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NECKLACES AND CONVEX k-GONS*

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A short proof of the Hansraj Gupta[1] formula that solves the Richard H. Reis problem about the number of the incongruent convex k-gons with the tops on circumference is given.

Key Words : Convex K-gons; Necklaces; Compositions; Dihedral Group; Cycle Index

1. INTRODUCTION

Professor Richard H. Reis (South-East University of Massachusetts, USA) in 1978 put the proboem: "Let a circumference is split by the same n parts. It is required to find the number $R(n, k)$ of the incongruent convex k-gons, which could be obtained by connection of some k from n dividing points. Two k-gons are considered congruent if they are coincided at the rotation of one relatively other along the circumference and (or) by reflection of one of the k-gons relatively some diameter.

In 1979 Hansraj Gupta[1] gave the solution of the Reis problem.

Theorem* 1 — *(Hansraj Gupta[1])

$$R(n,k) = \frac{1}{2}\left(\binom{\left[\frac{n-h_k}{2}\right]}{\left[\frac{k}{2}\right]} + \frac{1}{k}\sum_{d\,|\,(k,n)} \varphi(d)\binom{\frac{n}{d}-1}{\frac{k}{d}-1}\right), \qquad \text{... (1)}$$

where $h_k \equiv k(\text{mod } 2)$, $h_k = 0$ *or* 1, $\varphi(n)$*-the Euler function.*

Based on another idea, diverse than in[1], proof of formula (1) is given.

2. SOME BIJECTIONS

(1) Let's consider some convex polygon with the tops in the circumference splitting points, "1" or "0" is put in accordance to each splitting point depending on whether a top of the polygon is in the point. Thus, there is the mutual one-to-one correspondence between the set of convex polygons with the tops in the circumference splitting points and the set of all (0, 1)-configurations with the elements in these points.

(2) As it is known[2], the composition of the number n is the partition with taking into account the sequence of its members. Imagine that there are n colors (color 1, color 2, ..., color n). Put in accordance to each n composition with k members the necklace with k beads, each bead painted by

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color j if the appropriate part of the composition has length j. For example, the appropriate necklace of the composition (3, 1, 8, 1) of number 13 contains 4 beads. The first bead is painted by the 3rd color, the second and the fourth are painted by the 1st color and the third by the 8th. Two necklaces are imagined as equivalent if they coincide at the rotation along the circumference in a chosen direction or at the reflection relatively a circumference diameter.

(3) It is clear that in accordance to each considered above (0, 1)-configuration there is the composition n with length of intervals between neighbour ones of the circumference. So "1" corresponds to two ones following succession; "2" corresponds two ones split by one zero etc. Thus there is a bijection between all incongruent (0, 1)-configurations contain k ones with k_i intervals of length i, $i = 1, 2, ..., n$ and the set of non-equivalent necklaces consisted of k beads and each of the beads is painted by one of n colors, if there k_i beads are painted by color i.

3. Cyclic Index of Dihedral Group

Mac Mahon (see[3&4]) for the first time has enumerated necklaces with congruences only by rotation, as to equivalence considered we use the following performance of the generating polynomial $N_k(x_1, x_2, ..., x_n)$ for numbers of non-equivalent necklaces from k beads each of them painted by one on n colors. This polynomial is the cyclic index of the Dihedral group[3].

Theorem 2 — *(Page 162[3].)*

$$N_k(x_1, x_2, ..., x_n) = N_k^{(1)} + N_k^{(2)}, \qquad \text{... (2)}$$

where

$$N_k^{(1)} = \frac{1}{2k} \sum_{i \mid k} \varphi(i) \left(x_1^i + \ldots + x_n^i\right)^{k/i}, \qquad \text{... (3)}$$

$$N_k^{(2)} = \begin{cases} \frac{1}{2}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{k-1}{2}}, & \text{if } k \text{ is odd}, \\ \frac{1}{4}\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{k}{2}-1}\left(\left(\sum_{i=1}^{n} x_i\right)^2 + \sum_{i=1}^{n} x_i^2\right) & \text{if } k \text{ is even.} \end{cases} \qquad \text{... (4)}$$

Thus, the number of necklaces consisted from k beads, where k_i are painted by i color, $i = 1, 2, ..., n$, is equal to $coef_{x_1^{k_1} \cdots x_n^{k_n}} N_k(x_1, ..., x_n)$ and the number of all non-equivalent necklaces from k beads is equal to $\sum coef_{x_1^{k_1} \cdots x_n^{k_n}} N_k(x_1, ..., x_n)$, where sum is taken over all solutions $k_i (\geq 0)$ of equations

$$k_1 + k_2 + \ldots + k_n = k,$$

$$k_1 + 2k_2 + \ldots + nk_n = n. \quad \ldots (5)$$

Now directly from the Theorem 2, it follows

Lemma 1 —

$$R(n, k) = \sigma(n, k) + \theta(n, k), \quad \ldots (6)$$

where

$$\sigma(n, k) = \sum coef_{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}} N_k^{(1)}, \quad \ldots (7)$$

$$\theta(n, k) = \sum coef_{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}} N_k^{(2)}, \quad \ldots (8)$$

where the sums (7) and (8) are taken over all solutions $k_i \geq 0$ of (5).

4. Evaluation of Sums (7) and (8)

Further part of this article is concerned with obtaining of the formulas for sums (7) and (8).

Lemma 2 — (See for example [5, p. 68]).

$$\sum \frac{k!}{k_1 ! k_2 ! \ldots k_n !} = \binom{n-1}{k-1}, \quad \ldots (9)$$

where the sum is taken over all solutions $k_i \geq 0$, $i = 1, 2, \ldots, n$ of (5).

One of the possible combinatorial proofs of the Lemma 2 is given in following.

Let's consider the set M_k of square (0, 1) matrixes of order k consisting of n ones and do not have zero rows. Let's say that type of $A \in M_k$ is determined by its rows' sums vector. Apparently, the number of all matrix types from M_k is equal to the number of all compositions of n with k parts that is equal to $\binom{n-1}{k-1}$. On the other hand, if at the fixed $k_1, k_2, \ldots, k_n$ the rows' sums vector has k_1 ones, k_2 twos and so on, then the conditions (5) are satisfied and the number of such vectors is equal to $\frac{k!}{k_1 ! k_2 ! \ldots k_n !}$. Summing this by all $k_1, k_2, \ldots, k_n$ satisfying the conditions (5) we obtain the total number the matrix types from M_k, that is $\binom{n-1}{k-1}$ □.

Lemma 3 —

$$\sigma(n, k) = \frac{1}{2k} \sum_{d \mid (k, n)} \varphi(d) \binom{\frac{n}{d} - 1}{\frac{k}{d} - 1}. \qquad \text{... (10)}$$

PROOF : Using the polynomial formula

$$\left(\sum_{i=1}^{n} x_i \right)^m = \sum_{k_1 + k_2 + \ldots + k_n = m} \frac{m!}{k_1 ! k_2 ! \ldots k_n !} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}, \qquad \text{... (11)}$$

on the ground of formulas (7) and (3) we conclude that

$$\sigma(n, k) = \frac{1}{2k} \sum_{d \mid k} \varphi(d)\, coef_{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}} \left(x_1^d + x_2^d + \ldots + x_n^d \right)^{k/d}$$

$$= \frac{1}{2k} \sum_{d \mid k} \varphi(d) \sum \frac{(k/d)!}{k_1 ! k_2 ! \ldots k_n !}, \qquad \text{... (12)}$$

where the sum is taken over all solutions $k_1, k_2, ..., k_n$ of equations

$$k_1 + k_2 + \ldots + k_n = \frac{k}{d},$$

$$k_1 + 2k_2 + \ldots + nk_n = n. \qquad \text{... (13)}$$

The formula (10) follows from formulas (12), (13) and Lemma 1 □

Lemma 4 —

$$\theta(n, k) = \frac{1}{2} \binom{\left[\frac{n - h_k}{2}\right]}{\left[\frac{k}{2}\right]}, \qquad \text{... (14)}$$

where $h_k \equiv k (\text{mod } 2)$, $h_k = 0$ or 1.

PROOF : The treatment depends on parity of k.

(1) Let $k \geq 1$ is odd. In accordance to (8) and (4)

$$\theta(n, k) = \frac{1}{2} \sum coef_{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}} \sum_{i=1}^{n} x_i$$

$$\sum \frac{\left(\frac{k-1}{2}\right)!}{j_1!j_2!\dots j_n!} x_1^{2j_1} x_2^{2j_2} \dots x_n^{2j_n}, \quad \dots (15)$$

where by (11) the internal summing is taken over $j_1, j_2, \dots, j_n$ are satisfying the condition

$$j_1 + j_2 + \dots + j_n = \frac{k-1}{2}, \quad \dots (16)$$

and the external summing is taken over $k_1, k_2, \dots, k_n$ are satisfying the conditions (5). The sum of exponents of each monomial of $\sum_{i=1}^{n}$, that is $x_i x_1^{2j_1} \dots x_n^{2j_n}$ is equal to $1 + 2\,(j_1 + j_2 + \dots + j_n) = 1 + k - 1 = k$ that is satisfying the first condition of (5).

Further let's equate to n the sum $k_1 + 2k_2 + \dots + nk_n$ corresponds to the ith addend of this sum:

$$2j_1 + 2(2j_2) + 3(2j_3) + \dots + (i-1)\,(2j_{i-1})$$

$$+ i(2j_i + 1) +)i + 1)\,(2j_{i+1}) + \dots + n(2j_n) = n,$$

whence

$$j_1 + 2j_2 + \dots + nj_n = \frac{n-i}{2}. \quad \dots (17)$$

From here it follows that i must have the same parity as n and $j_1, j_2, \dots, j_n$ must satisfy the conditions (16) and (17).

In the total from (15) and for each odd $k \geq 1$ follows

$$\sigma(n, k) = \frac{1}{2} \sum_{i \equiv n \,(\mathrm{mod}\, 2)} \sum \frac{\left(\frac{k-1}{2}\right)!}{j_1!j_2!\dots j_n!}. \quad \dots (18)$$

where the internal summing is taken over all solutions $j_1, j_2, \dots, j_n$ of eqs. (16) and (17).

Comparing the conditions (16) and (17) with (5), we conclude that by (9) the internal sum in (18) is

$$\begin{pmatrix} \frac{n-i}{2} - 1 \\ \frac{k-1}{2} - 1 \end{pmatrix}.$$

Therefore supposing in (18) $n - i = 2t$, we find

$$\theta(n,k) = \frac{1}{2}\sum_{t=0}^{\left[\frac{n-1}{2}\right]} \begin{pmatrix} t-1 \\ \frac{k-3}{2} \end{pmatrix}, \ k \geq 1 \qquad \text{... (19)}$$

(in case of $k = 1$ the first addend $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ is supposed equal to 1, the remaining addends are equal to 0).

However, by induction it is easy to show that for $m \geq -1, p \geq 0$

$$\sum_{t=0}^{p} \begin{pmatrix} t-1 \\ m \end{pmatrix} = \begin{pmatrix} p \\ m+1 \end{pmatrix}, \qquad \text{... (20)}$$

and then it follows from (19) that for odd $k \geq 1$

$$\theta(n,k) = \frac{1}{2}\begin{pmatrix} \left[\frac{n-1}{2}\right] \\ \left[\frac{k-1}{2}\right] \end{pmatrix},$$

as it is required.

(2) Now let $k \geq 2$ is even. As far as

$$\left(\sum_{i=1}^{n} x_i\right)^2 + \sum_{i=1}^{n} x_i^2 = 2\left(\sum_{i=1}^{n} x_i^2 + \sum_{i \leq i_1 < i_2 \leq n} x_{i_1} x_{i_2}\right),$$

that as well as above we receive

$$\theta(n,k) = \frac{1}{2}\left(\sum_{i=1}^{n/2} \sum_{1} \frac{\left(\frac{k}{2}-1\right)!}{j_1!\ldots j_n!} + \sum_{\substack{1 \leq i_1 < i_2 \leq n \\ i_1 + i_2 \equiv n \pmod 2}} \sum_{2} \frac{\left(\frac{k}{2}-1\right)!}{l_1!\ldots l_2!}\right), \qquad \text{... (21)}$$

where in $\sum_1$ the sum is taken over all solutions $j_1, j_2, \ldots, j_n$ of the equations

$$j_1 + j_2 + \ldots + j_n = \frac{k}{2} - 1,$$

$$j_1 + 2j_2 + \ldots + nj_n = \frac{n}{2} - i, \qquad \text{... (22)}$$

and therefore $\sum_1 = 0$ if n is odd, and in $\sum_2$ the sum is taken over all solutions $l_1, l_2, \ldots, l_n$ of equations

$$l_1 + l_2 + \ldots + l_n = \frac{k}{1} - 1,$$

$$l_1 + 2l_2 + \ldots + nl_n = \frac{n - i_1 - i_2}{2}, \qquad \ldots (23)$$

and therefore $i_1 + i_2$ must have the same parity as n.

At the beginning let n be odd. Therefore, by (21) and (9)

$$\theta(n,k) = \frac{1}{2} \sum_{\substack{1 \le i_1 < i_2 \le n \\ i_1 + i_2 \equiv 1 \pmod 2}} \binom{\frac{n - i_1 - i_2}{2} - 1}{\frac{k}{2} - 1}$$

$$= \frac{1}{2} \sum_{\substack{j=3 \\ j \text{ is odd}}}^{n-k+2} \binom{\frac{n-j}{2} - 1}{\frac{k}{2} - 2} \frac{j-1}{2}, \qquad \ldots (24)$$

where the factor $\frac{j-1}{2}$ corresponds to the number of representations of odd j as sum of two positive integers ($j = 1 + (j - 1) = 2 + (j - 2) = \ldots$).

Supposing in (24) $n - j = 2(t + 1)$, we find

$$\theta(n,k) = \frac{1}{2} \sum_{t=\frac{k-4}{2}}^{\frac{n-5}{2}} \binom{t}{\frac{k}{2} - 2} \left(\frac{n-1}{2} - (t+1)\right)$$

$$= \frac{n-1}{4} \sum_{t=\frac{k-4}{2}}^{\frac{n-5}{2}} \binom{t}{\frac{k}{2} - 2} - \frac{k-2}{4} \sum_{t=\frac{k-4}{2}}^{\frac{n-5}{2}} \binom{t+1}{\frac{k}{2} - 1}$$

$$= \frac{n-1}{4} \sum_{t=\frac{k-4}{2}}^{\frac{n-5}{2}} \binom{t}{\frac{k}{2} - 2} - \frac{k-2}{4} \sum_{t=\frac{k-2}{2}}^{\frac{n-3}{2}} \binom{t}{\frac{k}{2} - 1}. \qquad \ldots (25)$$

Finally, using the identity easily received from (20)

$$\sum_{t=b}^{c} \binom{t}{a} = \binom{c+1}{a+1} - \binom{b}{a+1}, \qquad \text{... (26)}$$

from (25) we find:

$$\theta(n,k) = \frac{n-1}{4}\binom{\frac{n-3}{2}}{\frac{k}{2}-1} - \frac{k-2}{4}\binom{\frac{n-1}{2}}{\frac{k}{2}} = \frac{k}{4}\binom{\frac{n-1}{2}}{\frac{k}{2}}$$

$$-\frac{k-2}{4}\binom{\frac{n-1}{2}}{\frac{k}{2}} = \frac{1}{2}\binom{\frac{n-1}{2}}{\frac{k}{2}},$$

as it is required.

Let n be even. Then from (21) and by (9) we have

$$\theta(n,k) = \frac{1}{2}\sum_{i=1}^{n/2}\binom{\frac{n}{2}-i-1}{\frac{k}{2}-2} + \frac{1}{2}\sum_{\substack{1 \le i_1 < i_2 \le n \\ i_1+i_2 \equiv 1 \,(\mathrm{mod}\, 2)}}\binom{\frac{n-i_1-i_2}{2}-1}{\frac{k}{2}-2}$$

$$= \frac{1}{2}\sum_{t=0}^{\frac{n}{2}-1}\binom{t-1}{\frac{k}{2}-2} + \frac{1}{2}\sum_{\substack{j=4 \\ j \text{ is even}}}^{n-k+2}\binom{\frac{n-j}{2}-1}{\frac{k}{2}-2}\frac{j-2}{2}, \qquad \text{... (27)}$$

where the factor $\frac{j-2}{2}$ corresponds to the number of representations even j as sum of two positive integers $i_1 < i_2$.

It follows from (20) that

$$\sum_{t=0}^{\frac{n}{2}-1}\binom{t-1}{\frac{k}{2}-2} = \binom{\frac{n}{2}-1}{\frac{k}{2}-1}. \qquad \text{... (28)}$$

Supposing that, in the second sum of (27), $n-j = 2(t+1)$, by (28) we find

$$\theta(n,k) = \frac{1}{2}\sum_{t=\frac{k-4}{2}}^{\frac{n-6}{2}} \binom{t}{\frac{k}{2}-2}\left(\frac{n-2}{2}-(t+1)\right) + \frac{1}{2}\binom{\frac{n}{2}-1}{\frac{k}{2}-1}$$

$$= \frac{n-2}{4}\sum_{t=\frac{k-4}{2}}^{\frac{n-6}{2}} \binom{t}{\frac{k}{2}-2} - \frac{k-2}{4}\sum_{t=\frac{k-4}{2}}^{\frac{n-6}{2}} \binom{t+1}{\frac{k}{2}-1}$$

$$+ \frac{1}{2}\binom{\frac{n}{2}-1}{\frac{k}{2}-1} = \frac{n-2}{4}\sum_{t=\frac{k-4}{2}}^{\frac{n-6}{2}} \binom{t}{\frac{k}{2}-2} - \frac{k-2}{4}\sum_{t=\frac{k-2}{2}}^{\frac{n-4}{2}} \binom{t}{\frac{k}{2}-1}$$

$$+ \frac{1}{2}\binom{\frac{n}{2}-1}{\frac{k}{2}-1}.$$

Now using (26), we finally find

$$\theta(n,k) = \frac{n-2}{4}\binom{\frac{n-4}{2}}{\frac{k}{2}-1} - \frac{k-2}{4}\binom{\frac{n-2}{2}}{\frac{k}{2}} + \frac{1}{2}\binom{\frac{n-2}{2}}{\frac{k}{2}-1}$$

$$= \frac{k}{4}\binom{\frac{n-2}{2}}{\frac{k}{2}} - \frac{k-2}{4}\binom{\frac{n-2}{2}}{\frac{k}{2}} + \frac{1}{2}\binom{\frac{n-2}{2}}{\frac{k}{2}-1}$$

$$= \frac{1}{2}\binom{\frac{n-2}{2}}{\frac{k}{2}} + \frac{1}{2}\binom{\frac{n-2}{2}}{\frac{k}{2}-1} = \frac{1}{2}\binom{\frac{n}{2}}{\frac{k}{2}}.$$

Lemma 4 is proved completely.

Theorem 1 now follows from formula (6), Lemma 3 and Lemma 4 □.

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