Conservative Rewritability of Description Logic TBoxes: First Results

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Abstract. We want to understand when a given TBox $\mathcal{T}$ in a description logic $\mathcal{L}$ can be rewritten into a TBox $\mathcal{T}'$ in a weaker description logic $\mathcal{L}'$. Two notions of rewritability are considered: model-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and all models of $\mathcal{T}$ can be expanded to models of $\mathcal{T}'$) and $\mathcal{L}$-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and every $\mathcal{L}$-consequence of $\mathcal{T}'$ in the signature of $\mathcal{T}$ is a consequence of $\mathcal{T}$) and investigate rewritability of TBoxes in $\mathcal{ALCI}$ to $\mathcal{ALC}$, $\mathcal{ALCQ}$ to $\mathcal{ALC}$, $\mathcal{ALC}$ to $\mathcal{EL}_\bot$, and $\mathcal{ALCI}$ to *DL-Lite*$_{horn}$. We compare conservative rewritability with equivalent rewritability, give model-theoretic characterizations of conservative rewritability, prove complexity results for deciding rewritability, and provide some rewriting algorithms.

Over the past 30 years, a multitude of different description logics (DLs) have been designed, investigated, and used in practice as ontology languages. The introduction of new DLs has been driven both by the need for additional expressive power (such as transitive roles in the 1990s) and by applications that require efficient reasoning of a novel type (such as ontology-based data access in the 2000s). While the resulting flexibility in choosing DLs has had the positive effect of making DLs available for a large number of domains and applications, it has also led to the development of ontologies with language constructors that are not really required to axiomatize their knowledge. For a constructor to be 'not required' can mean different things here, ranging from the high-level 'this domain can be represented in an adequate way in a weaker DL' to the very concrete 'this ontology is logically equivalent to an ontology in a weaker DL'. In this paper, we take the latter understanding as our starting point. Equivalent rewritability of a given DL ontology (TBox) to a weaker DL has been investigated in [17], where model-theoretic characterizations and the complexity of deciding rewritability were investigated. For example, equivalent rewritability of an $\mathcal{ALC}$ TBox to an $\mathcal{EL}_\bot$ TBox has been characterized in terms of preservation under products and global equisimilations, and a NExpTime upper bound for deciding equivalent rewritability has been established. Equivalent rewritability is a very strong notion, however, that appears to apply to a very small number of real-world TBoxes. A more practically relevant notion we propose in this paper is *conservative* rewritability, which allows one to use new concept and role names when rewriting a given ontology into a weaker DL. In this case, we clearly cannot demand that the new TBox is logically equivalent to the original one, but only that it entails the original TBox. To avoid uncontrolled additional consequences of the new TBox, we can also require that (i) it does not entail any new consequences in the language of the original TBox, or even that (ii) every model of the original TBox can be expanded a model of the new TBox. The latter type of conservative extension is known as *model-conservative extension* [16], and we call a TBox $\mathcal{T}$ model-conservatively $\mathcal{L}$-rewritable if a model-conservative rewriting of $\mathcal{T}$ in the DL $\mathcal{L}$ exists. The former type of conservative extension is known as a *language-conservative extension* or *deductive conservative extension* [12] and, given a DL $\mathcal{L}$ in which $\mathcal{T}$ is formulated and a weaker DL $\mathcal{L}'$, we call $\mathcal{T}$ $\mathcal{L}$-conservatively $\mathcal{L}'$-rewritable if there is a TBox $\mathcal{T}'$ in $\mathcal{L}'$ such that $\mathcal{T}'$ has the same $\mathcal{L}$-consequences as $\mathcal{T}$ in the signature of $\mathcal{T}$. Model-conservative rewritability is the more robust notion as it is language-independent and does not only leave unchanged the entailed concept inclusions of the original TBox but also, for example, certain answers if the ontologies are used to access data.

The main result of this paper is that there are important DLs for which model-conservative and $\mathcal{L}$-conservative rewritability can be transparently characterized, effectively decided, and for which rewriting algorithms can be designed. This is in contrast to the undecidability of the problem whether one TBox is a model-conservative extension of another one even for weak DLs such as $\mathcal{EL}$ [18, 16]. In particular, we show that, given an $\mathcal{ALCI}$ TBox, one can compute in polynomial time its model-conservative $\mathcal{ALC}$-rewriting provided that such a rewriting exists, which can be decided in ExpTime. We characterize model-conservative $\mathcal{ALC}$-rewritability in terms of preservation under generated subinterpretations and show that $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability coincides with model-conservative one. For $\mathcal{ALCQ}$ TBoxes, we show that model-conservative $\mathcal{ALC}$-rewritability coincides with equivalent rewritability, but is different from $\mathcal{ALCQ}$-conservative rewritability. The latter can be characterized using bounded morphisms, and all these notions of rewritability are decidable in 2ExpTime. Unlike the $\mathcal{ALCI}$ case, we currently do not have polynomial rewritings for $\mathcal{ALCQ}$ TBoxes. As to rewritability from $\mathcal{ALCI}$ to $\textit{DL-Lite}_{horn}$, we observe that all our notions of rewritability coincide and are ExpTime-complete. In contrast, for rewritability from $\mathcal{ALC}$ to $\mathcal{EL}_\bot$ they are all distinct and, in fact, rather intricate and difficult to analyse. We prove decidability of model-conservative rewritability and give necessary semantic conditions for both $\mathcal{ALC}$-conservative and model-conservative $\mathcal{EL}_\bot$-rewritability.

Related work. Conservative rewritings of TBoxes are ubiquitous in the DL research. For example, many rewritings of TBoxes into normal forms are model-conservative [14, 4]. Regarding rewritability of TBoxes into weaker DLs, the focus has been on polynomial satisfability preserving rewritings as a pre-processing step to reasoning [11, 9, 8] or to prove complexity results for reasoning [10]. Such rewritings are mostly not conservative. There has been significant work on rewritings of ontology-mediated queries (pairs of ontologies and queries), which preserve their certain answers, into datalog or ontology-mediated queries based on weaker DLs [13, 5]. It seems, however, that this problem is different from TBox conservative rewritability. In [2], the expressive power of DLs and corresponding notions of rewritability are introduced based on a variant of model-conservative extension, and the relationship to $\mathcal{L}$-conservative extensions is discussed.

For omitted proofs, see http://cgi.csc.liv.ac.uk/~frank/publ/publ.html.

1 Conservative Rewritability

We consider the standard description logics $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQ}$, $\mathcal{EL}_\bot$, and *DL-Lite*$_{horn}$ [3, 4, 7, 1], where $\mathcal{EL}_\bot$ is $\mathcal{EL}$ extended with the concept $\bot$, and *DL-Lite*$_{horn}$ is *DL-Lite*$_{core}$ extended with conjunctions of basic concepts on the left-hand side of concept inclusions. As usual, the alphabet of DLs consists of countably infinite sets $\mathsf{N_C}$ of *concept names* and $\mathsf{N_R}$ of *role names*. By a *signature*, Σ, we mean any set of concept and role names. The *signature* $\mathsf{sig}(\mathcal{T})$ *of a TBox* $\mathcal{T}$ is the set of concept and role names occurring in $\mathcal{T}$.

Before introducing our notions of conservative rewritability, we remind the reader of a simpler notion of TBox rewritability. Suppose $\mathcal{L}$ and $\mathcal{L}'$ are DLs; we typically assume that $\mathcal{L}$ is more expressive than $\mathcal{L}'$.

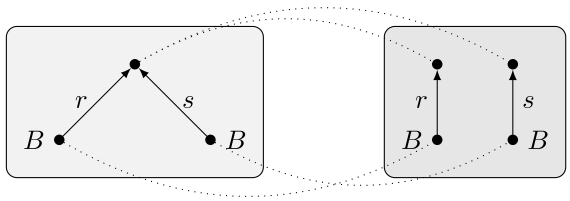
Definition 1 (equivalent $\mathcal{L}$-to-$\mathcal{L}'$ rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called an *equivalent $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \models \mathcal{T}$ (in other words, if $\mathcal{T}$ and $\mathcal{T}'$ have the same models). An $\mathcal{L}$ TBox is called *equivalently $\mathcal{L}'$-rewritable* if it has an equivalent $\mathcal{L}'$-rewriting.

Equivalent $\mathcal{L}$-to-$\mathcal{L}'$ rewritability has been studied in [17], where semantic characterizations are given and complexity results for deciding equivalent rewritability are obtained for various DLs $\mathcal{L}$ and $\mathcal{L}'$. For example, if $\mathcal{L}$ is $\mathcal{ALCI}$ or $\mathcal{ALCQ}$ and $\mathcal{L}'$ is $\mathcal{ALC}$, then an $\mathcal{L}$ TBox $\mathcal{T}$ is equivalently $\mathcal{L}'$-rewritable just in case its class of models is preserved under global bisimulations, which are defined as follows. Given interpretations $\mathcal{I}_i = (\Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}_i})$, for $i = 1, 2$, and a signature Σ, we call a relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ a *Σ-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$* if

- for any $A \in \Sigma$, whenever $(d_1, d_2) \in S$ then $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$;
- for any $r \in \Sigma$ and $(d_1, d_2) \in S$,
  - if $(d_1, e_1) \in r^{\mathcal{I}_1}$ then there is e_2 such that $(e_1, e_2) \in S$ and $(d_2, e_2) \in r^{\mathcal{I}_2}$,
  - if $(d_2, e_2) \in r^{\mathcal{I}_2}$ then there is e_1 such that $(e_1, e_2) \in S$ and $(d_1, e_1) \in r^{\mathcal{I}_1}$.

S is a *global* Σ-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ if $\Delta^{\mathcal{I}_1}$ is the domain of S and $\Delta^{\mathcal{I}_2}$ its range. $\mathcal{I}_1$ and $\mathcal{I}_2$ are *globally Σ-bisimilar* if there is a global Σ-bisimulation between them, in which case we write $\mathcal{I}_1 \sim^{\Sigma}_{\mathcal{ALC}} \mathcal{I}_2$. For $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$, we say that $(\mathcal{I}_1, d_1)$ is *Σ-bisimilar to* $(\mathcal{I}_2, d_2)$ if there is a Σ-bisimulation S between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $(d_1, d_2) \in S$. If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we omit Σ, write $\mathcal{I}_1 \sim_{\mathcal{ALC}} \mathcal{I}_2$ and say simply '(global) bisimulation.'

Example 1. The $\mathcal{ALCI}$ TBox $\{\exists r^-.B \sqsubseteq A\}$ can be equivalently rewritten to the $\mathcal{ALC}$ TBox $\{B \sqsubseteq \forall r.A\}$. However, the $\mathcal{ALCI}$ TBox $\mathcal{T} = \{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ is not equivalently $\mathcal{ALC}$-rewritable. Indeed, the interpretation on the right-hand side in the picture below is a model of $\mathcal{T}$ and globally bisimilar to the interpretation on the left-hand side, which is not a model of $\mathcal{T}$.



We now introduce two subtler notions of TBox rewritability, which allow the use of fresh concept and role names in rewritings. For an interpretation $\mathcal{I}$ and signature Σ, the *Σ-reduct of* $\mathcal{I}$ is the interpretation $\mathcal{I}_{|\Sigma}$ coinciding with $\mathcal{I}$ on the names in Σ and having $X^{\mathcal{I}_{|\Sigma}} = \emptyset$ for all $X \notin \Sigma$. We say that interpretations $\mathcal{I}$ and $\mathcal{J}$ *coincide on* Σ and write $\mathcal{I} =_\Sigma \mathcal{J}$ if the Σ-reducts of $\mathcal{I}$ and $\mathcal{J}$ coincide. A TBox $\mathcal{T}'$ is a *model-conservative extension* of $\mathcal{T}$ if an interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ just in case there is a model $\mathcal{I}'$ of $\mathcal{T}'$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

Definition 2 (model-conservative $\mathcal{L}$-to-$\mathcal{L}'$-rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called a *model-conservative $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is a model-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is *model-conservatively $\mathcal{L}'$-rewritable* if a model-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$ exists.

Clearly, any equivalent $\mathcal{L}'$-rewriting of a TBox $\mathcal{T}$ is also a model-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$. The next example shows that the converse does not hold.

Example 2. The $\mathcal{ALCI}$ TBox $\mathcal{T} = \{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ from Example 1 is model-conservatively $\mathcal{ALC}$-rewritable to

$$\mathcal{T}' = \{B \sqsubseteq \forall r.B_{\exists r^-.B},\ B \sqsubseteq \forall s.B_{\exists s^-.B},\ B_{\exists r^-.B} \sqcap B_{\exists s^-.B} \sqsubseteq A\},$$

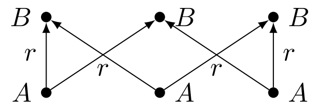
where $B_{\exists r^-.B}$, $B_{\exists s^-.B}$ are *fresh* concept names.

A TBox $\mathcal{T}'$ is called an *$\mathcal{L}$-conservative extension* of $\mathcal{T}$ if $\mathcal{T}' \models \mathcal{T}$ and $\mathcal{T}' \models C \sqsubseteq D$ implies $\mathcal{T} \models C \sqsubseteq D$, for every $\mathcal{L}$-concept inclusion $C \sqsubseteq D$ formulated in $\mathsf{sig}(\mathcal{T})$.

Definition 3 ($\mathcal{L}$-conservative $\mathcal{L}'$-rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called an *$\mathcal{L}$-conservative $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is an $\mathcal{L}$-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is *$\mathcal{L}$-conservatively $\mathcal{L}'$-rewritable* if an $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$ exists.

It should be clear that every model-conservative $\mathcal{L}'$-rewriting of an $\mathcal{L}$ TBox $\mathcal{T}$ is also an $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$. The next example shows that the converse implication does not hold.

Example 3. The $\mathcal{ALCQ}$ TBox $\mathcal{T} = \{A \sqsubseteq\ \geq 2\, r.B\}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable to $\mathcal{T}' = \{A \sqsubseteq \exists r.C,\ A \sqsubseteq \exists r.D,\ C \sqsubseteq \neg D,\ C \sqcup D \sqsubseteq B\}$, where C and D are *fresh* concept names. However, $\mathcal{T}'$ is not a model-conservative rewriting of $\mathcal{T}$ because the model of $\mathcal{T}$ shown below is not the $\mathsf{sig}(\mathcal{T})$-reduct of any model of $\mathcal{T}'$. Note that $\mathcal{T}$ is not equivalently $\mathcal{ALC}$-rewritable.



In our examples so far, we have used fresh concept names but no fresh role names. This is no accident: it turns out that, for the DLs considered in this paper, fresh role names in conservative rewritings are not required. More precisely, we call a model-conservative or $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting $\mathcal{T}'$ of $\mathcal{T}$ a model-conservative or, respectively, $\mathcal{L}$-conservative $\mathcal{L}'$-*concept rewriting* of $\mathcal{T}$ if $\mathsf{sig}_R(\mathcal{T}) = \mathsf{sig}_R(\mathcal{T}')$, where $\mathsf{sig}_R(\mathcal{T})$ is the set of role names in $\mathcal{T}$.

Say that a DL $\mathcal{L}$ *reflects disjoint unions* if, for any $\mathcal{L}$ TBox $\mathcal{T}$, whenever the *disjoint* union $\bigcup_{i \in I} \mathcal{I}_i$ of interpretations $\mathcal{I}_i$ is a model of $\mathcal{T}$, then each $\mathcal{I}_i$, $i \in I$, is also a model of $\mathcal{T}$. All the DLs considered in this paper reflect disjoint unions.

Theorem 1. *Let $\mathcal{L}$ be a DL reflecting disjoint unions, $\mathcal{T}$ an $\mathcal{L}$ TBox, and let $\mathcal{L}' \in \{\mathcal{ALC}, \mathcal{EL}_\bot, \text{DL-Lite}_{horn}\}$. Then $\mathcal{T}$ is model-conservatively (or $\mathcal{L}$-conservatively) $\mathcal{L}'$-rewritable if and only if it is model-conservatively (or, respectively, $\mathcal{L}$-conservatively) $\mathcal{L}'$-concept rewritable.*

2 $\mathcal{ALCI}$-to-$\mathcal{ALC}$ Rewritability

Equivalent $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability was studied in [17], where the characterization in terms of global bisimulations was used to design a 2ExpTime algorithm for checking this property. Here, we give a characterization of model-conservative $\mathcal{ALC}$ rewritability of $\mathcal{ALCI}$ TBoxes in terms of generated subinterpretations and use it to show that (i) model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritings are of polynomial size and can be constructed in polynomial time (if they exist), and that (ii) deciding model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is ExpTime-complete. We also observe that $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability coincides with model-conservative rewritability.

We remind the reader that an interpretation $\mathcal{I}$ is a *subinterpretation* of an interpretation $\mathcal{J}$ if $\Delta^\mathcal{I} \subseteq \Delta^\mathcal{J}$, $A^\mathcal{I} = A^\mathcal{J} \cap \Delta^\mathcal{I}$ for all concept names A, and $r^\mathcal{I} = r^\mathcal{J} \cap (\Delta^\mathcal{I} \times \Delta^\mathcal{I})$ for all role names r. $\mathcal{I}$ is a *generated subinterpretation* of $\mathcal{J}$ if, in addition, whenever $d \in \Delta^\mathcal{I}$ and $(d, d') \in r^\mathcal{J}$, r a role name, then $d' \in \Delta^\mathcal{I}$. We say that a TBox $\mathcal{T}$ is *preserved under generated subinterpretations* if every generated subinterpretation of a model of $\mathcal{T}$ is also a model of $\mathcal{T}$. As well known, every $\mathcal{ALC}$ TBox is preserved under generated subinterpretations.

Suppose we want to find a model-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$. Without loss of generality, we assume that $\mathcal{T} = \{\top \sqsubseteq C_\mathcal{T}\}$ and $C_\mathcal{T}$ is built using $\neg$, $\sqcap$ and $\exists$ only. Let $\mathsf{sub}(\mathcal{T})$ be the closure under single negation of the set of (subconcepts) of concepts in $\mathcal{T}$. For every role name r in $\mathcal{T}$, we take a fresh role name $\bar{r}$ and, for every $\exists r.C$ in $\mathsf{sub}(\mathcal{T})$ (where r is a role name or its inverse), we take a fresh concept name $B_{\exists r.C}$. Denote by $D^\sharp$ the $\mathcal{ALC}$-concept obtained from any $D \in \mathsf{sub}(\mathcal{T})$ by replacing every top-most occurrence of a subconcept of the form $\exists r.C$ in it with $B_{\exists r.C}$. Now, let $\mathcal{T}^\dagger$ be an $\mathcal{ALC}$ TBox comprised of the following concept inclusions, for $r \in \mathsf{N_R}$: $\top \sqsubseteq C^\sharp_\mathcal{T}$,

$$C^\sharp \sqsubseteq \forall \bar{r}.B_{\exists r.C}, \quad B_{\exists r.C} \equiv \exists r.C^\sharp, \qquad \text{for every } \exists r.C \in \mathsf{sub}(\mathcal{T}),$$

$$C^\sharp \sqsubseteq \forall r.B_{\exists r^-.C}, \quad B_{\exists r^-.C} \equiv \exists \bar{r}.C^\sharp, \qquad \text{for every } \exists r^-.C \in \mathsf{sub}(\mathcal{T}).$$

Clearly, $\mathcal{T}^\dagger$ can be constructed in polynomial time in the size of $\mathcal{T}$.

Theorem 2. *An $\mathcal{ALCI}$ TBox $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved generated subinterpretations. Moreover, if $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is its model-conservative $\mathcal{ALC}$-rewriting.*

It is now easy to show that model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is decidable in ExpTime. By Theorem 2, this amounts to deciding whether $\mathcal{T}^\dagger$ is a model-conservative extension of $\mathcal{T}$. In general, this is an undecidable problem. It is, however, easy to see that, for every model $\mathcal{I}$ of $\mathcal{T}$, there is a model $\mathcal{I}'$ of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. It thus remains to decide whether every interpretation $\mathcal{I}$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$, for some model $\mathcal{I}'$ of $\mathcal{T}^\dagger$, is a model of $\mathcal{T}$. In other words, this means to decide whether $\mathcal{T}^\dagger \models \mathcal{T}$, which can be done in ExpTime. A matching lower bound is easily obtained by reducing satisfiability in $\mathcal{ALC}$.

Corollary 1. *The problem of deciding model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is* ExpTime*-complete.*

$\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCI}$ TBoxes coincides with model-conservative $\mathcal{ALC}$-rewritability. This can be proved using the characterization via subinterpretations and robustness under replacement of $\mathcal{ALCI}$ TBoxes, an important property in the context of modular ontology design [15, Theorem 4].

Theorem 3. *An $\mathcal{ALCI}$ TBox $\mathcal{T}$ is $\mathcal{ALCI}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable.*

3 $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ Rewritability

Equivalent $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability was characterized in [17] in terms of preservation under global bisimulations. Below, we use this characterization to give a 2ExpTime algorithm for checking equivalent $\mathcal{ALC}$-rewritability.

We first prove a characterization of $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability in terms of preservation under inverse bounded morphisms and use it to show that one can (*i*) decide $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability in 2ExpTime and (*ii*) construct effectively an $\mathcal{ALCQ}$-conservative rewriting if it exists. We also show that, unlike $\mathcal{ALCI}$-to-$\mathcal{ALC}$-rewritability, model-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQ}$ TBoxes coincides with equivalent rewritability.

A *bounded Σ-morphism* from an interpretation $\mathcal{I}_1$ to an interpretation $\mathcal{I}_2$ is a global Σ-bisimulation S between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that S is a function from $\Delta^{\mathcal{I}_1}$ to $\Delta^{\mathcal{I}_2}$. A class $\mathcal{K}$ of interpretations is *preserved under inverse bounded Σ-morphisms* if whenever there is a bounded Σ-morphism from an interpretation $\mathcal{I}_1$ to some $\mathcal{I}_2 \in \mathcal{K}$, then $\mathcal{I}_1 \in \mathcal{K}$. The following lemma provides the fundamental property of bounded morphisms:

Lemma 1. *Suppose* $f\colon \mathcal{I}_1 \to \mathcal{I}_2$ *is a bounded* Σ*-morphism, where* $\mathcal{I}_2$ *is a model of an* $\mathcal{ALC}$ *TBox* $\mathcal{T}$ *and* $\mathsf{sig}_R(\mathcal{T}) \subseteq \Sigma$. *Then there is* $\mathcal{J}_1 \models \mathcal{T}$ *such that* $\mathcal{J}_1 =_\Sigma \mathcal{I}_1$.

Proof. We define $\mathcal{J}_1$ in the same way as $\mathcal{I}_1$ except that $B^{\mathcal{J}_1} := f^{-1}(B^{\mathcal{I}_2})$ for all concept names $B \in \mathsf{sig}(\mathcal{T}_1) \setminus \Sigma$. Then f is a bounded $\mathsf{sig}(\mathcal{T})$-morphism from $\mathcal{J}_1$ to $\mathcal{I}_2$. Thus, $\mathcal{J}_1$ is a model of $\mathcal{T}$ since $\mathcal{I}_1$ is a model of $\mathcal{T}$. ❏

An interpretation $\mathcal{I}$ is a *directed tree interpretation* if $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$, for $r \neq s$, and the directed graph with nodes $\Delta^\mathcal{I}$ and edges E defined by setting $(d, d') \in E$ iff $(d, d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}$ is a directed tree. We start our investigation with the observation that $\mathcal{ALCQ}$-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability can be regarded as a principled approximation of model-conservative rewritability:

Lemma 2. *An* $\mathcal{ALC}$ *TBox* $\mathcal{T}'$ *is an* $\mathcal{ALCQ}$*-conservative rewriting of an* $\mathcal{ALCQ}$ *TBox* $\mathcal{T}$ *iff* $\mathcal{T}'$ *is a model-conservative rewriting of* $\mathcal{T}$ *over the class of directed tree interpretations of finite outdegree.*

Suppose we want to find an $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$. Without loss of generality, we assume that $\mathcal{T}$ is of the form $\{\top \sqsubseteq C_\mathcal{T}\}$ and that $C_\mathcal{T}$ is built using $\neg$, $\sqcap$, $(\geqslant n\ r\ C)$ only. Construct a TBox $\mathcal{T}^\dagger$ as follows. Take fresh concept names $B_D, B_1^D, \ldots, B_n^D$ for every $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$. We use Σ to denote $\mathsf{sig}(\mathcal{T})$ extended with all fresh concept names of the form B_i^D. For each $C \in \mathsf{sub}(\mathcal{T})$, $C^\sharp$ denotes the $\mathcal{ALC}$-concept that results from C by replacing all top-most occurrences of any $D = (\geqslant n\ r\ C)$ in $\mathcal{T}$ with B_D. Now, define $\mathcal{T}^\dagger$ to be the infinite TBox that consists of the following inclusions:

- $\top \sqsubseteq C_\mathcal{T}^\sharp$,
- $B_D \sqsubseteq \exists r.(C^\sharp \sqcap B_1^D) \sqcap \cdots \sqcap \exists r.(C^\sharp \sqcap B_n^D)$,
- $B_i^D \sqsubseteq \neg B_j^D$, for $i \neq j$, and
- for all $\mathcal{ALC}$-concepts $C_1, \ldots, C_n$ in Σ and all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$,

$$\bigsqcap_{1 \leq i \leq n} (\exists r.(C^\sharp \sqcap C_i^\sharp \sqcap \bigsqcap_{j \neq i} \neg C_j^\sharp)) \sqsubseteq B_D.$$

The next theorem characterizes $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability.

Theorem 4. *An* $\mathcal{ALCQ}$ *TBox* $\mathcal{T}$ *is* $\mathcal{ALCQ}$*-conservatively* $\mathcal{ALC}$*-rewritable iff* $\mathcal{T}$ *is preserved under inverse bounded* $\mathsf{sig}(\mathcal{T})$*-morphisms. Moreover, if* $\mathcal{T}$ *is* $\mathcal{ALCQ}$*-conservatively* $\mathcal{ALC}$*-rewritable, then* $\mathcal{T}^\dagger$ *is an (infinite) rewriting.*

The semantic characterization of Theorem 4 can be employed to prove the following complexity result using a type elimination argument. We assume that numbers in number restrictions are given in unary.

Theorem 5. *For* $\mathcal{ALCQ}$ *TBoxes,* $\mathcal{ALCQ}$*-conservative* $\mathcal{ALC}$*-rewritability is decidable in 2*ExpTime.

It follows that, given an $\mathcal{ALCQ}$ TBox $\mathcal{T}$, one can first decide $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability and then, in case of a positive answer, effectively construct a rewriting by going through the finite subsets of $\mathcal{T}^\dagger$ in a systematic way until a finite $\mathcal{T}' \subseteq \mathcal{T}^\dagger$ with $\mathcal{T}' \models \mathcal{T}$ is reached. By compactness, such a set $\mathcal{T}'$ exists.

We finally show that every model-conservatively $\mathcal{ALC}$-rewritable $\mathcal{ALCQ}$ TBox is equivalently $\mathcal{ALC}$-rewritable.

Theorem 6. *An $\mathcal{ALCQ}$ TBox is model-conservatively $\mathcal{ALC}$-rewritable iff it is equivalently $\mathcal{ALC}$-rewritable, which is decidable in 2*ExpTime.

4 $\mathcal{ALCI}$-to-*DL-Lite$_{horn}$* and $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ Rewritability

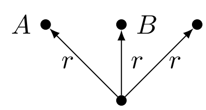
We first observe that all notions of rewritability introduced in this paper coincide in the case of $\mathcal{ALCI}$-to-*DL-Lite$_{horn}$* rewritability. Deciding rewritability is ExpTime-complete in all cases since deciding equivalent $\mathcal{ALCI}$-to-*DL-Lite$_{horn}$* rewritability is ExpTime-complete [17]:

Theorem 7. *For $\mathcal{ALCI}$ TBoxes, equivalent* DL-Lite$_{horn}$*-rewritability, model-conservative* DL-Lite$_{horn}$*-rewritability, and $\mathcal{ALCI}$-conservative* DL-Lite$_{horn}$*-rewritability coincide and are* ExpTime*-complete.*

We now provide separating examples for all three notions of $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ rewritability and then prove decidability of model-conservative $\mathcal{EL}_\bot$-rewritability. While we have not yet been able to find purely model-theoretic characterizations of model- and $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$-rewritability, we then give necessary model-theoretic conditions for these two notions of rewritability.

Equivalent $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ rewritability has been characterized in [17] in terms of preservation under products and global equisimulations. A *simulation* between interpretations $\mathcal{I}$ and $\mathcal{J}$ is a relation $S \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ such that, for any $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $(d_1, d_2) \in S$, if $d_1 \in A^{\mathcal{I}_1}$ then $d_2 \in A^{\mathcal{I}_2}$, and if $(d_1, e_1) \in r^\mathcal{I}$ then there exists e_2 with $(e_1, e_2) \in S$ and $(d_2, e_2) \in r^\mathcal{J}$. $(\mathcal{I}, d)$ is *simulated* by $(\mathcal{J}, e)$ if there is a simulation S between $\mathcal{I}$ and $\mathcal{J}$ such that $(d, e) \in S$. Interpretations $\mathcal{I}$ and $\mathcal{J}$ are *globally equisimilar* if, for any $d \in \Delta^\mathcal{I}$, there exists $e \in \Delta^\mathcal{J}$ such that $(\mathcal{I}, d)$ is simulated by $(\mathcal{J}, e)$ and $(\mathcal{J}, e)$ is simulated by $(\mathcal{I}, d)$. According to [17, Theorem 17], an $\mathcal{ALC}$ TBox is equivalently $\mathcal{EL}_\bot$-rewritable if its models are preserved under products and global equisimulations.

Example 4. The TBox $\mathcal{T} = \{\exists r.A \sqcap \exists r.B \sqcap \forall r.(A \sqcup B) \sqsubseteq E \sqcup F,\ A \sqcap B \sqsubseteq \bot\}$ is not equivalently $\mathcal{EL}_\bot$-rewritable because its models are not preserved under global equisimulations. Indeed, the interpretation $\mathcal{I}$ shown below is clearly a model of $\mathcal{T}$. However, by removing the rightmost r-arrow from $\mathcal{I}$, we obtain an interpretation which is globally equisimilar to $\mathcal{I}$ but not a model of $\mathcal{T}$.

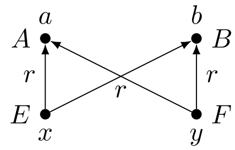


On the other hand, the $\mathcal{EL}_\bot$ TBox

$$\{\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.G,\ \exists r.(G \sqcap A) \sqsubseteq E,\ \exists r.(G \sqcap B) \sqsubseteq F,\ A \sqcap B \sqsubseteq \bot\}$$

is easily seen to be an $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$ rewriting of $\mathcal{T}$. We now show that $\mathcal{T}$ is not model-conservatively $\mathcal{EL}_\bot$-rewritable. For suppose $\mathcal{T}$ has such a rewriting $\mathcal{T}'$ given in standard normal form (with inclusions of the form $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$, $\exists r.B \sqsubseteq A$, or $A \sqsubseteq \exists r.B$ where $A_1, \ldots, A_n, A, B \in \mathsf{N_C} \cup \{\bot\}$). Consider the model

$\mathcal{I}$ of $\mathcal{T}$ depicted below, and let $\mathcal{I}'$ be a model of $\mathcal{T}'$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.



Let $\mathcal{J}$ be the same as $\mathcal{I}'$ except that $x, y \in M^{\mathcal{J}}$ iff both $x \in M^{\mathcal{I}'}$ and $y \in M^{\mathcal{I}'}$, for every $M \in \mathsf{N_C}$. Since $x \notin E^{\mathcal{J}}$ and $y \notin F^{\mathcal{J}}$, $\mathcal{J}$ is not a model of $\mathcal{T}'$. Since the restriction of $\mathcal{I}'$ to $\{a, b\}$ is a model of $\mathcal{T}'$, and the restrictions of $\mathcal{I}'$ to $\{a, b, x\}$ and $\{a, b, y\}$ coincide, there is $(C \sqsubseteq D) \in \mathcal{T}'$ such that $x, y \in C^{\mathcal{J}}$ but $x, y \notin D^{\mathcal{J}}$. As $\mathcal{I}'$ is a model of $\mathcal{T}'$, which is in standard normal form, and by the definition of $\mathcal{J}$, D must be a concept name. Since clearly $x, y \in C^{\mathcal{I}'}$, we must also have $x, y \in D^{\mathcal{I}'}$, and so $x, y \in D^{\mathcal{J}}$, which is a contradiction.

The following modified version of $\mathcal{T}$

$$\mathcal{T}_m = \{\exists r.A \sqcap \exists r.B \sqcap \forall r.(A \sqcup B) \sqsubseteq \exists r.(A \sqcap E) \sqcup \exists r.(B \sqcap F),\ A \sqcap B \sqsubseteq \bot\}$$

is not equivalently $\mathcal{EL}_\bot$-rewritable, but has a model-conservative $\mathcal{EL}_\bot$-rewriting

$$\mathcal{T}'_m = \{\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.M,\ \exists r.(M \sqcap A) \sqsubseteq \exists r.(M \sqcap E),\ \exists r.(M \sqcap B) \sqsubseteq \exists r.(M \sqcap F),\ A \sqcap B \sqsubseteq \bot\}.$$

The difference from the previous example is that if d is an instance of $\exists r.A \sqcap \exists r.B$, then we can place the 'marker' M onto an r-successor of d which is either in $A \sqcap E$ or in $B \sqcap F$, whereas in the previous example the decision on where to put the 'marker' G was not determined by the r-successors of d but by d itself.

We now prove that if there exists an $\mathcal{EL}_\bot$-rewriting of an $\mathcal{ALC}$ TBox $\mathcal{T}$, then there is one without any 'recursion' for the newly introduced symbols. Let $\Sigma = \mathsf{sig}(\mathcal{T})$. We say that an $\mathcal{EL}_\bot$ TBox $\mathcal{T}'$ is in *Σ-layered form of depth n* if there are mutually disjoint sets $\Gamma_0, \dots, \Gamma_n$ of concept names such that $\Gamma_i \cap \Sigma = \emptyset$ $(0 \le i \le n)$ and the inclusions of $\mathcal{T}'$ take the following form, where $r \in \Sigma$:

level i atom inclusions: $A_1 \sqcap \dots \sqcap A_n \sqsubseteq B$, for $A_1, \dots, A_n, B \in \Sigma \cup \Gamma_i \cup \{\bot\}$,
level i right-atom inclusions: $\exists r.A \sqsubseteq B$ for $A \in \Sigma \cup \Gamma_{i+1}$, $B \in \Sigma \cup \Gamma_i \cup \{\bot\}$,
level i left-atom inclusions: $A \sqsubseteq \exists r.B$, for $A \in \Sigma \cup \Gamma_i$, $B \in \Sigma \cup \Gamma_{i+1} \cup \{\bot\}$.

The *depth* of a concept C is the maximal number of nestings of existential restrictions in C. The *depth* of a TBox is the maximal depth of its concepts.

Lemma 3. *If an $\mathcal{ALC}$ TBox $\mathcal{T}$ of depth n is model- (or $\mathcal{ALC}$-) conservatively $\mathcal{EL}_\bot$-rewritable, then there exists a model- (respectively, $\mathcal{ALC}$-) conservative $\mathcal{EL}_\bot$-rewriting $\mathcal{T}'$ of $\mathcal{T}$ in $\mathsf{sig}(\mathcal{T})$-layered form of depth n.*

We use Lemma 3 to prove decidability of model-conservative $\mathcal{EL}_\bot$-rewritability. An $\mathcal{ALC}$ ABox $\mathcal{A}$ is a finite set of assertions of the form $C(a)$ and $r(a, b)$, where C is an $\mathcal{ALC}$ concept and a, b are *individual names*. The set of individual names that occur in an ABox $\mathcal{A}$ is denoted by $\mathsf{ind}(\mathcal{A})$. When interpreting ABoxes, we adopt the *standard name assumption*: $a^{\mathcal{I}} = a$, for all $a \in \mathsf{ind}(\mathcal{A})$.

Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth $n > 0$ (the case $n = 0$ is trivial). By $\mathsf{sub}^{n-1}(\mathcal{T})$ we denote the closure under single negation of the set of subconcepts of concepts in $\mathcal{T}$ of depth at most $n-1$. By $\Theta^{n-1}(\mathcal{T})$ we denote the set of maximal subsets $\boldsymbol{t}$ of $\mathsf{sub}^{n-1}(\mathcal{T})$ that are satisfiable in a model of $\mathcal{T}$. A $\mathcal{T}$-*ABox* is an ABox such that $\boldsymbol{t}_{\mathcal{A}}(a) = \{D \mid D(a) \in \mathcal{A}\} \in \Theta^{n-1}(\mathcal{T})$ for all $a \in \mathsf{ind}(\mathcal{A})$. Let $\mathcal{A}$ be a directed tree ABox of depth at most n (that is, all nodes in it are at distance $\leq n$ from the root). We say that $\mathcal{A}$ is n-*strongly satisfiable* w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}$ such that the $r^{\mathcal{I}}$-successors of $a^{\mathcal{I}}$, for every $a \in \mathsf{ind}(\mathcal{A})$ of depth $< n$ in $\mathcal{A}$, coincide with the r-successors of a in $\mathcal{A}$.

We now define inductively $(\mathcal{T}, i)$-*bisimilarity relations* $\sim_{i,\mathcal{T}}$ between pairs $(\mathcal{A}_1, a_1)$ and $(\mathcal{A}_2, a_2)$, where the $\mathcal{A}_i$ are $\mathcal{T}$-ABoxes and $a_i \in \mathsf{ind}(\mathcal{A}_i)$:

- $(\mathcal{A}_1, a_1) \sim_{0,\mathcal{T}} (\mathcal{A}_2, a_2)$ if $\boldsymbol{t}_{\mathcal{A}_1}(a_1) = \boldsymbol{t}_{\mathcal{A}_2}(a_2)$;
- $(\mathcal{A}_1, a_1) \sim_{i+1,\mathcal{T}} (\mathcal{A}_2, a_2)$ if $(\mathcal{A}_1, a_1) \sim_{0,\mathcal{T}} (\mathcal{A}_2, a_2)$ and, for every $r \in \mathsf{sig}(\mathcal{T})$, if $r(d_1, e_1) \in \mathcal{A}_1$ then there is $r(d_2, e_2) \in \mathcal{A}_2$ such that $(\mathcal{A}_1, e_1) \sim_{i,\mathcal{T}} (\mathcal{A}_2, e_2)$, and vice versa.

For every $i \geq 0$, one can determine a finite set AT_i of finite directed tree $\mathcal{T}$-ABoxes $\mathcal{A}$ with root $\rho_{\mathcal{A}}$ and of depth $\leq i$ such that:

- for every $\mathcal{I} \models \mathcal{T}$ and every $d \in \Delta^{\mathcal{I}}$, $(\mathcal{I}, d)$ is $(\mathcal{T}, i)$-bisimilar to exactly one $(\mathcal{A}, \rho_{\mathcal{A}}) \in \mathsf{AT}_i$;[4]
- every $\mathcal{A} \in \mathsf{AT}_i$ is strongly i-satisfiable w.r.t. $\mathcal{T}$.

We assume that all ABoxes in $\mathsf{AT}_0, \dots, \mathsf{AT}_n$ have mutually distinct roots. We define the *canonical ABox* $\mathcal{A}_{\mathcal{T}}$ with individuals $\{\rho_{\mathcal{A}} \mid \mathcal{A} \in \mathsf{AT}_i, i \leq n\}$ as follows:

- for $\mathcal{A}_i \in \mathsf{AT}_i$, $\mathcal{A}_{i+1} \in \mathsf{AT}_{i+1}$ and $r \in \mathsf{sig}(\mathcal{T})$, we have $r(\rho_{\mathcal{A}_{i+1}}, \rho_{\mathcal{A}_i}) \in \mathcal{A}_{\mathcal{T}}$ if there exists $r(\rho_{\mathcal{A}_{i+1}}, b) \in \mathcal{A}_{i+1}$ such that the subtree of $\mathcal{A}_{i+1}$ rooted at b is $(i, \mathcal{T})$-bisimilar to $\mathcal{A}_i$;
- for $\mathcal{A}_i \in \mathsf{AT}_i$ and $A \in \mathsf{sig}(\mathcal{T})$, we have $A(\rho_{\mathcal{A}_i}) \in \mathcal{A}_{\mathcal{T}}$ iff $A(\rho_{\mathcal{A}_i}) \in \mathcal{A}_i$.

Note that $\mathcal{A}_{\mathcal{T}}$ is acyclic (but not a directed tree ABox).

Lemma 4. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth n. An $\mathcal{EL}_{\bot}$ TBox $\mathcal{T}'$ in $\mathsf{sig}(\mathcal{T})$-layered form of depth n is a model-conservative $\mathcal{EL}_{\bot}$-rewriting of $\mathcal{T}$ iff*

- *$\mathcal{T}' \models \mathcal{T}$ and*
- *there exists $\mathcal{A}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{A}_{\mathcal{T}}$ such that, for all $i = 0, \dots, n$, $\mathcal{A}'$ satisfies all level i inclusions in $\mathcal{T}'$ at all $\rho_{\mathcal{A}_i}$ with $\mathcal{A}_i \in \mathsf{AT}_{n-i}$.*

Theorem 8. *Model-conservative $\mathcal{EL}_{\bot}$-rewritability of $\mathcal{ALC}$ TBoxes is decidable.*

Proof. Given an $\mathcal{ALC}$ TBox $\mathcal{T}$, we first construct the canonical ABox $\mathcal{A}_{\mathcal{T}}$. If an $\mathcal{EL}_{\bot}$ TBox $\mathcal{T}'$ in Σ-layered form of depth n satisfies the conditions of Lemma 4, then there exists such a TBox with at most $2^{|\mathcal{A}_{\mathcal{T}}|}$ distinct fresh concept names. As the number of such $\mathcal{EL}_{\bot}$ TBoxes is finite, one can check for each of them whether the conditions of Lemma 4 are satisfied. ❑

[4] Here we identify $\mathcal{I}$ with the ABox with assertions $r(a, b)$, for $(a, b) \in r^{\mathcal{I}}$, and $D(a)$, for $D \in \mathsf{sub}^{n-1}(\mathcal{T})$ and $a \in D^{\mathcal{I}}$.

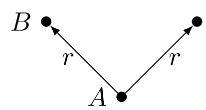
We now give necessary conditions for $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$-rewritability of $\mathcal{ALC}$ TBoxes. First, we still have the preservation under products:

Theorem 9. *Every $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox is preserved under products.*

Theorem 9 can be used to show that TBoxes such as $\{A \sqsubseteq B \sqcup E\}$ are not $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable. To separate equivalently rewritable TBoxes from $\mathcal{ALC}$-conservatively rewritable TBoxes, we generalize the construction of Example 4. In that case, we removed an r-arrow (d_0, d) from a tree-shaped model $\mathcal{I}$ of $\mathcal{T}$ and obtained a model that is globally equisimilar to the original model but not a model of $\mathcal{T}$. It turns out that $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBoxes of depth 1 are preserved under the inverse of this operation. We say that $(\mathcal{I}, d)$ is *$\subseteq_1$-simulated by* $(\mathcal{J}, e)$ if (*i*) $d \in A^\mathcal{I}$ iff $e \in A^\mathcal{J}$, for all $A \in \mathsf{N_C}$; (*ii*) for all $r \in \mathsf{N_R}$, if $(e, e') \in r^\mathcal{J}$ then there exists d' with $(d, d') \in r^\mathcal{I}$ and, for all $A \in \mathsf{N_C}$, if $e' \in A^\mathcal{J}$ then $d' \in A^\mathcal{I}$; (*iii*) for all $r \in \mathsf{N_R}$, if $(d, d') \in r^\mathcal{I}$ then there exists e' with $(e, e') \in r^\mathcal{J}$ and, for all $A \in \mathsf{N_C}$, we have $d' \in A^\mathcal{I}$ iff $e' \in A^\mathcal{J}$. Say that $\mathcal{I}$ is *globally $\subseteq_1$-simulated* by $\mathcal{J}$ if, for every $e \in \Delta^\mathcal{J}$, there exists $d \in \Delta^\mathcal{I}$ such that $(\mathcal{I}, d)$ is $\subseteq_1$-simulated by $(\mathcal{J}, e)$. An $\mathcal{ALC}$ TBox is *preserved under $\subseteq_1$-simulations* if every interpretation that globally $\subseteq_1$-simulates a model of $\mathcal{T}$ is a model of $\mathcal{T}$.

Theorem 10. *Every $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox of depth 1 is preserved under global $\subseteq_1$-simulations.*

This result can be used to show, for example, that $\mathcal{T} = \{A \sqsubseteq \forall r.B\}$ is not $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable. For the interpretation below is *not* a model



of $\mathcal{T}$, but by removing from it the rightmost r-arrow, we obtain an interpretation which is globally $\subseteq_1$-simulated by $\mathcal{J}$ and is a model of $\mathcal{T}$. It remains open whether preservation under products and global $\subseteq_1$-simulations is sufficient for an $\mathcal{ALC}$ TBox of depth 1 to be $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable.

5 Conclusion

Conservative rewritings of ontologies provide more flexibility than equivalent rewritings and are more natural in practice. However, they are also technically much more challenging to analyse. For future work, we are particularly interested in better understanding conservative rewritings to $\mathcal{EL}$ and related logics. For example, can we find transparent model-theoretic characterizations and explicit axiomatizations of the rewritten TBoxes? The results in Section 4 should provide a good starting point. Another challenging problem could be to investigate rewritability to OWL 2 QL—essentially *DL-Lite$_{core}$* extended with role inclusions—which preserves answers to conjunctive queries over all possible ABoxes. (Recall [6] that conjunctive query inseparability for OWL 2 QL TBoxes is ExpTime-complete.)

References

1. Artale, A., Calvanese, D., Kontchakov, R., Zakharyaschev, M.: The DL-Lite family and relations. Journal of Artificial Intelligence Research 36, 1–69 (2009)
2. Baader, F.: A formal definition for the expressive power of terminological knowledge representation languages. Journal of Logic and Computation 6(1), 33–54 (1996)
3. Baader, F.: The description logic handbook: Theory, implementation, and applications. Cambridge University Press, Cambridge (2007)
4. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ Envelope. In: Proceedings of IJCAI. pp. 364–369 (2005)
5. Bienvenu, M., ten Cate, B., Lutz, C., Wolter, F.: Ontology-based data access: A study through disjunctive datalog, CSP, and MMSNP. ACM Transactions of Database Systems 39(4), 33 (2014)
6. Botoeva, E., Kontchakov, R., Ryzhikov, V., Wolter, F., Zakharyaschev, M.: Query inseparability for description logic knowledge bases. In: Proceedings of KR (2014)
7. Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., Rosati, R.: Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. Journal of Automated Reasoning 39(3), 385–429 (2007)
8. Carral, D., Feier, C., Grau, B.C., Hitzler, P., Horrocks, I.: *EL*-ifying ontologies. In: Proceedings of IJCAR. pp. 464–479 (2014)
9. Carral, D., Feier, C., Romero, A.A., Grau, B.C., Hitzler, P., Horrocks, I.: Is your ontology as hard as you think? Rewriting ontologies into simpler DLs. In: Proceedings of DL. pp. 128–140 (2014)
10. De Giacomo, G.: Decidability of Class-Based Knowledge Representation Formalisms. Ph.D. thesis, Università di Roma (1995)
11. Ding, Y., Haarslev, V., Wu, J.: A new mapping from ALCI to ALC. In: Proceedings of DL (2007)
12. Ghilardi, S., Lutz, C., Wolter, F.: Did I damage my ontology? A case for conservative extensions in description logics. In: Proceedings of KR. pp. 187–197 (2006)
13. Kaminski, M., Grau, B.C.: Sufficient conditions for first-order and datalog rewritability in $\mathcal{ELU}$. In: Proceedings of DL. pp. 271–293 (2013)
14. Kazakov, Y.: Consequence-driven reasoning for horn SHIQ ontologies. In: Proceedings of IJCAI. pp. 2040–2045 (2009)
15. Konev, B., Lutz, C., Walther, D., Wolter, F.: Formal properties of modularisation. In: Modular Ontologies: Concepts, Theories and Techniques for Knowledge Modularization, pp. 25–66 (2009)
16. Konev, B., Lutz, C., Walther, D., Wolter, F.: Model-theoretic inseparability and modularity of description logic ontologies. Artificial Intelligence 203, 66–103 (2013)
17. Lutz, C., Piro, R., Wolter, F.: Description logic TBoxes: Model-theoretic characterizations and rewritability. In: Proceedings of IJCAI (2011)
18. Lutz, C., Wolter, F.: Deciding inseparability and conservative extensions in the description logic $\mathcal{EL}$. Journal of Symbolic Computation pp. 194–228 (2010)
19. Lutz, C., Wolter, F.: Foundations for uniform interpolation and forgetting in expressive description logics. In: Proceedings of IJCAI. pp. 989–995 (2011)