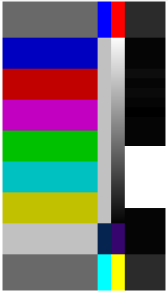
Dynamical algebraic combinatorics and homomesy: An action-packed introduction



Tom Roby (UConn)

AlCoVE: an Algebraic Combinatorics Virtual Expedition
(Hosted on Zoom)

15 June 2020

This talk is being recorded! (You have been warned.)

Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

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Abstract

Abstract: Dynamical Algebraic Combinatorics explores actions on sets of discrete combinatorial objects, many of which can be built up by small local changes, e.g., Schützenberger's promotion and evacuation, or the rowmotion map on order ideals. There are strong connections to the combinatorics of representation theory and with Coxeter groups. Some of these actions can be extended to piecewise-linear maps on polytopes, then detropicalized to the birational setting. Here the dynamics have the flavor of cluster algebras, but this connection is still relatively unexplored.

The term "homomesy" describes the following widespread phenomenon: Given a group action on a set of combinatorial objects, a statistic on these objects is called "homomesic" if its average value is the same over all orbits. Along with its intrinsic interest as a kind of "hidden invariant", homomesy can be used to help understand certain properties of the action. This notion can be lifted to the birational setting, and the resulting identities are somewhat surprising. Proofs of homomesy often involve developing tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection.

This talk will be a introduction to these ideas, giving a number of examples of such actions and pointing out connections to other areas.

Acknowledgments

This talks discusses joint work, mostly with Jim Propp and Mike Joseph.

I'm grateful to Mike Joseph and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

Thanks also to Drew Armstrong, Arkady Berenstein, Anders Björner, Barry Cipra, Karen Edwards, Robert Edwards, David Einstein, Darij Grinberg, Shahrzad Haddadan, Sam Hopkins, Mike La Croix, Svante Linusson, Gregg Musiker, Nathan Williams, Vic Reiner, Jessica Striker, Richard Stanley, Ralf Schiffler, Hugh Thomas, and Ben Young.

Please feel free to put questions and comments in the chat, and the moderator will convey them with appropriate timing and finese. Or someone else may answer them!

Some themes in Dynamical Algebraic Combinatorics

1. Periodicity/order;
2. Orbit structure;
3. Homomesy;
4. Equivariant bijections; and
5. Lifting from combinatorial to piecewise-linear and birational settings.

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \overset{C_R}{\longmapsto} 010100$$

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Example

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$$101000 \underset{C_R}{\longmapsto} 010100$$

- *Periodicity* is clear here. The map has order $n = 6$.
- *Orbit structure* is very nice; every orbit size must divide n.
- *Homomesy?* Need a statistic, first.
- *Equivariant bijection?* No need.

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$*.*

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$*.*

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ∎

Example

$n = 6,\ k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
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Cyclic rotation of binary strings

Example

$n = 6, \ k = 2$

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100010	5	000011	0	010010	4
010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\mathbb{1}_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Coxeter Toggling

Independent Sets

of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the factor 11.

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases} .$$

Toggles

Proposition

Each toggle τ_i is an involution, i.e., τ_i^2 is the identity. Also, τ_i and τ_j commute if and only if $|i-j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Order & Orbits

- The order of this action grows quite fast as n increases and is difficult to describe in general. It is the LCM of the orbit sizes, which are not all divisors of some small number (relative to n):
  2, 3, 6, 15, 24, 231, 210, 1989, 240, 72105, 18018, 3354725, 3360
- For $n = 6$ orbit sizes are 3, 7, and 11, so order is LCM(3,7,11)$=$ 231.
- The number of orbits appeared to be OEIS A000358 , but we didn't understand why at first.
- This means that this action is unlikely to exhibit interesting Cyclic Sieving.
- But we can still find homomesy.

Homomesy

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

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	1	**2**	**3**	**4**	**5**	**6**	**7**
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$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph–R. [JR18])

Define $\mathbb{1}_i : \mathcal{I}_n \to \{0,1\}$ to be the indicator function of vertex i.

For $1 \leq i \leq n$, $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic on φ-orbits of $\mathcal{I}_n$.

Also $2\mathbb{1}_1 + \mathbb{1}_2$ and $\mathbb{1}_{n-1} + 2\mathbb{1}_n$ are 1-mesic on φ-orbits of $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
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$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
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$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
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$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
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$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
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$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
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$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1) \bmod 4$.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Antichain Rowmotion on Posets

Rowmotion: an invertible operation on antichains

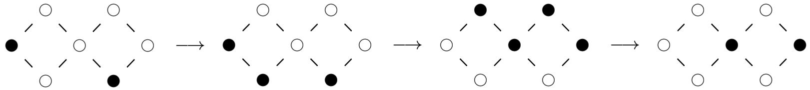
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the *downward-saturation* of A (the smallest downset containing A).

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

$\rho_{\mathcal{A}}$:



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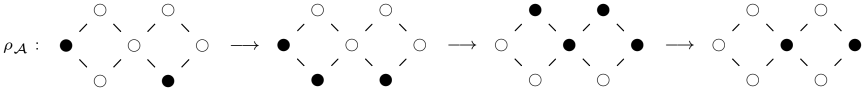
$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$



This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Picture of root posets

Here are the classes of posets included in Panyushev's conjecture.

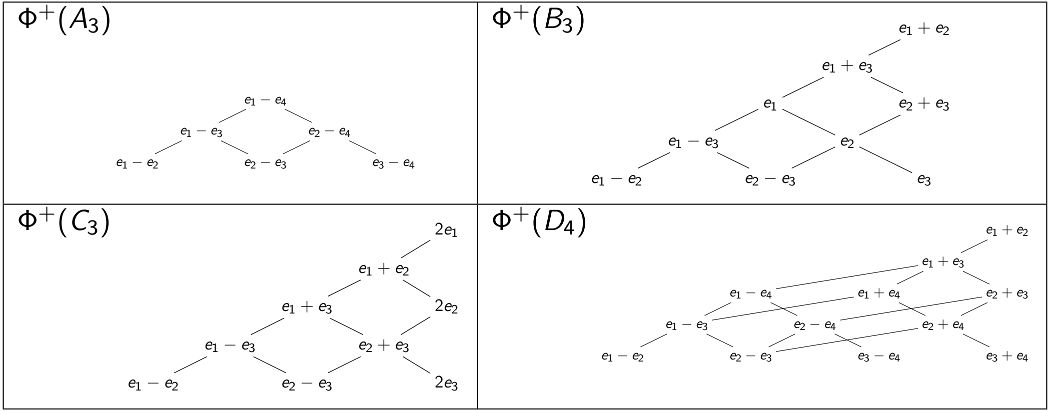
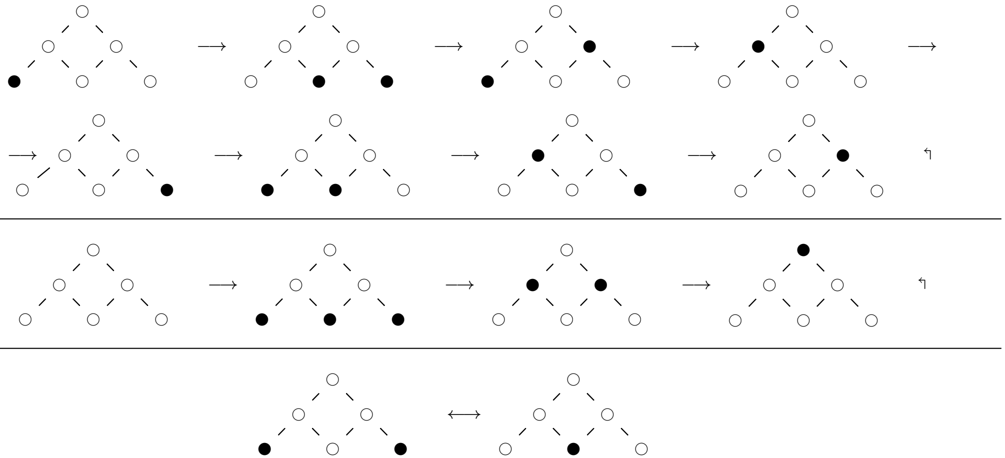


Figure: The positive root posets A_3, B_3, C_3, and D_4.

(Graphic courtesy of Striker-Williams.)

Example of antichain rowmotion on A_3 root poset

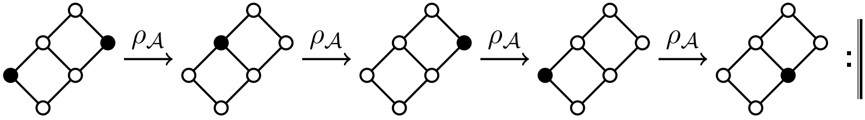
For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

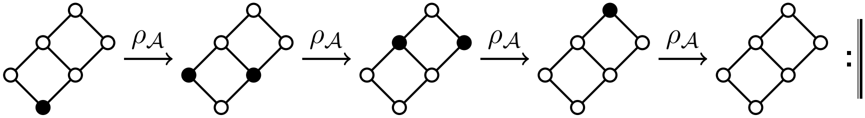
$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Orbits of rowmotion on antichains of $[2] \times [3]$



2 1 1 1 1

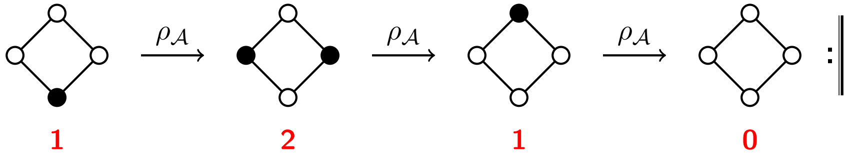
Average cardinality: 6/5

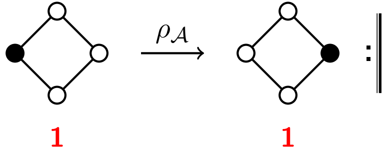


1 2 2 1 0

Average cardinality: 6/5

Orbits of rowmotion on antichains of $[2] \times [2]$





Average cardinality: 1

For antichain rowmotion on this poset, periodicity has been known for a long time:

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Antichains in $[a] \times [b]$: cardinality is homomesic

For rectangular posets $[a] \times [b]$ (the type A *minuscule* poset, where $[k] = \{1, 2, \ldots, k\}$), the homomesy is easier to show than for root posets.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{A \in \mathcal{O}} \#A = \dfrac{ab}{a+b}$.

Antichains in $[a] \times [b]$: cardinality is homomesic

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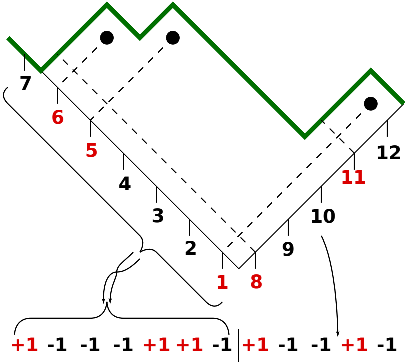

(Graphic courtesy of Ben Young.)

This proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between antichains in $[a] \times [b]$ and binary strings, which carries the $\rho_{\mathcal{A}}$ map to cyclic rotation of bitstrings.

The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

Antichains in $[a] \times [b]$: cardinality is homomesic

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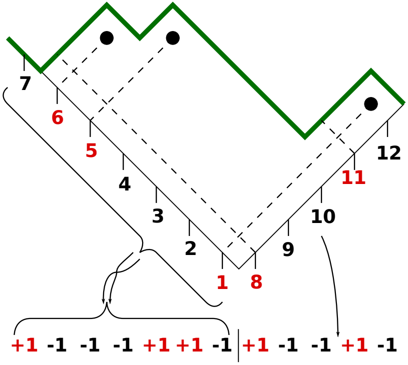

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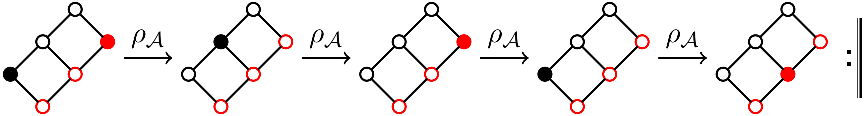
The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

This bijection also allowed Propp–R. to derive refined homomesy results for fibers and antipodal points in $[a] \times [b]$.

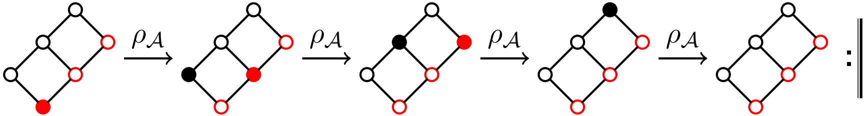
Orbits of rowmotion on antichains of $[2] \times [3]$

Look at the cardinalities across a **positive fiber** such as the one highlighted in red.



1 0 1 0 1

Average: 3/5

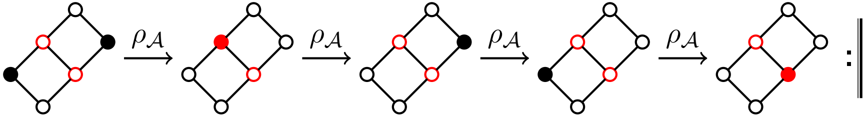


1 1 1 0 0

Average: 3/5

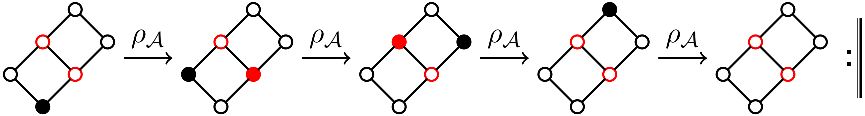
Orbits of rowmotion on antichains of $[2] \times [3]$

How about across a **negative fiber** such as the one highlighted in red.



0 **1** **0** **0** **1**

Average: 2/5



0 **1** **1** **0** **0**

Average: 2/5

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i,j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\mathbb{1}_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i,j).

Also, let $f_i(A) = \sum_{j \in [b]} \mathbb{1}_{i,j}(A) \in \{0,1\}$ (the cardinality of the intersection of A with the fiber $\{(i,1),(i,2),\ldots,(i,b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \mathbb{1}_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

For all i,j,

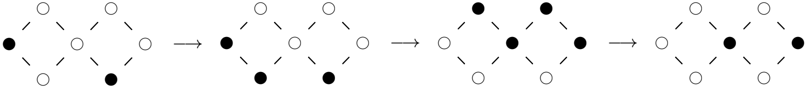
$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\mathbb{1}_{i,j}$ aren't.

Rowmotion on order ideals

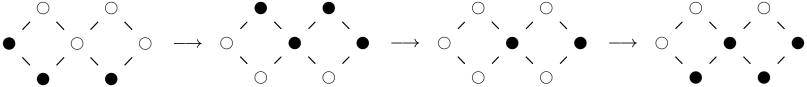
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:

$\rho_{\mathcal{A}}$:



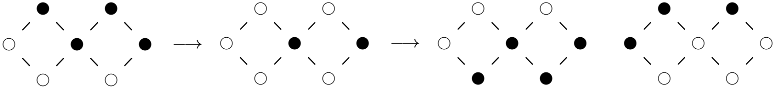
We can also define it as an operator ρ_J on $J(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:

ρ_J :



Or as an operator on the *up-sets (order filters)* $\mathcal{U}(P)$, of P:

$\rho_{\mathcal{U}}$:



Rowmotion via Toggling (Rowmotion in Slow motion)

Toggling Up-sets

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $\mathcal{U}(P)$ be the set of up-sets of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : \mathcal{U}(P) \to \mathcal{U}(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in \mathcal{U}(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in \mathcal{U}(P), \\ U & \text{otherwise.} \end{cases}$$

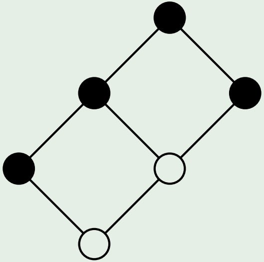
Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on up-sets of P.

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on up-sets of P.

Example



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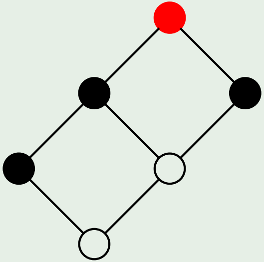

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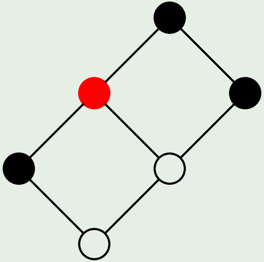

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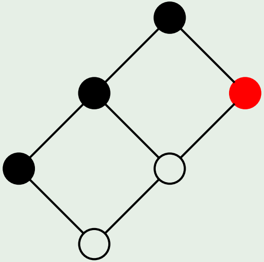

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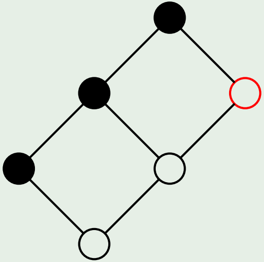

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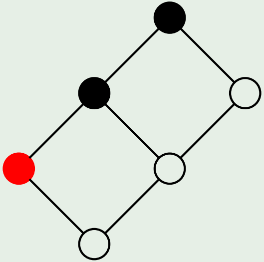

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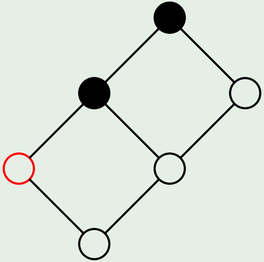

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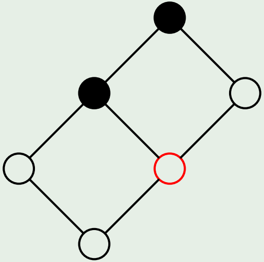

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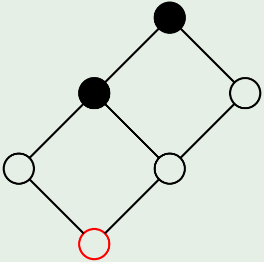

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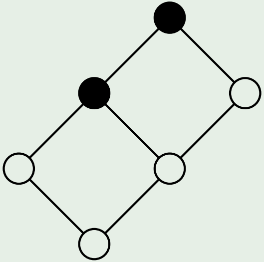


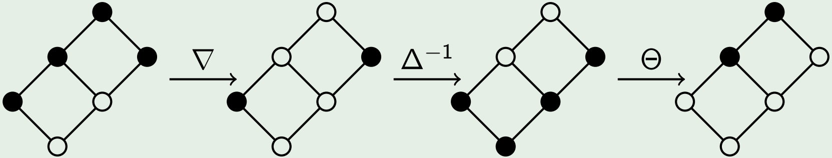
Rowmotion

We define the group action of **rowmotion** on the set of up-sets $\mathcal{U}(P)$ via the map $\mathrm{Row} : \mathcal{U}(P) \to \mathcal{U}(P)$ given by the following three step process.

Start with an up-set, and

1. ∇: Take the minimal elements (giving an antichain)
2. Δ^{-1}: Saturate downward (giving a down-set)
3. Θ: Take the complement (giving an up-set)

Example



Antichain toggles

Striker has generalized the notion of toggles relative to any set of “allowed” subsets, not necessarily up-sets.

Definition

Let $e \in P$. Then the **antichain toggle** corresponding to e is the map $\tau_e : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_e(A) = \begin{cases} A \cup \{e\} & \text{if } e \notin A \text{ and } A \cup \{e\} \in \mathcal{A}(P), \\ A \setminus \{e\} & \text{if } e \in A, \\ A & \text{otherwise.} \end{cases}$$

Let $\mathrm{Tog}_{\mathcal{A}}(P)$ denote the **toggle group** of $\mathcal{A}(P)$ generated by the toggles $\{\tau_e \mid e \in P\}$.

Theorem (Joseph 2017)

Applying the antichain toggles τ_e from bottom to top along a linear extension of P gives $\rho_{\mathcal{A}}$, rowmotion on antichains of P.

Antichain toggles

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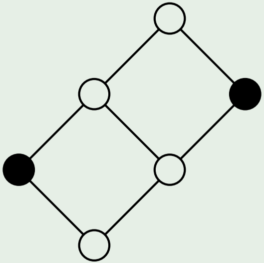

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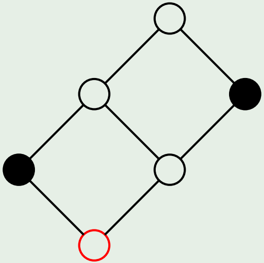

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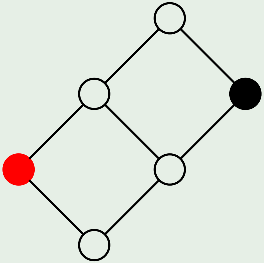

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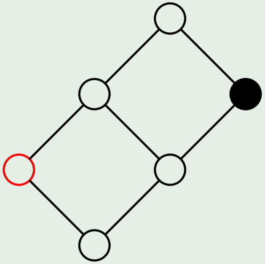

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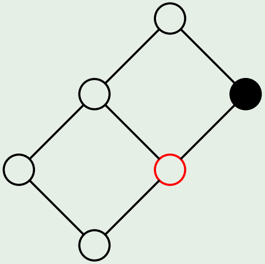

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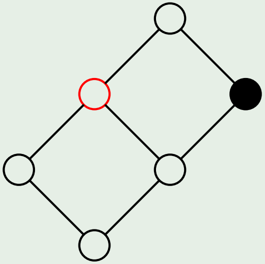

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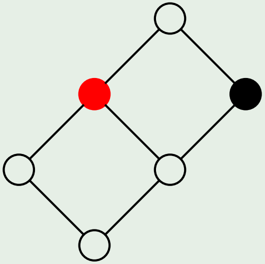

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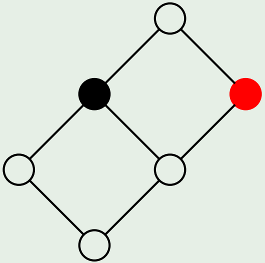

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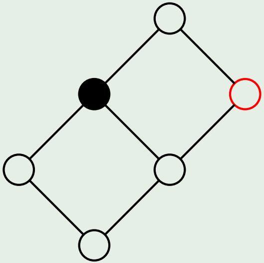

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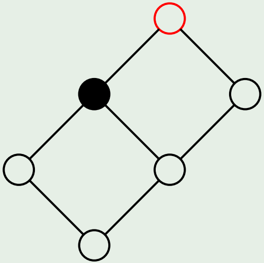

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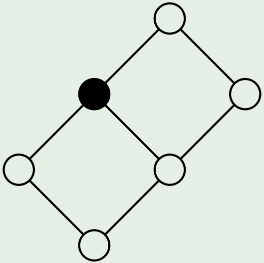
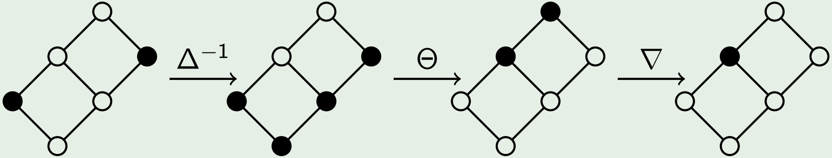


This gives the same result as the 3-step process

1. Δ^{-1}: Saturate downward (giving a down-set)
2. Θ: Take the complement (giving an up-set)
3. ∇: Take the minimal elements (giving an antichain)

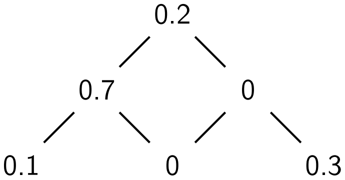
Example



Generalization to the piecewise-linear realm

Stanley (1986) defined some polytopes associated with posets.

- $\mathcal{C}(P)$ is the **chain polytope** of P, the set of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.
- $\mathcal{O}(P)$ is the **order polytope** of P, the set of all order-preserving labelings $f \in [0,1]^P$. Saying f is order-preserving means $f(x) \leq f(y)$ when $x \leq y$ in P.

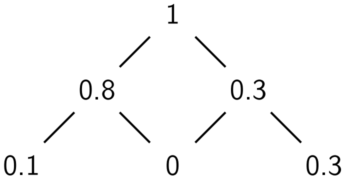


$\in \mathcal{C}(P)$

Generalization to the piecewise-linear realm

Stanley (1986) defined some polytopes associated with posets.

- $\mathcal{C}(P)$ is the **chain polytope** of P, the set of $f \in [0,1]^P$ such that $\sum_{i=1}^{n} f(x_i) \leq 1$ for all chains $x_1 < x_2 < \cdots < x_n$.
- $\mathcal{O}(P)$ is the **order polytope** of P, the set of all order-preserving labelings $f \in [0,1]^P$. Saying f is order-preserving means $f(x) \leq f(y)$ when $x \leq y$ in P.



$\in \mathcal{O}(P)$

Generalization to the piecewise-linear realm

- Up-sets of P correspond to elements of the order polytope $\mathcal{O}(P)$ for which every label is 0 or 1. These are the vertices of the order polytope.
- Antichains of P correspond to elements of the chain polytope $\mathcal{C}(P)$ for which every label is 0 or 1. These are the vertices of the chain polytope.

Generalization to the piecewise-linear realm

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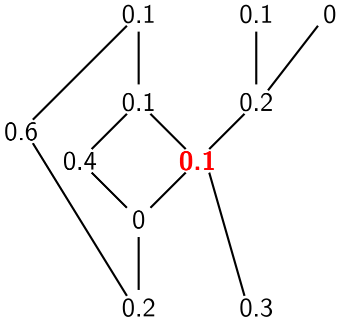
- Antichains of P correspond to elements of the chain polytope $\mathcal{C}(P)$ for which every label is 0 or 1. These are the vertices of the chain polytope.

Einstein and Propp have defined and analyzed **piecewise-linear** toggles on the order polytope that correspond exactly to up-set toggles when restricted to the vertices of the order polytope.

So why not define toggles on the chain polytope that correspond to antichain toggles when restricted to the vertices?

To define $\tau_e : \mathcal{C}(P) \to \mathcal{C}(P)$, given $g \in \mathcal{C}(P)$ and $e \in P$, $\tau_e(g)$ can only differ from g at the value of e.

$$\big(\tau_e(g)\big)(e) = 1 - \max\left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \dots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}$$



Toggles on the chain polytope $\mathcal{C}(P)$

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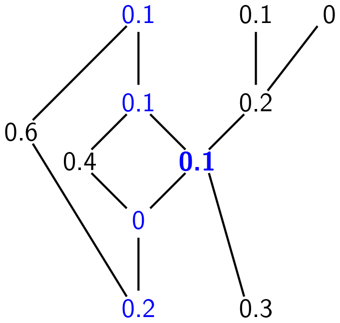

$$0.2 + 0 + 0.1 + 0.1 + 0.1 = 0.5$$

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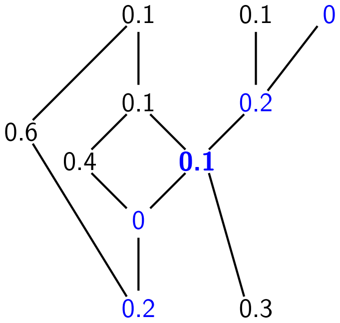

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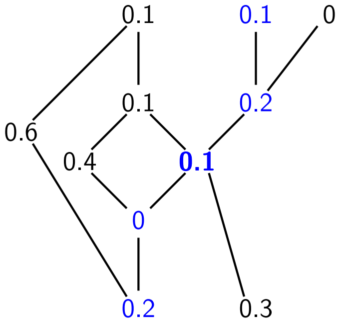

$$0.2 + 0 + 0.1 + 0.2 + 0.1 = 0.6$$

Toggles on the chain polytope $\mathcal{C}(P)$

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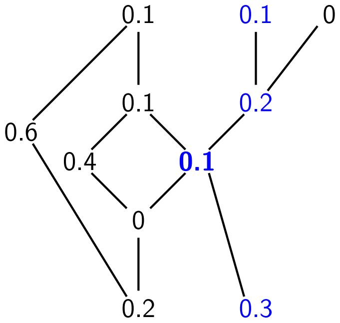

$$0.3 + 0.1 + 0.2 + 0.1 = 0.7$$

Toggles on the chain polytope $\mathcal{C}(P)$

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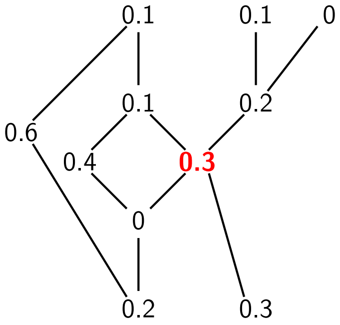
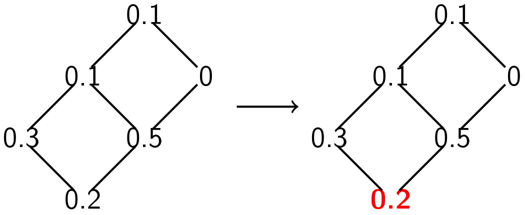
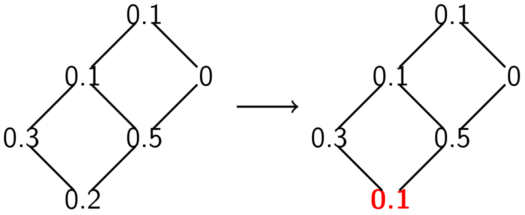


0.7 is max and $1 - 0.7 = 0.3$

PL Rowmotion on the chain polytope $\mathcal{C}(P)$



PL Rowmotion on the chain polytope $\mathcal{C}(P)$



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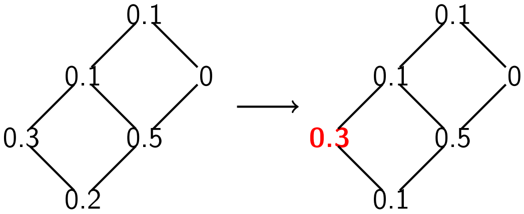

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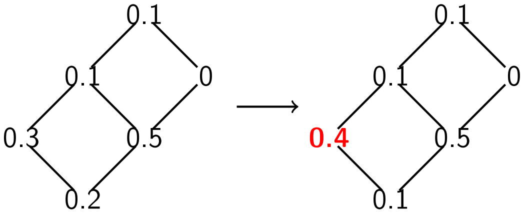

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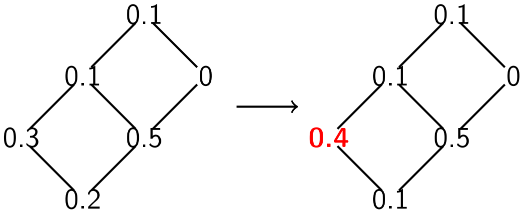

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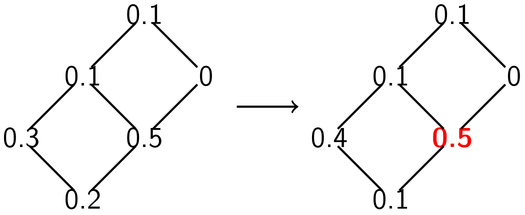

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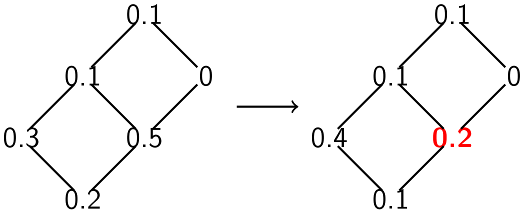

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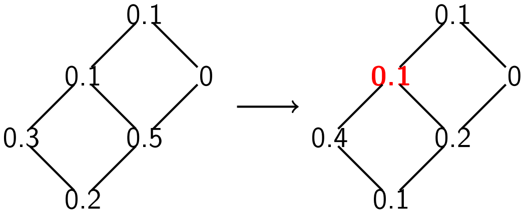

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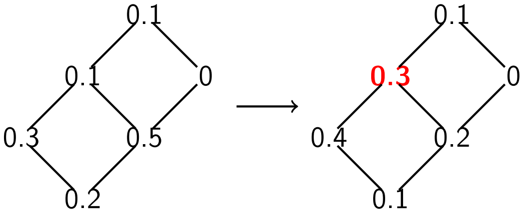

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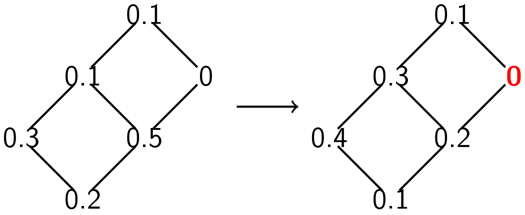

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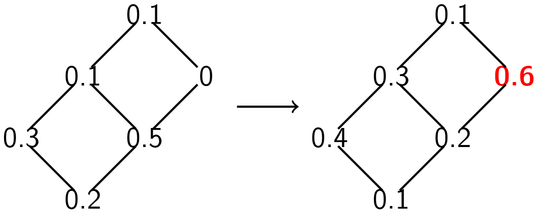

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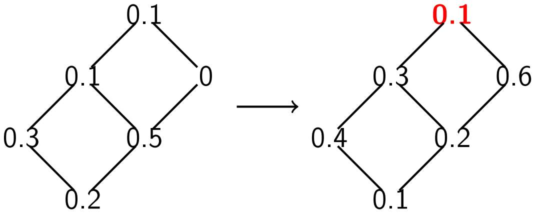

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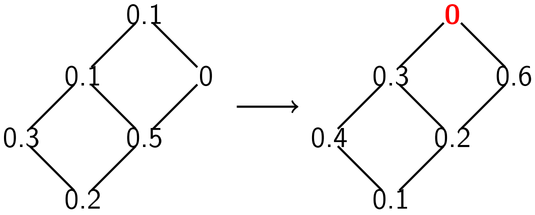

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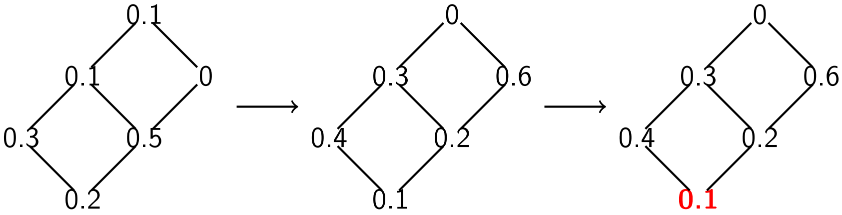

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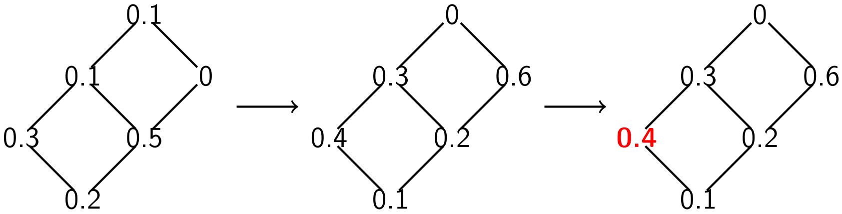

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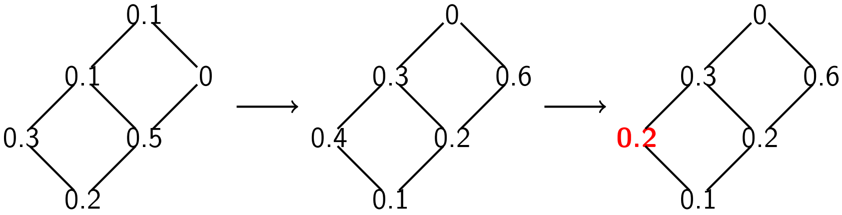

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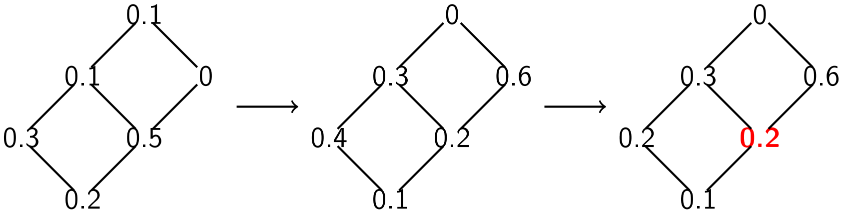

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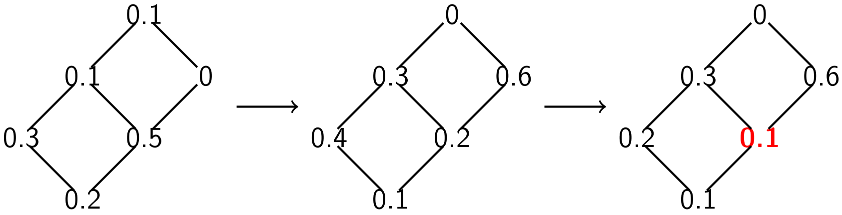

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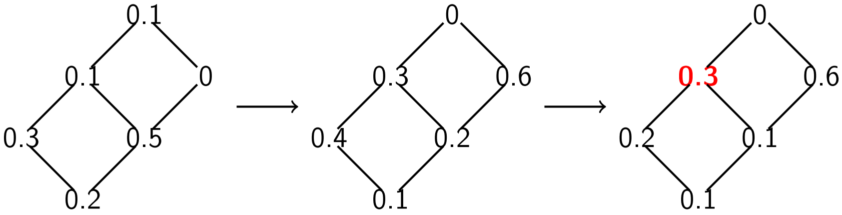

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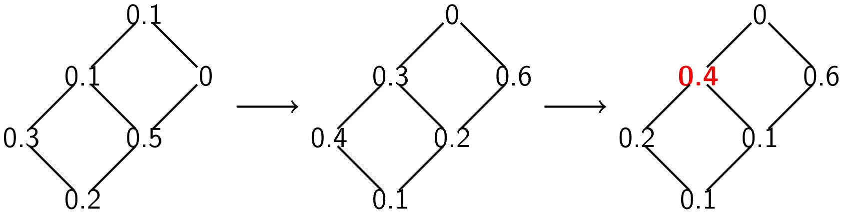

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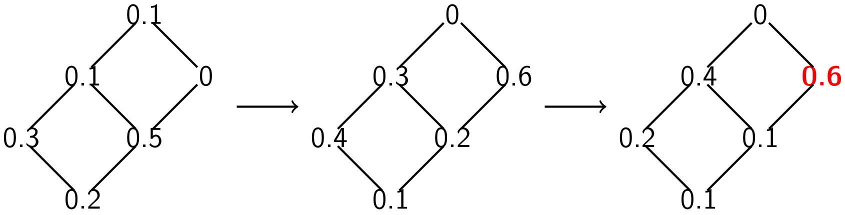

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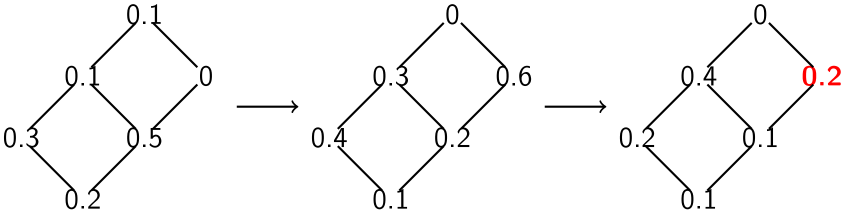

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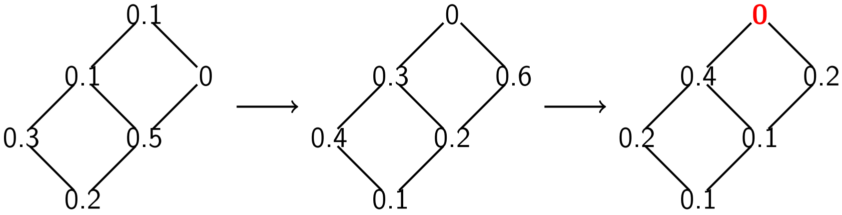

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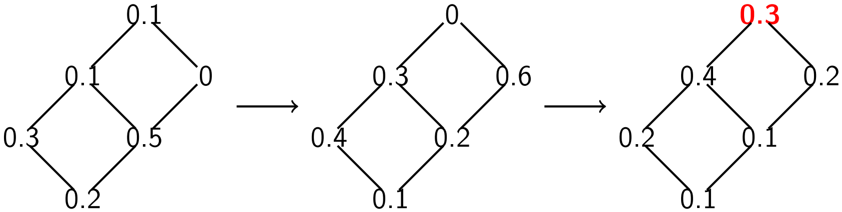

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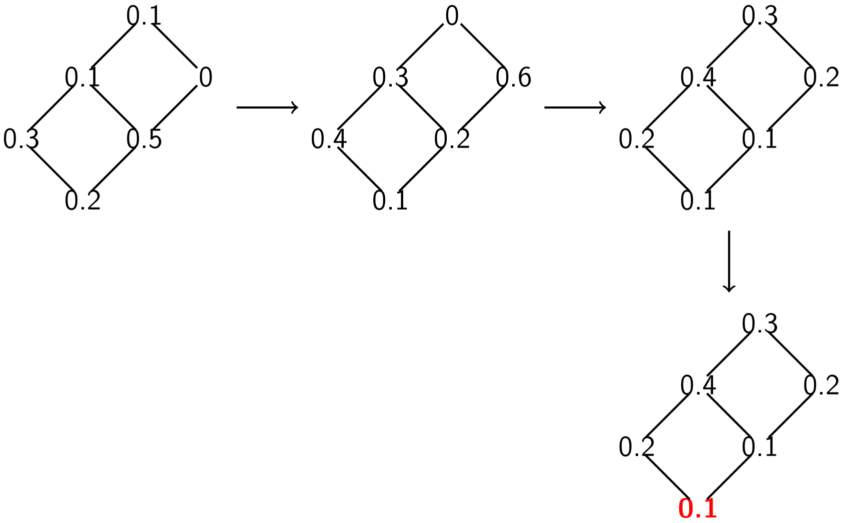

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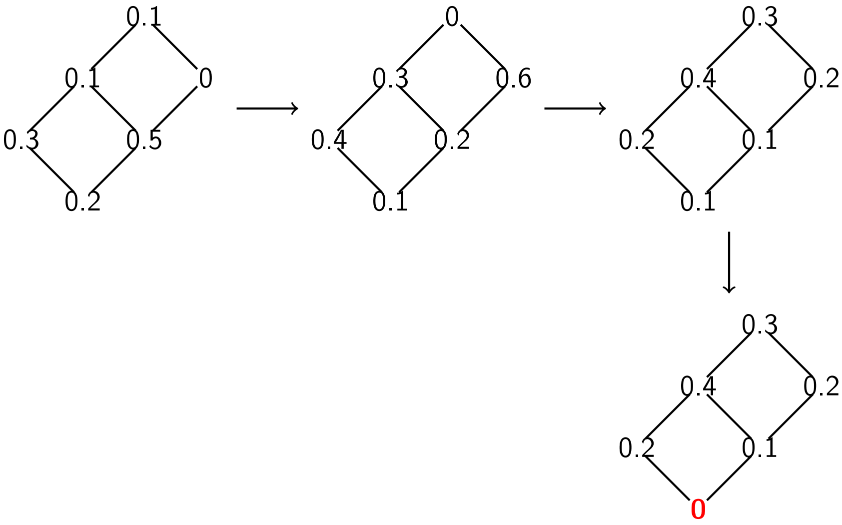

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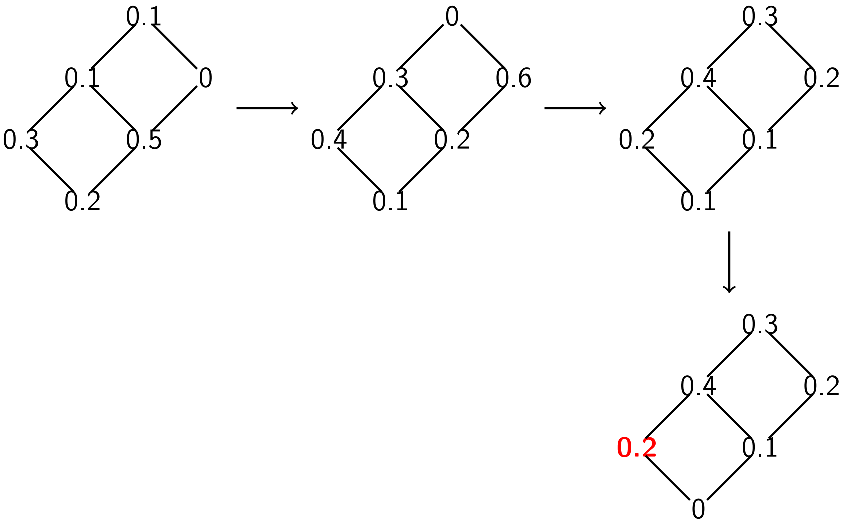

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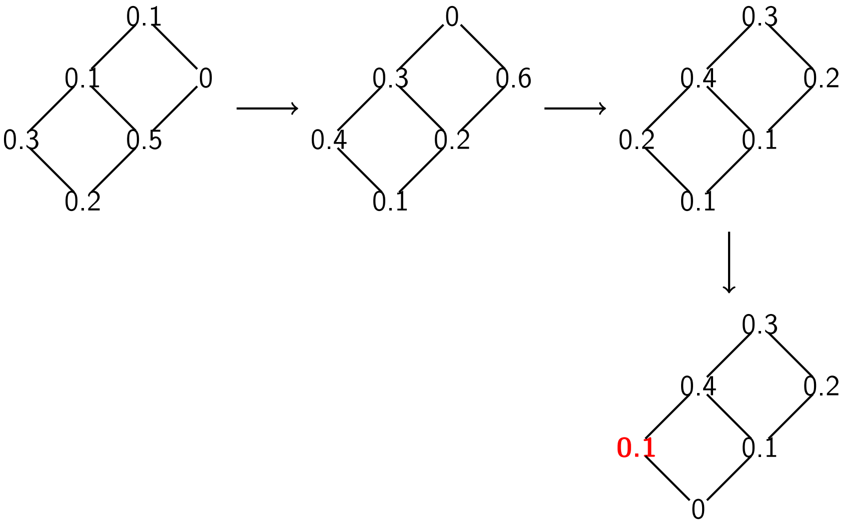

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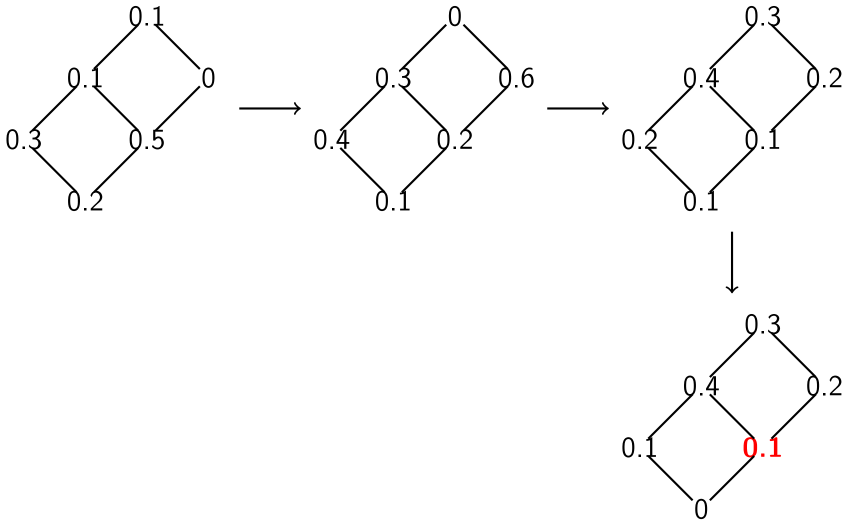

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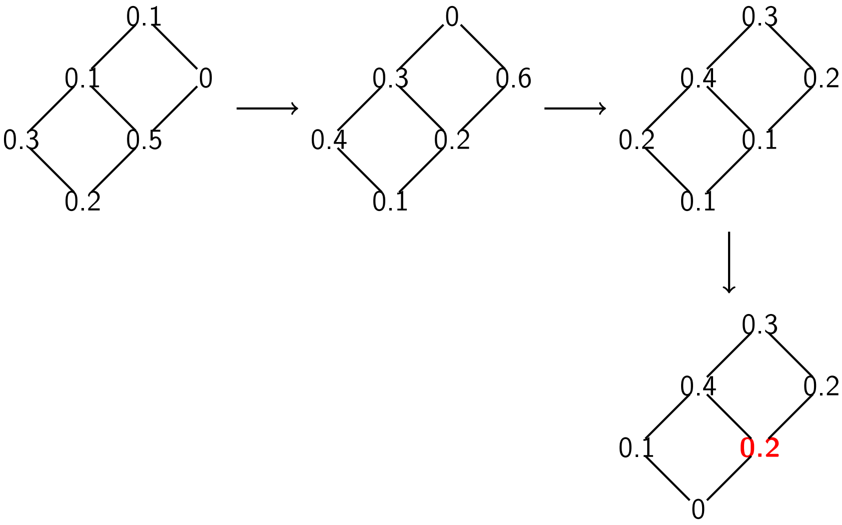

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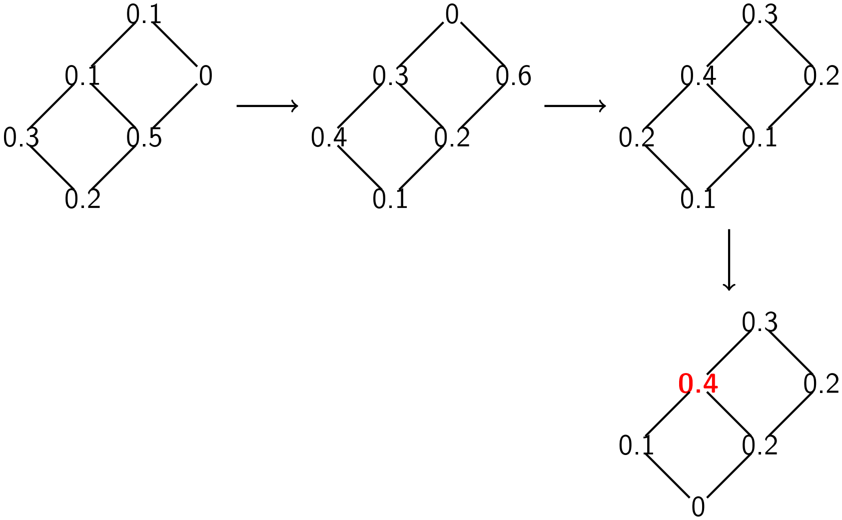

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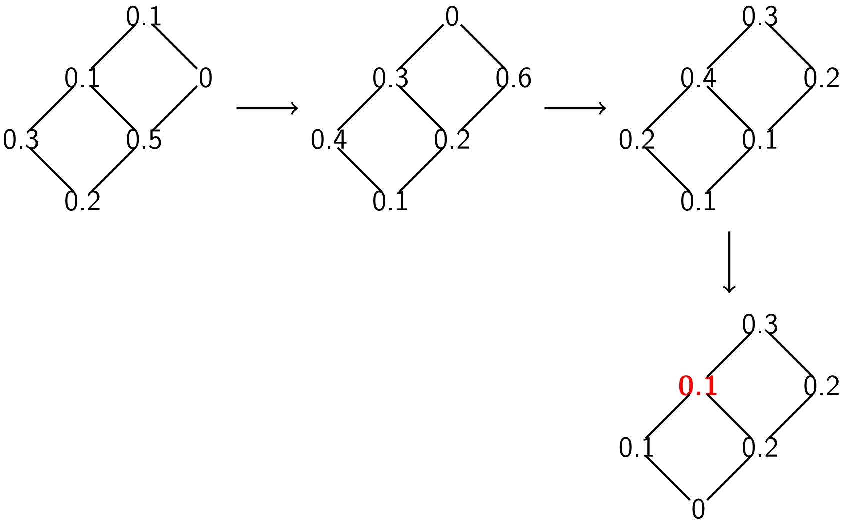

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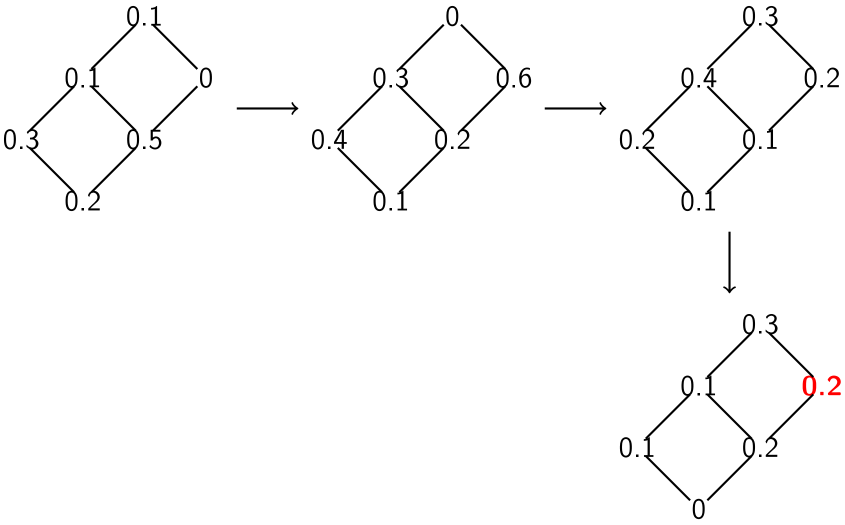

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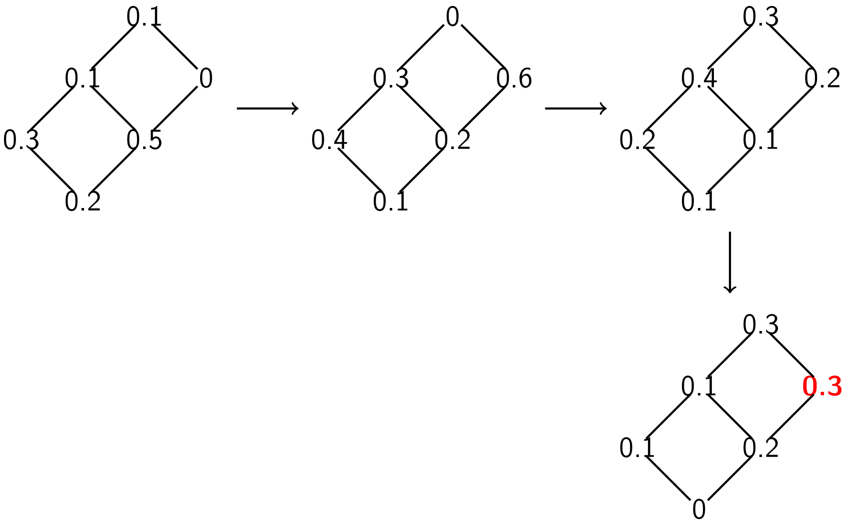

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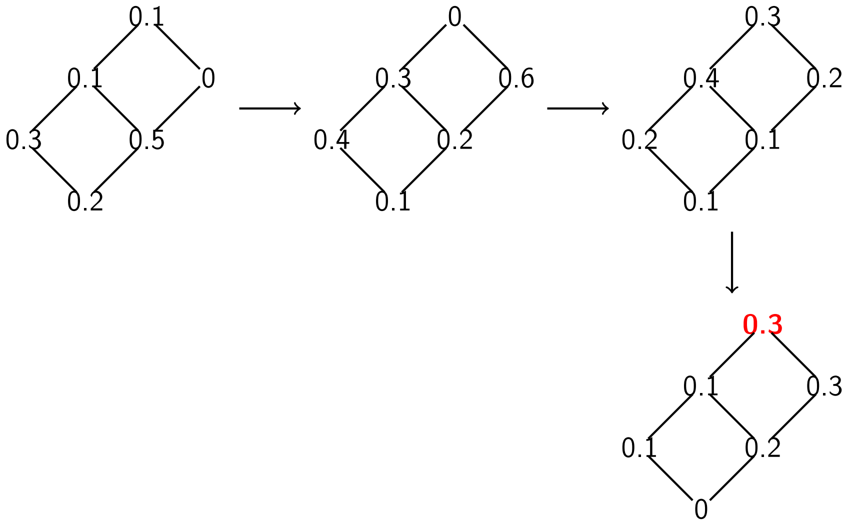

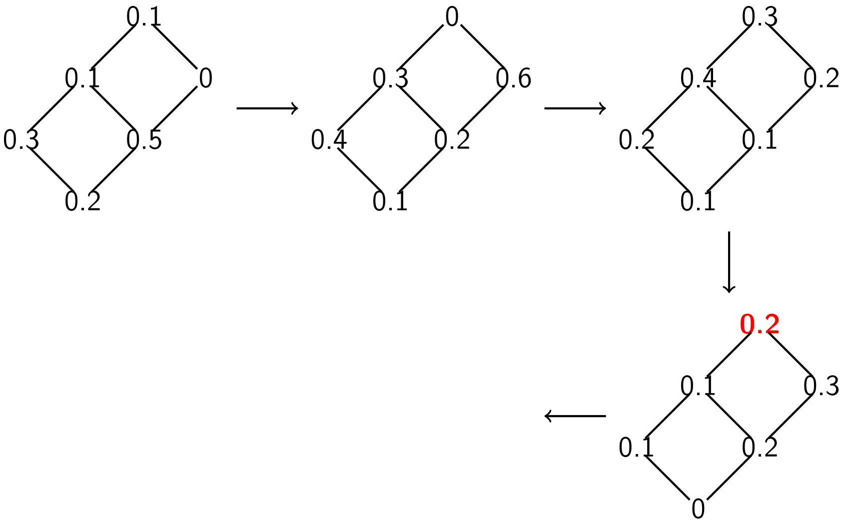
0.1
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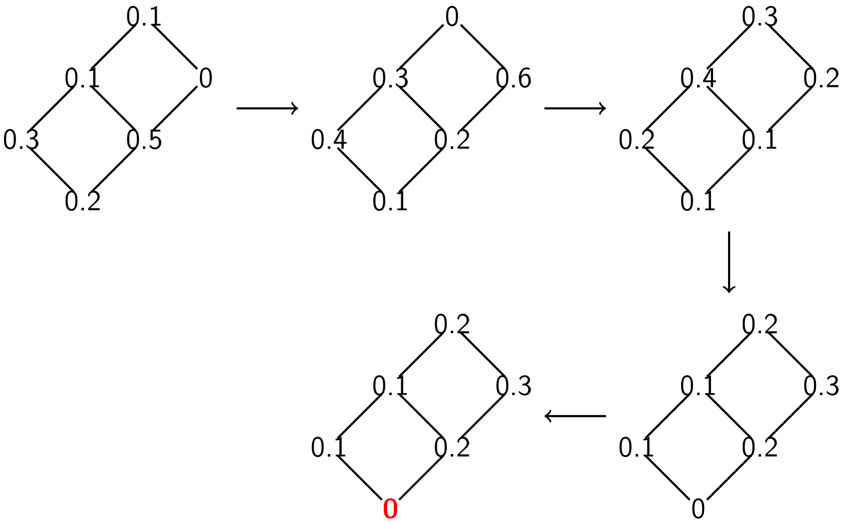
PL Rowmotion on the chain polytope $\mathcal{C}(P)$



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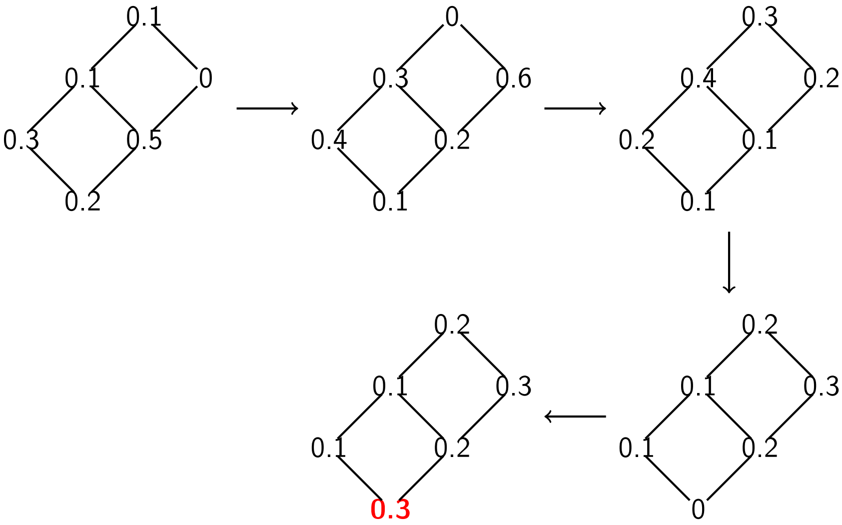

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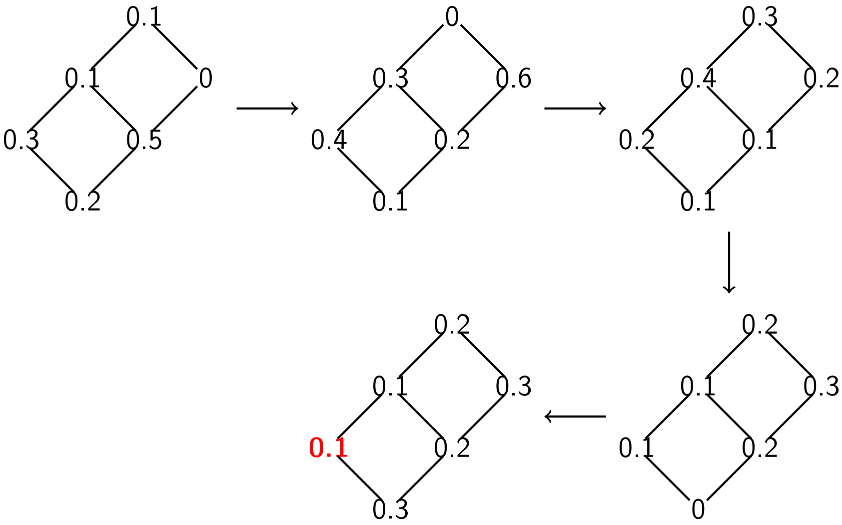

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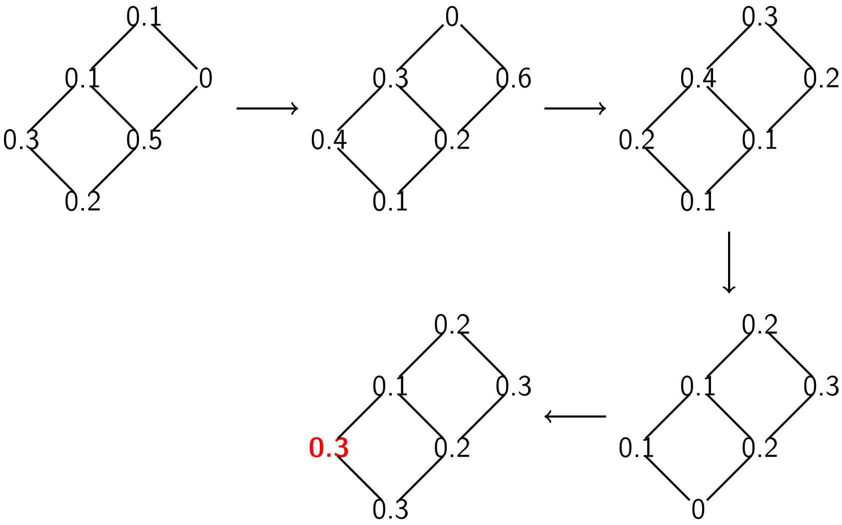

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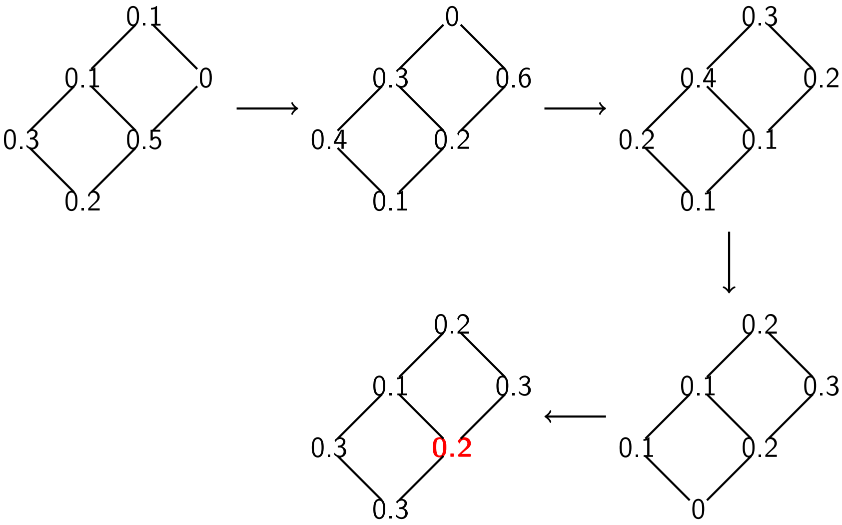

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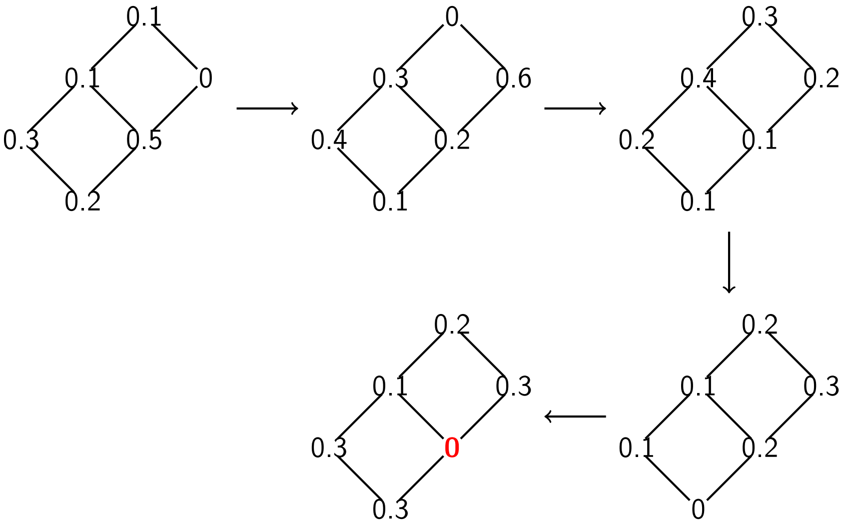

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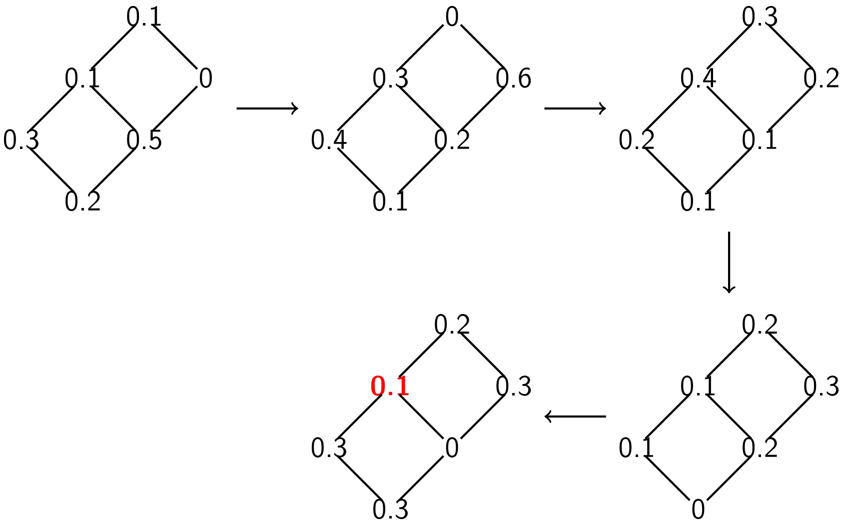

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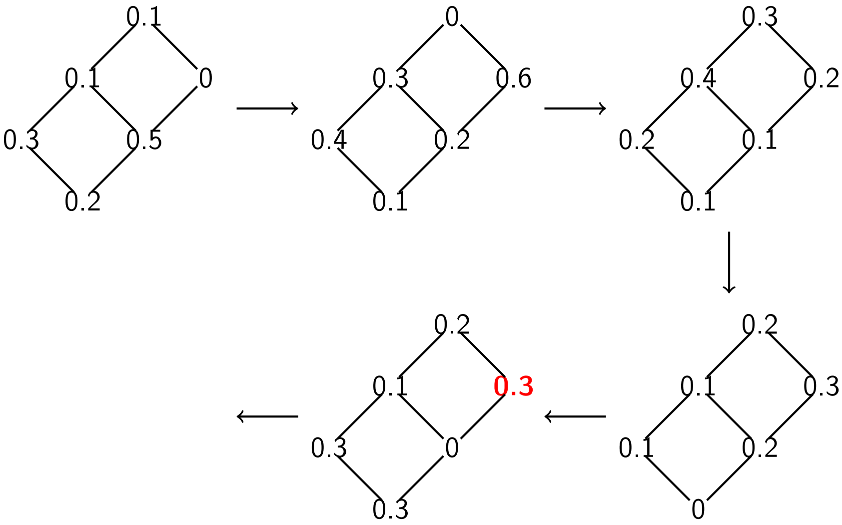

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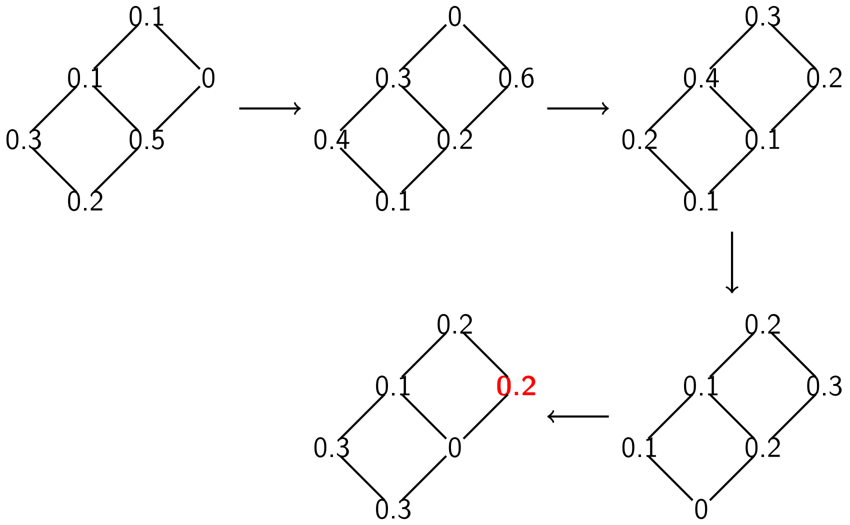

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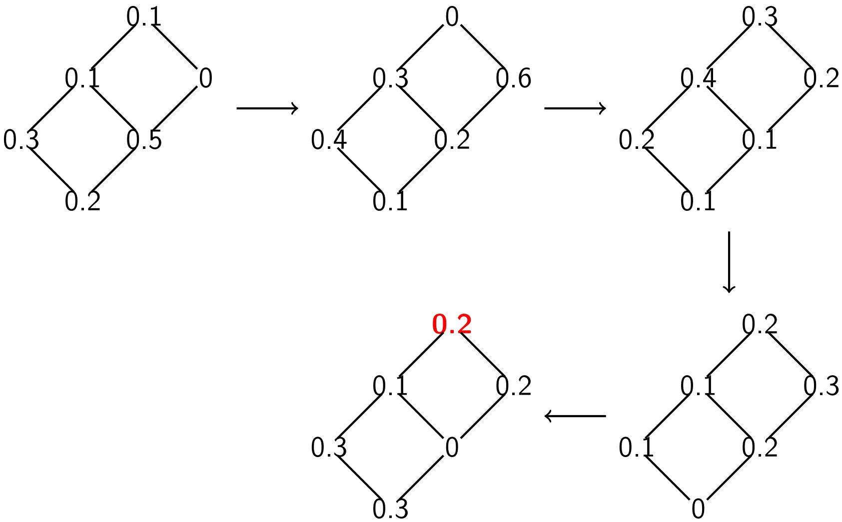

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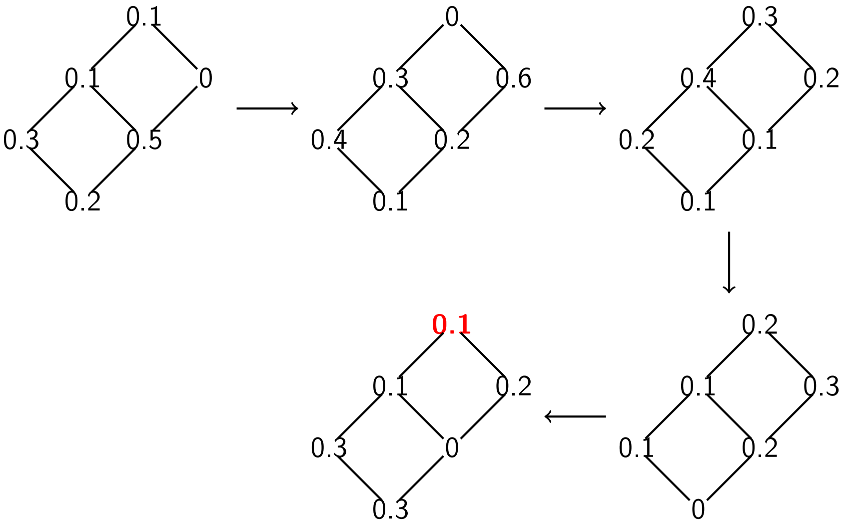
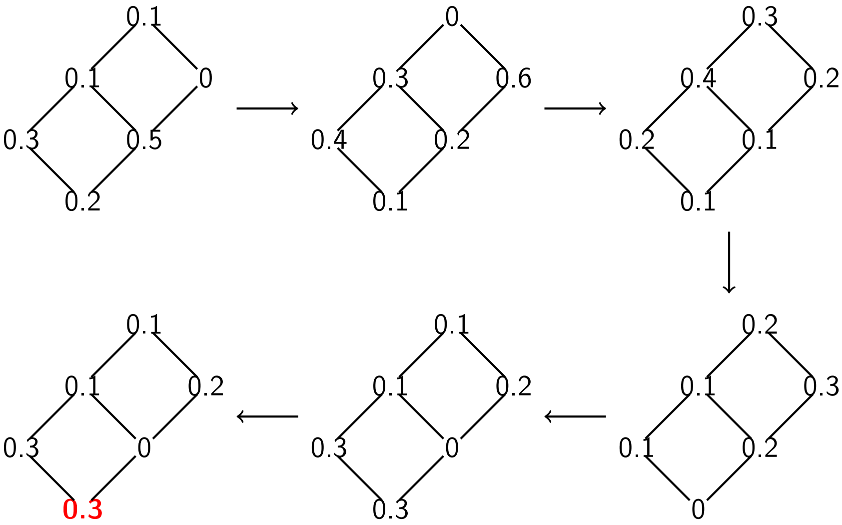


PL Rowmotion on the chain polytope $\mathcal{C}(P)$



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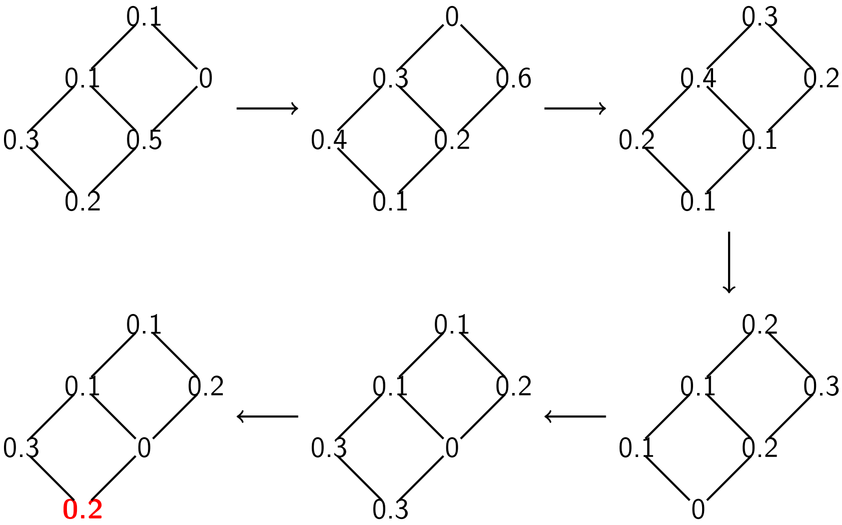

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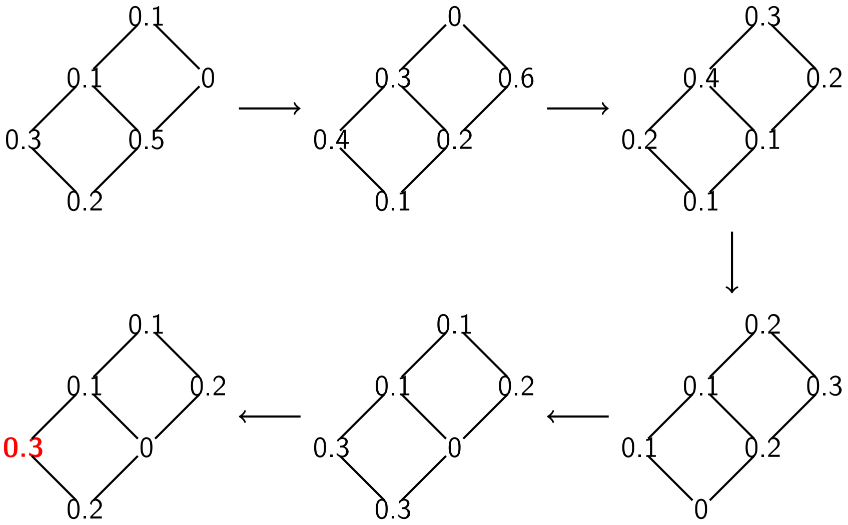

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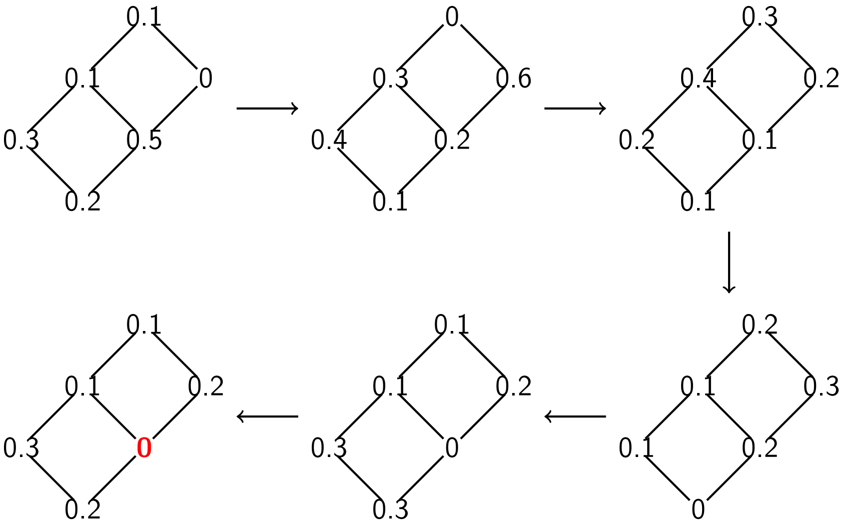

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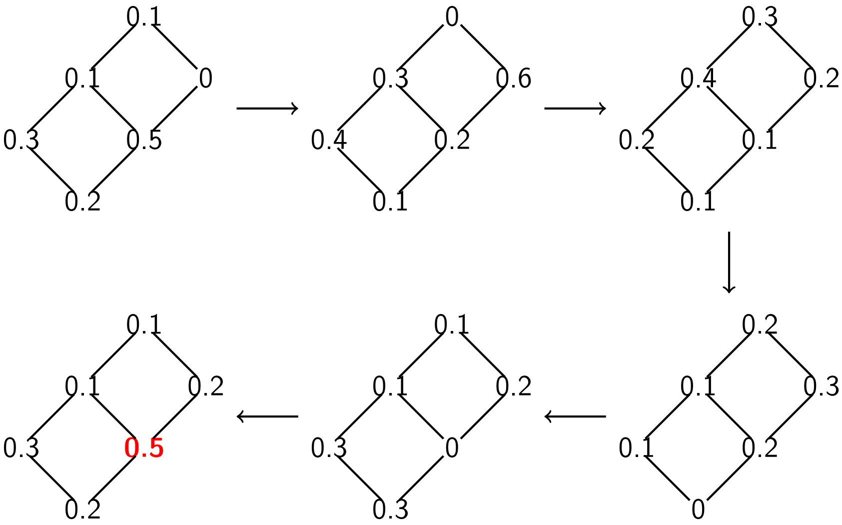

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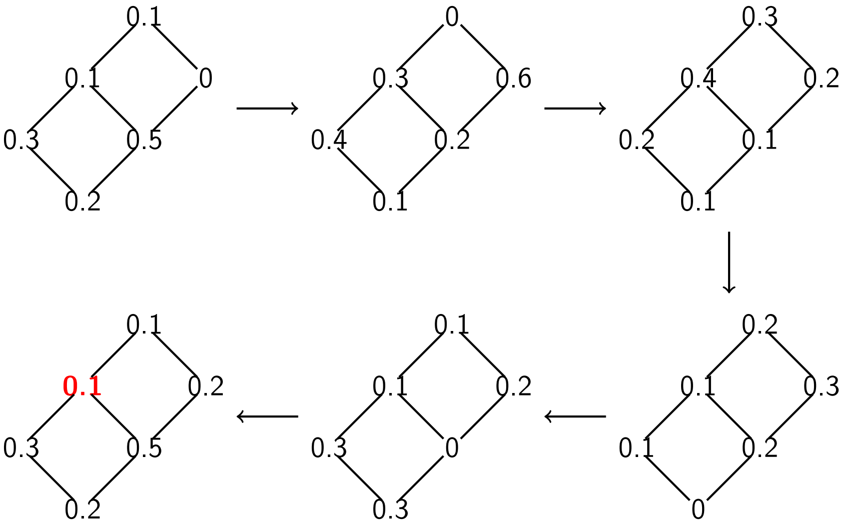

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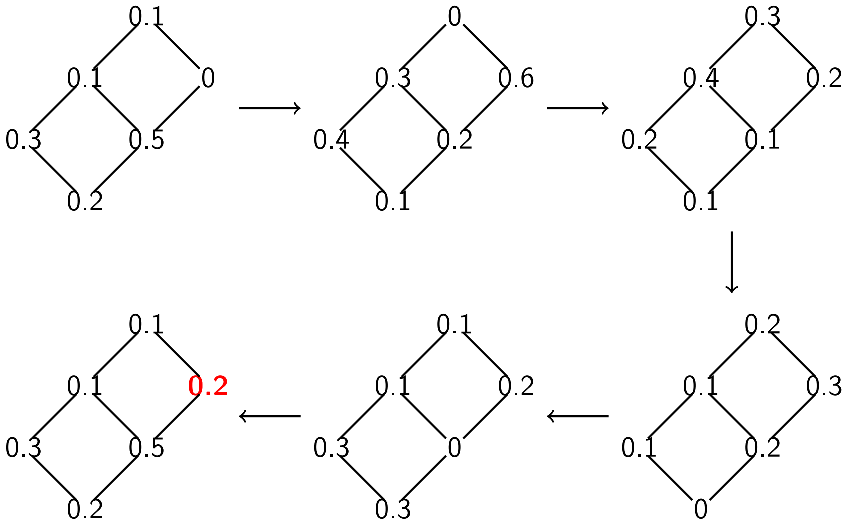

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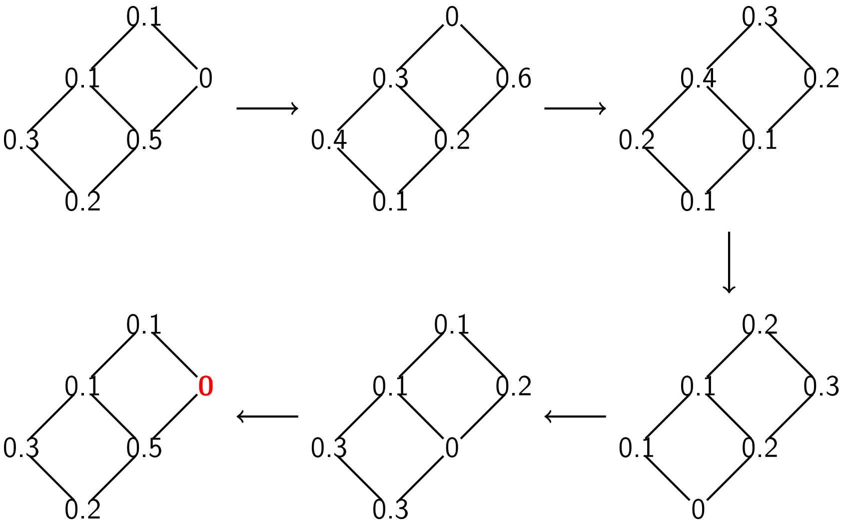

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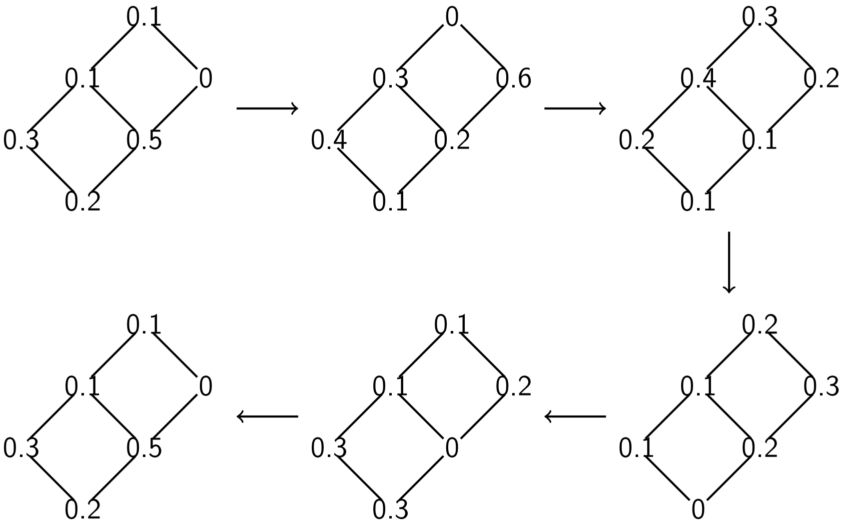


Detropicalizing from the piecewise-linear realm to the birational realm

- Einstein and Propp showed how to lift of order-ideal toggling and rowmotion on $\mathcal{O}(P)$ to the birational realm.

Detropicalizing from the piecewise-linear realm to the birational realm

- Einstein and Propp showed how to lift of order-ideal toggling and rowmotion on $\mathcal{O}(P)$ to the birational realm.
- Now let's do the same lifting of antichain toggling and rowmotion on $\mathcal{C}(P)$ to the birational realm.
- To do this, we replace max with $+$ and $+$ with multiplication. Under this dictionary

$$(\tau_e(g))(e) = 1 - \max \left\{ \sum_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}$$

becomes

$$(\tau_e(g))(e) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \;\middle|\; \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}}$$

Now we'll define the **birational antichain toggle** corresponding to $e \in P$.

Definition

For $e \in P$, and field F, let $\tau_e : P^F \to P^F$ be defined as the birational map that only changes the value at e in the following way.

$$(\tau_e(g))(e) = \frac{C}{\sum \left\{ \prod_{i=1}^{k} g(y_i) \,\middle|\, \begin{array}{c} (y_1, \ldots, y_k) \text{ is a maximal} \\ \text{chain in } P \text{ that contains } e \end{array} \right\}}$$

Definition

BAR-motion (birational antichain rowmotion) is the birational map obtained by applying the birational antichain toggles from the bottom to the top.

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & \color{red}{w} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & \color{red}{\dfrac{C}{wz(x+y)}} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$
g = \begin{tikzcd}[column sep=small]
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & w &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
\color{red}{x} \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & \dfrac{C}{wz(x+y)} &
\end{tikzcd}
$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ \color{red}{\dfrac{w(x+y)}{x}} \arrow[dr, dash] & & y \arrow[dl, dash] \\ & \dfrac{C}{wz(x+y)} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ \dfrac{w(x+y)}{x} \arrow[dr, dash] & & {\color{red} y} \arrow[dl, dash] \\ & \dfrac{C}{wz(x+y)} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$
g = \begin{tikzcd}[column sep=small]
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & w &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
\dfrac{w(x+y)}{x} \arrow[dr, dash] & & \color{red}{\dfrac{w(x+y)}{y}} \arrow[dl, dash] \\
 & \dfrac{C}{wz(x+y)} &
\end{tikzcd}
$$

BAR-motion on $[2] \times [2]$

$$
g = \begin{tikzcd}[column sep=small]
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[dr, dash] & & y \arrow[dl, dash] \\
 & w &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \color{red}{z} \arrow[dl, dash] \arrow[dr, dash] & \\
\dfrac{w(x+y)}{x} \arrow[dr, dash] & & \dfrac{w(x+y)}{y} \arrow[dl, dash] \\
 & \dfrac{C}{wz(x+y)} &
\end{tikzcd}
$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\mathrm{BAR}(g) = \begin{tikzcd}[column sep=small] & \color{red}{\dfrac{xy}{x+y}} \arrow[dl, dash] \arrow[dr, dash] & \\ \dfrac{w(x+y)}{x} \arrow[dr, dash] & & \dfrac{w(x+y)}{y} \arrow[dl, dash] \\ & \dfrac{C}{wz(x+y)} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd}[arrows=dash] & z \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & w & \end{tikzcd}$$

$$\mathrm{BAR}^2(g) = \begin{tikzcd}[arrows=dash] & w \arrow[dl] \arrow[dr] & \\ \frac{C}{wyz} \arrow[dr] & & \frac{C}{wxz} \arrow[dl] \\ & z & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

$$g = \begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\mathrm{BAR}^3(g) = \begin{tikzcd} & \dfrac{C}{wz(x+y)} \arrow[dl, dash] \arrow[dr, dash] & \\ \dfrac{z(x+y)}{x} \arrow[dr, dash] & & \dfrac{z(x+y)}{y} \arrow[dl, dash] \\ & \dfrac{xy}{x+y} & \end{tikzcd}$$

BAR-motion on $[2] \times [2]$

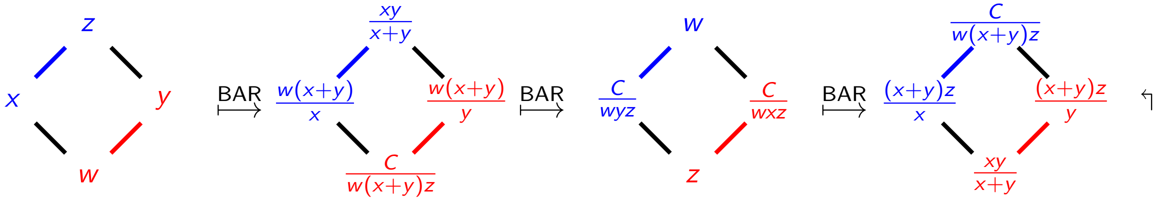
$$g = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

$$\mathrm{BAR}^4(g) = \begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & w & \end{tikzcd}$$

Properties of BAR-motion

- The order of BAR on $[a] \times [b]$ is $a + b$.
- The homomesy results for antichain cardinality in the combinatorial $\rho_{\mathcal{A}}$ setting lift to this setting.
- We can lift the *Stanley–Thomas* word to this setting as an equivariant *surjection*, cyclically rotating with *BAR*. It can be used to prove homomesy, but not periodicity [JR20+].

Here is the full orbit of BAR on a generic labeling for $P = [2] \times [2]$, with ST-words.



$$\left(wy, xz, \frac{C}{wx}, \frac{C}{yz}\right) \qquad \left(\frac{C}{yz}, wy, xz, \frac{C}{wx}\right) \qquad \left(\frac{C}{wx}, \frac{C}{yz}, wy, xz\right) \qquad \left(xz, \frac{C}{wx}, \frac{C}{yz}, wy\right)$$

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to our THEMES:
1) *Periodicity/order*; 2) *Orbit structure*; 3) *Homomesy* 4) *Equivariant bijections*

- Examples of cyclic sieving are also ripe for homomesy hunting.

- Situations in which maps can be built out of toggles seem particularly fruitful.

- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at this level.

Slides for this talk are available online at

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Thanks very much for coming to this talk!

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