On One Multicriteria Optimal Control Problem of Economic Growth

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Abstract. An optimal control problem of an economic system with two criteria is considered in the paper. The Solow model of economic growth is used. Rate of consumption is a control variable, quality vector criterion is a discounted utility in the long-term and near-term. Pontryagin's maximum principle is applied to the single criteria problem with weighted sum of two functionals. The existence of the solution is investigated. The law of Pareto-optimal control, such that the system converges to the balanced growth path, is derived.

Keywords: muticriteria problem of optimal control, the Solow model, the Pareto set.

1 Introduction

The model of economic growth, proposed by R. M. Solow, plays an important role in neoclassical theory [1, 4, 15] and originally is determined by labor and capital. Later the contribution of endogenous parameters, technology and population, was investigated by P. M. Romer [13], P. Aghion and P. Howitt [2], G. M. Grossman and E. Helpman [6], C. I. Jones [7], D. Acemoglu [1]. According to these research technological advance is the result of R&D activity, governmental actions (e.g., taxation, regulations of international trade), financial markets, and imperfect competition.

Golden Rule saving rate was established by E. S. Phelps [10], and it could be expressed in terms of the Solow growth model as follows: it is possible to maximize the size of the consumption in the steady–state mode by choosing the rate of saving. For this purpose we actually use the rate of consumption, which is naturally connected with the rate of saving.

The model considered in the paper is based on the differential equation described changes in capital, where development of technology and population are not taken into account. Single and multiple criteria problems of optimal control built on the Solow model were studied in [9, 3, 14]. Quality factors characterized a discounted utility of consumption and an ecological impact were introduced, and a consumption or a rate of consumption played the role of control variable in the aforementioned references. To

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solve these problems Pontryagin's maximum principle is used. As a result, the system converges to a balanced growth path under constructed optimal control laws.

In the paper we investigate an optimal control problem with two criteria. First component of vector criterion describes a discounted utility from an infinite planning horizon to the present time. However, in such case a near-term outlook is not taken into account. A discounted utility from a given finite time T in the future to the present time is introduced as a second component of vector criterion. Then we transform derived multiple criteria problem to a single criteria problem with weighted sum, which is a piecewise functional. So, the problem is investigated first on interval $[0, T]$ and second on interval $[T, +\infty)$.

Firstly, for each interval we study the existence of solution to the problem and obtain constraints on parameters of the model. Secondly, we use Pontryagin's maximum principle to derive a form of Pareto-optimal control and then establish a stationary point under this control. Summing up, we give a theorem that shows a Pareto-optimal control law on the entire interval $[0, +\infty)$, and thus it propels the system to the balanced growth path.

2 Model of Optimal Control with Two Quality Criteria

Let us consider a single-sector Solow model of optimal growth. We suppose that one product, which could be consumed and invested, is produced. The following external parameters describe the model: $Y(t)$ is a gross domestic product (GDP), $C(t)$ is a nonproduction consumption, $K(t)$ is basic production assets (capital), and $\rho(t)$ is a rate of consumption, i. e., the part of GDP going to the consumption: $C(t) = \rho(t)Y(t)$, where $t \geqslant 0$. In such notation the difference $1 - \rho(t)$ means a saving rate. It should be mentioned that variables are not time invariant and technology is not changed. Here, function $\rho(\cdot)$ plays a role of control variable, and the set of admissible control is $U_\rho = \{\rho(t) \in PC([0,\, 1]) \;\mid\; t \geqslant 0\}$.

The relation between GDP and capital is carried out via production function $F(\cdot)$. Here we will use the Cobb-Douglas production function $F(K) = AK^\alpha(t)$, where $A > 0$ is capital productivity and $\alpha \in (0,\, 1]$ is the elasticity of GDP with respect to capital.

According to [9] a capital is changed by the following equation:

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t),\; K(0) = K_0, \tag{1}$$

where $K \in G$, G is a given bounded open set, $t \geqslant 0$, $\delta > 0$ is a depreciation rate of the stock of capital, $K_0 > 0$ is the initial value of capital. From formula (1) one could obtain the equation of Solow model

$$\dot{K}(t) = (1 - \rho(t))AK^\alpha(t) - \delta K(t),\; K(0) = K_0,\; t \geqslant 0. \tag{2}$$

Consider the logarithmic function of utility

$$u(\rho, K) = \ln(\rho A K^\alpha) = u_0 + \ln(\rho) + \alpha \ln(K),$$

where $u_0 = \ln(A)$ is a constant.

In book [9] Solow model was investigated with integral criterion of quality in the long-term outlook. Let us introduce a multicriterion problem with vector criterion $\mathbf{I}(\rho) = (I_1(\rho), I_2(\rho))$:

$$I_1(\rho) = \int_0^{+\infty} e^{-rt} u(\rho, K) dt = \int_0^{+\infty} e^{-rt} (u_0 + \ln(\rho) + \alpha \ln(K)) dt,$$
$$I_2(\rho) = \int_0^{T} e^{-rt} u(\rho, K) dt = \int_0^{T} e^{-rt} (u_0 + \ln(\rho) + \alpha \ln(K)) dt,$$

where $r > 0$ is a discount rate and $T > 0$ is a fixed given time. The first component of vector criterion $I_1(\cdot, \cdot)$ indicates a discounted utility in the long-term outlook, and the second component $I_2(\cdot, \cdot)$ indicates a discounted utility in the near-term outlook (until time T).

Thus, we obtain the following optimal control problem with two criteria:

$$\begin{gathered}
\mathbf{I}(\rho) = (I_1(\rho), I_2(\rho)) \longrightarrow \max_{\rho(\cdot) \in U_\rho} \\
I_1(\rho) = \int_0^{+\infty} e^{-rt} (\ln(\rho) + \alpha \ln(K)) dt, \\
I_2(\rho) = \int_0^{T} e^{-rt} (\ln(\rho) + \alpha \ln(K)) dt, \\
\dot{K}(t) = (1 - \rho(t)) A K^{\alpha}(t) - \delta K(t),\ K(0) = K_0, \\
\rho(t) \in U_\rho, \quad t \geqslant 0.
\end{gathered} \tag{3}$$

Without loss of generality we omit the component u_0, because it is constant and does not effect on the result. Here, the notation *max* is formal.

Problem (3) with only criterion I_1 was solved in book [9], i. e., a control $\rho^*(t)$, which maximizes the functional I_1 and corresponding balanced growth path $K^*(t)$ were established. To solve problem (3) one should find the Pareto set $P_{\mathbf{I}}(U_\rho)$ (see, e. g., [11]), since the problem is multicriterion. The Pareto set $P_{\mathbf{I}}(U_\rho)$ is the set of control $\rho^*(t)$ from admissible set U_ρ, such that it does not exist any other control $\rho'(t)$ from admissible set U_ρ which satisfies inequality $\mathbf{I}(\rho'(t)) \geq \mathbf{I}(\rho^*(t))$. Thus,

$$P_{\mathbf{I}}(U_\rho) = \{\rho^*(t) \in U_\rho \mid \nexists\ \rho'(t) \in U_\rho :\ \mathbf{I}(\rho'(t)) \geq \mathbf{I}(\rho^*(t))\}.$$

The control $\rho^*(t)$ is called Pareto-optimal control. Here, inequality $\mathbf{I}(\rho'(t)) \geq \mathbf{I}(\rho^*(t))$ means that inequalities $I_1(\rho'(t)) \geqslant I_1(\rho^*(t)),\ I_2(\rho'(t)) \geqslant I_2(\rho^*(t))$ hold, that is more $\mathbf{I}(\rho'(t)) \neq \mathbf{I}(\rho^*(t))$.

So, to solve problem (3) we should find the Pareto set $P_{\mathbf{I}}(U_\rho)$. For that purpose we use Pontryagin's maximum principle, which will be considered in the next section.

3 Use of Pontryagin's Maximum Principle to Weighted Sum of Criteria

Consider the set $\mathcal{I} = \mathbf{I}(U_\rho)$. It could be easily proved that the set $\mathcal{I}$ is convex. According to Proposition III.1.5 from [8] if the set $\mathcal{I}$ is convex, then for any function $\rho^*(t) \in P_{\mathbf{I}}(U_\rho)$

there exists such parameter $\mu = (\mu_1, \mu_2) \geq 0_2$ that $\rho^*(t) = \arg \max_{\rho(t) \in U_\rho} \varphi(\rho(t))$, where $\varphi(\rho(t)) = \mu_1 I_1(\rho(t)) + \mu_2 I_2(\rho(t))$.

Also one could establish that any admissible control $\rho^*(t)$ maximizing functional $\varphi(\rho)$ is a Pareto-optimal control with respect to criterion $\mathbf{I}$. Thus, multicriteria problem (3) is equivalent to the problem with one criterion $\varphi(\rho)$ with nonnegative parameters $\mu_1,\ \mu_2$:

$$\begin{gathered} \varphi(\rho) \longrightarrow \max_{\rho(\cdot) \in U_\rho} \\ \text{subject to (2)},\ \rho(t) \in U_\rho,\ \mu \geq 0_2,\ t \geqslant 0. \end{gathered} \tag{4}$$

3.1 Solution to the Problem on Interval $[0, T]$

The weighted sum $\varphi(\rho)$, as an optimization criterion of problem (4), is piece-wise defined: on interval $[0, T]$ and on interval $[T, +\infty)$. Let us initially consider problem (4) on finite interval $[0, T]$. Put $\varphi_T(\rho) = \varphi(\rho)\Big|_{t \in [0,T]}$.

$$\begin{gathered} \varphi_T(\rho) = \int_0^T e^{-rt}(\mu_1 + \mu_2)(\ln(\rho) + \alpha \ln(K))dt \longrightarrow \max_{\rho(\cdot) \in U_\rho} \\ \text{subject to: } \dot{K}(t) = (1 - \rho(t))AK^\alpha(t) - \delta K(t),\ K(0) = K_0, \\ \rho(t) \in U_\rho,\ \mu \geq 0_2,\ t \in [0, T]. \end{gathered} \tag{5}$$

Conditions of existence of solution to optimal control problem on infinite interval was established in [3], [14]. Let us investigate the corresponding conditions for problem (5). Firstly, one could obtain that there exists such constant $L > 0$ that inequality

$$K((1 - \rho)AK^\alpha - \delta K) \leqslant L(1 + K^2)$$

holds for all $K \in G$, $\rho \in U_\rho$ (condition (A1) in [3]). Secondly, condition (A2) [3] of convexity of set

$$\begin{gathered} Q(K) = \{(z^0, z) :\ z^0 \leqslant (\mu_1 + \mu_2)(\ln(\rho) + \alpha \ln(K)); \\ z = (1 - \rho(t))AK^\alpha(t) - \delta K(t);\ \rho \in U_\rho\ \} \end{gathered}$$

is valid for all $K \in G$ due to linearity of the equality with respect to ρ and convexity of logarithmic function. Then from conditions (A1), (A2) and Filippov's theorem (Theorem 9.3.i from [5]) there exists an optimal pair (K^*, ρ^*) for problem (5).

As aforesaid, if some control $\rho = \rho^*(t)$ is an optimal control with respect to functional $\varphi(\rho)$, and hence it is a Pareto-optimal solution with respect to criterion $\mathbf{I}$, then according to Pontryagin's maximum principle [12] such control maximizes Hamilton function.

Let ψ_0 and ψ be the adjoint variables. Construct a Hamilton function to problem (5):

$$H(t, K(t), \psi_0, \hat{\psi}(t), \rho) = ((1 - \rho)AK^\alpha - \delta K)\hat{\psi} + \psi_0(\mu_1 + \mu_2)(\ln(\rho) + \alpha \ln(K)),$$

where $\hat{\psi}(t) = e^{rt}\psi(t)$. Also without loss of generality we can assume that $\psi_0 = 1$. Hamilton function could be written in this way according to [3].

Let $\rho_{opt} \in P_{\mathbf{I}}(U_\rho)$. From equation $\partial H/\partial\rho = 0$ we obtain constrained maximum

$$\rho_{opt} = \begin{cases} 0, & K^\alpha \hat{\psi} < 0, \\ \frac{\mu_1+\mu_2}{AK^\alpha \hat{\psi}}, & K^\alpha \hat{\psi} \geqslant \frac{\mu_1+\mu_2}{A}, \\ 1, & 0 \leqslant K^\alpha \hat{\psi} \leqslant \frac{\mu_1+\mu_2}{A}, \end{cases} \tag{6}$$

and also the adjoint equation is the following:

$$\dot{\hat{\psi}} = ((r+\delta) - \alpha(1-\rho)AK^{\alpha-1})\hat{\psi} - \alpha(\mu_1+\mu_2)\frac{1}{K},\ \hat{\psi}(T) = 0. \tag{7}$$

Using the same arguments as in Theorem 10.1 [3] we can establish that the adjoint variable $\hat{\psi}(t) > 0$ for all $t \in [0, T)$. It leads to the impossibility of the equality $\rho_{opt} = 0$ in formula (6).

As mentioned above derived control (6), which depends on nonnegative parameters μ_1, μ_2 ($\mu_1 + \mu_2 > 0$), is a Pareto-optimal control of problem (5).

3.2 Constructing Balanced Growth Path on Interval $[0, T]$

From the economic point of view consideration of steady-state mode $K(t) == K^*$, $C(t) = C^*$ has a practical interest since it shows an equilibrium state of the economic system.

Consider the case when $\rho = \frac{\mu_1+\mu_2}{AK^\alpha \hat{\psi}}$. Then using formula (6) we change variable $\hat{\psi}$ by variable $C(t) = \rho(t)Y(t)$ in equation (7). So, we have the following Hamilton system:

$$\begin{aligned} \dot{K}(t) &= AK^\alpha(t) - C(t) - \delta K(t), \\ \dot{C}(t) &= C(t)(\alpha AK^{\alpha-1}(t) - (r+\delta)). \end{aligned} \tag{8}$$

Conditions $\dot{K}(t) = 0$, $\dot{C}(t) = 0$ give a stationary solution (K^*, C^*) of equations (8)

$$K^* = \left(\frac{r+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}},\ C^* = \left(\frac{r+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} \cdot \left(A\left(\frac{r+\delta}{\alpha A}\right)^{\alpha} - \delta\right). \tag{9}$$

It is easy to see that $K^* > 0$ and $C^* > 0$, if the inequality $K^* < (A/\delta)^{1/(1-\alpha)}$ is valid. One can show that if $\rho \equiv 1$ on interval $[0, T]$, then steady-state mode is $K^* \equiv 0$, $C^* \equiv 0$. Obviously, this case does not have a practical sense.

Lemma 1. *Stationary point (K^*, C^*) (9) of Hamilton system (8) is a saddle point.*

The proof is based on linearization of Hamilton system (8) in a neighborhood of stationary point (K^*, C^*). Moreover, eigenvalues of Jacobian of system (8) are real, and its product is negative.

3.3 Solution to the Problem on Interval $[T, +\infty)$

Now consider problem (4) on interval $[T, +\infty)$. Put $\varphi_{+\infty}(\rho) = \varphi(\rho)\Big|_{t\in[T,+\infty)}$.

$$\begin{aligned} &\varphi_{+\infty}(\rho) = \int_T^{+\infty} e^{-rt}\mu_2(\ln(\rho) + \alpha\ln(K))dt \longrightarrow \max_{\rho(\cdot)\in U_\rho} \\ &\text{subject to: } \dot{K}(t) = (1-\rho(t))AK^\alpha(t) - \delta K(t),\ K(T) = K_1(T), \\ &\qquad \rho(t) \in U_\rho,\ \mu_2 \geqslant 0,\ t \in [T, +\infty), \end{aligned} \tag{10}$$

where $K_1(t)$ is the solution of equation (2) on interval $[0, T]$. According to [14] one could derive that the following result is valid.

Lemma 2. *A solution to optimal control problem (10) exists if $A > \delta$ and*

$$0 \leqslant K_0 \leqslant \left(\frac{A}{\delta}\right)^{\frac{1}{1-\alpha}}. \tag{11}$$

The proof is based on checking the implementation of conditions (A1), (A2), and (A3) from [3] and conditions of Filippov's theorem (Theorem 9.3.i from [5]).

Construct a Hamilton function to problem (10):

$$H(t, K(t), \psi_0, \hat{\psi}(t), \rho) = ((1-\rho)AK^\alpha - \delta K)\hat{\psi} + \mu_2(\ln(\rho) + \alpha\ln(K)),$$

Let $\rho_{opt} \in P_{\mathbf{I}}(U_\rho)$. Simultaneously Subsection 3.1 we establish constrained maximum

$$\rho_{opt} = \begin{cases} 0, & K^\alpha\hat{\psi} < 0, \\ \frac{\mu_2}{AK^\alpha\hat{\psi}}, & K^\alpha\hat{\psi} \geqslant \frac{\mu_2}{A}, \\ 1, & 0 \leqslant K^\alpha\hat{\psi} \leqslant \frac{\mu_2}{A}, \end{cases} \tag{12}$$

and the adjoint equation

$$\dot{\hat{\psi}} = ((r+\delta) - \alpha(1-\rho)AK^{\alpha-1})\hat{\psi} - \alpha\mu_2\frac{1}{K}. \tag{13}$$

Also if conditions $A > \delta$ and (11) are hold, the adjoint variable $\hat{\psi}(t) > 0$ for all $t \in [T, +\infty)$ (see Theorem 10.1 [3] and Lemma 4.1 in [14]). So, the equality $\rho_{opt} = 0$ in formula (12) is impossible. We establish that derived control (12), which depends only on parameter $\mu_2 > 0$, is a Pareto-optimal control of problem (10).

3.4 Law of Optimal Control for System (3)

Similarly to Subsection 3.2 if we assume that $\rho = \frac{\mu_2}{AK^\alpha\hat{\psi}}$, then we will have the same Hamilton system (8) with stationary solution (K^*, C^*) (9).

Since a stationary solution $K^* \equiv 0$, $C^* \equiv 0$ is yielded if $\rho \equiv 1$, this control could not propel the system to the balanced growth path $K(t) = K^*$, $C(t) = C^*$, which is assigned by formula (9).

We prove that the asymptotic behavior of Pareto-optimal control is determined by the following theorem.

Theorem 1. *Let inequalities $A > \delta$ and (11) hold, point $(K^*(\cdot), \rho^*(\cdot))$ is a Pareto-optimal point of problem (3), and variable $\hat{\psi}(\cdot)$ is the adjoint variable. Then there exists such $\tau \geqslant 0$, that*

$$\begin{aligned} \rho^*(t) &= \frac{\mu_1 + \mu_2}{A\hat{\psi}(t)(K^*)^\alpha(t)} \text{ for all } t \in [\tau, T], \text{ if } \tau \leqslant T; \\ \rho^*(t) &= \frac{\mu_2}{A\hat{\psi}(t)(K^*)^\alpha(t)} \text{ for all } t \in [T, +\infty), \text{if } \tau \leqslant T \\ &\qquad (\text{ for all } t \in [\tau, +\infty) \text{ , if } \tau > T), \end{aligned} \tag{14}$$

where parameters $\mu_1, \mu_2 \geqslant 0$, $\mu_1 + \mu_2 > 0$.

The proof is based on Lemma 4.2 from [14].

Thus, Pareto-optimal control (14) asymptotically propels the system (3) to the balanced growth path $K(t) \equiv K^*$,$C(t) \equiv C^*$ (9). Moreover, the situation, in which the control $\rho \equiv 1$ on interval $[0, \tau]$ could also be used, is to be investigated.

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References

1. Acemoglu, D.: Introduction to Modern Economic Growth. Princeton University Press (2009)
2. Aghion, P., Howitt, P.: Endogenous Growth Theory. MIT Press, Cambridge MA (1998)
3. Aseev, S.M., Kryazhimskii, A.V.: The pontryagin maximum principle and optimal economic growth problems. In: Proceedings of the Steklov Institute of Mathematics. vol. 257 (1), pp. 1–255 (2007)
4. Barro, R.J., Sala-i Martin, X.: Economic Growth. The MIT Press, London, England (2004)
5. Cesari, L.: Optimization – theory and applications. Problems with ordinary differential equations. Springer, New York (1983)
6. Grossman, G.M., Helpman, E.: Innovation and Growth in the Global Economy. MIT Press, Cambridge, MA (1991)
7. Jones, C.I.: Growth:with or without scale effects. American Economic Review 89, 139–144 (1999)
8. Makarov, I.M., Vinogradskaya, T.M., Rubchinskiy, A.A., Sokolov, V.B.: Teoriya vybora i prinyatiya recheniy (Theory of choice and decision making) (In Russian). Nauka, Moscow (1982)
9. Noghin, V.D.: Vvedenie v optimal'noe upravlenie (Introduction to optimal control) (In Russian). Yutas, Saint-Petersburg (2008)
10. Phelps, E.S.: Golden Rules of Economic Growth: Studies of Efficient and Optimal Investment. W. W. Norton, New York (1966)

11. Podinovskiy, V.V., Noghin, V.D.: Pareto-optimal'nye resheniya mnogocriterial'nyh zadach (Pareto-optimal solutions of multicriteria problems) (In Russian). Fizmatlit, Moscow (2007)
12. Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishechenko, E.F.: The Mathematical Theory of Optimal Processes. John Wiley & Sons, New York / London (1962)
13. Romer, P.M.: Endogenous technological change. Journal of Political Economy 98, part II, 71–102 (1990)
14. Rovenskaya, E.A.: Model' economicheskogo rosta i svyazannogo s nim kachestva okruzhayushchey sredy (economic growth model and related quality of environment) (in russian). Matematicheskaya teoriya igr i ee prilozheniya 3(3), 67–84 (2011)
15. Solow, R.M.: Growth Theory: An Exposition. Oxford University Press, 2 edn. (2000)