Distributed approximation algorithms for k-dominating set in graphs of bounded genus and linklessly embeddable graphs Regular Submission

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Abstract. A k-dominating set in a graph $G = (V, E)$ is a set $U \subseteq V$ such that ever vertex of G is either in U or has at least k neighbors in U. In this paper we give simple distributed approximation algorithms in the local model for the minimum k-dominating set problem for $k \geq 2$ in graphs with no $K_{3,h}$-minor and graphs with no $K_{4,4}$-minor. In particular, this gives fast distributed approximations for graphs of bounded genus and linklessly embeddable graphs. The algorithms give a constant approximation ratio and run in a constant number of rounds. In addition, we will give a $(1 + \epsilon)$-approximation for an arbitrary fixed $\epsilon > 0$ which runs in $O(\log^* n)$ rounds where n is the order of a graph.

1 Introduction

The minimum dominating set (MDS) problem is one of the most extensively studied graph-theoretic questions. On one hand, it plays an important role in graph theory, on the other, it admits many important applications. Its complexity has been studied in many different computational models and for various classes of graphs. In general, the problem is NP-hard [7], and it is even NP-hard to find a $(C \log \Delta(G))$-approximation for some constant C [16]. In addition to the above hardness results for the sequential model, similar restrictions apply to the distributed complexity of the problem in general graphs. At the same time, there has been much more success in devising efficient approximations for specific classes of graphs. In particular, the problem is much more tractable when restricted to the family of planar graphs and an efficient deterministic approximation algorithm for the minimum dominating set problem in planar graphs was given in [13]. The algorithm from [13] gives a constant approximation of a

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minimum dominating set, runs in a constant number of rounds but uses long messages (more than $O(\log n)$ bits).In [20], Wawrzyniak improved the analysis from [13] and showed that the algorithm from [13] gives a 52-approximation. In addition, in [19], Wawrzyniak proposed a local (constant-time) algorithm which gives a 694-approximation of the minimum dominating set problem in planar graphs which uses messages of length at most $O(\log n)$ and works in the port numbering model. (In this model, which is similar to the Local model, no unique identifiers are required but edges which are incident to a vertex are numbered, see [18].)

In [2], the authors showed that essentially the same algorithm gives a constant approximation of a minimum dominating set in graphs of bounded genus. In fact, as proved in [2], no geometrical considerations are needed when analyzing the procedure from [13] as the only requirement is that the underlying graph has no $K_{3,h}$ as a minor.

Often a stronger notion of a domination is needed. For example, in some applications it can be desirable to require that every vertex in a graph is dominated not by one vertex from a set but by k vertices for some $k \in \mathcal{Z}^+$. This can be interpreted as a fault tolerance requirement; Servers are placed in a network so that each vertex/client has direct access to k servers.

To define the problem formally we need some notation and terminology. Let $G = (V, E)$ be a graph with $|V| = n$. By $N_G(v) = \{u \in V : uv \in E\}$ we denote the neighborhood of a vertex $v \in V$ and by $N_G[v] = N_G(v) \cup \{v\}$ the closed neighborhood of v. We use $d_G(v)$ (or $d(v)$ if G is clear from the context) to denote $|N_G(v)|$. Set $D \subseteq V$ is called a *dominating set* in G if every vertex in $V \setminus D$ is adjacent to a vertex in D. *A minimum dominating set* in a graph G is a dominating set in G of the smallest size. For a positive integer k, a k*-dominating set* in G is a subset $D \subseteq V$ such that for every $v \in V \setminus D$, $|N(v) \cap D| \geq k$, that is, for every vertex $v \in V$, v is in D or v has at least k neighbors in D. In particular, a 1-dominating set is simply a dominating set. In addition, we will say that a set W k-dominates set U in G, if every vertex from $U \setminus W$ has at least k neighbors in W. Note that other notions of k-domination have been considered and many variants of the problem have been studied (see [8]). The k-domination problem was proposed in [6] as a natural generalization of the minimum dominating set problem. It has been studied extensively in graph theory, where the main line of research is to establish upper bounds for $\gamma_k(G)$, the minimum size of a k-dominating set. It is known that the problem is NP-complete in general graphs but can be solved in linear time in trees and series-parallel graphs ([9]). Further generalizations of the k-dominating set were considered and many slightly different variants have been proposed. The distributed complexity of finding $\gamma_k(G)$ (the k-MDS problem) were studied in [10] and [12]. In addition, in [3] distributed approximation algorithms for the minimum k-dominating set problem in planar graphs are proposed. In this paper, we will study distributed complexity of the problem in two classes of graphs, graphs of bounded genus and linklessly embeddable graphs. Both of these classes are proper minor-closed families of graphs and in our algorithms and analysis we will not need any geometric properties of the graphs as we will work in a slightly more general setting by forbidding some graphs as minors. Let G be a graph, we say that H is a minor of G (or that G has an H-minor) if H can be obtained from a subgraph of G by sequence of edge deletions and contractions. We say that G contains a subdivision of H (or that TH is a subgraph of G) if G contains a graph obtained from H by replacing edges of H by independent paths of length at least one.

Let S_h denote the surface obtained from the sphere by adding h handles (see [5] for a formal definition of this process). The *genus* of a graph G, $g(G)$, is the least integer h such that there is an embedding of a graph G into S_h. In what follows we will not make any geometric arguments as our argument applies to a bigger class of graphs than the class of graphs of genus at most g. Note that if G is embeddable in a surface S, then the graphs obtained from G by deleting an edge or contracting an edge are embeddable as well. Consequently the class of graphs of genus at most g is a proper minor-closed family of graphs and can be characterized by a finite set of forbidden minors. In addition, since $g(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$ (see [14]), we have that if $g(G) \leq g$, then G has no $K_{3,4g+3}$-minor. As in [2], we will work with the family of graphs $\mathcal{B}_h$ which have no $K_{3,h}$-minor.

In addition to the graphs with no $K_{3,h}$-minor we will give a distributed approximation algorithm for graphs with no $K_{4,4}$-minor. This, in particular, yields an algorithms for *linklessly embeddable* graphs which are sometimes viewed as 3-dimensional analogs of planar graphs. An embedding of a graph in R^3 is called *linkless* if every pair of (vertex) disjoint cycles are unlinked closed curves. It is not difficult to see that $K_{4,4}$ is not linklessly embeddable (the only fact that will be needed).

We will consider the *Local* model ([15]), which is a synchronous message-passing model in which vertices have unique identifiers from $\{1, \ldots, n\}$ where n is the order of the graph. Computations proceed in rounds and in each round a vertex can send messages to all its neighbors, can receive messages from its neighbors and can perform some local computations. Neither the size of the messages nor the amount of local computations is restricted in any way. An algorithm is executed by the vertices of a network and its objective is to find a solution to a problem (in our case the k-MDS problem) in this underlying network.

Our contribution is twofold. First, we show that for graphs of bounded genus there is an easy distributed algorithm which gives a constant approximation of the minimum k-dominating set for $k \geq 2$. This mirrors the corresponding result for planar graphs. In addition, the same approach gives a constant approximation for $k \geq 3$ in linklessly embeddable graphs. The case $k = 2$ is different, and to address it we use a modification of the algorithm from [13] and some ideas from [2] and [1]. The above approximation algorithms run in a constant number of rounds and give a constant approximation ratio. Further improvement of the approximation ratio are left for future work. Second, we propose a distributed approximation algorithm with approximation ratio of $(1+\epsilon)$ for every fixed $\epsilon > 0$.

This algorithm runs in $O(\log^* n)$ rounds where n is the order of the graph. This algorithm combines the main procedure from [4] with some additional ideas.

The paper is structured as follows. In the next section, we state some preliminary facts. In the following section we give the first (simple) approximation algorithm for the minimum k-dominating set in graphs with no $TK_{h,k+1}$. This yields an algorithm for graphs of bounded genus for $k \geq 2$ and for linklessly embeddable graphs when $k \geq 3$. In Section 4 we address the $2MDS$ problem for graphs with no $TK_{4,4}$ which gives, in particular, an algorithm for linklessly embeddable graphs when $k = 2$. Finally, in Section 5, we discuss the $(1+\epsilon)$-approximation.

2 Preliminaries

We will start our technical discussion with a few facts which will be used later. We have the following bound for the number of edges in a graph with TK_h (see [5]).

Lemma 1. *If G is a graph such that $|E(G)| \geq 5h^2n$, then G has TK_h.*

We will also need a more general bound for $E(G)$ in the case G has no K_p-minor. The following was proved by Kostochka in [11].

Lemma 2 (Kostochka). *There exists $c \in R$ such that for every $r \in \mathcal{Z}^+$ if G is a graph on n vertices with no K_r-minor, then $|E(G)| \leq cr\sqrt{\log r}n$.*

One of the main tools used in the proofs will be lemmas from [1]. Let v be a vertex in G and let $S \subseteq V(G) \setminus \{v\}$. Then a set of paths is called a v, S-*fan* if every path starts at v, ends in a vertex from S which is the only vertex from S on the path, and any two distinct paths have only v in common. For a dominating set D in a graph G let $D_{k,l}$ be the set of vertices $v \in V(G) \setminus D$ such that there is a v, D-fan subgraph in G consisting of k paths, each of length at most l. The following two facts are proved in [1].

Lemma 3. *For $h, l \in \mathcal{Z}^+$ there is c such that the following holds. Let G be a graph with no TK_h and let D be a dominating set in G. Then $|D_{h-1,l}| \leq c|D|$.*

In addition, we have the following fact.

Lemma 4. *Let $h \in \mathcal{Z}^+$. For $l, m \in \mathcal{Z}^+$ there is $c = c(m,l)$ such that if G is graph with no $TK_{m,h}$ and D is a dominating set in G then $|D_{m,l}| \leq c|D|$.*

3 First algorithm for graphs with bounded genus and linklessly embeddable graphs

In our first algorithm (Procedure 1) we simply add to the solution all vertices of sufficiently large degree and claim that this gives a constant approximation of a k-dominating set. Specifically, we will show that in the case when $k \geq 2$ and G has bounded genus or $k \geq 3$ and G is linklessly embeddable, this gives a constant approximation of a minimum k-dominating set. We will need the following lemma in the analysis.

Lemma 5. *Let $k, h, q \in \mathcal{Z}^+$ be such that $k \leq q$. If $H = H[U, X]$ is a bipartite graph such that $|U| \geq \binom{q}{k}(h-1)+1$, $|X| \leq q$ and for every vertex $u \in U$, $d_H(u) \geq k$, then H contains $K_{h,k}$.*

Proof. The proof is similar to a proof of an upper bound for the extremal number of a complete bipartite graph. Suppose H has no $K_{h,k}$. We will count the number of k-stars (stars of degree k) with centers in U. On one hand, the number of such k-stars is at least $\sum_{u \in U} \binom{d(u)}{k} \geq |U|$. On the other hand, since H has no $K_{h,k}$, the number of k-stars is at most $\binom{|X|}{k}(h-1)$. Consequently, if $|U| \geq \binom{|X|}{k}(h-1)+1$, then H contains $K_{h,k}$. $\square$

Let $h, k \in \mathcal{Z}^+$, set $L := \binom{k+h}{k}(h-1)+2(k+h)+1$ and consider the following procedure.

PROCEDURE 1

1. For every $v \in V(G)$, if $d(v) \geq L$, then add v to D.
2. For every $v \in V(G) \setminus D$, if $|N(v) \cap D| < k$, then add v to D.
3. Return D.

In fact we will proves something slightly more general.

Fact 1 *For $h, k \in \mathcal{Z}^+$ let $L := \binom{k+h}{k}(h-1)+2(k+h)+1$. If G is a graph with no TK_{h+k+1} and no $K_{h,k+1}$, then Procedure 1 applied to G with L returns a k-dominating set of size $O(\gamma_k(G))$.*

Proof. Let $W := \{v \in V(G) : d(v) \geq L\}$ and let D^* be an optimal k-dominating set. Then the number of vertices in G which have a neighbor in $D^* \setminus W$ is at most $(L-1)|D^*|$. Suppose that $v \in V(G) \setminus W$ has $|N(v) \cap W| < k$ and so it is added to D in the second step of the algorithm. Then v is either in D^* or has at least one neighbor in $D^* \setminus W$ and so the number of vertices added in step 2 is at most $L|D^*|$. Thus to show that $|D| = O(\gamma_k(G))$ it is enough to show that $|W| = O(|D^*|)$. To that end, note that for every $v \in W \setminus D^*$, $N(v) \setminus D^*$ is k-dominated by D^*. If $|N(v) \cap D^*| \geq h+k$, then $v \in D^*_{h+k,1}$. Since G has no TK_{h+k+1}, D^* is a k-dominating set and so a 1-dominating set, by Lemma 3, the number of such vertices is $O(|D^*|)$. Thus we may assume that $|N(v) \setminus D^*| > L-(h+k)$ and recall that $N(v) \setminus D^*$ is k-dominated by D^*. Consider a maximum matching M in $G[N(v) \setminus D^*, D^*]$ and note that if $|M| \geq h+k$, then $v \in D^*_{h+k,2}$. Thus we may assume that $|M| < h+k$. Since M is a maximum matching in $G[N(v) \setminus D^*, D^*]$, if $w \in N(v) \setminus (D^* \cup V(M)$, then all neighbors of w which are in D^* are in $V(M)$ as otherwise we could increase M. Consequently, $|N(N(v) \setminus (D^* \cup V(M))) \cap D^*| \leq |V(M) \cap D^*| < h+k$. In addition, $|E(N(v) \setminus (D^* \cup V(M)), V(M) \cap D^*)| \geq k|N(v) \setminus (D^* \cup V(M))|$ because D^* is a k-dominating set,and , by definition of L, $|N(v) \setminus (D^* \cup V(M))| \geq \binom{k+h}{k}(h-1)+1$. Thus, by Lemma 5, $G[N(v) \setminus (D^* \cup V(M)), V(M) \cap D^*]$ contains $K_{h,k}$, which in connection with v gives $K_{h,k+1}$ contradicting the fact that G has no $TK_{h,k+1}$. The contradiction shows that $|M| \geq h+k$ and so $v \in D^*_{h+k,2}$. By Lemma 3, $|D^*_{h+k,2}| = O(\gamma_k(G))$. $\square$

We immediately get constant approximations for $\gamma_k(G)$ in linklessly embeddable graphs for $k \geq 3$.

Corollary 2 *Let G be a linklessy embeddable graph and let $k \geq 3$. Then Procedure 1 applied to G returns a k-dominating set of size $O(\gamma_k(G))$ when applied with $h = 4$.*

Proof. Since G is linklessly embeddable, G has no $K_{4,4}$-minor and therefore no $K_{4,4}$ and no TK_8. We can apply Procedure 1 with $h = 4$. □

If G is a graph with $g(G) \leq g$, then G has no $K_{4g+3,3}$-minor. Consequently Procedure 1 can be applied to G for every $k \geq 2$ gives a constant approximation.

Corollary 3 *Let $g \in \mathcal{Z}^+$ and $k \geq 2$. For every graph G with $g(G) \leq g$, Procedure 1 returns a k-dominating set of size $O(\gamma_k(G))$ when applied with $h = 4g+3$.*

Proof. Since G has genus bounded by g, $G \in \mathcal{B}_{4g+3}$, and so G has no $TK_{4g+3,3}$. □

4 Graphs with no $TK_{4,4}$ for $k = 2$

In the case G has no $K_{4,4}$-minor and $k = 2$ it is no longer possible to claim that Procedure 1 gives a constant approximation of $\gamma_k(G)$ and a slightly more involved algorithm is needed. We will show that a modification of the algorithm for the minimum dominating set in planar graphs from [13] gives a constant approximation in this case. Recall that we say that a vertex v is 2-dominated by a set S if either $v \in S$ or v has at least two neighbors in S.

We will start with the following observation.

Lemma 6. *Let $K \in \mathcal{Z}^+$. If U is a subset of $V(G)$ such that $|U| \geq 4K^2$ and there is a set $S \subseteq V(G) \setminus U$ of size $|S| \leq K$ which 2-dominates U, then there exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $|N(s_1) \cap N(s_2) \cap U| \geq 4$.*

Proof. Clearly there exists $s_1 \in S$ such that $|N(s_1) \cap U| \geq 4K^2/K = 4K$ and there exists $s_2 \in S \setminus \{s_1\}$ such that $|N(s_2) \cap (N(s_1) \cap U)| \geq 4K/K \geq 4$. □

In what follows, let $K := 642$, $C := 4K^2 + K$ and $L := 4C + 3$. We will assume that G has no $TK_{4,4}$.

Procedure 2

1. For every $v \in V(G)$: If there is no set $S \subseteq V \setminus \{v\}$ such that $|S| \leq K$ and S 2-dominates $N(v)$, then add v to D_1. In addition if $d(v) < 2$, then add v to D_1.
2. Let Z be the set of vertices which are 2-dominated by D_1 and let $d'(v) := |N(v) \setminus Z|$ for $v \in V$.
3. For every $u \in V \setminus Z$:
   - If $|N(u) \cap D_1| = 1$, then choose $v \in N(u) \setminus D_1$ such that $d'(v)$ is maximum and add it to D_2.
   - If $|N(u) \cap D_1| = 0$, then choose two distinct $v, v' \in N(u)$ with the largest degrees d' and add them to D_2.
4. Return $D_1 \cup D_2$.

Note that Procedure 2 works in the *Local* model as in the second step a vertex v can obtain locally information about the ball of radius two centered at v to determine if it should be added to D_1.

We first note that $D_1 \cup D_2$ is a 2-dominating set.

Fact 4 *The set $D_1 \cup D_2$ is a 2-dominating set in G.*

Proof. Suppose $u \notin D_1$ and $|N(u) \cap D_1| = i$ for $i = 0, 1$. Then $d(u) \geq 2$, and so u has at least $2 - i$ neighbors in $V \setminus D_1$. In the second step of the procedure $2 - i$ of them will be added to D_2. □

Let D^* be an optimal 2-dominating set in G. We will now show that $|D_1 \cup D_2| \leq C|D^*|$ for some constant C.

Fact 5 $|D_1| = O(|D^*|)$.

Proof. First note that if $d(v) < 2$, then $v \in D^*$. In addition, for $v \in D_1 \setminus D^*$, we certainly have $|N(v)| > K$ because $N(v)$ is an option for S. Let $V_1 := \{v \in D_1 \setminus D^* : |N(v) \cap D^*| \geq K/2\}$. We have $K|V_1|/2 \leq |E_G(V_1, D^*)| < 5 \cdot 64(|V_1| + |D^*|)$ by Lemma 1, and so $|V_1| < 320|D^*|$. Let $V_2 := \{v \in D_1 \setminus D^* : |N(v) \cap D^*| < K/2\}$. Take $v \in V_2$ and let $W \subseteq D^*$ be a minimum set which 2-dominates $N(v) \setminus D^*$. We have $|W| \geq K/2$ because $W \cup (N(v) \cap D^*)$ 2-dominates $N(v)$. In addition, $|N(v) \setminus D^*| = |N(v)| - |N(v) \cap D^*| > K/2$. Therefore, by minimality of W, there is a matching of size at least $|W|/2 \geq K/4 \geq 4$ between W and $N(v) \setminus D^*$, and consequently there is a v, W_2-fan of size four. Therefore, $v \in D^*_{4,2}$ and, by Lemma 4, $|V_2| = O(|D^*|)$. □

To show that $|D_1 \cup D_2| = O(|D^*|)$ we will now prove that $|D_2| = O(|D^*|)$.

Fact 6 $|D_2| = O(|D^*|)$.

Proof. If $D^* \setminus D_1 = \emptyset$, then every vertex in $V \setminus D_1$ has two neighbors in D_1 and $D_2 = \emptyset$. For a vertex $u \in V \setminus (Z \cup D^*)$ which has one neighbor in D_1, there is at least one $v \in D^* \setminus D_1$ such that $u \in N(v)$. We fix one such v and say that u *belongs* to v. If u has no neighbors in D_1, then there exist at least two vertices $v, v' \in D^* \setminus D_1$ such that $u \in N(v) \cap N(v')$. We fix such a pair (v, v') and say that u belongs to v if $d'(v) < d'(v')$ (if the degrees are equal then we select one of v, v' arbitrarily.)

Note that every vertex $u \in V \setminus Z$ is either in $D^* \setminus D_1$ or belongs to exactly one vertex from $D^* \setminus D_1$.

In order to show that $|D_2| = O(|D^*|)$ we will argue that for every vertex $v \in D^* \setminus D_1$ there can be at most a constant number of vertices added to D_2 by v and by vertices which belong to v. Let $v \in D^* \setminus D_1$. Note that v can select at most two vertices to be added to D_2. In addition, there exists a set $S \subseteq V \setminus \{v\}$ such that $|S| \leq K$ and S 2-dominates $N(v)$. Fix such a set S and call it S_v. Let $\tilde{N}(v)$ be the set of $u \in N(v)$ such that u belongs v. First suppose $|\tilde{N}(v)| \leq L$. Since each $u \in \tilde{N}(v)$ selects at most two vertices in step 3, the number of vertices added to D_2 by vertices from $\tilde{N}(v)$ is at most $2L$. Now assume $|\tilde{N}(v)| > L$. Let $w \in V \setminus (D_1 \cup D^* \cup S_v)$ and suppose w has a neighbor in $\tilde{N}(v)$. If $|N(w) \cap \tilde{N}(v)| > C$, then S_v 2-dominates $(N(w) \cap \tilde{N}(v)) \setminus S_v$ which has size at least $4K^2$, and so by Lemma 6, G contains $K_{4,4}$ which is not possible. Thus $0 < |N(w) \cap \tilde{N}(v)| \leq C$ for every vertex v. For a vertex x, if $u \in N(w) \cap \tilde{N}(x)$, u belongs to x and $d'(x) > L$. Thus if u selects w in step 3, then $d'(w) > L$. All neighbors of w in $G[V \setminus Z]$ are either in $D^* \setminus D_1$ or in one of the sets $\tilde{N}(x)$. If w has four neighbors in D^*, then $w \in D^*_{4,1}$. Otherwise, w has at least one neighbor in $\tilde{N}(x)$ for at least $(L-3)/C \geq 4$ vertices $x \in D^* \setminus D_1$. Consequently, by Lemma 4, the number of such vertices is $O(|D^*|)$. Finally, since $|S_v| \leq K$, there are at most K vertices in S_v which can be added to D_2 by vertices in $\tilde{N}(v)$ in step 3. □

Thus we have the following theorem.

Theorem 7. *There exists $C \in \mathcal{Z}^+$ such that the following holds. Let G be a graph with no $TK_{4,4}$. Then* PROCEDURE 2 *finds a 2-dominating set D such that $|D| \leq C\gamma_2(G)$.*

From Theorem 7, we get a constant approximation for the 2-minimum dominating set problem in linklessly embeddable graphs.

Corollary 8 *Let G be a linklessy embeddable graph. Then* PROCEDURE 2 *applied to G returns a 2-dominating set of size $O(\gamma_2(G))$.*

5 $(1+\epsilon)$-approximation

In this section we will show that a modification of the approach from [4] gives a $(1+\epsilon)$-approximation of the k-dominating set for $k \geq 2$ in the class of graphs of bounded genus as well as in the class of linklessly embeddable graphs. In fact, our main algorithm is much more general and applies to the class of graphs with no K_p-minor for some fixed positive integer p. Let G be a graph on n vertices with no K_p as a minor and suppose $\omega : E(G) \to \mathcal{Z}^+$ is a weight function. For a partition $(V_1, \ldots, V_l)$ of $V(G)$, we will denote by $\tilde{G} = \tilde{G}(V_1, \ldots, V_l)$ the graph obtained from G by contracting each of the sets V_i to a vertex v_i and setting the weight $\omega(v_i v_j) = \sum_{e \in E(V_i, V_j)} \omega(e)$ when $E(V_i, V_j)$, the set of edges with one endpoint in V_i and another in V_j, is non-empty. A straightforward generalization of the main clustering procedure from [4] gives the following theorem.

Theorem 9. *Let $p \in \mathcal{Z}^+$ and let $\epsilon > 0$. There exists C such that the following holds. Suppose G is a graph on n vertices with no K_p-minor and let $\omega : E(G) \to \mathcal{Z}^+$. There is a distributed algorithm which finds a partition $(V_1, \ldots, V_l)$ such that $G[V_i]$ has diameter $O(C)$ and*

$$\sum_{e \in \tilde{G}} \omega(e) \leq \epsilon \sum_{e \in G} \omega(e).$$

The algorithm runs in $C \log^* n$ rounds.

By combining Theorem 9 with some additional analysis we obtain the following fact.

Theorem 10. *Let $p, k \in \mathcal{Z}^+$ and let $\delta > 0$. Let G be a graph on n vertices with no K_p-minor and suppose Q is a k-dominating set for G. There is a distributed algorithm which finds a k-dominating set D in G such that*

$$|D| \leq \gamma_k(G) + \delta|Q|.$$

The algorithm runs in $C \log^* n$ rounds where C depends on p, k and δ only.

Note that the case $k = 1$ is proved in [4] but the algorithm from [4] doesn't address the situation $k > 1$ and a slightly different approach is needed. Before proving Theorem 10 let us note that in view of the results from the previous section we have the following corollaries.

Corollary 11 *Let $g, k \in \mathcal{Z}^+$ such that $k \geq 2$ and let $\epsilon > 0$. Let G be a graph on n vertices with genus g. There is a distributed algorithm which finds a k-dominating set D in G such that $|D| \leq (1+\epsilon)\gamma_k(G)$. The algorithm runs in $C \log^* n$ rounds where C depends only on g, k and ϵ.*

Proof. In view of Corollary 3, there is a distributed algorithm which in a constant number of rounds finds a k-dominating set Q in G such that $|Q| \leq L\gamma_K(G)$ for some constant L. Let $\delta := \epsilon/L$. The by Theorem 10 finds a dominating set D such that $|D| \leq (1+\epsilon)\gamma_k(G)$. □

In addition, using the algorithms for linklessly embeddable graphs from previous sections we obtain the following fact.

Corollary 12 *Let $k \geq 2$ and let $\epsilon > 0$. Let G be a linklessly embeddable graph on n vertices. There is a distributed algorithm which finds a k-dominating set D in G such that $|D| \leq (1+\epsilon)\gamma_k(G)$. The algorithm runs in $C \log^* n$ rounds where C depends only on k and ϵ.*

Proof of Theorem 10. Let Q be a k-dominating set in G. Let G' be the following oriented graph obtained form the bipartite graph $G[Q, V(G) \setminus Q]$. We put the arc from v to w for every edge vw with $v \in V(G) \setminus Q$ and $w \in Q$. Since Q is k-dominating, every vertex $v \in V(G) \setminus Q$ has at least k out-neighbors in G'. By choosing one such out-neighbor arbitrarily, we obtain a set of stars $S_1, \ldots, S_l$ with centers in vertices from Q. Let $U_j := V(S_j)$ and let c_{U_j} denote the center of S_j, that is $\{c_{U_j}\} = U_j \cap Q$. We will refer to sets U_j as small clusters. Let H_0 be the digraph obtained from G' by contracting each S_j to a vertex. (Note that it is possible to have both arcs U_jU_i and U_iU_j in H_0.) Let $D := \emptyset$ and let $K := 2c \cdot p\sqrt{\log p}k/\delta$ where c is the constant in Lemma 2.

We will modify the original small clusters and construct a sequence of digraphs $H_1, \ldots, H_{k-1}$ by increasing D in each step so that $|D|$ is small with respect to $|Q|$ and clusters in H_{k-1} obtained by deleting vertices which are k-dominated by D have out-neighbors in at most K other vertices from Q. Some of the centers of small clusters will be added to D, some vertices will be reassigned from one cluster to another, and, in addition, some vertices will become k-dominated by D and will become inactive. Although we no longer need to k-dominate inactive vertices (as they are k-dominated by D), they can still play an important role because they themselves can be inside an optimal k-dominating set. Although the sets U_j will change as we modify them, we will use U_j to refer to the cluster determined by c_{U_j} during the execution of the algorithm. For cluster U we use U'' to denote the set of inactive vertices in $U \setminus \{c_U\}$ and we set $U' := U \setminus U''$. Initially, $U'' := \emptyset$ for every cluster U and we set $B_0 := V(G) \setminus Q$. We will now describe our procedure which modifies the small clusters. We will use $d_F^+(X)$ to denote the number of out-neighbors of a cluster X in F. Recall that clusters are associated with their centers and so cluster X at step i can be different than X at step j for $i \neq j$.

For the general step, assume $i \geq 1$ and let $X_{i-1} := \{U | d^+_{H_{i-1}}(U') > K\}$. If $U \in X_{i-1}$, then add c_U to D. Consider $v \in B_{i-1} \cap U'$. If v becomes k-dominated by D, then move v to U'' and otherwise, add v to B_i. Every vertex $v \in B_i$ has at most $k-1$ neighbors in D and so there exists a small cluster W such that $c_W \notin D$ and $vc_W \in G'$. Let v join one such cluster W.

Let H_i be obtained by contracting each small cluster U to a vertex and by adding the arc UW if the set of arcs from U' to W' in G' is non-empty.

We will prove the following lemma.

Lemma 7. *The following holds:*

(a) $\sum_{i=1}^{k} |X_{i-1}| < \delta|Q|/2$.
(b) For every $v \in B_k$, $|N_G(v) \cap D| \geq k$.
(c) For every small cluster U *there exists a set of small clusters* $Q' \subseteq Q$ *such that* $|Q'| \leq K$ *and every vertex* $v \in U \setminus (B_k \cup \{c_U\})$ *has* $N_G(v) \cap Q \subseteq Q' \cup D$.

To prove the lemma we will show a sequence of claims. Part (a) follows from the following observation.

Claim. $|X_{i-1}| < \delta|Q|/(2k)$.

Proof. From Lemma 2, $\sum_{U \in H_{i-1}} d^+_{H_{i-1}}(U) = |E(H_{i-1})| \leq cp\sqrt{\log p}|Q|$. Thus $|X_{i-1}| < cp\sqrt{\log p}|Q|/K = \delta|Q|/(2k)$. □

We will now verify part (b).

Claim. If $v \in B_i$, then $|N_G(v) \cap D| \geq i$.

Proof. This is certainly true when $i = 0$. Suppose $i \geq 1$. If $v \in B_i$, then $v \in B_{i-1}$ and in the ith step of the procedure $v \in U'$ for some $U \in X_{i-1}$. Since c_U is added to D, $|N_G(v) \cap D|$ increases by at least one. □

Part (c) is slightly more involved and we split the argument into two claims. First note the following.

Claim. If $U \in X_i$, then $U \notin X_{i+1}$.

Proof. We proceed by induction on i. If $U \in X_0$, then after adding c_u to D and partitioning U into U' and U'', every vertex from U' is reassigned and so in the next step U' is empty. Suppose $U \in X_i$. Then, by induction, $U \notin X_{i-1}$. Thus some vertices from B_{i-1} joined U, as this is the only way for the out-degree of a cluster to increases. Each such vertex is either inactive or is in B_i after c_U is added to D in the ith step, and every vertex from $B_i \cap U$ joins a cluster different than U with center which is not in D. Consequently, none of these added vertices is in U' in the iteration $i+1$, and we have $d^+_{H_{i+1}}(U') \le d^+_{H_{i-1}}(U')$. □

Claim. Let $i \ge 1$. For every U there exists a set $Q' \subseteq Q$ such that $|Q'| \le K$ and for every vertex $v \in U \setminus (B_i \cup \{c_U\})$, $N_G(v) \cap Q \subseteq Q' \cup D$.

Proof. If $U \notin X_i$ then $d^+_{H_i}(U') \le K$ and so $N_G(U' \setminus (B_i \cup \{c_U\})) \cap Q \subseteq Q'$ for some set Q' of size at most K. Suppose $U \in X_i$. Then $U \notin X_{i-1}$ by Claim 5. Therefore some vertices from B_{i-1} joined U and each of them is either inactive (and dominated by D) or is in B_i. Consider U' in the ith iteration of the procedure. The set U' is a subset of the union of X and Y, where X is the set $U' \setminus B_{i-1}$ from the iteration $i-1$ and Y is the set of vertices added to the small cluster U in the ith iteration. The number of distinct out-neighbors of X in Q is therefore at most K, and the vertices from Y are either dominated by D or are in B_i. □

This proves part (c) of the lemma.

We will now continue with the proof of Theorem 10.

Let $U_1, \ldots, U_l$ denote the clusters in H_k and recall that $l = |Q|$. Let G^* be obtained from G by contracting each U_i to a vertex u_i. In addition, set $\omega(u_i u_j) = 1$ for every edge $u_i u_j \in G^*$. Let $\epsilon := \delta/(6c \cdot p\sqrt{\log p}K)$ and use the clustering procedure from Theorem 9 to find a partition P of $\{u_1, \ldots, u_l\}$ into $V_1, \ldots, V_s$. Then

$$\omega(\tilde{G}) \le \epsilon |E(G)| \le \epsilon \cdot cp\sqrt{\log p}\, l < \delta l/(6K).$$

Let ∂P be the set of $u \in \{u_1, \ldots, u_l\}$ such that for some $i \ne j$, $u \in V_i$ and $N_{G^*}(u) \cap V_j \ne \emptyset$. We have $|\partial P| \le 2\omega(\tilde{G}) \le \delta l/(3K)$.

Let $u \in \partial P$ and consider the small cluster U which was contracted to u. Add c_U to D and note that by Lemma 7 (b) every vertex $v \in B_k \cap U$ has k-neighbors in D. In addition, by Lemma 7 (c), there is a set Q' of at most K vertices in Q such that every vertex $v \in U \setminus (B_k \cup \{c_U\})$ has $N_G(v) \cap Q \subseteq Q' \cup D$. Since every vertex $v \in V(G) \setminus Q$ has at least k neighbors in Q, by adding $Q' \cup \{c_U\}$ to D we k-dominate all vertices from U. Add $Q' \cup \{c_U\}$ to D for every $U \in \partial P$. By Lemma 7 (a), we have

$$|D| \le \delta|Q|/2 + \delta l(K+1)/(3K) < \delta|Q|.$$

Let $W_i := \bigcup_{U \in V_i \setminus \partial P} U$. Since the diameter of $G[V_i]$ is $O(1)$, it is possible to find in $O(1)$ rounds an optimal set $D_i \subseteq V_i$ such that $D_i \cup D$ k-dominates W_i. Let $D' := \bigcup_{i=1}^{l} D_i \cup D$ and let D^* denote an optimal k-dominating set in G. Then

$$|D'| \le \sum |D_i| + |D| \le \sum |D^* \cap V_i| + |D| \le |D^*| + |D| < \gamma_k(G) + \delta|Q|$$

where the second inequality follows from the fact that every vertex in W_i can be k-dominated only by vertices in $V_i \cup D$ and every vertex in $\bigcup_{U \in \partial P} U$ is k-dominated by D. □

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