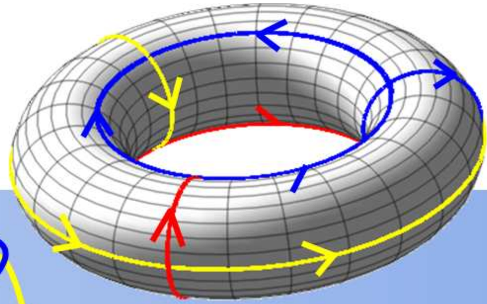
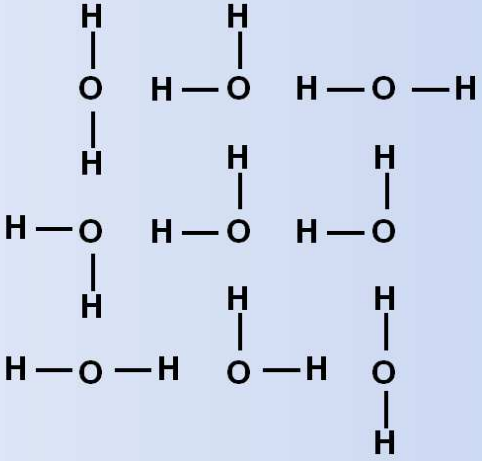
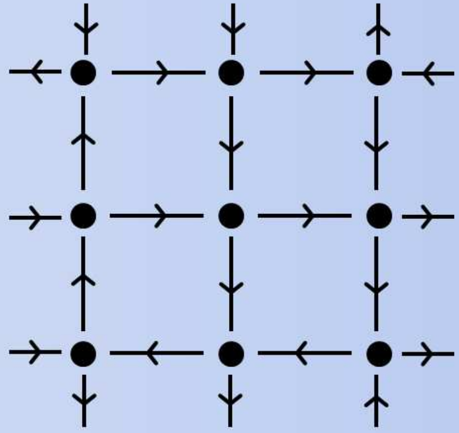
# THEOREM OF THE DAY

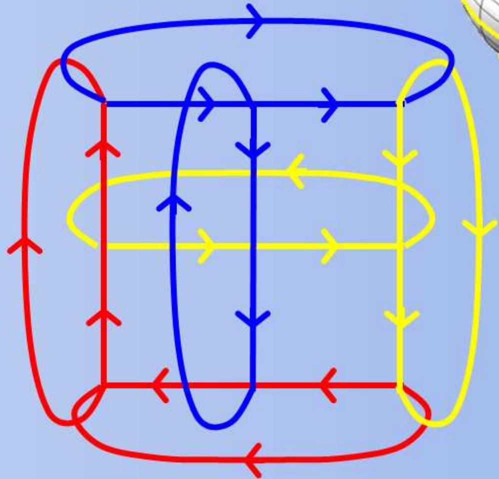
**Lieb's Square Ice Theorem** *For the graph of the $n \times n$ toroidal lattice, let $f_n$ denote the number of Eulerian orientations. Then*

$$\lim_{n\to\infty} f_n^{1/n^2} = \frac{8\sqrt{3}}{9}.$$









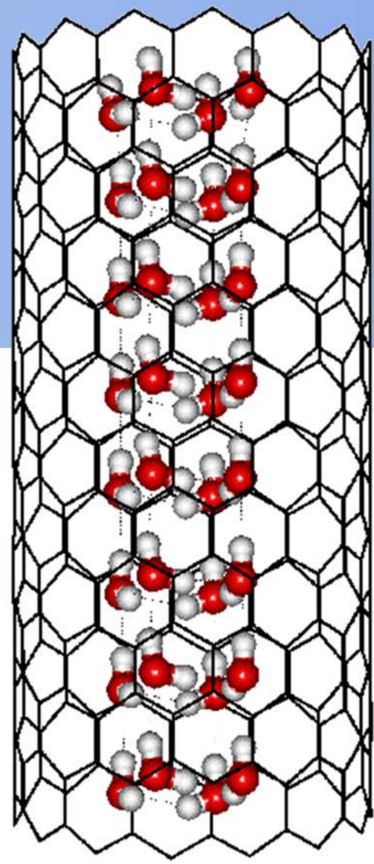


Image by Martin Chaplin

Square ice is a model of how water molecules, consisting of one oxygen atom and two hydrogen atoms, may combine as a lattice to form a solid (above left). With the oxygen atoms aligned on lattice points, the hydrogen atom bondings may be represented by arrows directed towards these points (centre); the rule is that precisely two hydrogens must be close to each oxygen and this is indicated by the arrows; an inward pointing arrow indicates a close hydrogen. In a standard simplification, the lattice, rather than being unbounded, is assumed to wrap around a torus in each direction (far top right). Then the possible configurations of hydrogen bonds become orientations of a graph, with each vertex have two edges directed towards it (centre right, the colours are merely to help compare with the torus). This is an *Eulerian orientation*.

The problem of counting Eulerian orientations for arbitrary graphs is ♯P, or 'number P', -complete and can thereby, with reasonable confidence, be said to be intractably difficult; this was proved in 1992 by Milena Mihail and Peter Winkler. However, intricate analysis of lattice graphs had enabled the mathematical physicist Elliott Lieb, in 1967, to produce a solution in this special case, exhibiting a limit value that is constant up to scaling by the number of lattice points. This value, $8\sqrt{3}/9 \approx 1.539601$, is known as Lieb's Square Ice Constant and relates to the 'residual entropy' of square ice.

The image far right is of 'confined water', with square ice crystals each comprising four molecules confined in a carbon nanotube.

**Web link:** arxiv.org/abs/math/0208125

**Further reading:** *Mathematical Constants* by Steven R. Finch, Cambridge University Press, 2003, chapter 5.