^n h(i) = n + n^{1/2} \zeta(3/2) / \zeta(3) + n^{1/3} (\zeta(2/3) / \zeta(2) + c_0) + o(n^{1/3})$,

where $c_0 = \prod_p (1 + p^{-4/3} + p^{-5/3})$.

Finally we shall give the formula of sum of $H(n)$.

Proof of Theorem. Define $S_2 = \{k^2 | k \in N\}$, $S_3 = \{k^3 | k \in N\}$ and

$S_3(n) = \sum_{\substack{m \leq n \\ m \in S_3}} 1$, and define $T_k = \{m | h(m) \geq k \text{ and } m \in N\}$ and $T_k(n) = \sum_{\substack{m \leq n \\ m \in T_k}} 1$.

It is clear that

$$(1) \quad n + T_2(n) + T_3(n) \leq \sum_{i=1}^n h(i) \leq n + T_2(n) + T_3(n) + T_4(n) \log_2 n.$$

In this paper, p_i is the i -th prime.

From (1), we know that

$$T_3(n) = \sum S_3(n/(p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r})),$$

where the sum is over the 3^r terms with $\beta_i = 0, 4$ or 5 and r chosen so that $p_{r+1} > n$.

By $S_3(n) = [\sqrt[3]{n}] < \sqrt[3]{n}$, we have

$$T_3(n) \leq \sum (n/(p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}))^{1/3} \leq n^{1/3} \prod_{i=1}^r (1 + p_i^{-4/3} + p_i^{-5/3})$$

and

$$(2) \quad \lim_{n \rightarrow \infty} T_3(n)/n^{1/3} \leq c_0.$$

In the other hand, let $\{n_j\}$ be any sequence of integers with $32 < n_1 < n_2 < n_3 \dots$ and $\lim n_j = \infty$. Let r_j be the largest positive integer such that $n_j \geq (p_1 p_2 \dots p_{r_j})^5$, we have $r_j \rightarrow \infty$ when $n_j \rightarrow \infty$.
We have

$$T_3(n_j) \geq \sum S_3(n_j/(p_1^{\beta_1} p_2^{\beta_2} \dots p_{r_j}^{\beta_{r_j}})),$$

where the sum is over the 3^{r_j} terms with $\beta_i = 0, 4$, or 5 .

By $S_3(n) > \sqrt[3]{n} - 1$, we get

$$T_3(n_j) > \sum (n_j/(p_1^{\beta_1} p_2^{\beta_2} \dots p_{r_j}^{\beta_{r_j}}))^{1/3} - 3^{r_j}$$

and

$$T_3(n_j)/n_j^{1/3} > \prod_{i=1}^{r_j} (1 + p_i^{-4/3} + p_i^{-5/3}) - 3^{r_j}/n_j^{1/3}.$$

Note that

$$3^{r_j}/n_j^{1/3} < 3^{r_j}/(p_1 p_2 \dots p_{r_j}) < 3/2(3/5)^{r_j-2}.$$

Hence we get

$$(3) \quad \lim_{n \rightarrow \infty} T_3(n)/n^{1/3} \geq c_0.$$

By (2) and (3) we get

$$(4) \quad T_3(n) = c_0 n^{1/3} + o(n^{1/3}).$$

From [1], we know that any element m of T_2 not in S_2 can be written uniquely in the form

$$m = k^2 q_1 q_2 \dots q_t (q_1 q_2 \dots q_t) | k, t \geq 1,$$

where q_1, q_2, \dots, q_t are distinct primes. Hence we have

$$\begin{aligned}
 (5) \quad T_2(n) &= \sum_{\substack{m \leq n \\ m \in T_2}} 1 = \sum_{m^2 r^3 \leq n} |\mu(r)| = \sum_{m^2 r^3 \leq n} \sum_{k^2 | r} \mu(k) = \sum_{m^2 r^3 \leq n} \sum_{hk^2=r} \mu(k) \\
 &= \sum_{m^2 h^3 k^6 \leq n} \mu(k) = \sum_{k \leq n^{1/6}} \mu(k) \sum_{m^2 h^3 \leq n/k^6} 1,
 \end{aligned}$$

where m, r are positive integers and $\mu(n)$ is the Möbius function.

By an elementary argument, E. Landau [2] proved that if d_1 and d_2 are fixed positive integers and $d_1 \neq d_2$, then

$$(6) \quad \sum_{m^d n^{d_2} \leq x} 1 = \zeta(d_2/d_1) x^{1/d_1} + \zeta(d_1/d_2) x^{1/d_2} + O(x^{1/(d_1+d_2)}).$$

By (5) and (6), we get

$$\begin{aligned}
 T_2(n) &= \sum_{k \leq n^{1/6}} \mu(k) (\zeta(3/2)(n/k^6)^{1/2} + \zeta(2/3)(n/k^6)^{1/3} + O((n/k^6)^{1/5})) \\
 (7) \quad &= n^{1/2} \zeta(3/2)/\zeta(3) + n^{1/3} \zeta(2/3)/\zeta(2) + O(n^{1/2} \sum_{k > n^{1/6}} 1/k^3) + O(n^{1/3} \sum_{k > n^{1/6}} 1/k^2) \\
 &\quad + O(n^{1/5}) = n^{1/2} \zeta(3/2)/\zeta(3) + n^{1/3} \zeta(2/3)/\zeta(2) + O(n^{1/5}).
 \end{aligned}$$

By (1), (4) and (7), and since $T_k(n) = O(n^{1/k})$ for any integer $k \geq 3$ (see [1], p. 358), this completes the proof.

We discuss the formula of sum of $H(n)$ below.

Let $Q_k(x)$ denote the number of $n \leq x$ such that n is a k -power free integer. From [1], we know that when $k \geq 2$

$$\sum_{i=1}^n H(i) = [\log_2 n]n - \sum_{k=2}^{[\log_2 n]} Q_k(n)$$

and

$$|Q_k(n) - n\zeta(k)^{-1}| < 3n^{1/k}.$$

Hence we get

$$\begin{aligned}
 \sum_{i=1}^n H(i) &= [\log_2 n]n - n \sum_{k=2}^{[\log_2 n]} \zeta(k)^{-1} + O(n^{1/2} \log n) \\
 (8) \quad &= (1 + \sum_{k=2}^{[\log_2 n]} \{1 - \zeta(k)^{-1}\})n + O(n^{1/2} \log n) \\
 &= (1 + \sum_{k=2}^{\infty} \{1 - \zeta(k)^{-1}\})n + O(n \sum_{k > [\log_2 n]} \{1 - \zeta(k)^{-1}\}) + (n^{1/2} \log n).
 \end{aligned}$$

We have

$$\begin{aligned}
 \sum_{k > [\log_2 n]} \{1 - \zeta(k)^{-1}\} &= - \sum_{k > [\log_2 n]} \sum_{h=2}^{\infty} \mu(h)/h^k = - \sum_{h=2}^{\infty} \mu(h) \sum_{k > [\log_2 n]} 1/h^k \\
 (9) \quad &= - \sum_{h=2}^{\infty} \mu(h) \cdot h^{-([\log_2 n] + 1)/(1 - h^{-1})} = O(2^{1 - [\log_2 n]} \sum_{h=2}^{\infty} h^{-1}(h-1)^{-1}) = O(1/n).
 \end{aligned}$$

By (8) and (9), we obtain

$$\sum_{i=1}^n H(i) = c'_0 n + O(n^{1/2} \log n),$$

where $c'_0 = 1 + \sum_{k=2}^{\infty} \{1 - \zeta(k)^{-1}\}$, and the series $\sum_{k=2}^{\infty} \{1 - \zeta(k)^{-1}\}$ converges to 0.7 approximately.

From the above proof, it is easy to see that making use of the result $Q_2(n) = n/\zeta(2) + O(n^{1/2} \exp\{-c_1(\log n)^{3/5}(\log \log n)^{-1/5}\})$ (see [3]) we can obtain

$$\sum_{i=1}^n H(i) = c'_0 n + O(n^{1/2} \exp\{-c_2(\log n)^{3/5}(\log \log n)^{-1/5}\}).$$

References

1. Ivan Niven. Averages of exponents in factoring integers. *Proc. Amer. Math. Soc.*, 22, 1969, 356-360.
2. Edmund Landau. Au sujet d'une certaine expression asymptotique. *L'Intermédiaire des Math.*, vol. 20, 1913, 155.
3. A. Walfisz. *Weylsche Exponentialsummen in der neueren Zahlentheorie*. Berlin, 1963.

Dept. of Mathematics
Shandong University
Jinan 250100
Shandong
CHINA

Received 16.07.1990