Materialization Calculus for Contexts in the Semantic Web

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Abstract. Representation of context dependent knowledge in the Semantic Web is becoming a recognized issue and a number of DL-based formalisms have been proposed in this regard: among them, in our previous works we introduced the Contextualized Knowledge Repository (CKR) framework. In CKR, contexts are organized hierarchically according to a broader-narrower relation and knowledge propagation across contexts is limited among contexts hierarchically related. In several applications, however, this structure is too restrictive, as they demand for a more flexible and scalable framework for representing and reasoning about contextual knowledge.
In this work we present an evolution of the original CKR (based on OWL RL), where contexts can be organized in any graph based structure (declared as a meta-knowledge base) and knowledge propagation is allowed among any pair of contexts via a new "evaluate-in-context" operator. In particular, we detail a materialization calculus for reasoning over the revised CKR framework and prove its soundness and completeness. Moreover, we outline the current implementation of the calculus on top of SPRINGLES, an extension of standard RDF triple stores for representing and rule-based inferencing over multiple RDF named graphs.

1 Introduction

Representation of context dependent knowledge in the Semantic Web is becoming a recognized issue and a number of DL-based formalisms have been proposed in this regard: among them, in our previous works we introduced the Contextualized Knowledge Repository (CKR) framework [13, 4, 2]. In a CKR, each context is associated with a set of formulae that hold under the same circumstances, which are specified by a set of dimensional attributes associated to the context (e.g. *time*, *space*, *topic*, *author* or *access-permissions*). In CKR, as proposed in [13], the values of contextual attributes were organized hierarchically according to a broader-narrower relation. For instance the geographical value "Italy" is narrower than "Europe", the time attribute "2010" is narrower than "21st century", and the topic attribute "football" is narrower than "sport". This structure induces a hierarchical (*coverage*) relation between contexts. For example, the context associated to Italian football during 2010 is covered by the context for European sports in the 21st century. Coverage relation regulates the propagation of knowledge across contexts, which is limited among contexts hierarchically related.

Practical applications, however, show that this structure might be too restrictive and demand for a more flexible and scalable representation of contextual knowledge. For example, it is often necessary to: associate a type (e.g. Event) to a set of contexts and characterize it by a set of axioms; use in a context the interpretation of a complex expression from a possibly unrelated class of contexts. For this reason, in this work we extend and generalize the original CKR framework by the following two aspects:

1. Generalizing contextual structure: hierarchical contextual attributes are replaced with a generic graph structure specified by a DL knowledge base, containing individual names to denote contexts (i.e. contexts IDs), identifiers for contextual attributes values, and binary relations that allow to specify the relations and contextual attributes for each context. This generalization enables also the introduction of context classes, prototypical contexts that represent homogeneous classes of contexts.

2. Generalizing knowledge propagation: we introduce the $eval$ operator, a primitive that allows to refer to the extension of a concept (or role) in another set of contexts. For instance the subsumption $c_1 : eval(A, \mathsf{C}) \sqsubseteq A$ can be used to state that the extension of concept A in contexts of type C is contained in the extension of A in c_1. This subsumption enable to propagate the information of A from contexts in C to c_1.

Over this generalized version of CKR, we detail a materialization calculus for instance checking and prove its soundness and completeness[1]. Moreover, we outline its current implementation on top of SPRINGLES, an extension of standard RDF triple stores for representing and rule-based inferencing over multiple RDF named graphs.

2 Preliminaries: $\mathcal{SROIQ}$-RL

In the following we assume the usual presentation of description logics [1] and definitions[2] for the logic $\mathcal{SROIQ}$ [6].

OWL RL [12] is basically defined on a restriction of $\mathcal{SROIQ}$ syntax: in the following we call such restricted language $\mathcal{SROIQ}$-RL. The language is obtained by limiting the form of General Concept Inclusion axioms (GCIs) and concept equivalence of $\mathcal{SROIQ}$ to $C \sqsubseteq D$ where C and D are concept expressions, called *left-side concept* and *right-side concept* respectively, and defined by the following grammar:

$$C := A \,|\, \{a\} \,|\, C_1 \sqcap C_2 \,|\, C_1 \sqcup C_2 \,|\, \exists R.C_1 \,|\, \exists R.\{a\} \,|\, \exists R.\top$$
$$D := A \,|\, \neg C_1 \,|\, D_1 \sqcap D_2 \,|\, \exists R.\{a\} \,|\, \forall R.D_1 \,|\, \leqslant nR.C_1 \,|\, \leqslant nR.\top$$

where $n \in \{0, 1\}$. Finally, a *both-side concept* E is a concept expression which is both a left- and right-side concept. TBox axioms can only take the form $C \sqsubseteq D$ or $E \equiv E$. RBox for $\mathcal{SROIQ}$-RL can contain every role axiom of $\mathcal{SROIQ}$ but $\mathrm{Ref}(R)$. ABox concept assertions can be only stated in the form $D(a)$, where D is a right-side concept.

3 Contextualized Knowledge Repositories

A CKR is a two layered structure. The upper layer is composed of a knowledge base $\mathfrak{G}$, which describes two types of knowledge: (i) the structure and the properties of contexts

[1] Complete proofs for soundness and completeness are presented in the Appendix.

[2] For ease of reference, the intended presentation of syntax and semantics for $\mathcal{SROIQ}$ constructors and axioms is presented in Table 4 in the Appendix.

of the CKR (called *meta-knowledge*), and (ii) the knowledge that is context independent, i.e., that holds in every context (called *global knowledge*). The lower layer, is constituted by a set of (local) contexts, each containing (locally valid) facts, that can also refer to what holds in other contexts. To favor knowledge reuse, the knowledge of each context is organized in multiple knowledge modules. The association between contexts and modules is represented in the meta-knowledge via a binary relation, and can be either explicitly asserted or inferred via meta-reasoning. The knowledge in a CKR can be expressed by means of any DL language: in the following we provide the parametric definition to any DL language and successively we instantiate it to $\mathcal{SROIQ}$-RL.

The meta-knowledge of a CKR is expressed by a DL language defined on a meta-vocabulary, containing the elements that define the contextual structure[3].

Definition 1 (Meta-vocabulary). *A* meta-vocabulary *is a DL vocabulary Γ composed of a set of atomic concepts NC_Γ, a set atomic roles NR_Γ and a set of individual constants NI_Γ that are mutually disjoint and contain the following sets of symbols*

1. $\mathsf{N} \subseteq \mathrm{NI}_\Gamma$ *of* context names.
2. $\mathsf{M} \subseteq \mathrm{NI}_\Gamma$ *of* module names.
3. $\mathsf{C} \subseteq \mathrm{NC}_\Gamma$ *of* context classes, *including the class* Ctx[4].
4. $\mathsf{R} \subseteq \mathrm{NR}_\Gamma$ *of* contextual relations.
5. $\mathsf{A} \subseteq \mathrm{NR}_\Gamma$ *of* contextual attributes.
6. *For every attribute* $\mathsf{A} \in \mathbf{A}$*, a set* $\mathsf{D}_\mathsf{A} \subseteq \mathrm{NI}_\Gamma$ *of* attribute values *of* A.

The meta-language of a CKR, denoted by $\mathcal{L}_\Gamma$, is a DL language built starting from the meta-vocabulary Γ with the following syntactic conditions on the application of role restrictions: for every $\bullet \in \{\forall, \exists, \leqslant n, \geqslant n\}$, (i) if the concept $\bullet\mathsf{A}.\mathsf{B}$ occurs in a concept expression, then $\mathsf{B} = \{\mathsf{a}\}$ with $\mathsf{a} \in \mathsf{D}_\mathsf{A}$; (ii) if the concept $\bullet\mathsf{mod}.\mathsf{B}$ occurs in a concept expression, then $\mathsf{B} = \{\mathsf{m}\}$ with $\mathsf{m} \in \mathbf{M}$.

The context (in)dependent knowledge of a CKR is expressed via a DL language called *object-language*, based on an object-vocabulary $\Sigma = \mathrm{NC}_\Sigma \uplus \mathrm{NR}_\Sigma \uplus \mathrm{NI}_\Sigma$. The object-language $\mathcal{L}_\Sigma$ is the DL language defined starting from the Σ. The expressions of the object language will be evaluated locally to each context: namely each context can interpret each symbol independently. However, there are cases in which one want to constrain the meaning of a symbol in a context with the meaning of a symbol in some other context. For instance if John likes all Indian restaurants in Trento, then the extension of the concept $GoodRestaurant$ in the context of John preferences, contains the extension of $IndianRestaurant$ in the context of tourism in Trento. To represent such external references, we extend the object language $\mathcal{L}_\Sigma$ with the so called *eval* expressions. Given a concept or role expression X of $\mathcal{L}_\Sigma$, and a context expression C of $\mathcal{L}_\Gamma$, an *eval expression* is an expression of the form $eval(X, \mathsf{C})$. The DL language $\mathcal{L}^e_\Sigma$ extends $\mathcal{L}_\Sigma$ with the set of eval-expressions in $\mathcal{L}_\Sigma$. Intuitively, the expression $eval(C, \{\mathsf{c}\})$ represents the extension of the concept C in the context c. Generalizing, $eval(C, \mathsf{C})$ with C a context class represent the union of the extensions of C in each context of type C. Similar intuition can be given for $eval(R, \mathsf{C})$. The above example can be formalized by adding the following axiom to the context of John's preferences: $eval(IndianRestaurant, \{\mathsf{trento\_tourism}\}) \sqsubseteq GoodRestaurant$. Finally,

[3] To support readability we use sans-serif typeface to denote elements of the meta-vocabulary.
[4] Intuitively, Ctx will be used to denote the class of all contexts.

notice that we do not allow nested $eval$ expressions: every expression occurring inside an $eval$ should be an expression in $\mathcal{L}_\Sigma$.

Definition 2 **(Contextualized Knowledge Repository, CKR).** *Given a meta-vocabulary Γ and an object vocabulary Σ, a* Contextualized Knowledge Repository (CKR) *over $\langle \Gamma, \Sigma \rangle$ is a structure $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_{\mathsf{m}}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ where:*

- *$\mathfrak{G}$ is a DL knowledge base over $\mathcal{L}_\Gamma \cup \mathcal{L}_\Sigma$*
- *for every module name $\mathsf{m} \in \mathbf{M}$, K_{m} is a DL knowledge base over $\mathcal{L}^e_\Sigma$.*

Definition 3 **($\mathcal{SROIQ}$-RL CKR).** *A CKR is a $\mathcal{SROIQ}$-RL CKR, if $\mathfrak{G}$ is a knowledge base in $\mathcal{SROIQ}$-RL, and for each $\mathsf{m} \in \mathbf{M}$, K_{m} is a $\mathcal{SROIQ}$-RL knowledge base, where eval-expressions only occur in left-concepts and contain left-concepts or roles.*

Example 1. We introduce an example in the tourism recommendation domain[5]. In this scenario we use CKR to implement a knowledge base, called $\mathfrak{K}_{tour}$, that is populated with touristic events, locations, organizations and tourists' preferences and profiles, and that is capable to identify events that are interesting for a tourist (or tourists class) starting from his/her preferences. A simplified version of the structure and the con-

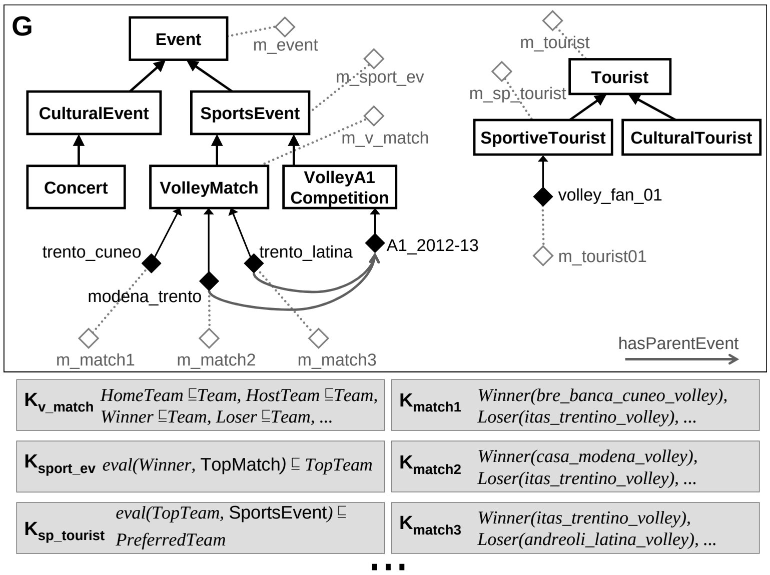

Fig. 1. Example CKR knowledge base $\mathfrak{K}_{tour}$

texts of $\mathfrak{K}_{tour}$ is shown in Figure 1. In this example we focus on sportive events and in particular on volley matches. Intuitively, in the global context $\mathfrak{G}$, every sport event and tourist is modelled with a context: in the figure, these are depicted as black diamonds and we show some of the official volley matches and a tourist. Contexts are

[5] This example is a simplified version of a real case scenario in which we apply CKR to a tourist event recommendation system in Trentino (see `http://www.investintrentino.it/News/Trentour-Trentino-platform-for-smart-tourism`).

grouped by types and organized in hierarchies by means of context classes: in the figure (depicted as boxes) we show a distinct context class hierarchy for event types (e.g. $\mathsf{SportiveEvent}, \mathsf{VolleyMatch}$) and for tourists types (e.g. $\mathsf{SportiveTourist}$). The metaknowledge in $\mathfrak{G}$ associates to contexts and context classes a set of knowledge modules, by axioms of the kind $\mathsf{Event} \sqsubseteq \exists \mathsf{mod}.\{\mathsf{m\_event}\}$ or $\{\mathsf{modena\_trento}\} \sqsubseteq \exists \mathsf{mod}.\{\mathsf{m\_match2}\}$: in the figure these associations are represented by dotted lines to the gray empty diamonds depicting module name individuals. Knowledge bases associated to modules are depicted as the corresponding gray boxes in the lower part of Figure 1: for example, in $\mathrm{K}_{\mathsf{v\_match}}$ we have general axioms about the structure of volley matches, while in modules for specific matches as $\mathrm{K}_{\mathsf{match1}}$ we store assertions about the actual results of the match. Intuitively, the semantics will enforce a form of inheritance of modules via context class hierarchy. Contextual relations across events and tourists are depicted as bold arrows in the figure: the only relation $\mathsf{hasParentEvent}$ connects matches with the competition in which they occour. Note that in some of the knowledge modules, we use $eval$ expressions to define references across contexts. For example, in $\mathrm{K}_{\mathsf{sport\_ev}}$ we can state[6] that "Winners of major volley matches are top teams" with:

$$eval(Winner, \mathsf{VolleyMatch} \sqcap \exists \mathsf{hasParentEvent}.\mathsf{VolleyA1Competition}) \sqsubseteq TopTeam$$

Similarly, in $\mathrm{K}_{\mathsf{sp\_tourist}}$ we say that for each sportive tourist "top teams are preferred teams" with $eval(TopTeam, \mathsf{SportsEvent}) \sqsubseteq PreferredTeam$. ◇

Definition 4 (CKR interpretation). *A* CKR interpretation *for $\langle \Gamma, \Sigma \rangle$, is a structure $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ s.t. $\mathcal{M}$ is a DL interpretation of $\Gamma \cup \Sigma$ and, for every $x \in \mathsf{Ctx}^{\mathcal{M}}$, $\mathcal{I}(x)$ is a DL interpretation over Σ.*

According to the previous definition, a CKR interpretation is composed by an interpretation for the "upper-layer" (which includes the global knowledge and the metaknowledge) and an interpretation of the object language for each instance of type context (i.e., for all $x \in \mathsf{Ctx}^{\mathcal{M}}$), providing a semantics of the object-vocabulary in x.

Definition 5 (CKR Model). *A CKR interpretation $\mathfrak{I}$ is a* CKR model *of $\mathfrak{K}$ (in symbols, $\mathfrak{I} \models \mathfrak{K}$) iff the following conditions hold:*

1. Global interpretation: *(i) $\mathbf{N}^{\mathcal{M}} \subseteq \mathsf{Ctx}^{\mathcal{M}}$ and, for every $\mathsf{C} \in \mathbf{C}$, $\mathsf{C}^{\mathcal{M}} \subseteq \mathsf{Ctx}^{\mathcal{M}}$; (ii) $\mathcal{M} \models \mathfrak{G}$.*
2. Local interpretations: *if $\langle x, y \rangle \in \mathsf{mod}^{\mathcal{M}}$ and $y = \mathsf{m}^{\mathcal{M}}$, then $\mathcal{I}(x) \models \mathrm{K}_{\mathsf{m}}$.*
3. Knowledge propagation: *for every $x \in \mathsf{Ctx}^{\mathcal{M}}$: (i) for $a \in \mathrm{NI}_{\Sigma}$ and any $y \in \mathsf{Ctx}^{\mathcal{M}}$, $a^{\mathcal{I}(x)} = a^{\mathcal{I}(y)} = a^{\mathcal{M}}$; (ii) for $\alpha \in \mathfrak{G}$ with $\alpha \in \mathcal{L}_{\Sigma}$, $\mathcal{I}(x) \models \alpha$.*
4. Interpretation of $eval$ expressions: *given $x \in \mathsf{Ctx}^{\mathcal{M}}$,*

$$eval(X, \mathsf{C})^{\mathcal{I}(x)} = \bigcup_{\mathsf{e} \in \mathsf{C}^{\mathcal{M}}} X^{\mathcal{I}(\mathsf{e})}$$

Note that the interpretation of an $eval$ expression is independent from the context in which it is evaluated.

We adapt to CKR models the definition of the classic reasoning problem of entailment: intuitively, the problem is specialized by indicating the context of reference.

[6] For space reasons, in Figure 1 and in following examples, we write $\mathsf{TopMatch}$ as a shortcut for the corresponding context expression.

Definition 6 **(c-entailment).** *Given a CKR* $\mathfrak{K}$ *over* $\langle \Gamma, \Sigma \rangle$ *with* $\mathsf{c} \in \mathbf{N}$ *and an axiom* $\alpha \in \mathcal{L}^{e}_{\Sigma}$*, we say that* α *is* c*-entailed by* $\mathfrak{K}$ *(denoted by* $\mathfrak{K} \models \mathsf{c} : \alpha$*) if, for every CKR model* $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ *of* $\mathfrak{K}$*, we have* $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \models \alpha$*.*

We extend this definition to CKR knowledge bases as follows.

Definition 7 **(Global entailment).** *Given a CKR* $\mathfrak{K}$ *over* $\langle \Gamma, \Sigma \rangle$ *and an axiom* α*, we say that* α *is (globally) entailed by* $\mathfrak{K}$ *(denoted by* $\mathfrak{K} \models \alpha$*) if:*

- $\alpha \in \mathcal{L}^{e}_{\Sigma}$ *and, for every* $\mathsf{c} \in \mathbf{N}$ *and model* $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ *of* $\mathfrak{K}$*, we have* $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \models \alpha$*;*
- $\alpha \in \mathcal{L}_{\Gamma}$ *and, for every model* $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ *of* $\mathfrak{K}$*, we have* $\mathcal{M} \models \alpha$*.*

Example 2. We can now consider again the $\mathfrak{K}_{tour}$ CKR of Example 1 and provide a formal interpretation for it. Let $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ be the model of $\mathfrak{K}_{tour}$ directly induced by its assertions. By the definition of the model, we can now find the knowledge base associated to each context: for example, for the context of the match modena_trento, we have that $\mathcal{I}(\mathsf{modena\_trento}^{\mathcal{M}}) \models \mathrm{K}_{\mathsf{event}} \cup \mathrm{K}_{\mathsf{sport\_ev}} \cup \mathrm{K}_{\mathsf{v\_match}} \cup \mathrm{K}_{\mathsf{match2}}$ and similarly for the other matches. In particular, we have that the local interpretation satisfies the axiom $eval(Winner, \mathsf{TopMatch}) \sqsubseteq TopTeam \in \mathrm{K}_{\mathsf{sport\_ev}}$. We can now give the formal reading of such axiom: we have that $eval(Winner, \mathsf{TopMatch})^{\mathcal{I}(\mathsf{modena\_trento}^{\mathcal{M}})} = \bigcup_{\mathsf{e} \in \mathsf{TopMatch}^{\mathcal{M}}} Winner^{\mathcal{I}(\mathsf{e})}$. From the assertions in the ABox of $\mathfrak{G}$, this corresponds to $\bigcup_{\mathsf{e} \in \{\mathsf{match\_2},\, \mathsf{match\_3}\}} Winner^{\mathcal{I}(\mathsf{e})}$. Now, by assertions on $Winner$ inside $\mathrm{K}_{\mathsf{match2}}$ and $\mathrm{K}_{\mathsf{match3}}$, we obtain $\{itas\_trentino, casa\_modena\} \subseteq TopTeam^{\mathcal{I}(\mathsf{modena\_trento}^{\mathcal{M}})}$.

We can do a similar reasoning for the context describing the tourist volley_fan_01. We have that $\mathcal{I}(\mathsf{volley\_fan\_01}^{\mathcal{M}}) \models \mathrm{K}_{\mathsf{tourist}} \cup \mathrm{K}_{\mathsf{sp\_tourist}} \cup \mathrm{K}_{\mathsf{tourist01}}$. Thus, the interpretation satisfies $eval(TopTeam, \mathsf{SportsEvent}) \sqsubseteq PreferredTeam$ from $\mathrm{K}_{\mathsf{sp\_tourist}}$. As in the case above, $eval(TopTeam, \mathsf{SportsEvent})^{\mathcal{I}(\mathsf{volley\_fan\_01}^{\mathcal{M}})}$ is interpreted as $\bigcup_{\mathsf{e} \in \mathsf{SportsEvent}^{\mathcal{M}}} TopTeam^{\mathcal{I}(\mathsf{e})}$. From the assertions in $\mathfrak{G}$, we obtain that this is equal to $\bigcup_{\mathsf{e} \in \{\mathsf{match\_1},\, \mathsf{match\_2},\, \mathsf{match\_3}\}} TopTeam^{\mathcal{I}(\mathsf{e})}$. Finally, from the reference axiom above, we obtain $\{itas\_trentino, casa\_modena\} \subseteq PreferredTeam^{\mathcal{I}(\mathsf{volley\_fan\_01}^{\mathcal{M}})}$. ◇

4 Materialization Calculus K_{ic}

In this section we extend the materialization calculus K_{inst} for instance checking on $\mathcal{SROEL}(\sqcap, \times)$ (OWL-EL) proposed by [10] in order to reason over the two layered structure of a CKR in $\mathcal{SROIQ}$-RL. As we will discuss in the following, this calculus basically formalizes the operation of forward closure we realized in the implementation.

To simplify the presentation of the calculus rules, we introduce a normal form for the considered axioms. We say that a CKR $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_{\mathsf{m}}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ is in *normal form* if:

- $\mathfrak{G}$ contains axioms in $\mathcal{L}_{\Gamma}$ of the form of Table 1 or in the form $\mathsf{C} \sqsubseteq \exists \mathsf{mod}.\{\mathsf{m}\}$, $\mathsf{C} \sqsubseteq \exists \mathsf{A}.\{\mathsf{d_A}\}$ for $A, B, C \in \mathbf{C}$, $R, S, T \in \mathbf{R}$, $a, b \in \mathbf{N}$, $\mathsf{m} \in \mathbf{M}$, $\mathsf{A} \in \mathbf{A}$ and $\mathsf{d_A} \in \mathsf{D_A}$.
- $\mathfrak{G}$ and every K_{m} contain axioms in $\mathcal{L}_{\Sigma}$ of the form of Table 1 and every K_{m} contain axioms in $\mathcal{L}^{e}_{\Sigma}$ of the form $eval(A, \mathsf{C}) \sqsubseteq B$, $eval(R, \mathsf{C}) \sqsubseteq T$ for $A, B, C \in \mathrm{NC}$, $a, b \in \mathrm{NI}$, $R, S, T \in \mathrm{NR}$ and $\mathsf{C} \in \mathbf{C}$.

$A(a)$	$R(a,b)$	$\neg R(a,b)$	$a = b$	$a \neq b$
$A \sqsubseteq B$	$\{a\} \sqsubseteq B$	$A \sqsubseteq \neg B$	$A \sqcap B \sqsubseteq C$	
$\exists R.A \sqsubseteq B$	$A \sqsubseteq \exists R.\{a\}$	$A \sqsubseteq \forall R.B$	$A \sqsubseteq \leqslant 1 R.B$	
$R \sqsubseteq T$	$R \circ S \sqsubseteq T$	$\mathrm{Dis}(R,S)$	$\mathrm{Inv}(R,S)$	$\mathrm{Irr}(R)$

Table 1. Normal form axioms

We can give a set of rules[7] that can be used to transform any $\mathcal{SROIQ}$-RL CKR in an "equivalent" CKR in normal form. As in [10], we assume that rule chain axioms in input are already decomposed in binary role chains. The correctness of this translation can be shown by the following lemma.

Lemma 1. *For every CKR $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_\mathsf{m}\}_{\mathsf{m} \in \mathsf{M}} \rangle$ over $\langle \Gamma, \Sigma \rangle$, a CKR $\mathfrak{K}'$ over extended vocabularies $\langle \Gamma', \Sigma' \rangle$ can be computed in linear time s.t. all axioms in $\mathfrak{K}'$ are in normal form and, for all axioms α only using symbols from $\langle \Gamma, \Sigma \rangle$, $\mathfrak{K} \models \alpha$ iff $\mathfrak{K}' \models \alpha$.* □

Let us introduce the basic concepts of the proposed calculus. We follow the same presentation given in [10], and we will thus express our rules in the language of *datalog*. A *signature* is a tuple $\langle \mathbf{C}, \mathbf{P} \rangle$, with $\mathbf{C}$ a finite set of *constants* and $\mathbf{P}$ a finite set of *predicates*. We assume a set $\mathbf{V}$ of *variables* and we call *terms* the elements of $\mathbf{C} \cup \mathbf{V}$. An *atom* over $\langle \mathbf{C}, \mathbf{P} \rangle$ is in the form $p(t_1, \dots, t_n)$ with $p \in \mathbf{P}$ and every $t_i \in \mathbf{C} \cup \mathbf{V}$ for $i \in \{1, \dots, n\}$. A *rule* is an expression in the form $B_1, \dots, B_m \to H$ where H and $B_1, \dots, B_m$ are datalog atoms (the *head* and *body* of the rule). A *fact* H is a ground rule with empty body. A *program* P is a finite set of datalog rules. A *ground substitution* σ for $\langle \mathbf{C}, \mathbf{P} \rangle$ is a function $\sigma : \mathbf{V} \to \mathbf{C}$. We define as usual substitutions on atoms and *ground instances* of atoms. A *proof tree* for P is a structure $\langle N, E, \lambda \rangle$ where $\langle N, E \rangle$ is a finite directed tree and λ is a labelling function assigning a ground atom to each node, where: for each $v \in N$, there exists a rule $B_1, \dots, B_m \to H$ in P and a ground substitution σ s.t. (i) $\lambda(v) = \sigma(H)$ and (ii) v has m child nodes w_i in E, with $\lambda(w_i) = \sigma(B_i)$ for $i \in \{1, \dots, m\}$. A ground atom H is a *consequence* of P (denoted $P \models H$) if there exists a proof tree for P with root node r and with $\lambda(r) = H$.

We can now present the definition of our materialization calculus: we instantiate and adapt the general definition given by [10] in order to meet the structure of CKR.

Definition 8 (Materialization calculus K_{ic}). *The materialization calculus K_{ic} is composed by the* input translations I_{glob}, I_{loc}, I_{rl}, *the* deduction rules P_{loc}, P_{rl}, *and* output translation O, *such that:*

- *every input translation I and output translation O are partial functions (defined over axioms in normal form) while deduction rules P are sets of datalog rules;*
- *given an axiom or signature symbol α (and $\mathsf{c} \in \mathsf{N}$), each $I(\alpha)$ (or $I(\alpha, \mathsf{c})$) is either undefined or a set of datalog facts;*
- *given an axiom α and $\mathsf{c} \in \mathsf{N}$, $O(\alpha, \mathsf{c})$ is either undefined or a single datalog fact;*
- *the set of predicates used by each I, P, O is fixed and finite;*
- *all constant symbols in input or output translations for α are signature symbols appearing in (or equal to) α.*

[7] Presented in Table 5 in the Appendix.

Global input rules $I_{glob}(\mathfrak{G})$	
(igl-subctx1)	$\mathsf{C} \in \mathbf{C} \mapsto \{\mathtt{subClass}(\mathsf{C}, \mathsf{Ctx}, \mathsf{gm})\}$
(igl-subctx2)	$\mathsf{c} \in \mathbf{N} \mapsto \{\mathtt{inst}(\mathsf{c}, \mathsf{Ctx}, \mathsf{gm})\}$
Local input rules $I_{loc}(\mathrm{K}_\mathsf{m}, \mathsf{c})$	
(ilc-subevalat)	$eval(A, \mathsf{C}) \sqsubseteq B \mapsto \{\mathtt{subEval}(A, \mathsf{C}, B, \mathsf{c})\}$
(ilc-subevalr)	$eval(R, \mathsf{C}) \sqsubseteq T \mapsto \{\mathtt{subEvalR}(R, \mathsf{C}, T, \mathsf{c})\}$
Local deduction rules P_{loc}	
(plc-subevalat)	$\mathtt{subEval}(a, c_1, b, c), \mathtt{inst}(c', c_1, \mathsf{gm}), \mathtt{inst}(x, a, c') \rightarrow \mathtt{inst}(x, b, c)$
(plc-subevalr)	$\mathtt{subEvalR}(r, c_1, t, c), \mathtt{inst}(c', c_1, \mathsf{gm}), \mathtt{triple}(x, r, y, c') \rightarrow \mathtt{triple}(x, t, y, c)$
(plc-eq)	$\mathtt{nom}(x, c), \mathtt{eq}(x, y, c') \rightarrow \mathtt{eq}(x, y, c)$
Output translation $O(\alpha, \mathsf{c})$	
(o-concept)	$A(a) \mapsto \{\mathtt{inst}(a, A, \mathsf{c})\}$
(o-role)	$R(a, b) \mapsto \{\mathtt{triple}(a, R, b, \mathsf{c})\}$

Table 2. K_{ic} translation and deduction rules

We extend the definition of input translations to knowledge bases (set of axioms) S with their signature Σ, with $I(S) = \bigcup_{\alpha \in S} I(\alpha) \cup \bigcup_{s \in \Sigma, I(s) \text{ defined}} I(s)$ (similarly $I(S, \mathsf{c}) = \bigcup_{\alpha \in S} I(\alpha, \mathsf{c}) \cup \bigcup_{s \in \Sigma, I(s) \text{ defined}} I(s, \mathsf{c})$). The sets of rules defining input translation I_{glob} for global context, input translation I_{loc} and deduction rules P_{loc} for local contexts and output rules O are presented in Table 2. Finally, $\mathcal{SROIQ}$-RL input I_{rl} and deduction P_{rl} rules are presented in Table 3.

In order to introduce the notion of entailment, we define in the following the "translation process" to produce a program that represents the knowledge of the complete input CKR. Let $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_\mathsf{m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ be an input CKR in normal form. Then, let:

$$PG(\mathfrak{G}) = I_{glob}(\mathfrak{G}) \cup P_{rl} \cup I_{rl}(\mathfrak{G}_\Gamma, \mathsf{gm}) \cup I_{rl}(\mathfrak{G}_\Sigma, \mathsf{gk})$$

with gm, gk new, $\mathfrak{G}_\Gamma = \{\alpha \in \mathfrak{G} \mid \alpha \in \mathcal{L}_\Gamma\}$ and $\mathfrak{G}_\Sigma = \{\alpha \in \mathfrak{G} \mid \alpha \in \mathcal{L}_\Sigma\}$. We define the set of contexts $\mathbf{N}_\mathfrak{G} = \{\mathsf{c} \in \mathbf{C} \mid PG(\mathfrak{G}) \models \mathtt{inst}(\mathsf{c}, \mathsf{Ctx}, \mathsf{gm})\}$. For every $\mathsf{c} \in \mathbf{N}_\mathfrak{G}$, we define its associated knowledge base as:

$$\mathrm{K}_\mathsf{c} = \bigcup \{\mathrm{K}_\mathsf{m} \in \mathfrak{K} \mid PG(\mathfrak{G}) \models \mathtt{triple}(\mathsf{c}, \mathsf{mod}, \mathsf{m}, \mathsf{gm})\}$$

We define the program for c as:

$$PC(\mathsf{c}) = P_{loc} \cup I_{loc}(\mathrm{K}_\mathsf{c}, \mathsf{c}) \cup I_{rl}(\mathrm{K}_\mathsf{c}, \mathsf{c}) \cup I_{rl}(\mathfrak{G}_\Sigma, \mathsf{c})$$

Finally, the program for $\mathfrak{K}$ can be encoded as $PK(\mathfrak{K}) = PG(\mathfrak{G}) \cup \bigcup_{\mathsf{c} \in \mathbf{N}_\mathfrak{G}} PC(\mathsf{c})$.

We say that $\mathfrak{G}$ *entails* an axiom $\alpha \in \mathcal{L}_\Gamma$ (denoted $\mathfrak{G} \vdash \alpha$) if $PG(\mathfrak{G})$ and $O(\alpha, \mathsf{gm})$ are defined and $PG(\mathfrak{G}) \models O(\alpha, \mathsf{gm})$. The same can be stated if $\alpha \in \mathcal{L}_\Sigma$, substituting gm with gk. We say that $\mathfrak{K}$ *entails* an axiom $\alpha \in \mathcal{L}^e_\Sigma$ in a context $\mathsf{c} \in \mathbf{N}$ (denoted $\mathfrak{K} \vdash \mathsf{c} : \alpha$) if the elements of $PK(\mathfrak{K})$ and $O(\alpha, \mathsf{c})$ are defined and $PK(\mathfrak{K}) \models O(\alpha, \mathsf{c})$.

The presented rules and translation provide a sound and complete materialization calculus for instance checking (with respect to c-entailment) in $\mathcal{SROIQ}$-RL CKRs in normal form. The result can be verified by extending the proofs in [10] to $\mathcal{SROIQ}$-RL and to the CKR structure. Soundness w.r.t. the global knowledge is established as:

Lemma 2. *Given $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_\mathsf{m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ a CKR in normal form, and $\alpha \in \mathcal{L}_\Gamma$ or $\alpha \in \mathcal{L}_\Sigma$ with α an atomic concept or role assertion, then $\mathfrak{G} \vdash \alpha$ implies $\mathfrak{G} \models \alpha$.* $\square$

RL input translation $I_{rl}(S, c)$

(irl-nom)	$a \in \mathrm{NI} \mapsto \{\mathtt{nom}(a, c)\}$	(irl-not)	$A \sqsubseteq \neg B \mapsto \{\mathtt{supNot}(A, B, c)\}$
(irl-cls)	$A \in \mathrm{NC} \mapsto \{\mathtt{cls}(A, c)\}$	(irl-subcnj)	$A_1 \sqcap A_2 \sqsubseteq B \mapsto \{\mathtt{subConj}(A_1, A_2, B, c)\}$
(irl-rol)	$R \in \mathrm{NR} \mapsto \{\mathtt{rol}(R, c)\}$	(irl-subex)	$\exists R.A \sqsubseteq B \mapsto \{\mathtt{subEx}(R, A, B, c)\}$
(irl-inst1)	$A(a) \mapsto \{\mathtt{inst}(a, A, c)\}$	(irl-supex)	$A \sqsubseteq \exists R.\{a\} \mapsto \{\mathtt{supEx}(A, R, a, c)\}$
(irl-triple)	$R(a, b) \mapsto \{\mathtt{triple}(a, R, b, c)\}$	(irl-forall)	$A \sqsubseteq \forall R.B \mapsto \{\mathtt{supForall}(A, R, B, c)\}$
(irl-ntriple)	$\neg R(a, b) \mapsto \{\mathtt{negtriple}(a, R, b, c)\}$	(irl-leqone)	$A \sqsubseteq \leqslant 1 R.B \mapsto \{\mathtt{supLeqOne}(A, R, B, c)\}$
(irl-eq)	$a = b \mapsto \{\mathtt{eq}(a, b, c)\}$	(irl-subr)	$R \sqsubseteq S \mapsto \{\mathtt{subRole}(R, S, c)\}$
(irl-neq)	$a \neq b \mapsto \{\mathtt{neq}(a, b, c)\}$	(irl-subrc)	$R \circ S \sqsubseteq T \mapsto \{\mathtt{subRChain}(R, S, T, c)\}$
(irl-inst2)	$\{a\} \sqsubseteq B \mapsto \{\mathtt{inst}(a, B, c)\}$	(irl-dis)	$\mathrm{Dis}(R, S) \mapsto \{\mathtt{dis}(R, S, c)\}$
(irl-subc)	$A \sqsubseteq B \mapsto \{\mathtt{subClass}(A, B, c)\}$	(irl-inv)	$\mathrm{Inv}(R, S) \mapsto \{\mathtt{inv}(R, S, c)\}$
(irl-top)	$\top(a) \mapsto \{\mathtt{inst}(a, \mathtt{top}, c)\}$	(irl-irr)	$\mathrm{Irr}(R) \mapsto \{\mathtt{irr}(R, c)\}$
(irl-bot)	$\bot(a) \mapsto \{\mathtt{inst}(a, \mathtt{bot}, c)\}$		

RL deduction rules P_{rl}

(prl-ntriple)	$\mathtt{negtriple}(x, v, y, c), \mathtt{triple}(x, v, y, c) \to \mathtt{inst}(x, \mathtt{bot}, c)$
(prl-eq1)	$\mathtt{nom}(x, c) \to \mathtt{eq}(x, x, c)$
(prl-eq2)	$\mathtt{eq}(x, y, c) \to \mathtt{eq}(y, x, c)$
(prl-eq3)	$\mathtt{eq}(x, y, c), \mathtt{inst}(x, z, c) \to \mathtt{inst}(y, z, c)$
(prl-eq4)	$\mathtt{eq}(x, y, c), \mathtt{triple}(x, u, z, c) \to \mathtt{triple}(y, u, z, c)$
(prl-eq5)	$\mathtt{eq}(x, y, c), \mathtt{triple}(z, u, x, c) \to \mathtt{triple}(z, u, y, c)$
(prl-eq6)	$\mathtt{eq}(x, y, c), \mathtt{eq}(y, z, c) \to \mathtt{eq}(x, z, c)$
(prl-neq)	$\mathtt{eq}(x, y, c), \mathtt{neq}(x, y, c) \to \mathtt{inst}(x, \mathtt{bot}, c)$
(prl-top)	$\mathtt{inst}(x, z, c) \to \mathtt{inst}(x, \mathtt{top}, c)$
(prl-subc)	$\mathtt{subClass}(y, z, c), \mathtt{inst}(x, y, c) \to \mathtt{inst}(x, z, c)$
(prl-not)	$\mathtt{supNot}(y, z, c), \mathtt{inst}(x, y, c), \mathtt{inst}(x, z, c) \to \mathtt{inst}(x, \mathtt{bot}, c)$
(prl-subcnj)	$\mathtt{subConj}(y_1, y_2, z, c), \mathtt{inst}(x, y_1, c), \mathtt{inst}(x, y_2, c) \to \mathtt{inst}(x, z, c)$
(prl-subex)	$\mathtt{subEx}(v, y, z, c), \mathtt{triple}(x, v, x', c), \mathtt{inst}(x', y, c) \to \mathtt{inst}(x, z, c)$
(prl-supex)	$\mathtt{supEx}(y, r, x', c), \mathtt{inst}(x, y, c) \to \mathtt{triple}(x, r, x', c)$
(prl-supforall)	$\mathtt{supForall}(z, r, z', c), \mathtt{inst}(x, z, c), \mathtt{triple}(x, r, y, c) \to \mathtt{inst}(y, z', c)$
(prl-leqone)	$\mathtt{supLeqOne}(z, r, z', c), \mathtt{inst}(x, z, c), \mathtt{triple}(x, r, x_1, c),$ $\mathtt{inst}(x_1, z', c), \mathtt{triple}(x, r, x_2, c), \mathtt{inst}(x_2, z', c) \to \mathtt{eq}(x_1, x_2, c)$
(prl-subr)	$\mathtt{subRole}(v, w, c), \mathtt{triple}(x, v, x', c) \to \mathtt{triple}(x, w, x', c)$
(prl-subrc)	$\mathtt{subRChain}(u, v, w, c), \mathtt{triple}(x, u, y, c), \mathtt{triple}(y, v, z, c) \to \mathtt{triple}(x, w, z, c)$
(prl-dis)	$\mathtt{dis}(u, v, c), \mathtt{triple}(x, u, y, c), \mathtt{triple}(x, v, y, c) \to \mathtt{inst}(x, \mathtt{bot}, c)$
(prl-inv1)	$\mathtt{inv}(u, v, c), \mathtt{triple}(x, u, y, c) \to \mathtt{triple}(y, v, x, c)$
(prl-inv2)	$\mathtt{inv}(u, v, c), \mathtt{triple}(x, v, y, c) \to \mathtt{triple}(y, u, x, c)$
(prl-irr)	$\mathtt{irr}(u, c), \mathtt{triple}(x, u, x, c) \to \mathtt{inst}(x, \mathtt{bot}, c)$

Table 3. RL input and deduction rules

Soundness for global entailment is proved by extending the result to local knowledge.

Theorem 1 (Soundness). *Given* $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ *a CKR in normal form,* $\alpha \in \mathcal{L}_\Sigma$ *an atomic concept or role assertion and* $\mathsf{c} \in \mathbf{N}$*, then* $\mathfrak{K} \vdash \mathsf{c} : \alpha$ *implies* $\mathfrak{K} \models \mathsf{c} : \alpha$. □

Completeness can be verified in a similar way. We say that a CKR $\mathfrak{K}$ is *consistent* if there does not exist $\mathsf{c} \in \mathbf{N} \cup \{\mathsf{gm}, \mathsf{gk}\}$ and $a \in \mathrm{NI}_\Sigma \cup \mathrm{NI}_\Gamma$ s.t. $PK(\mathfrak{K}) \models \mathtt{inst}(a, \mathtt{bot}, \mathsf{c})$.

Lemma 3. *Given* $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ *a consistent CKR in normal form, and* $\alpha \in \mathcal{L}_\Gamma$ *or* $\alpha \in \mathcal{L}_\Sigma$ *with* α *an atomic concept or role assertion, then* $\mathfrak{G} \models \alpha$ *implies* $\mathfrak{G} \vdash \alpha$. □

Theorem 2 (Completeness). *Given* $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ *a consistent CKR in normal form,* $\alpha \in \mathcal{L}_\Sigma$ *an atomic concept or role assertion and* $\mathsf{c} \in \mathbf{N}$*, then* $\mathfrak{K} \models \mathsf{c} : \alpha$ *implies* $\mathfrak{K} \vdash \mathsf{c} : \alpha$. □

5 Concrete Syntax for RDF with Named Graphs

The forward reasoning over CKR expressed by the materialization calculus has been implemented in a prototype. Basically, the prototype accepts RDF input data expressing OWL-RL axioms and assertions for global and local knowledge modules: these different pieces of knowledge are represented as distinct named graphs, while contextual primitives (e.g. contexts and the $eval$ operator) have been encoded in a RDF vocabulary. The prototype is based on an extension of the Sesame 2.6 framework: the main component, called *CKR core* module exposes the CKR primitives and a SPARQL 1.1 endpoint for query and update operations on the contextualized knowledge. The module offers the ability to compute and materialize the inference closure of the input CKR, add and remove knowledge and execute queries over the complete CKR structure.

The distribution of knowledge in different named graphs asks for a module to compute inference over multiple graphs in a RDF store. This component has been realized as a general software layer called *SPRINGLES*[8]. Intuitively, the layer provides methods to demand a closure materialization on the RDF store data: rules are encoded as SPARQL queries and it is possible to customize both the input ruleset and the evaluation strategy.

In our case, the ruleset basically encodes the rules of the presented materialization calculus. As an example, we present the rule dealing with atomic concept inclusions:

```
:prl-subc a spr:Rule ;
  spr:head """ GRAPH ?mx { ?x rdf:type ?z } """ ;
  spr:body """ GRAPH ?m1 { ?y rdfs:subClassOf ?z }
               GRAPH ?m2 { ?x rdf:type ?y }
               GRAPH sys:dep { ?mx sys:derivedFrom ?m1,?m2 }
               FILTER NOT EXISTS
                 { GRAPH ?m0 { ?x rdf:type ?z }
                   GRAPH sys:dep { ?mx sys:derivedFrom ?m0 } } """ .
```

This code corresponds to the local rule (prl-subc) of P_{rl}, thus has the scope of a single context. When the condition in the body part of the rule is verified in graphs `?m1` and `?m2`, the head part is materialized in the inference graph `?mx`. In the rule we work at level of knowledge modules (i.e. named graphs): the first three lines directly correspond to the rule in P_{rl}; in the fourth line we require that module `?mx`, containing the inferences in the given context, depends on the modules of the assumptions. In other words, we require that both rule preconditions and results belong to the KB associated to the same context[9]. The filter expression checks that the results have not been derived yet.

The rules are evaluated with a strategy that basically follows the same steps of the translation process defined for the calculus. Intuitively, the plan goes as follows: (i) we compute the closure on the graph for global context $\mathfrak{G}$, by a fixpoint on rules corresponding to P_{rl}; (ii) we derive associations between contexts and their modules, by adding dependencies for every assertion of the kind $\mathsf{mod}(\mathsf{c}, \mathsf{m})$ in the global closure; (iii) we compute the closure the contexts, by applying rules encoded from P_{rl} and P_{loc} and resolving $eval$ expressions by the metaknowledge information in the global closure.

[8] *SParql-based Rule Inference over Named Graphs Layer Extending Sesame*.

[9] Statements `?mx sys:derivedFrom ?my` are generated from the closure of the global context. For every context, an additional module for its inferences is created and associated to it via `sys:derivedFrom` relation.

A demo of the prototype, containing RDF data that encodes the example CKR $\mathfrak{K}_{tour}$, can be found at `https://dkm.fbk.eu/index.php/CKR-TourismDemo`.

6 Related Works

The interest in representation of contexts in the Semantic Web favored the proposal of several DL-based formalisms. One relevant example is the Two-dimensional description logics of context [8, 9]. It is based on a multimodal extension of one DL with another: the composition of an object language $\mathcal{L}_O$ with a contextual language $\mathcal{L}_C$ results in a combined language $\mathfrak{C}_{\mathcal{L}_O}^{\mathcal{L}_C}$. This allows a clear separation of the metaknowledge and a complex definition of the contextual structure. Contextual modal operators introduced by this logic allow to define references to other contexts, similarly to the $eval$ operator. On the other hand, we note that most of the approaches (including [13]) only consider fixed structures for the representation of metaknowledge, possibly for dealing with the complexity of the combination of languages [7, 14–16]. A comparison of the approaches with respect to the needs for context in Semantic Web can be found in [3].

The presented materialization calculus extends the calculus for instance checking presented in [10] for $\mathcal{SROEL}(\sqcap, \times)$ to the language of OWL RL: in our extension, we also referred to the subsequent work [11] presenting a rule-based calculus for classification in OWL RL.

7 Conclusions

In this paper we presented a revised and extended version of the CKR framework, generalizing the previous formalization in aspects of contextual structure and knowledge propagation. We then proposed a sound and complete materialization calculus for reasoning over the new CKR definition and outlined its implementation based on SPARQL and a concrete syntax in RDF with named graphs.

We remark that the newly proposed version addresses several limitations of CKR in [13]: the global context $\mathfrak{G}$ extends the metaknowledge component of CKR with global object knowledge and, most of all, metaknowledge is not limited by a fixed structure but can be defined as a full DL knowledge base; contextual structure is now independent from the definition of context dimensions and the coverage relation, while general context relations can be explicitly asserted across contexts; the $eval$ operator generalizes the notion of qualified symbols to generic object expressions and contexts classes; finally, we allow multiple modules to be associated to contexts. On the other hand we have still to verify how this increase in representation potential affects the complexity of reasoning in the new representation.

We are currently realizing a different implementation of the calculus based on DL rewritings to Datalog programs [5]. Moreover we plan to study the complexity properties of the new version of the CKR with respect to the previous formalization. Finally, we want to evaluate the different realizations of CKR with respect to reasoning performances and ease of modelling over the same set of contextualized data, as in [2].

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A Appendix: Result Proofs

A.1 Soundness

Lemma 1. *Given $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathsf{M}} \rangle$ a CKR in normal form, and $\alpha \in \mathcal{L}_\Gamma$ or $\alpha \in \mathcal{L}_\Sigma$ with α an atomic concept or role assertion, then $\mathfrak{G} \vdash \alpha$ implies $\mathfrak{G} \models \alpha$.*

Proof. We can basically follow the same proof schema used for proving soundness in [10], by adapting it to the rules of our calculus. By definition, we have that $PG(\mathfrak{G}) = I_{glob}(\mathfrak{G}) \cup P_{rl} \cup I_{rl}(\mathfrak{G}_\Gamma, \mathsf{gm}) \cup I_{rl}(\mathfrak{G}_\Sigma, \mathsf{gk})$. We can assign an interpretation to the ground atoms derived from $PG(\mathfrak{G})$ as follows, where $\mathsf{g} = \mathsf{gm}$ or $\mathsf{g} = \mathsf{gk}$:

- $\mathtt{inst}(a, A, \mathtt{g})$ with $a \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, $A \in \mathrm{NC}_\Gamma \cup \mathrm{NC}_\Sigma$, then $\mathfrak{G} \models A(a)$;
- $\mathtt{inst}(a, \mathtt{top}, \mathtt{g})$ with $a \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, then $\mathfrak{G} \models \top(a)$;
- $\mathtt{inst}(a, \mathtt{bot}, \mathtt{g})$ with $a \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, then $\mathfrak{G} \models \bot(a)$;
- $\mathtt{triple}(a, R, b, \mathtt{g})$ with $a, b \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, $R \in \mathrm{NR}_\Gamma \cup \mathrm{NR}_\Sigma$, then $\mathfrak{G} \models R(a, b)$;
- $\mathtt{eq}(a, b, \mathtt{g})$ with $a, b \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, then $\mathfrak{G} \models a = b$;
- $\mathtt{neq}(a, b, \mathtt{g})$ with $a, b \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$, then $\mathfrak{G} \models a \neq b$;

We claim that, for any ground atom H of the above form with the corresponding semantic condition $C(H)$, $PG(\mathfrak{G}) \models H$ implies $\mathfrak{G} \models C(H)$. We can prove the claim by induction on the possible proof tree of the above atoms H.

- **(prl-ntriple):** then $H = \mathtt{inst}(a, \mathtt{bot}, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{negtriple}(a, R, b, \mathtt{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathtt{g})$. We have that $\neg R(a, b) \in \mathfrak{G}$ and, by the above interpretation of atoms, we obtain that $\mathfrak{G} \models R(a, b)$. This is an absurd, thus there cannot be an interpretation satisfying $\mathfrak{G}$, which justifies the consequence $\mathfrak{G} \models \bot(a)$.
- **(prl-eq1):** then $H = \mathtt{eq}(a, a, \mathtt{g})$ and, by I_{rl} rules, $a \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$. For any model $\mathcal{M}$ of $\mathfrak{G}$, for any $a \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$ it holds that $a^\mathcal{M} = a^\mathcal{M}$, thus this verifies $\mathfrak{G} \models (a = a)$.
- **(prl-eq2):** then $H = \mathtt{eq}(b, a, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{eq}(a, b, \mathtt{g})$. By the above interpretation of atoms, $\mathfrak{G} \models (a = b)$: by symmetricity of equality relation this directly implies that $\mathfrak{G} \models (b = a)$.
- **(prl-eq3):** then $H = \mathtt{inst}(b, B, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{eq}(a, b, \mathtt{g})$, $PG(\mathfrak{G}) \models \mathtt{inst}(a, B, \mathtt{g})$. By the above interpretation of atoms, $\mathfrak{G} \models (a = b)$ and $\mathfrak{G} \models B(a)$. This directly implies that $\mathfrak{G} \models B(b)$, thus proving the assertion.
- **(prl-eq4):** then $H = \mathtt{triple}(a, R, b, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{eq}(c, a, \mathtt{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(c, R, b, \mathtt{g})$. By the above interpretation of atoms, $\mathfrak{G} \models (c = a)$ and $\mathfrak{G} \models R(c, b)$. This directly implies that $\mathfrak{G} \models R(a, b)$. The case for **(prl-eq5)** can be verified similarly.
- **(prl-eq6):** then $H = \mathtt{eq}(a, b, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{eq}(a, c, \mathtt{g})$, $PG(\mathfrak{G}) \models \mathtt{eq}(c, b, \mathtt{g})$. By the above interpretation of atoms, $\mathfrak{G} \models (a = c)$ and $\mathfrak{G} \models (c = b)$: by transitivity of equality relation this directly implies that $\mathfrak{G} \models (a = b)$.
- **(prl-neq):** then $H = \mathtt{inst}(a, \mathtt{bot}, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{neq}(a, b, \mathtt{g})$, $PG(\mathfrak{G}) \models \mathtt{eq}(a, b, \mathtt{g})$. By induction hypothesis and the above semantic conditions, we obtain $\mathfrak{G} \models (a = b)$ and $\mathfrak{G} \models (a \neq b)$. This is an absurd, thus there cannot be an interpretation satisfying $\mathfrak{G}$: this justifies the consequence $\mathfrak{G} \models \bot(a)$.
- **(prl-top):** then $H = \mathtt{inst}(a, \mathtt{top}, \mathtt{g})$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, B, \mathtt{g})$. By induction hypothesis, $\mathfrak{G} \models B(a)$: for every model $\mathcal{M}$ of $\mathfrak{G}$, it holds that $a^\mathcal{M} \in \Delta^\mathcal{M} = \top^\mathcal{M}$, thus it is verified that $\mathfrak{G} \models \top(a)$.

Concept constructors	Syntax	Semantics
atomic concept	A	$A^{\mathcal{I}}$
complement	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
intersection	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
union	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \begin{array}{l} \exists y.\, \langle x, y\rangle \in R^{\mathcal{I}} \\ \wedge\, y \in C^{\mathcal{I}} \end{array} \right\}$
self restriction	$\exists R.\text{Self}$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \langle x, x\rangle \in R^{\mathcal{I}} \right\}$
universal restriction	$\forall R.C$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \begin{array}{l} \forall y.\, \langle x, y\rangle \in R^{\mathcal{I}} \\ \to y \in C^{\mathcal{I}} \end{array} \right\}$
min. card. restriction	$\geqslant nR.C$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \begin{array}{l} \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \\ \wedge\, y \in C^{\mathcal{I}}\} \geq n \end{array} \right\}$
max. card. restriction	$\leqslant nR.C$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \begin{array}{l} \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \\ \wedge\, y \in C^{\mathcal{I}}\} \leq n \end{array} \right\}$
cardinality restriction	$= nR.C$	$\left\{ x \in \Delta^{\mathcal{I}} \,\middle|\, \begin{array}{l} \sharp\{y \mid \langle x, y\rangle \in R^{\mathcal{I}} \\ \wedge\, y \in C^{\mathcal{I}}\} = n \end{array} \right\}$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
Role constructors	Syntax	Semantics
atomic role	R	$R^{\mathcal{I}}$
inverse role	R^{-}	$\left\{ \langle y, x\rangle \,\middle|\, \langle x, y\rangle \in R^{\mathcal{I}} \right\}$
role composition	$S \circ Q$	$\left\{ \langle x, z\rangle \,\middle|\, \langle x, y\rangle \in S^{\mathcal{I}}, \langle y, z\rangle \in Q^{\mathcal{I}} \right\}$
Axioms	Syntax	Semantics
concept inclusion (GCI)	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
concept definition	$C \equiv D$	$C^{\mathcal{I}} = D^{\mathcal{I}}$
role inclusion (RIA)	$S \sqsubseteq R$	$S^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
role disjointness	$\text{Dis}(P, R)$	$P^{\mathcal{I}} \cap R^{\mathcal{I}} = \emptyset$
reflexivity assertion	$\text{Ref}(R)$	$\{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} \subseteq R^{\mathcal{I}}$
irreflexivity assertion	$\text{Irr}(R)$	$R^{\mathcal{I}} \cap \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} = \emptyset$
symmetry assertion	$\text{Sym}(R)$	$\langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow \langle y, x\rangle \in R^{\mathcal{I}}$
asymmetry assertion	$\text{Asym}(R)$	$\langle x, y\rangle \in R^{\mathcal{I}} \Rightarrow \langle y, x\rangle \notin R^{\mathcal{I}}$
transitivity assertion	$\text{Tra}(R)$	$\{\langle x, y\rangle, \langle y, z\rangle\} \subseteq R^{\mathcal{I}} \Rightarrow \langle x, z\rangle \in R^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$R(a, b)$	$\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$
negated role assertion	$\neg R(a, b)$	$\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \notin R^{\mathcal{I}}$
equality assertion	$a = b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
inequality assertion	$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$

Table 4. Syntax and Semantics of $\mathcal{SROIQ}$

$C(a) \mapsto \{X(a), X \sqsubseteq C\}$	$\mathrm{Sym}(P) \mapsto \{P \sqsubseteq W, \mathrm{Inv}(P, W)\}$
$C \sqsubseteq D \mapsto \{C \sqsubseteq X, X \sqsubseteq D\}$	$\mathrm{Trans}(P) \mapsto \{P \circ P \sqsubseteq P\}$
$A \sqsubseteq \top \mapsto \emptyset$	$\mathrm{Asym}(P) \mapsto \{\mathrm{Dis}(P, W), \mathrm{Inv}(P, W)\}$
$\bot \sqsubseteq A \mapsto \emptyset$	
$A \sqsubseteq \neg C \mapsto \{A \sqsubseteq \neg X, C \sqsubseteq X\}$	$eval(D, \mathsf{C}) \sqsubseteq B \in \mathrm{K}_{\mathsf{m}} \mapsto \{eval(X, Y) \sqsubseteq B \in \mathrm{K}_{\mathsf{m}}, D \sqsubseteq X \in \mathrm{K}_{\mathsf{mx}}, \mathsf{C} \sqsubseteq Y \in \mathfrak{G}, Y \sqsubseteq \exists \mathsf{mod}.\{\mathsf{mx}\} \in \mathfrak{G}\}$
$C \sqcap A \sqsubseteq B \mapsto \{C \sqsubseteq X, X \sqcap A \sqsubseteq B\}$	
$A \sqsubseteq C \sqcap D \mapsto \{A \sqsubseteq C, A \sqsubseteq D\}$	$eval(R, \mathsf{C}) \sqsubseteq T \in \mathrm{K}_{\mathsf{m}} \mapsto \{eval(R, Y) \sqsubseteq T \in \mathrm{K}_{\mathsf{m}}, \mathsf{C} \sqsubseteq Y \in \mathfrak{G}\}$
$C \sqcup D \sqsubseteq B \mapsto \{C \sqsubseteq B, D \sqsubseteq B\}$	
$\exists R.C \sqsubseteq A \mapsto \{C \sqsubseteq X, \exists R.X \sqsubseteq A\}$	$\exists eval(R, \mathsf{C}).A \sqsubseteq B \in \mathrm{K}_{\mathsf{m}} \mapsto \{\exists W.A \sqsubseteq B \in \mathrm{K}_{\mathsf{m}}, eval(R, \mathsf{C}) \sqsubseteq W \in \mathrm{K}_{\mathsf{m}}\}$
$A \sqsubseteq \exists R.C \mapsto \{A \sqsubseteq \exists R.X, C \sqsubseteq X\}$	
$A \sqsubseteq \forall R.D \mapsto \{A \sqsubseteq \forall R.X, X \sqsubseteq D\}$	$eval(R, \mathsf{C}) \circ S \sqsubseteq T \in \mathrm{K}_{\mathsf{m}} \mapsto \{eval(R, \mathsf{C}) \sqsubseteq W \in \mathrm{K}_{\mathsf{m}}, W \circ S \sqsubseteq T \in \mathrm{K}_{\mathsf{m}}\}$
$A \sqsubseteq \leqslant 0R.D \mapsto \{A \sqsubseteq \forall R.\neg D\}$	
$A \sqsubseteq \leqslant 1R.D \mapsto \{A \sqsubseteq \leqslant 1R.X, D \sqsubseteq X\}$	$\mathrm{Dis}(eval(R, \mathsf{C}), S) \in \mathrm{K}_{\mathsf{m}} \mapsto \{eval(R, \mathsf{C}) \sqsubseteq W \in \mathrm{K}_{\mathsf{m}}, \mathrm{Dis}(W, S) \in \mathrm{K}_{\mathsf{m}}\}$

With $A, B \in \mathrm{NC}$, $R, S, T \in \mathrm{NR}$, $X, Y \in \mathrm{NC}$ fresh concept names, $W \in \mathrm{NR}$ a fresh role name, mx a fresh module name and K_{mx} its associated knowledge base, C, D, C concept expressions with $C, D, \mathsf{C} \notin \mathrm{NC}$.

Table 5. Normal form transformation

- **(prl-subc):** then $H = \mathtt{inst}(a, B, \mathrm{g})$, $A \sqsubseteq B \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$. By the above semantic conditions, $\mathfrak{G} \models A(a)$: this directly implies that $\mathfrak{G} \models B(a)$.
- **(prl-not):** then $H = \mathtt{inst}(a, \mathtt{bot}, \mathrm{g})$, $A \sqsubseteq \neg B \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$, $PG(\mathfrak{G}) \models \mathtt{inst}(a, B, \mathrm{g})$. By induction hypothesis, $\mathfrak{G} \models A(a)$ and $\mathfrak{G} \models B(a)$: this is an absurd, since the first consequence would imply that $\mathfrak{G} \models (\neg B)(a)$. Thus there can not be an interpretation satisfying $\mathfrak{G}$, which justifies the consequence $\mathfrak{G} \models \bot(a)$.
- **(prl-subcnj):** then $H = \mathtt{inst}(a, B, \mathrm{g})$, $A \sqcap C \sqsubseteq B \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$, $PG(\mathfrak{G}) \models \mathtt{inst}(a, C, \mathrm{g})$. By the above semantic conditions, $\mathfrak{G} \models A(a)$ and $\mathfrak{G} \models C(a)$: this directly implies that $\mathfrak{G} \models (A \sqcap C)(a)$, and thus $\mathfrak{G} \models B(a)$.
- **(prl-subex):** then $H = \mathtt{inst}(a, B, \mathrm{g})$, $\exists R.A \sqsubseteq B \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathrm{g})$, $PG(\mathfrak{G}) \models \mathtt{inst}(b, A, \mathrm{g})$. By induction hypothesis, this implies that $\mathfrak{G} \models R(a, b)$ and $\mathfrak{G} \models A(b)$: by definition of the semantics, this proves that $\mathfrak{G} \models (\exists R.A)(a)$ which implies $\mathfrak{G} \models B(a)$.
- **(prl-supex):** then $H = \mathtt{triple}(a, R, b, \mathrm{g})$, $A \sqsubseteq \exists R.\{b\} \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$. By induction hypothesis, $\mathfrak{G} \models A(a)$: this implies that, for every model $\mathcal{M}$ of $\mathfrak{G}$, $a^{\mathcal{M}} \in (\exists R.\{b\})^{\mathcal{M}}$, that is $\langle a^{\mathcal{M}}, b^{\mathcal{M}} \rangle \in R^{\mathcal{M}}$. This proves that $\mathfrak{G} \models R(a, b)$.
- **(prl-supforall):** then $H = \mathtt{inst}(b, B, \mathrm{g})$, $A \sqsubseteq \forall R.B \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathrm{g})$. By induction hypothesis, $\mathfrak{G} \models A(a)$ and $\mathfrak{G} \models R(a, b)$: this implies that, for every model $\mathcal{M}$ of $\mathfrak{G}$, $a^{\mathcal{M}} \in (\forall R.B)^{\mathcal{M}}$, and thus $b^{\mathcal{M}} \in B^{\mathcal{M}}$. This proves that $\mathfrak{G} \models B(b)$.
- **(prl-leqone):** then $H = \mathtt{eq}(b, c, \mathrm{g})$, $A \sqsubseteq \leqslant 1R.B \in \mathfrak{G}$. Moreover, $PG(\mathfrak{G}) \models \mathtt{inst}(a, A, \mathrm{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathrm{g})$ with $PG(\mathfrak{G}) \models \mathtt{inst}(b, B, \mathrm{g})$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, c, \mathrm{g})$ with $PG(\mathfrak{G}) \models \mathtt{inst}(c, B, \mathrm{g})$. By induction hypothesis, $\mathfrak{G} \models A(a)$ and thus $\mathfrak{G} \models (\leqslant 1R.B)(a)$. Moreover, $\mathfrak{G} \models R(a, b)$ and

$\mathfrak{G} \models R(a,c)$ with $\mathfrak{G} \models B(b)$ and $\mathfrak{G} \models B(c)$. By definition of the semantics, for every model $\mathcal{M}$ of $\mathfrak{G}$, it holds that $b^{\mathcal{M}} = c^{\mathcal{M}}$, which implies $\mathfrak{G} \models (b = c)$.
- **(prl-subr):** then $H = \mathtt{triple}(a, S, b, \mathsf{g})$, $R \sqsubseteq S \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathsf{g})$. By the above semantic constraints, $\mathfrak{G} \models R(a,b)$ which directly implies $\mathfrak{G} \models S(a,b)$.
- **(prl-subrc):** then $H = \mathtt{triple}(a, T, b, \mathsf{g})$, $R \circ S \sqsubseteq T \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, c, \mathsf{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(c, S, b, \mathsf{g})$. By the above semantic constraints, $\mathfrak{G} \models R(a,c)$ and $\mathfrak{G} \models S(c,b)$: by definition of the semantics, this implies that $\mathfrak{G} \models T(a,b)$.
- **(prl-dis):** then $H = \mathtt{inst}(a, \mathtt{bot}, \mathsf{g})$, $\mathrm{Dis}(R,S) \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, b, \mathsf{g})$, $PG(\mathfrak{G}) \models \mathtt{triple}(a, S, b, \mathsf{g})$. By the above semantic constraints, $\mathfrak{G} \models R(a,b)$ and $\mathfrak{G} \models S(a,b)$: this is an absurd, thus there cannot be an interpretation satisfying $\mathfrak{G}$, which justifies the consequence $\mathfrak{G} \models \bot(a)$.
- **(prl-inv1):** then $H = \mathtt{triple}(a, S, b, \mathsf{g})$, $\mathrm{Inv}(R,S) \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(b, R, a, \mathsf{g})$. By the above semantic constraints, $\mathfrak{G} \models R(b,a)$: this directly implies that $\mathfrak{G} \models S(a,b)$. The case for **(prl-inv2)** can be proved similarly.
- **(prl-irr):** then $H = \mathtt{inst}(a, \mathtt{bot}, \mathsf{g})$, $\mathrm{Irr}(R) \in \mathfrak{G}$ and $PG(\mathfrak{G}) \models \mathtt{triple}(a, R, a, \mathsf{g})$. By induction hypothesis this would imply that $\mathfrak{G} \models R(a,a)$, which is an absurd: thus there cannot be an interpretation satisfying $\mathfrak{G}$, which justifies the consequence $\mathfrak{G} \models \bot(a)$.

□

Theorem 1 (Soundness). *Given $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ a CKR in normal form, $\alpha \in \mathcal{L}_\Sigma$ an atomic concept or role assertion and $\mathsf{c} \in \mathbf{N}$, then $\mathfrak{K} \vdash \mathsf{c} : \alpha$ implies $\mathfrak{K} \models \mathsf{c} : \alpha$.*

Proof. To prove the assertion, we extend the construction of the previous Lemma 1 to the local interpretations for contexts and the definition of the program representing the whole input CKR.

By definition, $PK(\mathfrak{K}) = PG(\mathfrak{G}) \cup \bigcup_{\mathsf{c} \in \mathbf{N}_\mathfrak{G}} PC(\mathsf{c})$ where, for every $\mathsf{c} \in \mathbf{N}_\mathfrak{G}$, $PC(\mathsf{c}) = P_{loc} \cup I_{loc}(\mathrm{K_c}, \mathsf{c}) \cup I_{rl}(\mathrm{K_c}, \mathsf{c}) \cup I_{rl}(\mathfrak{G}_\Sigma, \mathsf{c})$.

For Lemma 1, we can also easily derive that, for every interpretation $\mathcal{M}$ such that $\mathcal{M} \models \mathfrak{G}$: if $\mathsf{c} \in \mathbf{N}_\mathfrak{G}$ (that is, if $PG(\mathfrak{G}) \models \mathtt{inst}(\mathsf{c}, \mathsf{Ctx}, \mathsf{gm})$) then $\mathsf{c}^\mathcal{M} \in \mathsf{Ctx}^\mathcal{M}$; if $\mathrm{K_m} \in \mathrm{K_c}$ (that is, if $PG(\mathfrak{G}) \models \mathtt{triple}(\mathsf{c}, \mathsf{mod}, \mathsf{m}, \mathsf{gm})$) then $\langle \mathsf{c}^\mathcal{M}, \mathsf{m}^\mathcal{M} \rangle \in \mathsf{mod}^\mathcal{M}$.

As in previous lemma, we can assign a semantic constraint to the ground atoms derived from $PK(\mathfrak{K})$ as follows, where $\mathsf{c} \in \mathbf{N}_\mathfrak{G}$:

- $\mathtt{inst}(a, A, \mathsf{c})$ with $a \in \mathrm{NI}_\Sigma$, $A \in \mathrm{NC}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : A(a)$;
- $\mathtt{inst}(a, \mathtt{top}, \mathsf{c})$ with $a \in \mathrm{NI}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : \top(a)$;
- $\mathtt{inst}(a, \mathtt{bot}, \mathsf{c})$ with $a \in \mathrm{NI}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : \bot(a)$;
- $\mathtt{triple}(a, R, b, \mathsf{c})$ with $a, b \in \mathrm{NI}_\Sigma$, $R \in \mathrm{NR}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : R(a,b)$;
- $\mathtt{eq}(a, b, \mathsf{c})$ with $a, b \in \mathrm{NI}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : a = b$;
- $\mathtt{neq}(a, b, \mathsf{c})$ with $a, b \in \mathrm{NI}_\Sigma$, then $\mathfrak{K} \models \mathsf{c} : a \neq b$;

We claim that, for any ground atom H of the above form with the corresponding semantic condition $C(H)$, $PK(\mathfrak{K}) \models H$ implies $\mathfrak{K} \models \mathsf{c} : C(H)$. We show the claim by induction on the possible proof tree of the above atoms H: the cases for the rules in P_{rl} are analogous to what has been shown in the previous lemma, thus we only have to prove the assertion for the rules in P_{loc}.

– **(plc-subeval):** then $H = \texttt{inst}(a, B, \mathsf{c})$ and $PK(\mathfrak{K}) \models \texttt{subEval}(A, \mathsf{C}, B, \mathsf{c})$, $PK(\mathfrak{K}) \models \texttt{inst}(\mathsf{c}', \mathsf{C}, B, \mathsf{gm})$ and $PK(\mathfrak{K}) \models \texttt{inst}(a, A, \mathsf{c}')$. By induction hypothesis and Lemma 1, $\mathfrak{G} \models \mathsf{C}(\mathsf{c}')$, $\mathfrak{K} \models \mathsf{c}' : A(a)$ and $\mathfrak{K} \models \mathsf{c} : eval(A, \mathsf{C}) \sqsubseteq B$. Hence, for every model $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ of $\mathfrak{K}$, $\bigcup_{e \in \mathsf{C}^{\mathcal{M}}} A^{\mathcal{I}(e)} \subseteq B^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})}$ then $A^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})} \subseteq B^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})}$ and thus $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \models B(a)$ which means $\mathfrak{K} \models \mathsf{c} : B(a)$.
– **(plc-subevalr):** then $H = \texttt{triple}(a, S, b, \mathsf{c})$ and $PK(\mathfrak{K}) \models \texttt{subEvalR}(R, \mathsf{C}, S, \mathsf{c})$, $PK(\mathfrak{K}) \models \texttt{inst}(\mathsf{c}', \mathsf{C}, B, \mathsf{gm})$ and $PK(\mathfrak{K}) \models \texttt{triple}(a, R, b, \mathsf{c}')$. By induction hypothesis and Lemma 1, $\mathfrak{G} \models \mathsf{C}(\mathsf{c}')$, $\mathfrak{K} \models \mathsf{c}' : R(a, b)$. and $\mathfrak{K} \models \mathsf{c} : eval(R, \mathsf{C}) \sqsubseteq S$. Hence, for every model $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ of $\mathfrak{K}$, $\bigcup_{e \in \mathsf{C}^{\mathcal{M}}} R^{\mathcal{I}(e)} \subseteq S^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})}$ then $R^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})} \subseteq S^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})}$ and thus $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \models S(a, b)$ which means $\mathfrak{K} \models \mathsf{c} : S(a, b)$.
– **(plc-eq):** then $H = \texttt{eq}(a, b, \mathsf{c})$, $PK(\mathfrak{K}) \models \texttt{nom}(a, \mathsf{c})$ and $PK(\mathfrak{K}) \models \texttt{eq}(a, b, \mathsf{c}')$. By induction hypothesis, by rules in I_{rl} we have $a \in \mathrm{NI}_\Sigma$ and $\mathfrak{K} \models \mathsf{c}' : (a = b)$. Then, for every model $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ of $\mathfrak{K}$, we have that $a^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})} = b^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})}$. By the condition on local interpretation of individuals in the definition of CKR model, we have that $a^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})} = a^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})} = b^{\mathcal{I}(\mathsf{c}'^{\mathcal{M}})} = b^{\mathcal{I}(\mathsf{c}^{\mathcal{M}})}$. Thus it holds that $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \models (a = b)$ which means $\mathfrak{K} \models \mathsf{c} : (a = b)$. □

A.2 Completeness

Lemma 2. *Let $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K}_{\mathsf{m}}\}_{\mathsf{m} \in \mathsf{M}} \rangle$ be a CKR in normal form and $PK(\mathfrak{K})$ its associated program. We define the equivalence relation $\approx$ on the Herbrand universe of $PK(\mathfrak{K})$ as the reflexive, symmetric and transitive closure of*

$$\{\langle a, b \rangle \mid PK(\mathfrak{K}) \models \texttt{eq}(a, b, c),\ \textit{for } a, b, c \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma\}$$

Given $a, b, c, d \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$ with $a \approx b$, it holds that:

(i) if $PK(\mathfrak{K}) \models \texttt{inst}(a, A, c)$, then $PK(\mathfrak{K}) \models \texttt{inst}(b, A, c)$;
(ii) if $PK(\mathfrak{K}) \models \texttt{triple}(a, R, d, c)$, then $PK(\mathfrak{K}) \models \texttt{triple}(b, R, d, c)$;
(iii) if $PK(\mathfrak{K}) \models \texttt{triple}(d, R, a, c)$, then $PK(\mathfrak{K}) \models \texttt{triple}(d, R, b, c)$;
(iv) if $PK(\mathfrak{K}) \models \texttt{inst}(d, A, a)$, then $PK(\mathfrak{K}) \models \texttt{inst}(d, A, b)$;
(v) if $PK(\mathfrak{K}) \models \texttt{triple}(c, R, d, a)$, then $PK(\mathfrak{K}) \models \texttt{triple}(c, R, d, b)$;

Proof. By rules (prl-eq2) and (prl-eq3), it follows immediately that if $PK(\mathfrak{K}) \models \texttt{eq}(a, b, c)$ then $PK(\mathfrak{K}) \models \texttt{inst}(a, A, c)$ iff $PK(\mathfrak{K}) \models \texttt{inst}(b, A, c)$. This also proves point (i) of the assertion. By rule (prl-eq2) and (prl-eq4), we can derive that if $PK(\mathfrak{K}) \models \texttt{eq}(a, b, c)$, then $PK(\mathfrak{K}) \models \texttt{triple}(a, R, d, c)$ iff $PK(\mathfrak{K}) \models \texttt{triple}(b, R, d, c)$, proving point (ii). Point (iii) can be proved similarly by rules (prl-eq2) and (prl-eq5).

For point (iv), let us assume that $PK(\mathfrak{K}) \models \texttt{inst}(d, A, a)$ with $a \neq c$ (otherwise the assertion is immediate). Then, by the definition of the program, it must be that $PG(\mathfrak{G}) \models \texttt{eq}(a, b, \mathsf{gm})$. By rules (prl-eq2)-(prl-eq5), in praticular this implies that $PG(\mathfrak{G}) \models \texttt{triple}(a, \mathsf{mod}, \mathsf{m}, \mathsf{gm})$ iff $PG(\mathfrak{G}) \models \texttt{triple}(b, \mathsf{mod}, \mathsf{m}, \mathsf{gm})$, meaning that, by definition of the translation, they have analogous local programs $PC(a)$ and $PC(b)$ (in which only the "context argument" in the atoms translated by I_{rl} and I_{loc} changes). Thus, we obtain that $PK(\mathfrak{K}) \models \texttt{inst}(d, A, b)$. The proof for point (v) follows from similar reasoning. □

Lemma 3. *Given $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m}\in\mathbf{M}}\rangle$ a consistent CKR in normal form, and $\alpha \in \mathcal{L}_\Gamma$ or $\alpha \in \mathcal{L}_\Sigma$ with α an atomic concept or role assertion, then $\mathfrak{G} \models \alpha$ implies $\mathfrak{G} \vdash \alpha$.*

Proof. We show by contrapositive that: $\mathfrak{G} \not\vdash \alpha$ implies $\mathfrak{G} \not\models \alpha$. Assuming that $\mathfrak{G} \not\vdash \alpha$, then we have by definition that $PG(\mathfrak{G}) \not\models O(\alpha, \mathsf{gm})$ if $\alpha \in \mathcal{L}_\Gamma$ or $PG(\mathfrak{G}) \not\models O(\alpha, \mathsf{gk})$ if $\alpha \in \mathcal{L}_\Sigma$. Let us assume that $\alpha \in \mathcal{L}_\Gamma$ (the other case can be proved similarly). Then there exists an Herbrand model $\mathcal{H}$ of $PG(\mathfrak{G})$ such that $\mathcal{H} \not\models O(\alpha, \mathsf{gm})$. We show that from this model for $PG(\mathfrak{G})$ we can build a model $\mathcal{M}$ for $\mathfrak{G}$ (meeting the definition of the CKR model for the global interpretation) such that $\mathcal{M} \not\models \alpha$, which allow us to derive that $\mathfrak{G} \not\models \alpha$.

Let us consider the equivalence relation $\approx$ as defined in Lemma 2. We define the equivalence classes $[c] = \{d \mid d \approx c\}$, that will be used to define the domain of the built interpretation.

Then $\mathcal{M} = \langle \Delta^\mathcal{M}, \cdot^\mathcal{M}\rangle$ is defined as follows:

- $\Delta^\mathcal{M} = \{[c] \mid c \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma\}$;
- For each $e \in \Delta^\mathcal{M}$, we define the projection function $\iota(e)$ such that, if $e = [c]$, then $\iota(e) = b$ with a fixed $b \in [c]$;
- $c^\mathcal{M} = [c]$, for every $c \in \mathrm{NI}_\Gamma \cup \mathrm{NI}_\Sigma$;
- $A^\mathcal{M} = \{d \in \Delta^\mathcal{M} \mid \mathcal{H} \models \mathtt{inst}(\iota(d), A, \mathsf{g})\}$, for every $A \in \mathrm{NC}_\Gamma \cup \mathrm{NC}_\Sigma$, with $\mathsf{g} = \mathsf{gm}$ or $\mathsf{g} = \mathsf{gk}$;
- $R^\mathcal{M}$ is the smallest set such that $\langle d, d'\rangle \in R^\mathcal{M}$ if one of the following conditions hold:
  - $\mathcal{H} \models \mathtt{triple}(\iota(d), R, \iota(d'), \mathsf{g})$ with $\mathsf{g} = \mathsf{gm}$ or $\mathsf{g} = \mathsf{gk}$;
  - $S \sqsubseteq R \in \mathfrak{G}$ and $\langle d, d'\rangle \in S^\mathcal{M}$;
  - $S \circ T \sqsubseteq R \in \mathfrak{G}$ and $\langle d, e\rangle \in S^\mathcal{M}$, $\langle e, d'\rangle \in T^\mathcal{M}$ for $e \in \Delta^\mathcal{M}$;
  - $\mathrm{Inv}(R, S) \in \mathfrak{G}$ or $\mathrm{Inv}(S, R) \in \mathfrak{G}$ and $\langle d', d\rangle \in S^\mathcal{M}$;

Note that by Lemma 2, the definition of $\mathcal{M}$ does not depend on the choice of the $\iota([c]) \in [c]$. It is easy to see that, given $\alpha \in \mathcal{L}_\Gamma$ with $\mathcal{H} \not\models O(\alpha, \mathsf{gm})$, then $\mathcal{M} \not\models \alpha$. For example, if $\alpha = \mathsf{C}(\mathsf{a})$, then $\mathcal{H} \not\models \mathtt{inst}(\mathsf{a}, \mathsf{C}, \mathsf{gm})$ which implies by definition that $\mathcal{M} \not\models \mathsf{C}(\mathsf{a})$.

In order to show that $\mathcal{M}$ is a model for $\mathfrak{G}$, we have to prove that $\mathcal{M}$ satisfies the definition of global model from the definition of CKR model, and in particular that $\mathcal{M} \models \mathfrak{G}$. We easily prove that $\mathbf{N}^\mathcal{M} \subseteq \mathsf{Ctx}^\mathcal{M}$: by the definition of rule (igl-subctx2), for every $\mathsf{c} \in \mathbf{N}$ we have $\mathcal{H} \models \mathtt{inst}(\mathsf{c}, \mathsf{Ctx}, \mathsf{gm})$, which implies $\mathsf{c}^\mathcal{M} \in \mathsf{Ctx}^\mathcal{M}$. The condition $\mathsf{C}^\mathcal{M} \subseteq \mathsf{Ctx}^\mathcal{M}$ for every $\mathsf{C} \in \mathbf{C}$ can be shown similarly by the rule (igl-subctx1). To prove that $\mathcal{M} \models \mathfrak{G}$, we proceed by cases and consider the form of all of the axioms $\beta \in \mathcal{L}_\Gamma$ or $\beta \in \mathcal{L}_\Sigma$ that can appear in $\mathfrak{G}$.

- Let $\beta = A(a) \in \mathfrak{G}$, then $\mathcal{H} \models \mathtt{inst}(a, A, \mathsf{g})$[10]. This directly implies that $a^\mathcal{M} = [a] \in A^\mathcal{M}$.
- Let $\beta = R(a, b) \in \mathfrak{G}$, then $\mathcal{H} \models \mathtt{triple}(a, R, b, \mathsf{g})$. By definition, we directly have that $\langle [a], [b]\rangle \in R^\mathcal{M}$.
- Let $\beta = \neg R(a, b) \in \mathfrak{G}$, then $\mathcal{H} \models \mathtt{negtriple}(a, R, b, \mathsf{g})$. Suppose that $\langle [a], [b]\rangle \in R^\mathcal{M}$, then $\mathcal{H} \models \mathtt{triple}(\iota([a]), R, \iota([b]), \mathsf{g})$. By rule (prl-ntriple) and Lemma 2, this would imply that $\mathcal{H} \models \mathtt{inst}(\iota([a]), \mathtt{bot}, \mathsf{g})$ contradicting our assumptions on the consistency of $\mathfrak{K}$. Thus $\langle [a], [b]\rangle \notin R^\mathcal{M}$ as required.

[10] In the proof of these cases, for simplicity of notation, we assume $\mathsf{g} = \mathsf{gm}$ or $\mathsf{g} = \mathsf{gk}$.

- Let $\beta = (a = b) \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{eq}(a, b, \texttt{g})$. By the definition of $\approx$, it holds that $a \approx b$, thus $\{a, b\} \subseteq [a]$ and $a^{\mathcal{M}} = b^{\mathcal{M}} = [a]$.
- Let $\beta = (a \neq b) \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{neq}(a, b, \texttt{g})$. Suppose that $a^{\mathcal{M}} = b^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{eq}(\iota([a]), \iota([b]), \texttt{g})$. By rule (prl-neq) and Lemma 2, we would obtain that $\mathcal{H} \models \texttt{inst}(\iota([a]), \texttt{bot}, \texttt{g})$. Again, this contradicts our assumptions on the consistency of $\mathfrak{K}$. Thus $a^{\mathcal{M}} \neq b^{\mathcal{M}}$ as required.
- Let $\beta = \{a\} \sqsubseteq B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{inst}(a, B, \texttt{g})$. This case can be proved similarly to the case $\beta = A(a)$.
- Let $\beta = A \sqsubseteq B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{subClass}(A, B, \texttt{g})$. If $d \in A^{\mathcal{M}}$, then by definition $\mathcal{H} \models \texttt{inst}(\iota(d), A, \texttt{g})$: by rule (prl-subc) we obtain that $\mathcal{H} \models \texttt{inst}(\iota(d), B, \texttt{g})$ and thus $d \in B^{\mathcal{M}}$.
- Let $\beta = \top(a) \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{inst}(a, \texttt{top}, \texttt{g})$. As in the case for $\beta = A(a)$ (in which this case is subsumed), we directly obtain that $a^{\mathcal{M}} = [a] \in \top^{\mathcal{M}}$.
- Let $\beta = \bot(a) \in \mathfrak{G}$. Assuming that $\mathfrak{K}$ in input is consistent, this case can not subsist as we would directly derive that $\mathcal{H} \models \texttt{inst}(a, \texttt{bot}, \texttt{g})$, showing the inconsistency of $\mathfrak{K}$.
- Let $\beta = A \sqsubseteq \neg B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{supNot}(A, B, \texttt{g})$. Suppose that $d \in A^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{inst}(\iota(d), A, \texttt{g})$. Moreover, suppose that $d \in B^{\mathcal{M}}$: this implies that $\mathcal{H} \models \texttt{inst}(\iota(d), B, \texttt{g})$. By rule (prl-not) and Lemma 2, we would obtain that $\mathcal{H} \models \texttt{inst}(\iota(d), \texttt{bot}, \texttt{g})$. This contradicts our assumptions on the consistency of $\mathfrak{K}$, thus $d \notin B^{\mathcal{M}}$ as required.
- Let $\beta = A_1 \sqcap A_2 \sqsubseteq B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{subConj}(A_1, A_2, B, \texttt{g})$. If $d \in A_1^{\mathcal{M}}$ and $d \in A_2^{\mathcal{M}}$, then by definition $\mathcal{H} \models \texttt{inst}(\iota(d), A_1, \texttt{g})$ and $\mathcal{H} \models \texttt{inst}(\iota(d), A_2, \texttt{g})$. By rule (prl-subconj), we directly obtain that $\mathcal{H} \models \texttt{inst}(\iota(d), B, \texttt{g})$: thus $d \in B^{\mathcal{M}}$ as required.
- Let $\beta = \exists R.A \sqsubseteq B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{subEx}(R, A, B, \texttt{g})$. Let $d \in (\exists R.A)^{\mathcal{M}}$: by definition of the semantics this means that there exists $d' \in A^{\mathcal{M}}$ such that $\langle d, d' \rangle \in R^{\mathcal{M}}$. Thus, $\mathcal{H} \models \texttt{inst}(\iota(d'), A, \texttt{g})$ and $\mathcal{H} \models \texttt{triple}(\iota(d), R, \iota(d'), \texttt{g})$. By rule (prl-subex), we obtain that $\mathcal{H} \models \texttt{inst}(\iota(d), B, \texttt{g})$: thus $d \in B^{\mathcal{M}}$ as required.
- Let $\beta = A \sqsubseteq \exists R.\{a\} \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{supEx}(A, R, a, \texttt{g})$. Let $d \in A^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{inst}(\iota(d), A, \texttt{g})$. By rule (prl-subex), this implies that $\mathcal{H} \models \texttt{triple}(\iota(d), R, a, \texttt{g})$: this proves $\langle d, a^{\mathcal{M}} \rangle \in R^{\mathcal{M}}$ as required.
- Let $\beta = A \sqsubseteq \forall R.B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{supForall}(A, R, B, \texttt{g})$. Let $d \in A^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{inst}(\iota(d), A, \texttt{g})$. Supposing that there exists $d' \in \Delta^{\mathcal{M}}$ such that $\langle d, d' \rangle \in R^{\mathcal{M}}$, we have that $\mathcal{H} \models \texttt{triple}(\iota(d), R, \iota(d'), \texttt{g})$. By rule (prl-supforall) this implies that $\mathcal{H} \models \texttt{inst}(\iota(d'), B, \texttt{g})$, thus proving $d' \in B^{\mathcal{M}}$ as required.
- Let $\beta = A \sqsubseteq \leqslant 1R.B \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{supLeqOne}(A, R, B, \texttt{g})$. Let $d \in A^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{inst}(\iota(d), A, \texttt{g})$. Suppose that there exist $d_1, d_2 \in \Delta^{\mathcal{M}}$ such that $\langle d, d_1 \rangle \in R^{\mathcal{M}}$ and $\langle d, d_2 \rangle \in R^{\mathcal{M}}$, and $\{d_1, d_2\} \subseteq B^{\mathcal{M}}$. Thus $\mathcal{H} \models \{\texttt{triple}(\iota(d), R, \iota(d_1), \texttt{g}), \texttt{triple}(\iota(d), R, \iota(d_2), \texttt{g}), \texttt{inst}(\iota(d_1), B, \texttt{g}), \texttt{inst}(\iota(d_2), B, \texttt{g})\}$. By (prl-leqone) rule we obtain that $\mathcal{H} \models \texttt{eq}(\iota(d_1), \iota(d_2), \texttt{g})$. This implies that $\iota(d_1) \approx \iota(d_2)$ and thus they are interpreted as the same domain element $d_1 = d_2$ in $\mathcal{M}$.
- The cases for $\beta = R \sqsubseteq S, R \circ S \sqsubseteq T$ and $\mathrm{Inv}(R, S)$ follow directly from the interpretation of roles in $\mathcal{M}$.
- Let $\beta = \mathrm{Dis}(R, S) \in \mathfrak{G}$, then $\mathcal{H} \models \texttt{dis}(R, S, \texttt{g})$. Suppose that $\langle d, d' \rangle \in R^{\mathcal{M}}$ and $\langle d, d' \rangle \in S^{\mathcal{M}}$. Then $\mathcal{H} \models \texttt{triple}(\iota(d), R, \iota(d')\texttt{g})$ and $\mathcal{H} \models \texttt{triple}(\iota(d), S, \iota(d')\texttt{g})$.

By rule (prl-dis) and Lemma 2, we would obtain that $\mathcal{H} \models \mathtt{inst}(\iota(d), \mathtt{bot}, \mathsf{g})$. This contradicts our assumptions on the consistency of $\mathfrak{K}$, thus there can not exists a pair $\langle d, d' \rangle \in R^{\mathcal{M}} \cap S^{\mathcal{M}}$ as required.
- Let $\beta = \mathrm{Irr}(R) \in \mathfrak{G}$, then $\mathcal{H} \models \mathtt{irr}(R, \mathsf{g})$. Suppose that $\langle d, d \rangle \in R^{\mathcal{M}}$, then $\mathcal{H} \models \mathtt{triple}(\iota(d), R, \iota(d), \mathsf{g})$. By rule (prl-irr) and Lemma 2, we would obtain that $\mathcal{H} \models \mathtt{inst}(\iota(d), \mathtt{bot}, \mathsf{g})$. This contradicts our assumptions on the consistency of $\mathfrak{K}$, thus $\langle d, d \rangle \notin R^{\mathcal{M}}$ as required.

□

Theorem 2 (Completeness). *Given $\mathfrak{K} = \langle \mathfrak{G}, \{\mathrm{K_m}\}_{\mathsf{m} \in \mathbf{M}} \rangle$ a consistent CKR in normal form, $\alpha \in \mathcal{L}_\Sigma$ an atomic concept or role assertion and $\mathsf{c} \in \mathbf{N}$, then $\mathfrak{K} \models \mathsf{c} : \alpha$ implies $\mathfrak{K} \vdash \mathsf{c} : \alpha$.*

Proof. As in the case of soundness, we prove the assertion by extending the previous construction on the global context to the whole structure of the input CKR.

We prove by contrapositive that $\mathfrak{K} \not\vdash \mathsf{c} : \alpha$ implies $\mathfrak{K} \not\models \mathsf{c} : \alpha$. If $\mathfrak{K} \not\vdash \mathsf{c} : \alpha$, then we have by definition that $PK(\mathfrak{K}) \not\models O(\alpha, \mathsf{c})$. Then there exists an Herbrand model $\mathcal{H}$ of $PK(\mathfrak{K})$ such that $\mathcal{H} \not\models O(\alpha, \mathsf{c})$. As in the previous lemma, from this model for $PK(\mathfrak{K})$ we build a CKR model $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ for $\mathfrak{K}$ such that $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \not\models \alpha$, implying that $\mathfrak{K} \not\models \mathsf{c} : \alpha$.

We consider again the equivalence relation $\approx$ defined in Lemma 2 and the equivalence classes $[c] = \{d \mid d \approx c\}$ as from the above lemma. Then we build $\mathfrak{I} = \langle \mathcal{M}, \mathcal{I} \rangle$ as follows: the global interpretation $\mathcal{M} = \langle \Delta^{\mathcal{M}}, \cdot^{\mathcal{M}} \rangle$ is a structure defined as in Lemma 3; for each $e \in \Delta^{\mathcal{M}}$, we define again the projection function $\iota(e)$ such that, if $e = [c]$, then $\iota(e) = b$ with a fixed $b \in [c]$. As in the case of Theorem 1, we can note that, since we can show $\mathcal{M} \models \mathfrak{G}$ then: if $\mathsf{c} \in \mathbf{N}_{\mathfrak{G}}$ (that is, if $PG(\mathfrak{G}) \models \mathtt{inst}(\mathsf{c}, \mathsf{Ctx}, \mathsf{gm})$) then $\mathsf{c}^{\mathcal{M}} \in \mathsf{Ctx}^{\mathcal{M}}$; if $\mathrm{K_m} \in \mathrm{K_c}$ (that is, if $PG(\mathfrak{G}) \models \mathtt{triple}(\mathsf{c}, \mathsf{mod}, \mathsf{m}, \mathsf{gm})$) then $\langle \mathsf{c}^{\mathcal{M}}, \mathsf{m}^{\mathcal{M}} \rangle \in \mathsf{mod}^{\mathcal{M}}$. For every $c \in \mathbf{N}_{\mathfrak{G}}$, we build $\mathcal{I}(c) = \langle \Delta_c, \cdot^{\mathcal{I}(c)} \rangle$ as follows:

- $\Delta_c = \{[d] \mid d \in \mathrm{NI}_\Sigma\}$;
- $a^{\mathcal{I}(c)} = [a]$, for every $a \in \mathrm{NI}_\Sigma$;
- $A^{\mathcal{I}(c)} = \{d \in \Delta_c \mid \mathcal{H} \models \mathtt{inst}(\iota(d), A, c)\}$, for every $A \in \mathrm{NC}_\Sigma$;
- $R^{\mathcal{I}(c)}$ is the smallest set such that $\langle d, d' \rangle \in R^{\mathcal{I}(c)}$ if one of the following conditions hold:
  - $\mathcal{H} \models \mathtt{triple}(\iota(d), R, \iota(d'), c)$;
  - $S \sqsubseteq R \in \mathrm{K}_c \cup \mathfrak{G}_\Sigma$ and $\langle d, d' \rangle \in S^{\mathcal{I}(c)}$;
  - $S \circ T \sqsubseteq R \in \mathrm{K}_c \cup \mathfrak{G}_\Sigma$ and $\langle d, e \rangle \in S^{\mathcal{I}(c)}$, $\langle e, d' \rangle \in T^{\mathcal{I}(c)}$ for $e \in \Delta_c$;
  - $\mathrm{Inv}(R, S) \in \mathrm{K}_c \cup \mathfrak{G}_\Sigma$ or $\mathrm{Inv}(S, R) \in \mathrm{K}_c \cup \mathfrak{G}_\Sigma$ and $\langle d', d \rangle \in S^{\mathcal{I}(c)}$;

As in the above lemma, we can see that, given $\mathcal{H} \not\models O(\alpha, \mathsf{c})$, then $\mathcal{I}(\mathsf{c}^{\mathcal{M}}) \not\models \alpha$ as required.

To show the assertion, we have to prove that $\mathfrak{I}$ meets the definition of CKR model and that $\mathfrak{I} \models \mathfrak{K}$. By Lemma 3 we directly obtain that the conditions on the global interpretation $\mathcal{M}$ are verified. Given $x, y \in \mathsf{Ctx}^{\mathcal{M}}$, we note also that, by the definition of $\iota(e)$, for every $a \in \mathrm{NI}_\Sigma$ it holds that $a^{\mathcal{I}(x)} = a^{\mathcal{I}(y)} = a^{\mathcal{M}} = [a]$.

To complete the proof, we have to show that for every $\mathrm{K_m}$ s.t. $\langle c, \mathsf{m}^{\mathcal{M}} \rangle \in \mathsf{mod}^{\mathcal{M}}$ (that is, every $\mathrm{K_m} \in \mathrm{K}_c$) we have $\mathcal{I}(c) \models \mathrm{K_m}$ and $\mathcal{I}(c) \models \mathfrak{G}_\Sigma$. As in the case above we proceed by cases and consider the form of all of the axioms $\beta \in \mathcal{L}_\Sigma$ that can appear in $\mathrm{K}_c \cup \mathfrak{G}_\Sigma$. The case for the axioms in the general normal form of Table 1 can be proved analogously as in the cases of Lemma 3: thus we have to prove the case of local reference axioms.

- Let $\beta = eval(A, \mathsf{C}) \sqsubseteq B \in \mathrm{K}_c$, then $\mathcal{H} \models \texttt{subEval}(A, \mathsf{C}, B, c)$. If $c' \in \mathsf{C}^{\mathcal{M}}$ and $d \in A^{\mathcal{I}(c')}$, then by definition $\mathcal{H} \models \texttt{inst}(\iota(d), A, \iota(c'))$ and $\mathcal{H} \models \texttt{inst}(\iota(c'), \mathsf{C}, \mathsf{gm})$. By rule (plc-evalat) we obtain that $\mathcal{H} \models \texttt{inst}(\iota(d), B, c)$: hence, by definition $d \in B^{\mathcal{I}(c)}$.
- Let $\beta = eval(R, \mathsf{C}) \sqsubseteq T \in \mathrm{K}_c$, then $\mathcal{H} \models \texttt{subEvalR}(R, \mathsf{C}, T, c)$. If there exists $c' \in \mathsf{C}^{\mathcal{M}}$, then $\mathcal{H} \models \texttt{inst}(\iota(c'), \mathsf{C}, \mathsf{gm})$. Moreover, suppose $\langle a, b\rangle \in R^{\mathcal{I}(c')}$, then $\mathcal{H} \models \texttt{triple}(\iota(a), R, \iota(b), \iota(c'))$. By rule (plc-evalr) and by Lemma 2 we obtain that $\mathcal{H} \models \texttt{triple}(\iota(a), T, \iota(b), c)$, and thus $\langle a, b\rangle \in T^{\mathcal{I}(c)}$. $\square$