The self-avoiding walk on the hexagonal lattice

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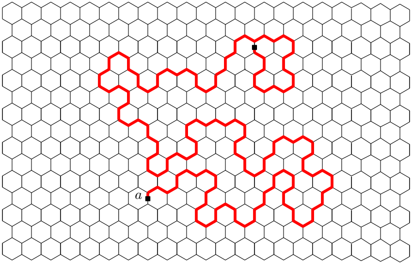
Self-Avoiding Walks on the hexagonal lattice $\mathbb{H}$:



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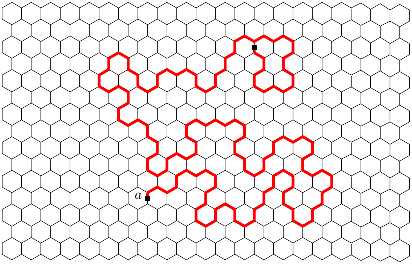

Conjecture (Flory, 1948; Nienhuis, 1982)

Precise asymptotics for the mean-square displacement c_n of SAWs of length n:

- $$\langle |\omega(n)|^2 \rangle \sim D n^{2\nu} \quad \text{as } n \longrightarrow \infty,$$

where $\nu := 3/4$.

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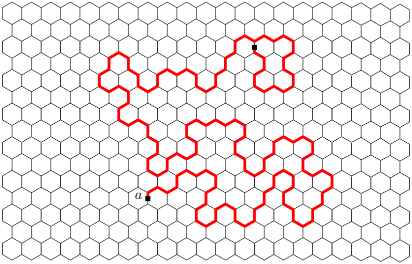

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Precise asymptotics for the mean-square displacement and for the number c_n of SAWs of length n:

- $\langle |\omega(n)|^2 \rangle \sim D n^{2\nu} \quad$ as $n \longrightarrow \infty$,
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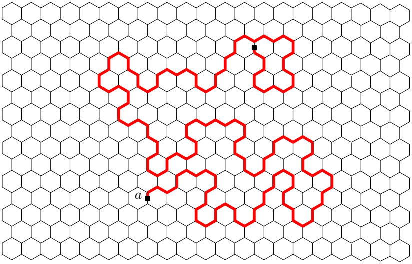

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γ and ν are universal; μ_c is lattice-dependent.

Theorem (H. Duminil-Copin, S. Smirnov, 2010)

The connective constant satisfies $\mu_c := \lim_{n\to\infty} c_n^{\frac{1}{n}} = \sqrt{2+\sqrt{2}}$.

- Easy observations:

$$c_{n+m} < c_n \cdot c_m \;\Rightarrow\; \exists\, \mu_c := \lim_{n\to\infty} c_n^{\frac{1}{n}}\ ,$$

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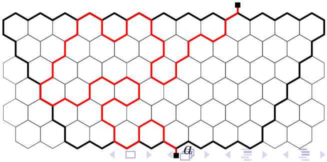
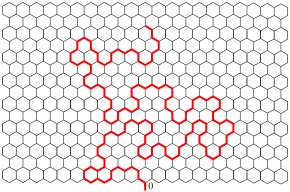
Try to count simpler objects, **bridges**: Walks that never go below the first step and above the last one. The number of bridges grows at the same (exponential) speed as walks.



Definition

A **self-avoiding bridge** is a SAW ω such that the first site is of minimal second coordinate and the last one of maximal second coordinate. Let b_n be the number of self-avoiding bridges of length n.



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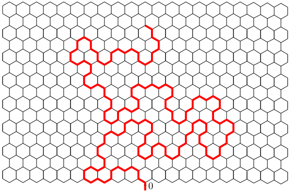

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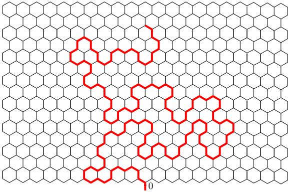

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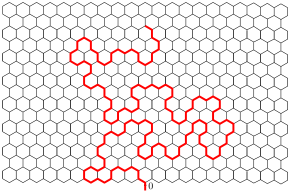

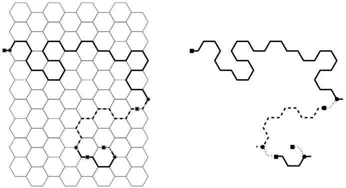

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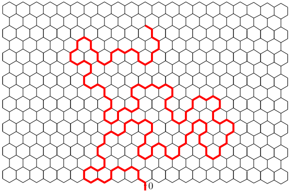

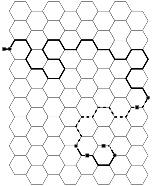

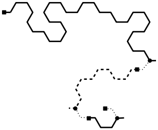

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$b_n \leq c_n$ for obvious reasons. Moreover, $c_n \leq r_n^2 b_n$ where r_n is the number of **partitions** of n into increasing positive integers. Since $r_n \leq Ce^{c\sqrt{n}}$, we obtain that b_n and c_n are logarithmically equivalent.

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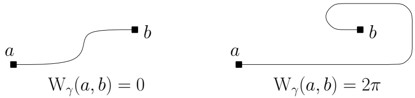
The winding $\mathrm{W}_\omega(a, b)$ of a curve ω between a and b is the rotation (in radians) of the curve between a and b.



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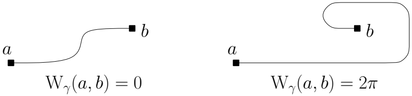

With this definition, we can define the *parafermionic operator* for $a\in\partial\Omega$ and $z\in\Omega$:

$$F(z)=F(a,z,\mu,\sigma):=\sum_{\omega\subset\Omega:\ a\to z}\mathrm{e}^{-\mathrm{i}\sigma\mathrm{W}_\omega(a,z)}\mu^{-\ell(\omega)}.$$

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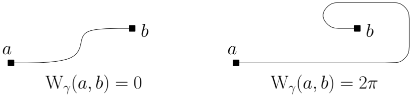

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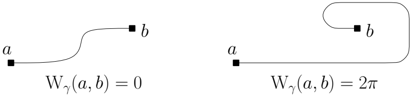

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Lemma (Discrete integrals on elementary contours vanish)

If $\mu = \mu_* = \sqrt{2+\sqrt{2}}$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$,

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0$$

where p, q, r are the mid-edges of the three edges adjacent to v.

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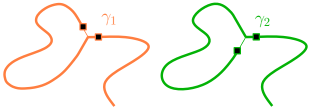

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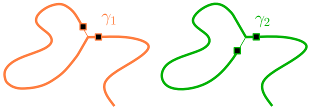

In the first case,

$$\begin{aligned} c(\omega_1) + c(\omega_2) &= (q - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\omega_1}(a,q)}\mu^{-\ell(\omega_1)} + (r - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\omega_2}(a,r)}\mu^{-\ell(\omega_2)} \\ &= (p - v)\mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_{\omega_1}(a,p)}\mu^{-\ell(\omega_1)}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\sigma\cdot\frac{-4\pi}{3}} + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\mathrm{e}^{-\mathrm{i}\sigma\cdot\frac{4\pi}{3}}\right) \end{aligned}$$

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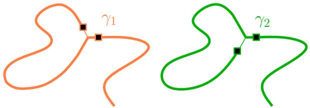

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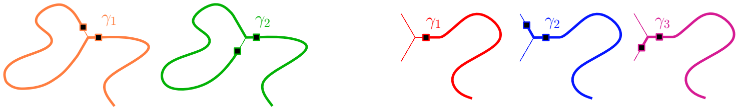

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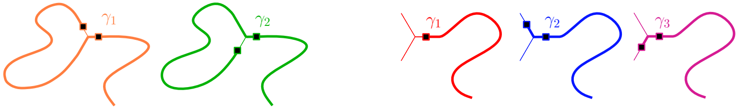

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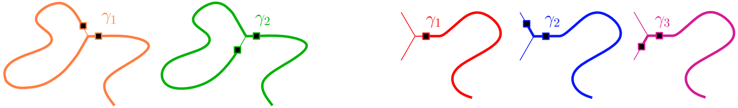

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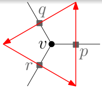

Proposition ((partial) Discrete holomorphicity)

If Ω is simply connected, then $\oint_\Gamma F(z)dz = 0$ **for any** *discrete contour Γ.*

Will be used to show $\mu_c = \mu_*$. Take a trapezoid contour $S_{T,L}$:

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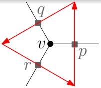

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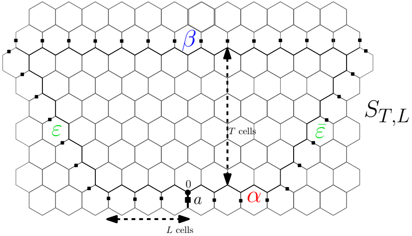

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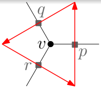

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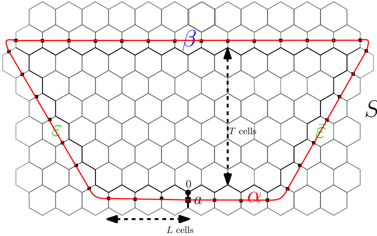
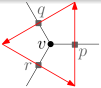


$S_{T,L}$

$$0 = -\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + e^{i\frac{2\pi}{3}} \sum_{z\in\varepsilon} F(z) + e^{-i\frac{2\pi}{3}} \sum_{z\in\bar{\varepsilon}} F(z)$$



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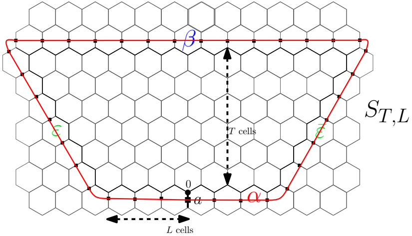
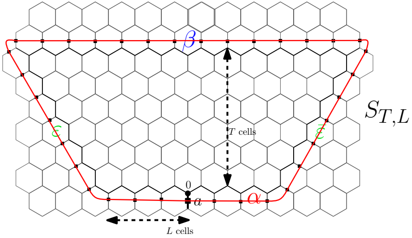


$$0 = -\sum_{z\in\alpha} F(z) + \sum_{z\in\beta} F(z) + \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} \sum_{z\in\varepsilon} F(z) + \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} \sum_{z\in\bar{\varepsilon}} F(z)$$



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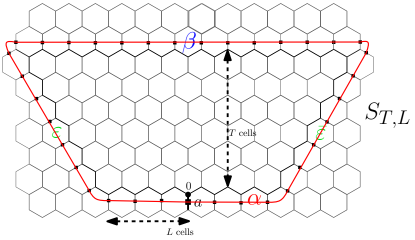

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$$1 = \cos\left(\frac{3\pi}{8}\right) \sum_{\omega:a\to\alpha} \mu_*^{-\ell(\omega)} + \sum_{\omega:a\to\beta} \mu_*^{-\ell(\omega)} + \cos\left(\frac{\pi}{4}\right) \sum_{\omega:a\to\varepsilon\cup\bar{\varepsilon}} \mu_*^{-\ell(\omega)}.$$

We know the winding on the boundary!
So we can replace F by the sum of Boltzman weights.

$$1 = \frac{\sqrt{2-\sqrt{2}}}{2} A(T,L,\mu_*) + B(T,L,\mu_*) + \frac{1}{\sqrt{2}} E(T,L,\mu_*).$$

An upper bound on μ_c:

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- Determined the connective constant.
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TO DO

- What to do next?
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What to do next? The case of the self-avoiding walk.

Conjecture (Nienhuis, 1982; Flory, 1948)

- **Combinatorial question:** Up to $n^{o(1)}$ (up to a multiplicative constant?) we have:

$$c_n \sim n^{\gamma-1}\left(\sqrt{2+\sqrt{2}}\right)^n \text{ as } n \longrightarrow \infty$$

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Would follow from the following conjecture

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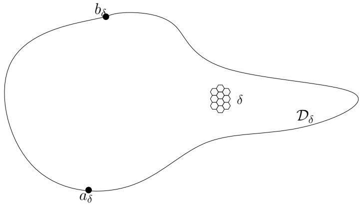

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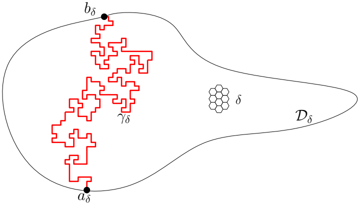

- For $\delta > 0$, we define a probability measure on self-avoiding paths from a_δ to b_δ by assigning a weight proportional to $\mu_c^{-\ell(\omega)}$. When $\delta \to 0$, the sequence converges to a **random continuous curve**.

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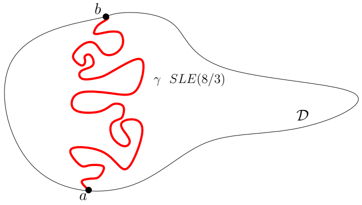

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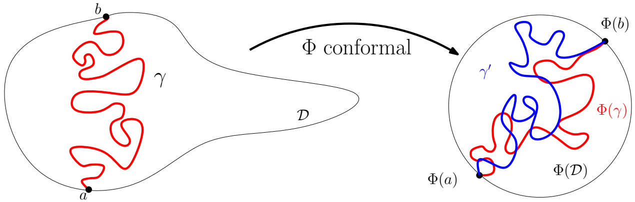

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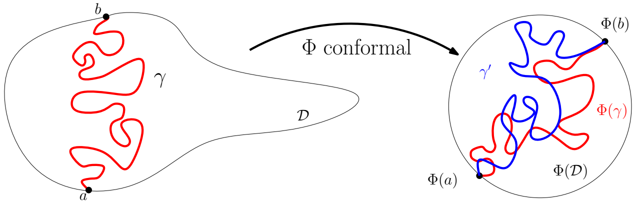

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A strategy to tackle this problem?

(1) Precompactness of the family of curves

(2) Conformally invariant martingales which are given by the ratio of two parafermionic observables: $F(a, z, \Omega)/F(a, b, \Omega)$.

Main missing point: show that F is fully discrete holomorphic

What to do next? $O(n)$ models (1).

The $O(n)$ model is a model on **closed loops** lying on a finite subgraph of the hexagonal lattice. The probability of a configuration is equal to

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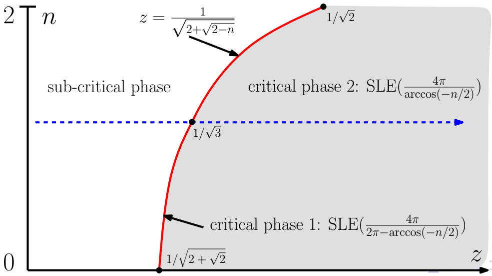

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For $O(n)$ models, the parafermionic observable

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should be discrete holomorphic for $x = x_c$ and $2\cos(\frac{4\sigma\pi}{3}) = -n$.

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For $n \in [0, 2]$ and $x = x_c(n)$, the interface between two points a and b (on the boundary) converges, as the lattice step goes to zero, to SLE(κ) where

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Known only for the critical percolation, $n = 1$, $x = 1$ (Smirnov) via a different observable.

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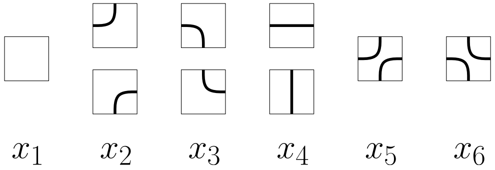
Consider a more general model on the square lattice, with the following weights



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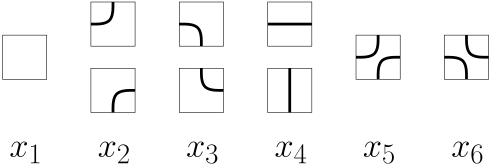

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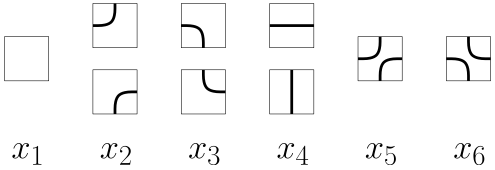

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- The solutions correspond to **integrable points** of the model (when the Yang-Baxter condition applies).

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- We can introduce parafermionic observables for a wide variety of models: $O(n)$-models, random-cluster models, self-avoiding walks...
- We can extract information from these operators in order to study the critical phase (example of the connective constant of the hexagonal lattice).
- In some cases, the information is total – universality class of the Ising model – and we can derive conformal invariance.

Question: Can we do the same for other models?

Thank you