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On How the 2 Set-Up Routley-Meyer Semantics Are a Specific Case of the Reduced General Routley-Meyer Semantics in the Context of Some 4-Valued Logics

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Abstract

Routley-Meyer ternary relational semantics can be introduced for models in different ways depending on how the set of regular elements of the model is defined. Two of the most prominent ones are the Reduced General semantics and the 2 Set-up semantics. On the other hand, Lti-logics are 4-valued logics characterized by variations of the conditional of the matrices upon which Brady's logic BN4, and Robles and Méndez's E4 are built. When Lti-logics are endowed with the Reduced General semantics they conform Lti-models; when endowed with 2 Set-up semantics, they conform 2 Set-up Lti-models. Then, it is shown that 2 Set-up Lti-models are actually a specific case of the more general structure that are the Lti-models.

Keywords: Routley-Meyer Semantics; 4-valued Logics; General Reduced Routley-Meyer Semantics; 2 Set-up Routley-Meyer Semantics

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1 Introduction

Routley-Meyer semantics, also known as ternary relational semantics, was introduced in the early 70s [30, 31, 32] to solve some long term problems associated with relevance logics, such as completeness [1]. Since its inception, it has been expanded beyond its original motivation and has in fact been applied to a wide range of systems with very satisfactory results such as [11], [23], and [26]. Going into detail, it is possible to point out two elements as the main characteristics of the Routley-Meyer semantics. The first one is the Routley star, a set-theoretical approach to negation [22] whose usefulness it is out of doubt [21]. The second characteristic is the ternary accessibility relation akin to that of Kripke semantics. However, instead of enabling access among two possible worlds [12], Routley-Meyer semantics extends it up to three elements [3]. The development of this ternary relation is due to Kripke-style semantics being unable to prevent the apparition of implicational paradoxes such as $B \to (A \to A)$, a characteristic axiom of S4 [28]. It is also worth noting that ternary relational semantics require a set of designated worlds, as otherwise certain paradoxes such as the rule *Verum ex quodlibet* ($A \Rightarrow B \to A$) would persist.

When defining Routley-Meyer semantics we can find multiple ways to tackle this according to the kind of model that we are introducing [9, 33]. With this in mind, there are two options that stick out: the Reduced General models and the 2 Set-up models[1]. The first one is an offspring of the general model used for Routley-Meyer semantics. The main difference lies on the approach to the set of designated worlds: in the case of reduced models, this set is not itself a subset of the set of possible worlds but rather an element of said set [10]. On the other hand, the 2 Set-up model relies on a restriction of the set of possible worlds. The set is restricted to just two different elements, instead of having, virtually, infinite elements as it is the case of general models [27]. The first version of the reduced models can be traced almost to the very inception of Routley-Meyer semantics, although the main treatise on them can be found in Chapter IV of [33]. Furthermore, the impact that the reduced models have had in recent research is undeniable, as it can be seen in [10]. This interest has motivated the application of Routley-Meyer semantics to systems that are borderline with relevance logics, such as 3 and 4-valued logics [24], or modal logics [11]. On the other hand, the inception of 2 Set-up models is much more unclear. It is possible to find a precedent of these models in [18], although Brady's paper points

[1]Let it be understood that the term Reduced General models is used to generally refer to any model endowed with the reduced general version of the Routley-Meyer semantics. On the other hand, 2 Set-up models may refer to any model endowed with the 2 Set-up version of the Routley-Meyer semantics.

towards a detailed definition of the 2 Set-up models semantics in [33][2]. It is also of importance to note that, even though they do not introduce 2 Set-up models *per se*, the work of [16] is also seminal for their further development. Additionally, it is quite possible that the most important work on the inception of the 2 Set-up models and their very first published definition, [17], is now lost according to the author[3].

Despite all the above, it is necessary to remark that there has been, to the best of our knowledge, no previous research exploring the relationship between these two kinds of semantics. We can only point out the work done in [5], where a very specific proof was given. Aside from this example, there are no published and widely available records of how this two different ways of interpreting the notions of Routley-Meyer semantics intertwine. Furthermore, the research has usually been focused on the Reduced General semantics rather than the 2 Set-up semantics.

R. T. Brady introduced the logic BN4 in [7] as a system built upon the matrix MBN4. This matrix was defined as a modification of Smiley's 4-valued matrix, the characteristic matrix of First Degree Entailment (FDE). According to Brady, BN4 was meant to be a 4-valued extension of Routley and Meyer's basic logic B [7]. Furthermore, it was J. Slaney who pointed out that the system seemed to be the adequate extension of FDE if it was to be endowed with a relevant conditional akin to that of R [34]. On the other hand, E4 was introduced in [29] by G. Robles and J. M. Méndez. They proposed the system as a companion to BN4, where BN4 could be understood as a 4-valued version of R (the system of relevant conditional), E4 would be a 4-valued version of E (the system of –relevant– entailment). Additionally, there are six different 4-valued conditional variants of the characteristics matrices of BN4 and E4 that verify B [13]. To name the logics characterized by these matrices, the term used is Lti-logics, where i refers to a numerical value assigned to each one of the logics considered. This way there are up to 8 Lti-logics. In particular Lt1 is BN4 and Lt5 is E4, while the other 6 logics do not have specific names with the exception of Lt2, which is known as EF4 [6].

Nowadays there seems to be a rising interest in 4-valued logics, as it can be seen in some recent papers such as [2, 20, 35]. One of the reasons for this rising interest is that they are useful for addressing philosophical topics [19], as well as topics from computer science [4]. Furthermore, as it can be seen below, there is a trend of endowing 4-valued logics with Routley-Meyer semantics, thus offering us a bridge to connect both together.

All of the Lti-logics have been endowed with both of the previously mentioned versions of the Routley-Meyer semantics, the Reduced General models in [14] and

[2]Let us state that by the time [7] was published, [33] was not. This is the main reason why the author states something that never happened until [8] was out.

[3]This was stated by the author in private correspondence.

the 2 Set-up models in [15]. In these papers it was shown that all these systems are both sound and complete in the strong sense with respect to both corresponding semantics. Nevertheless, the relation between these different semantics is, to the best of our knowledge, still unexplored. Therefore, the main aim of this paper is to study the relationship between the Reduced General models and the 2 Set-up models in the context of the Lti-logics. For that matter we will first introduce the logics themselves with their corresponding characteristic matrices, and then we will endow them with the two different models. Afterwards, we will proceed to show how the 2 Set-up models are indeed a specific case of the Reduced General models. This was already shown in [5] for Lt2/EF4.

With all of the above, this article is structured as follows: In Section 2 we introduce the Lti-logics and their characteristic matrices. In Section 3 we define the Reduced General models for the Lti-logics. In Section 4 we display the 2 Set-up models for the different Lti-logics. In Section 5 we provide the proof in which we show that the 2 Set-up models are a specific case of a more general structure that is the Reduced General models in the context of Lti-logics. Finally in Section 6 we recap all the work done in the article and sum up the conclusions to the paper.

2 Lti-logics

We begin this section by introducing the characteristic matrices of all the Lti-logics. Firstly we define the structure on which they are based on and all the common functions. Afterwards we introduce the notions that make each Lti-logic their own. Beforehand we define what is a language and what is a logic.

Definition 1 (Propositional Languages)**.** *A propositional language* $\mathfrak{L}$ *is a denumerable set of propositional variables* $p_1, p_2, ..., p_n, ...$ *and all or some of the connectives* $\wedge$ *(Conjunction),* $\vee$ *(Disjunction),* $\neg$ *(Negation) and* $\to$ *(Conditional)*[4]*. The set of well-formed formulas (wff) is defined as usual. Finally* A, $B, ...$ *are used to represent metalinguistic variables.*

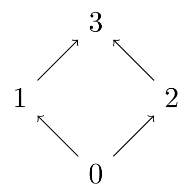
Definition 2 (Logics)**.** *A logic* S *is defined as a structure* $\langle \mathfrak{L}, \vdash_S \rangle$ *where* $\mathfrak{L}$ *is a propositional language from Definition 1 and* $\vdash_S$ *is a (proof-theoretical) consequence relation defined on* $\mathfrak{L}$ *by a set of axioms and rules of inference. The notions of proof and theorem are the usual ones of Hilbert-style axiomatic systems*[5]*.*

[4]We define $\leftrightarrow$ as is customary: $A \leftrightarrow B =_{df} (A \to B) \wedge (B \to A)$.

[5]$\Gamma \vdash_S A$ means that A is derivable from the set of wff Γ in S; $\vdash_S A$ means that A is a theorem of S.

Now we proceed unto defining the characteristic matrices of the Lti-logics. Let it be understood that these matrices are based on the notions of Definition 1.

Definition 3 (Characteristic Matrices of Lti-logics)**.** *With the propositional language $\mathfrak{L}$ consisting on the connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$, the matrices of the Lti-logics are structures $\langle \mathcal{V}, \mathcal{D}, \mathcal{F}\rangle$, where $\mathcal{V} = \{0, 1, 2, 3\}$ and its partially ordered according to the following lattice:*



Also, $\mathcal{D} = \{2, 3\}$, and $\mathcal{F} = \{f_\wedge, f_\vee, f_\neg, f_\rightarrow\}$, where $f_\wedge$ and $f_\vee$ are defined as the greatest lower bound (or lattice meet) and the lowest upper bound (or lattice join) respectively. $f_\neg$ is defined as an involutionary operation such that $f_\neg(0) = 3$, $f_\neg(1) = 1$, $f_\neg(2) = 2$, $f_\neg(3) = 0$. Finally, for $f_\rightarrow$ is defined for each matrix of the Lti-logics according to the following tables:

MLt1	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	1	2	3
3	0	1	0	3

MLt2	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	2	3
3	0	0	0	3

MLt3	0	1	2	3
0	3	3	3	3
1	1	3	1	3
2	0	0	2	3
3	0	0	0	3

MLt4	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	1	2	3
3	0	1	0	3

MLt5	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	0	2	3
3	0	0	0	3

MLt6	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2	0	1	2	3
3	0	0	0	3

MLt7	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	0	2	3
3	0	0	0	3

MLt8	0	1	2	3
0	3	3	3	3
1	0	2	1	3
2	0	1	2	3
3	0	0	0	3

Now that we have defined the matrices of the Lti-logics, it is time for us to present the Lti-logics themselves. For that matter we provide a Hilbert-style axiomatization of these logics based on the notions from Definitions 1 and 2.

Definition 4 (The Lti-logics)**.** *The logics considered in this paper are defined by means of a subset of the axioms as well as all the rules of inference displayed below:*

Axioms

A1. $A \to A$
A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A7. $\neg\neg A \to A$
A8. $A \to \neg\neg A$
A9. $\neg A \to [A \vee (A \to B)]$
A10. $B \to [\neg B \vee (A \to B)]$
A11. $(A \vee \neg B) \vee (A \to B)$
A12. $(A \to B) \vee [(\neg A \wedge B) \to (A \to B)]$
A13. $A \to [B \to [[(A \vee B) \vee \neg(A \vee B)] \vee (A \to B)]]$
A14. $(A \wedge \neg B) \to [(A \wedge \neg B) \to \neg(A \to B)]$
A15. $A \vee [\neg(A \to B) \to A]$
A16. $\neg B \vee [\neg(A \to B) \to \neg B]$
A17. $[A \wedge (A \to B)] \to B$
A18. $[(A \to B) \wedge \neg B] \to \neg A$
A19. $A \to [B \vee \neg(A \to B)]$
A20. $\neg B \to [\neg A \vee \neg(A \to B)]$
A21. $[\neg(A \to B) \wedge \neg A] \to A$
A22. $\neg(A \to B) \to (A \vee \neg B)$
A23. $[\neg(A \to B) \wedge B] \to \neg B$
A24. $B \to \{[B \wedge \neg(A \to B)] \to A]\}$
A25. $(A \to B) \vee \neg(A \to B)$
A26. $(\neg A \vee B) \vee \neg(A \to B)$
A27. $[(A \to B) \wedge (A \wedge \neg B)] \to \neg(A \to B)$
A28. $\neg(A \to B) \vee [(A \wedge \neg B) \to \neg(A \to B)]$
A29. $\{[\neg(A \to B) \wedge \neg A] \to \neg B\} \vee \neg B$
A30. $\{[\neg(A \to B) \wedge B] \to A\} \vee A$

Rules of inference

R1. A, $B \Rightarrow A \wedge B$
R2. A, $A \rightarrow B \Rightarrow B$
R3. $C \vee (A \rightarrow B) \Rightarrow C \vee [(B \rightarrow D) \rightarrow (A \rightarrow D)]$
R4. $C \vee (A \wedge \neg B) \Rightarrow C \vee \neg(A \rightarrow B)$
R5. $C \vee A$, $C \vee (A \rightarrow B) \Rightarrow C \vee B$
R6. $C \vee (A \rightarrow B) \Rightarrow C \vee [(D \rightarrow A) \rightarrow (D \rightarrow B)]$
R7. $C \vee (A \rightarrow B) \Rightarrow C \vee (\neg B \rightarrow \neg A)$

Where R1 is Adjunction, R2 is Modus Ponens, R3 is Disjunctive Suffixing, R4 is Disjunctive Counterexample, R5 is Disjunctive Modus Ponens, R6 is Disjunctive prefixing, and R7 is Disjunctive Contraposition.

In particular, each one of the Lti-logics are axiomatized by the subset A1-A13 plus the axioms of the following list and the rules of inference R1-R7:

Lt1: A14-A16
Lt2: A17-A23
Lt3: A14, A15, A18, A19, A22-A24
Lt4: A16, A17, A20-A22
Lt5: A17-A21, A23, A25-A27
Lt6: A17, A20, A21, A23, A26, A28, A29
Lt7: A14, A18, A19, A21, A23, A26, A30
Lt8: A14, A21, A23, A26, A29, A30

Thus we have presented all the Lti-logics. Furtheremore, we would like to point out that Lt1 is BN4, Lt2 is EF4, and Lt5 is E4, as we specified in Section 1. Moreover, it is obvious that each of the Lti-logics has a characteristic matrix from Defintion 3, and said matrix is the one whose name they bear.

3 Reduced General Routley-Meyer Semantics for Lti-logics

Now we proceed with the definition of the Reduced General models for the Lti-logics. For that matter we define the model generally with the whole set of semantic postulates and afterwards we show how each of the Lti-logics relates to a subset of said semantic postulates.

Definition 5 (Lti-models). *An Lti-model* M *is a structure* $< T, K, R, *, \models>$ *where* K *is a non-empty set,* $T \in K$*,* R *is a ternary relation on* K *and* $*$ *is an involutive unary operator on* K *subject to a subset of the following definitions and postulates for all* $a, b, c \in K$*:*

d1. $a \leq b =_{df} RTab$

d2. $a = b =_{df} a \leq b \,\&\, b \leq a$

d3. $R^2abcd =_{df} (\exists x \in K)\ (Rabx \,\&\, Rxcd)$

p1. $a \leq a$

p2. $(a \leq b \,\&\, Rbcd) \Rightarrow Racd$

p3. $R^2Tabc \Rightarrow (\exists x \in K)\ (RTbx \,\&\, Raxc)$

p4. $R^2Tabc \Rightarrow (\exists x \in K)\ (Rabx \,\&\, RTxc)$

p5. $a^{**} \leq a$

p6. $a \leq a^{**}$

p7. $a \leq b \Rightarrow b^* \leq a^*$

p8. RT^*TT^*

p9. $Rabc \Rightarrow (b \leq a^* \text{ or } b \leq a)$

p10. $Rabc \Rightarrow (a \leq c \text{ or } a^* \leq c)$

p11. $RTab \Rightarrow (T^* \leq b \text{ or } a \leq T)$

p12. $(RTab \,\&\, R^2Tcde) \Rightarrow (a \leq c^* \text{ or } d \leq c^* \text{ or } c \leq b \text{ or } c \leq e)$

p13. $(Rabc \,\&\, Rcde) \Rightarrow (a \leq c \text{ or } b \leq c \text{ or } c^* \leq c \text{ or } d \leq c \text{ or } b \leq e)$

p14. $Rabc \Rightarrow (Rc^*ab^* \text{ or } Rc^*ba^* \text{ or } Rc^*aa^* \text{ or } Rc^*bb^*)$

p15. $(RTab \,\&\, Ra^*cd) \Rightarrow (c \leq T \text{ or } c \leq b)$

p16. $(RTab \,\&\, Ra^*cd) \Rightarrow (T^* \leq d \text{ or } b^* \leq d)$

p17. $Raaa$

p18. Raa^*a^*

p19. Ra^*aa

p20. $Ra^*a^*a^*$

p21. $Ra^*bc \Rightarrow (b \leq a \text{ or } b \leq a^*)$

p22. $Ra^*bc \Rightarrow (a^* \leq c \text{ or } b \leq a)$

p23. $Ra^*bc \Rightarrow (a \leq c \text{ or } a^* \leq c)$

p24. $(Rabc \,\&\, Rb^*de) \Rightarrow (a \leq e \text{ or } b \leq e \text{ or } d \leq c)$

p25. $RTab \Rightarrow RT^*ab$

p26. RT^*T^*T

p27. $Raaa^* \text{ or } Ra^*aa^*$

p28. $RTab \Rightarrow (RT^*aa^* \text{ or } Rb^*aa^*)$

p29. $(RTab \,\&\, Ra^*cd) \Rightarrow (T^* \leq d \text{ or } b^* \leq d \text{ or } c \leq a^*)$

p30. $(RTab \,\&\, Ra^*cd) \Rightarrow (c \leq T \text{ or } c \leq b \text{ or } a \leq d)$

Finally, $\models$ *is a valuation relation from* K *to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable* p*, wffs* A*,* B *and* $a \in K$*:*

(i) $(a \leq b \;\&\; a \models p) \Rightarrow b \models p$

(ii) $a \models A \wedge B$ *iff* $a \models A \;\&\; a \models B$

(iii) $a \models A \vee B$ *iff* $a \models A$ *or* $a \models B$

(iv) $a \models A \rightarrow B$ *iff for all* b*,* $c \in K$*,* $(Rabc \;\&\; b \models A) \Rightarrow c \models B$

(v) $a \models \neg A$ *iff* $a^* \not\models A$

Every Lti-logic is subject to d1-d3, p1-p13 and they differ with each other in the additional characteristic subset of corresponding postulates listed above as p14-p30. In particular, for any axiom A_j *(where* $14 \leq j \leq 30$*) belonging to any of them, there is a corresponding postulate* p_j *from the list above.*

The postulates of the Lti-models are summarized as follows:

Remark 1 (Postulates for Lti-models)**.** *Each Lti-model relation R is characterized by d1-d3, p1-p13 plus:*

Lt1: p14-p16

Lt2: p17-p23

Lt3: p14, p15, p18, p19, p22-p24

Lt4: p16, p17, p20-p22

Lt5: p17-p21, p23, p25-p27

Lt6: p17, p20, p21, p23, p26, p28, p29

Lt7: p14, p18, p19, p21, p23, p26, p30

Lt8: p14, p21, p23, p26, p29, p30

To conclude this section, let us point out some of the most important results of these logics w.r.t. the model that we have just defined.

Remark 2 (Results for Lti-logics)**.** *All the logics from Definition 4 are sound and complete in the strong sense w.r.t. the Reduced General Routley-Meyer semantics and their corresponding model from 5 as shown in [5, 14, 25].*

4 2 Set-up Routley-Meyer Semantics for Lti-logics

We proceed unto defining the 2 Set-up models for the Lti-logics. We define the model the same way we did for the Lti-models of Definition 5; the main difference resides that in this case, instead of dealing with a set of semantic postulates, we deal with a set of accessibility relationships. This set of accessibility relationships will be tailored to fit each logic as it shown below.

Definition 6 (2 Set-up Lti-models)**.** *A 2 Set-up Lti-model* $\mathfrak{M}$ *is a structure* $<\mathfrak{K}$, $\mathfrak{R}$, $*_2$, $\models_2>$ *where* $\mathfrak{K}$ *is a set which contains two elements –labelled* O *and* O^{*_2} *– and no other elements.* $*_2$ *is an involutive unary operator defined on* $\mathfrak{K}$ *such that for any* $x \in \mathfrak{K}$, $x = x^{*_2 *_2}$. $\mathfrak{R}$ *is a ternary relation on* $\mathfrak{K}$ *defined as follows for each Lti-model class considered in this article: if* a, b, $c \in \mathfrak{K}$, *then* $\mathfrak{R}abc$ *iff:*

Lt1-models: $(a = O$ *&* $b = c)$ *or* $(a \neq b$ *&* $c = O^{*_2})$[6]

Lt2-models: $b = c$ *or* $(a = c = O^{*_2}$ *&* $b = O)$.

Lt3-models: $(a = O$ *&* $b = c)$ *or* $(a = O^{*_2}$ *&* $b = O)$.

Lt4-models: $a = b = c$ *or* $(c = O^{*_2}$ *&* $a \neq b)$.

Lt5-models: $a = O^{*_2}$ *or* $b = c$.

Lt6-models: $(a = O$ *&* $b = c)$ *or* $a = b = c$ *or* $(a = O^{*_2}$ *&* $b \neq c)$.

Lt7-models: $(a = O$ *&* $b = c)$ *or* $(b = c = O)$ *or* $(a = O^{*_2}$ *&* $b \neq c)$.

Lt8-models: $(a = O$ *&* $b = c)$ *or* $(a \neq O$ *&* $b \neq c)$.

$\models_2$ *is a (valuation) relation from* $\mathfrak{K}$ *to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable* p, *wffs* A, B *and* $a \in \mathfrak{K}$*:*

(i) $a \models_2 p$ *or* $a \not\models_2 p$

(ii) $a \models_2 A \wedge B$ *iff* $a \models_2 A$ *&* $a \models_2 B$

(iii) $a \models_2 A \vee B$ *iff* $a \models_2 A$ *or* $a \models_2 B$

(iv) $a \models_2 A \to B$ *iff for all* b, $c \in \mathfrak{K}$, $(\mathfrak{R}abc$ *&* $b \models_2 A) \Rightarrow c \models_2 B$

(v) $a \models_2 \neg A$ *iff* $a^{*_2} \not\models_2 A$

[6]This clause is equivalent to Brady's clause for BN4-models (i.e., our Lt1-models): $(a \neq O$ or $b = c)$ & $[a \neq O^*$ or $(b = O$ & $c = O^*)]$. Cf. [7, 15].

Let it be noted that we are writing O^* instead of O^{*_2} for the sake of the clarity of the text. We will also accept that $O^{**} = O$, as it is a common use that has been seen multiple times in references such as [5, 15, 27].

Additionally, we explicit the set of accessibility relations that each model associated to a Lti-logic has.

Remark 3 (Ternary relations in $\mathfrak{K}$)**.** *Suppose $O \neq O^*$. Now, given the definition of $\mathfrak{R}$ (cf. Definition 6), the following ternary relations are the only ones holding for each 2 Set-up Lti-model considered:*

Lt1 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*\}$.

Lt2 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O^*,\ \mathfrak{R}O^*OO\}$.

Lt3 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*OO\}$.

Lt4 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O^*\}$.

Lt5 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O^*,\ \mathfrak{R}O^*O^*O,\ \mathfrak{R}O^*OO\}$.

Lt6 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O^*,\ \mathfrak{R}O^*O^*O\}$.

Lt7 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O,\ \mathfrak{R}O^*OO\}$.

Lt8 $\mathfrak{R} = \{\mathfrak{R}OOO,\ \mathfrak{R}OO^*O^*,\ \mathfrak{R}O^*OO^*,\ \mathfrak{R}O^*O^*O\}$.

And in order to conclude the section, we do as we did before and point out some of the most interesting results of the Lti-logics w.r.t. this kind of models.

Remark 4 (Results for Lti-logics in 2 Set-up Lti-models)**.** *All the logics from Definition 4 are sound and complete in the strong sense w.r.t. the 2 Set-up Routley-Meyer semantics as shown in [5, 15].*

5 The 2 Set-up Lt*i*-models are a specific case of the Lt*i*-models

After we have introduced both models, Lti-models and 2 Set-up Lti-models, Definitions 5 and 6 respectively, we proceed to show how the latter is actually a specific case of the former. We prove this in the following theorem.

Theorem 1 (The 2 Set-up Lti-models are a specific case of the Lti-models)**.** *The 2 Set-up Lti-models of Definition 6 are a specific case of the Lti-models of Definition 5.*

Proof. We have to prove: (a) that whenever a postulate is included in a Lti-model, that postulate is verified in the corresponding 2 Set-up Lti-model according to the definition of the ternary relation in that Lti-model; (b) that there is no equivalence between Lti-models and 2 Set-up Lti-models; (c) that both clauses (i) of Definitions 5 and 6, are equivalent.

The fact (b) is easy to verify. It suffices to show that there are some relations in Lti-models which cannot be considered in 2 Set-up Lti-models. For any of the Lti-models in Definition 5, for a, b, $c \in K$, we could have $Rabc$, $a \neq b$, $a \neq c$ and $b \neq c$. This situation cannot be in any 2 Set-up Lti-model.

In the case of (c), let us remember that the corresponding clause, in Definition 5 (i) reads as $(a \leq b \;\&\; a \models p) \Rightarrow b \models p$, while in Definition 6 (i) reads as $a \models_2 p$ or $a \not\models_2 p$. To show that both clauses are equivalent we need to show that whenever $\mathfrak{R}Oab$ and $a \models_2 p$, then $b \models_2 p$. This follows automatically as whenever $\mathfrak{R}Oab$, then $a = b$ as it can be seen in Remark 3. And in that case, if $a \models_2 p$, necessarily $b \models_2 p$, as $a = b$.

Then, it remains to prove (a), this is, that postulates of Definition 5 are also verified in the 2 Set-up counterpart Lti-models. A few instances will suffice to illustrate (a).

p1, $a \leq a$, p5, $a^{**} \leq a$, p6, $a \leq a^{**}$ and p7, $a \leq b \Rightarrow b^* \leq a^*$, clearly hold in any 2 Set-up Lti-model given d1 and the fact that $\mathfrak{R}OOO$ and $\mathfrak{R}OO^*O^*$ are valid relations in any of those models.

p2, $(a \leq b \;\&\; Rbcd) \Rightarrow Racd$, holds in any 2 Set-up Lti-model. By d1, we have $(RTab \;\&\; Rbcd) \Rightarrow Racd$. Thus, we only need to consider cases $\mathfrak{R}OOO$ and $\mathfrak{R}OO^*O^*$ given Remark 3 and the fact that T (i.e., O in the 2 Set-up Lti-models) is the first element in the ternary relation. Consequently, we have $a = b$. Therefore, $(RTaa \;\&\; Racd) \Rightarrow Racd$, which is trivial.

p3, $R^2Tabc \Rightarrow (\exists x \in K)\,(RTbx \;\&\; Raxc)$, holds in any 2 Set-up Lti-model. Given that the first element in this (double) ternary relation is T, we just have to consider eight different cases, i.e., the cases when the first element in the 2 Set-up Lti-models is O. For each one of the following cases, we have to prove that whenever the antecedent of the postulate holds in a 2 Set-up Lti-model, the consequence also holds in the same model. The eight cases we initially have to consider are:

(1) $\mathfrak{R}^2OOOO \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}OxO)$

(2) $\mathfrak{R}^2OO^*OO \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}O^*xO)$
(3) $\mathfrak{R}^2OO^*O^*O \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}O^*xO)$
(4) $\mathfrak{R}^2OO^*OO^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}O^*xO^*)$
(5) $\mathfrak{R}^2OO^*O^*O^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}O^*xO^*)$
(6) $\mathfrak{R}^2OOO^*O \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}OxO)$
(7) $\mathfrak{R}^2OOO^*O^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OO^*x \;\&\; \mathfrak{R}OxO^*)$
(8) $\mathfrak{R}^2OOOO^* \Rightarrow (\exists x \in \mathfrak{K})\,(\mathfrak{R}OOx \;\&\; \mathfrak{R}OxO^*)$

Now, we note that double ternary relations are understood according to d3 in Definition 5. Then, there is no possibility for the antecedent of cases (6) and (8) to hold in any of the 2 Set-up Lti-models (see Remark 3). As for the rest of the cases, they will hold in some or all the 2 Set-up Lti-models. Let us first consider the situation for Lt5-models, where the antecedent of cases (1)-(5) and (7) does hold. For any of those cases, it is easy to see that the consequent also holds in 2 Set-up Lt5-models for some x: in particular, when (1) $x = O$; (2) $x = O$; (3) $x = O^*$; (4) $x = O$; (5) $x = O^*$; (7) $x = O^*$. Finally, the proof for the rest of the 2 Set-up Lti-models is similar. However, given d3 in Definition 5, Remark 3 and the specific antecedent of each case, a different subset of the six previously considered cases has to be contemplated for each 2 Set-up Lti-model. In particular, cases (1), (4) and (7) have to be considered in any 2 Set-up Lti-model. Case (2) must not be considered in 2 Set-up Lti-models where $i = \{1, 4, 6, 8\}$. Similarly, case (3) must not be considered in 2 Set-up Lti-models where $i = \{1, 2, 3, 4\}$. Lastly, case (5) must not be considered in 2 Set-up Lti-models where $i = \{1, 3, 7, 8\}$.

p10, $Rabc \Rightarrow (a \leq c$ or $a^* \leq c)$, holds in any of the 2 Set-up Lti-models. We note that, by d1 ($a \leq b =_{df} RTab$), p10 can also be read as $Rabc \Rightarrow (RTac$ or $RTa^*c)$. Let us consider the case of Lt5 since its ternary relation in the 2 Set-up Lti-models is the most complex among the Lti-logics. Then, we can simply obtain the proof for the rest of them by eliminating some considered cases. Given the definition of $\mathfrak{R}$ in Lt5 and assuming $\mathfrak{R}abc$, six different cases have to be considered: (1) $\mathfrak{R}OOO$, (2) $\mathfrak{R}OO^*O^*$, (3) $\mathfrak{R}O^*OO^*$, (4) $\mathfrak{R}O^*O^*O^*$, (5) $\mathfrak{R}O^*O^*O$ and (6) $\mathfrak{R}O^*OO$. By assuming each one of these, we obtain at least another valid relation in each case, $\mathfrak{R}Oac$ or $\mathfrak{R}Oa^*c$. Let us take case (2), this is, $a = O$ and $b = c = O^*$. Then, we have either $\mathfrak{R}OO^*O^*$ or $\mathfrak{R}OOO^*$ where $\mathfrak{R}OO^*O^*$ is a relation appearing in Lt5 –actually, in all the Lti-logics. The reader can easily check that results in the other five cases are similar.

p12, $(RTab \;\&\; R^2Tcde) \Rightarrow (a \leq c^*$ or $d \leq c^*$ or $c \leq b$ or $c \leq e)$, holds in any of the 2 Set-up Lti-models. Firstly, given d1 and d3, p12 can be more easily read as follows: $(RTab \;\&\; RTcx \;\&\; Rxde) \Rightarrow (RTac^*$ or $RTdc^*$ or $RTcb$ or $RTce)$. Let us show the case of Lt5. The proof for the rest of the Lti-logics is similar. In the case of the system Lt5, twelve different cases have to be considered. For the first six cases, we have $a = b = O$; for the other six: $a = b = O^*$. Let us consider now the first six, where $a = b = O$. (1) $\mathfrak{R}OOO$ & $\mathfrak{R}OOO$; (2) $\mathfrak{R}OOO$ & $\mathfrak{R}OO^*O^*$; (3) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*OO^*$; (4) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*O^*O^*$; (5) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*O^*O$; (6) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}O^*OO$. It is easy to see that ($\mathfrak{R}Oac^*$ or $\mathfrak{R}Odc^*$ or $\mathfrak{R}Ocb$ or $\mathfrak{R}Oce$) actually holds in any of them. On the one hand, we have $c = x = O$ for cases (1) and (2). Thus, we obtain at least one of those ternary relations (i.e., $\mathfrak{R}Ocb$) for both of these cases. On the other hand, we get $c = x = O^*$ for cases (3)-(6). Therefore, we obtain at least the relation $\mathfrak{R}Oac^*$. When we study the remaining six cases –i.e., the cases (1)-(6) written above plus a third ternary relation where $a = b = O^*$–, similar results are obtained. In particular, at least the relations $RTac^*$ and $RTcb$ are obtained for cases (1)-(2) and (3)-(6), respectively.

p14, $Rabc \Rightarrow (Rc^*ab^*$ or Rc^*ba^* or Rc^*aa^* or $Rc^*bb^*)$, holds in 2 Set-up Lti-models where $i = \{1, 3, 7, 8\}$. Let us prove the case where $i = 1$ –i.e., the case of Lt1-models– the rest of them are proved in a similar way. Given $i = 1$, we have to consider three different cases: (1) $\mathfrak{R}OOO$, (2) $\mathfrak{R}OO^*O^*$, (3) $\mathfrak{R}O^*OO^*$. For each one of these, the reader can easily see that at least another one of the valid ternary relations in Lt1-models is gotten: $\mathfrak{R}O^*OO^*$ in the first case and $\mathfrak{R}OO^*O^*$ in the other two.

p19, Ra^*aa, holds in Lti-models where $i = \{2, 3, 5, 7\}$, as the relation $\mathfrak{R}O^*OO$ also holds in the 2 Set-up Lti-models for these logics (see Definition 5 and Remark 3).

p25, $RTab \Rightarrow RT^*ab$, holds in the 2 Set-up Lt5-models. Given the fact that the first element in the ternary relation is T, only two cases need to be considered in the 2 Set-up Lt5-models: (1) $\mathfrak{R}OOO$ and (2) $\mathfrak{R}OO^*O^*$. Then, we get $\mathfrak{R}O^*OO$ and $\mathfrak{R}O^*O^*O^*$, respectively for each case. Both relations are included in 2 Set-up Lt5-models, therefore p25 holds in the 2 Set-up Lt5-models.

p29, $(RTab$ & $Ra^*cd) \Rightarrow (T^* \leq d$ or $b^* \leq d$ or $c \leq a^*)$, holds in the 2 Set-up Lti-models such that $i = \{6, 8\}$. Given d2 in Definition 5, p29 can also be read as follows: $(RTab$ & $Ra^*cd) \Rightarrow (RTT^*d$ or RTb^*d or $RTca^*)$. Thus, assuming $\mathfrak{R}Oab$ & $\mathfrak{R}a^*cd$ and Definition 6, only five different cases need to be considered in the 2 Set-up Lti-models: (1)$\mathfrak{R}OOO$ & $\mathfrak{R}O^*OO^*$; (2) $\mathfrak{R}OOO$ & $\mathfrak{R}O^*O^*O^*$; (3) $\mathfrak{R}OOO$ & $\mathfrak{R}O^*O^*O$; (4) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}OOO$; (5) $\mathfrak{R}OO^*O^*$ & $\mathfrak{R}OO^*O^*$. Now, each one of these cases should result in ($\mathfrak{R}OO^*d$ or $\mathfrak{R}Ob^*d$ or $\mathfrak{R}Oca^*$). For instance, let us take case (1). Then, we have $T = a = b = c = O$ and $d = O^*$. Then, $\mathfrak{R}OO^*O^*$

or $\mathfrak{R}OO^*O^*$ or $\mathfrak{R}OOO^*$. Thus, p29 can be correctly read in terms of 2 Set-up Lti-models because at least one of these relations ($\mathfrak{R}OO^*O^*$) holds in the models of the considered Lti-logics. Same could be said of the other four cases. In particular, by applying the same method to any of those cases we get at least one valid ternary relation in the considered Lti-model. For cases (2), (3) and (5), we also get $\mathfrak{R}OO^*O^*$. For case (4), we get $\mathfrak{R}OOO$.

Finally, let us note that p4 can be proved as p3. Also, proofs of p9, p11, p21, p22 and p23 are similar to that of p10 displayed above. Proofs of p8, p17, p18, p20, p26 and p27 are trivial –see the case of p19 showed above. Proof of p28 follows similar lines to that of p14. Lastly, proofs of p13, p15, p16, p24 and p30 are similar to the proof of p29. □

Thus we have shown that the 2 Set-up Lti-models are, indeed, a specific case of the Lti-models. Let us remind the reader that the former models have been introduced using a 2 Set-up Routley-Meyer semantics, while the latter were defined using the Reduced General Routley-Meyer semantics.

6 Conclusion

The main goal of the paper was to show that the 2 Set-up Routley Meyer models for the Lti-logics were a specific case of the corresponding Reduced General Routley-Meyer models for the same logics. And, as it has been shown in Theorem 1, we can easily conclude that, indeed, 2 Set-up Lti-models are instances or specific cases of the more general ones with an unrestricted number of set-ups.

The fact that the ternary accessibility relation needs three elements to operate over the Routley-Meyer semantics makes obvious the observation that, in the case of being evaluated in a 2 Set-up model at least two of the three members needed are equal to each other (being the third the same or its set-theoretical negation counterpart). Given that, it is necessary to remind that the requirement for a special set-up T (such that $T \in K$ and $a \leq b =_{df} RTab$) in Reduced General Routley-Meyer models is crucial for the development and feasibility of this proof. It is absolutely required that one of the set-ups in 2 Set-up Routley-Meyer models is equivalent to T and the other to the set-theoretical negation counterpart.

Under the Lti-logics –with these requirements well understood– and using 2 Set-up Routley-Meyer semantics, there are a limited amount of possible ternary accessibility relations [7]. Then it is concluded, following the proof of Theorem 1 here

[7] 8 possible relationships with $\mathfrak{K}$ cardinality restricted to 2.

presented, that any 2 Set-up Lti-model will be an instance or a special case of a more general one, such as the Reduced General Routley-Meyer models for the Lti-logics.

Thanks to the present work we would be able to study, in the future, a more abstract and general feature that would extrapolate the relationship between 2 Set-up Routley-Meyer models and Reduced General Routley-Meyer models. Proving this, not only for a given group of logics (like in this case), but for all the ones that can be modelled with Routley-Meyer semantics.

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On the Representation of Transposition Algebras and Non-commutative Cylindric Algebras

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Abstract

The two most successful algebraizations of first-order logic are the quasi-polyadic algebras and cylindric algebras. The algebras in the title are non-commutative versions (concerning the cylindrifications) of these algebras, respectively. These two algebra classes appear similar to each other. Nevertheless transposition algebras are relativized representable, moreover they have nice representations, but non-commutative cylindric algebras are not, in general. The difference between these algebra classes concerning representability is analysed in the paper.

1 Introduction

As it is well known, Boolean algebras provide the algebraization of classical *propositional logic* (i.e., the logic including the logical constants conjunction, disjunction, and negation). The two most successful algebraizations of classical *first order logic* (i.e., the extension of propositional logic including the logical constants quantifiers, equality relation and infinitary relations as non-logical constants) are the concepts of quasi-polyadic algebras and cylindric algebras. Transposition algebras and non-commutative cylindric algebras are remarkable superclasses of the previous classes, respectively. These classes are obtained from the classes of quasi-polyadic and cylindric algebras, if the property of the commutativity of cylindrifications is changed by that of single substitutions. So the algebra classes in the title are the "non-commutative versions" of the previous two, in a sense.

The history of research into polyadic algebras and cylindric algebras is exciting. There is even some rivalry between these two approaches. Having said that, cylindric algebras are undoubtedly the more popular research topic of the two: cylindric algebras have a long and thorough research history accompanied by a correspondingly sizable literature that even includes a big monograph (see [11]). The concept of quasi-polyadic algebra was invented by Paul Richard Halmos ([9] and [10]), that of transposition algebra was introduced by Ferenczi in [7]. The concept of cylindric algebra was developed by Alfred Tarski (see e.g.[14]), the non-commutative version was invented by Richard Thompson in [15]. For a long time, quasi-polyadic algebras have been considered only as a version (but not an equivalent version) of cylindric algebras. However, research into the representation theories of these two kinds of algebras has brought about an important shift in this view.

None of the above mentioned classes of algebras are representable in the classical sense ([11], [16], [8]), i.e. the classes of the usual set algebra versions of these classes are not finite-schema axiomatizable. But, it turned out that the quasi-polyadic algebras and their non-commutative versions, the transposition algebras, are *relativized representable* (i.e. the corresponding representant set algebra class is finite-schema axiomatizable), while the cylindric algebras and the non-commutative cylindric algebras are not.

The general relativized set algebra classes (cylindric-, or polyadic-relativized set algebras: Crs_α, or Prs_α) are essential enlargements of the foregoing classical representant classes, furthermore they are non-commutative. Certain subclasses of these relativized set algebras classes can be considered as algebraizations of some non-classical extensions of classical first order logic (e.g. guarded logics, arrow logics, etc, [11] II., [5]).

In this paper it is pointed out, that if the difference between the relativized representability of quasi-polyadic and cylindric algebras is analysed we can conclude that the main reason of the difference is the presence of the transposition operator p_{ij} in the polyadic case (presence of the polyadic paradigm). From the viewpoint of *logic* p_{ij} allows the exchange of two free variables in a formula without any restriction while in the cylindric case extra free variable is needed. This is why in cylindric algebras the so called "merry-go-round" (supplementing) axioms are so cumbersome. In the polyadic case p_{ij} allows the introduction of the finite general simultaneous substitution operator s_τ, where τ is a finite transformation on the ordinals. From the viewpoint of *algebra* p_{ij} can be regarded as a basic operator. For example, such an operator is the *converse* $\smile$ in the theory of relation algebras. To ignore p_{ij} in the language of cylindric algebras has certain negative consequences. We can state that the relativized representation theory of polyadic algebras is more elegant than that of the cylindric algebras. The subject of this survey paper is to focus on the above approach of the topic.

2 Basic concepts and their comparison

A *quasi-polyadic equality algebra of dimension* α (QEA_α) is a Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ enriched with a set of additional unary operations called "cylindrifications" c_i $(i < \alpha)$, "substitutions for i by j" s^i_j, "i, j transpositions" p_{ij} and the constants "i, j diagonals" d_{ij} $(i, j < \alpha)$, where $\alpha \geq 2$ is an ordinal, and the algebra satisfies the following axioms for every $i, j, k < \alpha$:

(Q_0) $s^i_i = p_{ii} = Id \restriction A$ and $p_{ij} = p_{ji}$ (where $Id \restriction A$ is the identity map on A)
(Q_1) $x \leq c_i x$
(Q_2) $c_i(x + y) = c_i x + c_i y$
(Q_3) $s^i_j c_i x = c_i x$
(Q_4) $c_i s^i_j x = s^i_j x \quad i \neq j$
(Q_5) $s^i_j c_k x = c_k s^i_j x \quad k \notin \{i, j\}$
(Q_6) s^i_j and p_{ij} are Boolean endomorphisms (i.e., $s^i_j(-x) = -s^i_j x$, etc.),
(Q_7) $p_{ij} p_{ij} x = x$
(Q_8) $p_{ij} p_{ik} x = p_{jk} p_{ij} x$ if i, j, k are distinct
(Q_9) $p_{ij} s^i_j x = s^j_i x$
(Q_{10}) $s^i_j d_{ij} = 1$
(Q_{11}) $x \cdot d_{ij} \leq s^i_j x$.

This definition of QEA_α is not the original definition of Halmos ([10], [11], [3]), but a version of that, by Sain and Thompson (see [12]).

A *transposition algebra of dimension* α (TA_α) is such an algebra which satisfies the quasi-polyadic axioms except for (Q_5) and, instead of (Q_5) the satisfaction of the following axiom $(Q_5)^*$ is required:

$$(Q_5)^* : \; s^i_j s^k_m x = s^k_m s^i_j x \tag{1}$$

where $i, j \notin \{k, m\}$ ([7]).

We remark that $\mathrm{QEA}_\alpha \subset \mathrm{TA}_\alpha$, furthermore the commutativity of cylindrifications ($c_i c_j x = c_j c_i x$) is valid in QEA_α, but it is false in TA_α.

A *cylindric algebra of dimension* α (CA_α) is

(C_0) a Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ enriched with a set of additional unary operations called "cylindrifications" c_i $(i < \alpha)$ and constants "diagonals" d_{ij} $(i, j < \alpha)$, where $\alpha \geq 2$ is an ordinal, and the algebra satisfies the following axioms (C_1)-(C_7) axioms for every $i, j < \alpha$:

(C_1) $c_i 0 = 0$

(C$_2$) $x \leq c_i x$
(C$_3$) $c_i(x \ \cdot c_i y) = c_i x \ \cdot c_i y$
(C$_4$) $c_i c_j x = c_j c_i x$
(C$_5$) $d_{ii} = 1$
(C$_6$) $c_j(d_{ji} \cdot d_{jk}) = d_{ik} \qquad j \notin \{i, k\}$
(C$_7$) $d_{ij} \cdot c_i(d_{ij} \cdot x) = d_{ij} x \qquad i \neq j$
(see [14], [11]).

The cylindric-type reducts of quasi-polyadic algebras are themselves cylindric algebras ([12]), but not every cylindric algebra can be obtained in this way.

Unlike the case of the algebraization of propositional logic, the Boolean algebras, where the axiomatization is finite, due to the nature of first order logic, the axiomatizations above are *finite-schema axiomatizations.* With the variables $v_1, v_2, \ldots v_i. \ldots$ of the logic are associated the ordinals $1, 2, 3, ...i, ...$ With the quantifiers $\exists v_i$, the substitutions v_i/v_j, the change of the variables v_i and v_j, and the formulas $v_i = v_j$, the abstract operations c_i, s^i_j, p_{ij} and the constants d_{ij} are associated, respectively.

Comparing the languages of cylindric- and quasi-polyadic algebras we can notice *two differences.* In the language of CA_α's, on the one hand, the operation symbols s^i_j and, on the other hand, the operation symbols p_{ij} do not occur.

But s^i_j can be defined in CA_α as

$$s^i_j x := c_i(d_{ij} \cdot x) \text{ if } i \neq j, \text{else } s^i_i x = x. \tag{2}$$

As regards the abstract transposition p_{ij}, only its weakened version ${}_k s(i, j)$ can be defined in CA_α in terms of (2) in this way: ${}_k s(i, j)x := s^k_i s^i_j s^j_k x$ (i, j, k are different), see [11]. Therefore the QEA_α properties (Q$_7$)-(Q$_9$) are valid only in a very limited way, i.e. under certain conditions, for ${}_k s(i, j)$ in cylindric algebras.

The presence of the operations p_{ij} is the *polyadic aspect* in the algebras QEA_α's.

A *non-commutative cylindric algebra of dimension* α (NA_α) is such an algebra which satisfies the cylindric axioms except for (C$_4$), furthermore, instead of (C$_4$) it satisfies the (Q$_5$)* axiom in (1) where the single substitution is defined as in (2) (see [15], do not confuse the notation NA_α with the notation NA used in the theory of relation algebras).

Let α be an arbitrary ordinal. A map $\tau : \alpha \to \alpha$ defined on α is *finite if* $\tau k = k$, except for finitely many k.

It can be proved that in terms of the operations s^i_j and p_{ij}, the so-called *general substitution operation* s_τ can be introduced in QEA_α and TA_α, where τ is an arbitrary *finite* transformation defined on α (see [12]). The logical meaning of s_τ is the simultaneous substitution of certain free variables with others. Such an operation s_τ can not be introduced in CA_α, in general. This shows an essential difference in the syntax between the quasi-polyadic and cylindric algebras (indicating the difference between the polyadic- and cylindric paradigms).

As for the properties of s_τ, see the following original definition of Halmos for QEA_α, which is based on s_τ, where τ is finite.

A Boolean algebra $\langle A, +, \cdot, -, 0, 1\rangle$ enriched with the operations $c_{(\Gamma)}$ for every finite subset Γ of α, with the operations s_τ for every finite transformation τ on α and with the constants d_{ij} for every $i, j < \alpha$, is said to be a *quasi-polyadic equality algebra of dimension* α, if it satisfies the following axioms for every possible i, j and finite $\Gamma, \Delta, \sigma, \tau$

(Q_0) $c_{(\Gamma)}0 = 0$
(Q_1) $x \le c_{(\Gamma)}\, x$
(Q_2) $c_{(\Gamma)}(x \cdot c_{(\Gamma)}y) = c_{(\Gamma)}x \cdot c_{(\Gamma)}y$
(Q_3) $c_{(\emptyset)}\, x = x$
(Q_4) $c_{(\Gamma)}c_{(\Delta)}\, x = c_{(\Gamma\cup\Delta)}\, x$
(Q_5) $s_{Id}\, x = x$ (where Id is the identity map on α)
(Q_6) $s_{\sigma\circ\tau}\, x = s_\sigma s_\tau x$
(Q_7) $s_\sigma(-x) = -s_\sigma\,(x)$ and $s_\sigma(x+y) = s_\sigma x + s_\sigma y$
(Q_8) if $(\alpha \sim \{i\}) \upharpoonleft \sigma \subseteq \tau$ then $s_\sigma c_i\, x = s_\tau c_i\, x$
(Q_9) if $\sigma^{-1*}\{i\} = \{j\}$ then $c_i s_\sigma\, x = s_\sigma c_j\, x$
where σ^{-1*} denotes the inverse image of a set via σ
(E_1) $d_{ii} = 1$
(E_2) $x \cdot d_{ij} \le s_{[i/j]}\, x$
where $[i/j]$ denotes the substitution of i for j
(E_3) $s_\tau d_{ij} = d_{\tau i, \tau j}$

([9], [11], [13]).

The following theorem (due to Sain, I., and Thompson, R.) deals with the connection between the above two definitions of quasi-polyadic algebra (the last and the first definition in this section) of QEA_α.

Theorem 1. *The above two definitions of quasi-polyadic equality algebras are equivalent (see [12]).*

The "equivalence" means here that with every structure satisfying Halmos' definition, a structure satisfying Sain and Thompson's definition can be associated in a unique way, and conversely.

An advantageous specificity of the polyadic approach of algebraization is that the transposition operator p_{ij} is present in relation algebras (as converse). Therefore the connection with the theory of relation algebras is closer in the case of polyadic algebras than in that of cylindric algebras.

3 On the representation theories

By Stone's theorem, Boolean algebras are representable by Boolean set algebras. It has been known for long that neither quasi-polyadic algebras, nor cylindric algebras are representable, in general (by set algebras). The same is true for transposition algebras and non-commutative cylindric algebras.

We need some definitions:

A Cylindric-relativized set algebra of dimension $\alpha, \alpha \geq 2$, with unit V (Crs_α) is a structure $\mathfrak{A}$ of the form:

$$\langle A,\ \cup,\ \cap,\ \sim_V,\ 0,\ V,\ C_i^V,\ D_{ij}^V\ \rangle_{\ i,j<\alpha}$$

where the unit V is a set of α–termed sequences, such that $V \subseteq {}^{\alpha}U$ for some base set U, A is a non-empty set of subsets of V, closed under the Boolean operations $\cup$, $\cap$, $\sim_V$ and under the cylindrifications

$$C_i^V X = \{y \in V\ :\ y_u^i \in X \ \text{ for some } u\}$$

where $i < \alpha$, $X \in A$, and A contains the sets $\emptyset$, V and the diagonals

$$D_{ij}^V = \{y \in V\ :\ y_i = y_j\}\,.$$

Here the definition of y_u^i is $(y_u^i)_j = y_j$ if $j \neq i$, and $(y_u^i)_j = u$ if $j = i$. The operator ${}^V S_j^i$ is defined like s_j^i for cylindric algebras. An algebra in Crs_α satisfies all the cylindric axioms, with the possible exceptions of (C_4) and (C_6) (see [11], [5], [2]).

The *subclass* D_α of Crs_α is the class:

$$\mathrm{Crs}_\alpha \cap \mathrm{Mod}\left\{C_i^V D_{ij}^V = V\right\}$$

([1]).

What does *representability* mean exactly, e.g., in the case of cylindric algebras? A possible definition is:

A cylindric algebra is said to be *representable* if it is isomorphic to a so-called *α-dimensional generalized weak cylindric set algebra* (Gws_α). $\mathcal{B} \in \mathrm{Gws}_\alpha$ (see [11]) if it is in Crs_α and the unit V is of the form

$$V = \bigcup_{k \in K} {}^{\alpha}U_k^{(pk)} \tag{3}$$

for some index set K, where the ${}^{\alpha}U_k^{(pk)}$ are mutually disjoint weak spaces. The set ${}^{\alpha}U^{(p)}$ is the weak space with base U determined by the point $p \in {}^{\alpha}U$ if it is of the form $\{y \in {}^{\alpha}U : \{t < \alpha : y_t \neq p_t\}$ is finite$\}$.
There are also other equivalent definitions of representability. The definition of representability for quasi-polyadic algebras is similar ([11], [13]).

A cylindric-type algebra $\mathcal{A}$ is *relativized representable*, if it is isomorphic to some set algebra in Crs_α.

A result of Resek and Thompson brought a break-through in the representation theory of cylindric algebras. They proved the following theorem (see [11] 3.2.88, [1]):

If the set of the cylindric axioms is supplemented by the following two axiom schemas (called merry-go-round axioms)

$$\mathrm{MGR1} : s_i^k s_j^i s_k^j \, c_k x = s_j^k s_i^j s_k^i \, c_k x$$

$$\mathrm{MGR2} : s_i^k s_j^i s_m^j s_k^m \, c_k x = s_j^k s_m^j s_i^m s_k^i \, c_k x$$

for distinct ordinals i, j, k and n, then the obtained cylindric algebras are *relativized representable.*

It can be checked that the MGR axioms together are equivalent to the property, that the weak transposition ${}_k s(i,j)$ satisfies (Q_8) (among the QEA_α axioms), (see [6]).

An even stronger theorem due to Resek, D. and Thompson, R. is:

Theorem 2. *The axioms of non-commutative cylindric algebras, supplemented by the MGR axioms, finite-schema axiomatize the class D_α.. (See [1] by Andréka, H. and Thompson, R.)*

In subsequent researches the concept of relativized representation was investigated with respect also to quasi-polyadic algebras ([3]). The concepts of *polyadic-relativized set algebra* (Prs_α) and *relativized representability in the polyadic sense* is completely analogous to the cylindric cases above [11].

With a *polyadic-relativized set algebra in the class* rGwt_α we can associate such a cylindric set algebra in Gws_α (see above) for which the disjointness of the weak spaces in (3) is rejected, furthermore the transpositions p_{ij} are the usual set operations $[i, j]$ (operation exchanging dimensions). It can be proved that quasi-polyadic algebras are *always* relativized representable by set algebras in rGwt_α ([7], [3]). Moreover the following even stronger theorem is true:

Theorem 3. *The axioms of transposition algebras finite-schema axiomatize the class $rGwt_\alpha$ (see [7]).*

Notice that in this representation theorem the representant class rGwt_α is nice, very concrete and the representability is analogous to the ordinary representability.

4 Conclusions

Transposition algebras have a nice relativized representation, but non- commutative cylindric algebras have not. Transposition algebras and quasi-polyadic algebras are relativized representable without further conditions, in contrast to cylindric algebras. The natural representants (e.g. algebras in the class rGwt_α) are non-commutative set algebras, therefore the non-commutative property is of central importance at investigating relativized representability.

The concept of cylindric algebra is a brilliant innovation of Alfred Tarski for the algebraization of the first order logic. This concept is simple, and requires only a minimal language. It is also interesting from a purely algebraic viewpoint. But, as the research into representation theories reached a more advanced state, it turned out that a *strong transposition operator* should be an inherent part of the algebraization itself. While transposition algebras, quasi-polyadic algebras a priori include transposition symbols p_{ij}, and simple axioms guarantee the power of the operations corresponding to p_{ij} as a part of the polyadic paradigm, in cylindric algebras such transposition symbols and strong transpositions are missing. In CA_α two quite complex axiom schemas (MGR axioms) must be assumed to ensure the existence of such a strong transposition. These axioms are different enough in their style from the other cylindric axioms and they are complicated enough. The creators of cylindric algebras did not consider the alternative of choosing the transpositions as basic symbols to begin with. The relativized representation theory of transposition algebras, and quasi-polyadic algebras is simpler and more natural than that of cylindric algebras.

We can state that the simple language of cylindric algebras, while being their strength, is also their deficiency. Nevertheless, one should keep in mind the fact that today transposition algebras, quasi-polyadic algebras, commutative and non-commutative cylindric algebras, all of them, are important in algebraic logic, and their theories are well-integrated.

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Logical Complexity of Spectra of Abelian ℓ-groups

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Abstract

In this paper we introduce an ad hoc variant of ESO, existential second-order logic (we call it $ESOW$, weakly extended existential second-order logic). As a main result, we show that the class of lattices isomorphic to the principal ideals of some abelian ℓ-group G is definable in $ESOW$ (but not in ESO itself, as it is known).

1 Introduction

The paper is a continuation of [2]. The main problem addressed in this paper is studying the logical complexity of the class of principal ℓ-ideal lattices $\mathrm{Id_c G}$ of an Abelian ℓ-group G.

Following [4], we say that a lattice L is ℓ-representable if there is an Abelian ℓ-group G such that $\mathrm{Id_c G} \cong L$, where $\cong$ denotes lattice isomorphism. ℓ-representability is the main lattice theoretic property we are interested in.

A *logical* approach to the spectrum problem is introduced in [4], where we have that ℓ-representable lattices are not definable in $L_{\infty,\omega}$, the extension of first-order logic with infinitary conjunctions and disjunctions. Wehrung then in [5] has improved his results to the effect that ℓ-representable lattices are not definable in $L_{\infty,\lambda}$ for every cardinal λ.

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This is an optimal result for infinitary first-order logic, in the sense that every class C of first-order structures, closed under isomorphic copy, is the class of models of some class of $L_{\infty,\infty}$ sentence. Proof: a model M belongs to C if and only if M is not isomorphic to X for every model X in the complement of C. Now isomorphism to X (and thus its negation) can be expressed by an $L_{\infty,\infty}$ sentence.

On the other hand, [4] also shows that *countable* ℓ-representable lattices are definable in first-order logic:

Theorem 1.1. *(see [4]) A countable lattice is ℓ-representable if and only if it has a minimum and is completely normal. So, ℓ-representability of countable lattices is definable by a first-order sentence.*

2 Logical framework

Unless stated otherwise, we will consider first-order (or higher-order) logic of partially ordered sets. The language of partially ordered sets consists only of logical symbols (including equality $x = y$) plus a binary relation symbol denoted by $x \leq y$. Lattices will be intended as particular partially ordered sets.

All formulas are supposed to be in prenex form, unless stated otherwise. That is, every formula is a finite list of quantifiers followed by a quantifier-free formula. In particular, we are not allowed to put negations in front of first-order (or higher-order) quantifiers. So, in our formalism, we cannot replace universal quantifiers with negations of existential quantifiers and conversely.

2.1 First-order logic

As primitive operators we choose $\neg, \wedge, \vee, \exists, \forall$. Relation symbols can have any finite arity. We denote by R^n a relation symbol of arity n, or just R when confusion does not arise. If R is an n-ary relation symbol, we write $R(x_1, \ldots, x_n)$ to mean that the tuple $(x_1, \ldots, x_n)$ is in the relation R.

2.2 Second-order logic

Second-order logic is obtained from first-order logic by adding arbitrary second-order existential quantifiers $\exists R^n$, and second-order universal quantifiers $\forall R^n$, where R^n is a relation symbol of arity n. A relational quantifier of arity one will be called monadic.

Given $n \geq 1$, we define Σ^1_n the fragment of second-order logic where we have at most n alternations between existential and universal second-order quantifiers, starting from an existential. Existential second-order logic (Σ^1_1, here denoted by *ESO*) is the class of formulas of second-order logic given by a finite sequence of existential quantifiers followed by a first-order formula.

It is well known that the models of any class of *ESO* sentences are closed under ultraproduct.

2.3 Weak second-order logic

Weak Second-order logic is obtained from first-order logic by adding weak second-order existential quantifiers $\exists_{\text{fin}} R^n$ or $\forall_{\text{fin}} R^n$, where R is a relation symbol of arity n. The difference with second-order logic is that this time R ranges over *finite* relations.

Unlike second-order logic, in weak second-order logic for convenience we do not require that all weak quantifiers precede all first-order quantifiers. So weak quantifiers and first-order quantifiers can be mixed.

2.4 *ESOW*, Weakly extended existential second-order logic

Now we introduce an ad hoc definition. The class *ESOW* (weakly extended existential second-order logic) is the class of the formulas of second-order logic of the form $\exists.\psi$, where $\exists$ is a sequence of second-order existential quantifiers, and ψ is a formula of weak second-order logic.

Lemma 2.1. *The set of second-order formulas equivalent to ESOW formulas is closed under conjunction, disjunction, existential and universal first-order quantifier.*

3 The relative difference in a lattice

In this section we recall some known lattice theoretical results.

In every lattice L, given $a, b \in L$, the relative difference between a and b is defined as the set

$$[a \ominus b] = \{x \in L \mid a \leq b \vee x\}.$$

We say that a lattice has singly based differences if every relative difference has a minimum.

We say that a lattice L has countably based differences if for every $a, b \in L$, the set $[a \ominus b]$ has a countable coinitial subset, that is, there is a countable set $S \subseteq [a \ominus b]$ such that for every $x \in [a \ominus b]$ there is $y \in S$ such that $y \leq x$. We note that, in this paper, a set is called countable if it is finite or denumerable.

Lemma 3.1. *(see [1] and [3] and [4]) Suppose L is a lattice such that: L has singly based differences (hence L has a minimum), L is completely normal, and L is distributive. Then L is ℓ-representable.*

Conversely, suppose L is an ℓ-representable lattice. Then: L has countably based differences, L has a minimum and is completely normal, and L is distributive.

4 The main result

Lemma 4.1. *Let L be an infinite lattice. If $L \cong \mathrm{Id_c G}$ for some ℓ-group G, then $L \cong \mathrm{Id_c H}$ for some ℓ-group H of at most the same size as L.*

Proof. For each principal ideal $I \in \mathrm{Id_c G}$ let $\gamma(I)$ be a generator of I and consider the ℓ-group $H = \langle \{\gamma(I) \mid I \in \mathrm{Id_c G}\} \rangle_G$.

For every $I \in \mathrm{Id_c G}$, we have $I \cap H = [\gamma(I)]_H \in \mathrm{Id_c H}$, so we have a map $f : \mathrm{Id_c G} \to \mathrm{Id_c H}$ such that $f(I) = I \cap H$.

Conversely, given $J \in \mathrm{Id_c H}$ let $g(J) = [J]_G$. If h is a generator of J, then h is also a generator of $g(J)$ in G, so $g(J) \in \mathrm{Id_c G}$ and we have a map $g : \mathrm{Id_c H} \to \mathrm{Id_c G}$ such that $g(J) = [J]_G$. Clearly both f and g are monotonic with respect to inclusion. Moreover we have to show that f and g are inverse to each other. To this aim, consider $I \in \mathrm{Id_c G}$. We have $g(f(I)) = g(I \cap H) = [I \cap H]_G$. Now, both I and $[I \cap H]_G$ are ideals of G generated by $\gamma(I)$, so they are equal, and $g(f(I)) = [I \cap H]_G = I$.

Conversely, consider $J \in \mathrm{Id_c H}$. We have $f(g(J)) = f([J]_G) = [J]_G \cap H$. Suppose some element $x \in H$ generates J; since J generates $[J]_G$ in G, we have that x generates $[J]_G$ in G, and since $[J]_G \cap H \subseteq [J]_G$, x generates $[J]_G \cap H$ in H. So, since $[J]_G \cap H$ and J are ideals of H with the same generator, they are equal, so $f(g(J)) = [J]_G \cap H = J$. So the pair (f, g) is an isomorphism of lattices between $\mathrm{Id_c G}$ and $\mathrm{Id_c H}$, and $L \cong \mathrm{Id_c G} \cong \mathrm{Id_c H}$. □

The main result of the paper is the following:

Theorem 4.2. *ℓ-representability of a lattice L is defined by a formula in ESOW (in the language of partially ordered sets).*

Proof. It is known (e.g. see [2] or [5]) that L is ℓ-representable if and only if either

(a) L has singly based differences and is completely normal, or

(b) there is an ℓ-group G of at most the same size as L and a surjective function $f : G \to L$, such that for every $x, y \in G$, $f(x) \le f(y)$ if and only if x is in the ℓ-ideal generated by y in G.

Now let us evaluate the logical complexity of (a) and (b).

Note that (a) is clearly expressible by a first-order formula χ. Moreover, in order to satisfy the $ESOW$ upper bound, we reformulate (b) as follows:

(b') there is a subset $G \subseteq L$, a ternary relation *Plus* on L, an element *Zero*$\in L$, three binary relations *Less*, *Minus*, f on L such that:

1. (G,*Plus*,*Zero*,*Minus*,*Less*) is an Abelian ℓ-group, where $Minus$ is the additive inverse of the group and *Less* is the lattice order,

2. $f : G \to L$ is a surjective function,

3. for all $x, y \in G$, for all $u, v \in L$, if $u = f(x)$, $v = f(y)$ and $u \leq v$, then x is in the ℓ-ideal generated by y

4. for all $x, y \in G$, for all $u, v \in L$, if $u = f(x)$ and $v = f(y)$ but it is not the case that $u \leq v$, then x is not in the ℓ-ideal generated by y.

Note that points 1 and 2 are expressible in first-order logic, say by formulas ψ_1 and ψ_2.

In order to evaluate the points 3 and 4, we first observe that x is in the ℓ-ideal generated by y if and only if there is $n \in N$ such that $|x| \leq n|y|$.

Note that in turn we have

there is n *such that* $|x| \leq n|y|$ if and only if the following statement holds: *there is a finite totally ordered subset* F *of* G *such that* $|y| = min(F)$*, every element of* F *different from* $|y|$ *is the sum of* $|y|$ *and the previous one, and* $|x| \leq max(F)$*.*

So, the property "x is in the ℓ-ideal generated by y" is expressible in weak second-order logic. Moreover, its negation is equivalent to the ESO formula "there is an ideal J such that $y \in J$ and $x \notin J$".

Therefore, points 3 and 4 are expressible in $ESOW$, by two formulas W_3 and W_4.

Summing up, we can rewrite the disjunction of (a) and (b) by a formula ϕ of the form

$$\phi \leftrightarrow \chi \vee \exists.\psi_1 \wedge \psi_2 \wedge W_3 \wedge W_4,$$

where $\exists$ is a sequence of second-order existential quantifiers, χ is a first-order formula and $\psi_1 \wedge \psi_2 \wedge W_3 \wedge W_4$ is an $ESOW$ formula. So ϕ belongs to the class $ESOW$.

□

A corresponding lower bound was proven in [2]: ℓ-representability is not definable by any class of ESO formulas.

5 Some results on countably based differences

We have seen that having countably based differences (c.b.d.) is a necessary condition for ℓ-representability. One may be interested in the logical complexity of this condition.

From the second-order logic point of view, we have bounds for this condition very similar to the ones we found for ℓ-representability. In fact:

Theorem 5.1. *Distributive lattices with countably based differences are definable by a formula in ESOW.*

Proof. We note that a distributive lattice L has countably based differences if and only if

(*) for every $a, b \in L$, (1) either $[a \ominus b]$ has a minimum, or (2) there is a countably infinite set $S \subseteq [a \ominus b]$ coinitial in $[a \ominus b]$.

Now let us evaluate the logical complexity of (*).

Clearly (1) is first-order expressible.

Moreover, (2) can be put in *ESOW* form. In fact, we can use the following equivalence:

a set S is countably infinite if and only if *there is a total order $\leq$ on S without maximum such that for every $a \in S$, the set $\{x \in S | x \leq a\}$ is finite.*

So, countable infiniteness is *ESOW* expressible, therefore also (2) is *ESOW*, and the disjunction of (1) and (2) is *ESOW*. Now the *ESOW* expressibility of (*) follows from Lemma 2.1. So the resulting formula is *ESOW*.

□

Theorem 5.2. *Distributive lattices with c.b.d. are not definable by any class of ESO formulas.*

Proof. The lattice D considered in [2], Theorem 7.4 has c.b.d., but any ultrapower of the form D^ω/U, where U is a nonprincipal ultrafilter over ω, does not. □

Now we turn to infinitary logic:

Theorem 5.3. *(See [6] and [4], Example 10.5) Distributive lattices with c.b.d. are definable by a formula of L_{ω_1,ω_1}, but not by any class of formulas in $L_{\infty,\omega}$.*

The paper [6] is also strongly related to both second-order logic and ℓ-representability, and thus to the present paper. For example, it is easy to see that a class of models is ESO-definable if and only if it is over first-order logic, in symbols

$ESO = PC(L_{\omega,\omega})$ (denoting there by the same symbol a class of sentences with its class of models). Also, obviously $ESOW \subseteq PC(L_{\omega_1,\omega})$. Wehrung proves in that paper that the complement of the class of all ℓ-representable lattices is not $PC(L_{\infty,\infty})$-definable; thus it is also not ESOW-definable.

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Rabbi Ishmael's Thirteen Hermeneutic Rules as a Kind of Logic

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Abstract

Since the Middle Ages, Judaic hermeneutics, containing certain rules of inference, called *middôṯ* in Hebrew, has been regarded as a kind of logic. For example, Maimonides argued this assertion. In the 19th century, some rabbis with a good German philosophical background attempted for the first time to formalize the most important *middôṯ* logically. These attempts have been continued into the 21st century. However, the question arises to what extent the early applications of the *middôṯ* by Hillel and Rabbi Ishmael in their historical-cultural contexts can be assessed as a kind of logic in the modern sense. In this paper, I argue that a simple formalization of Rabbi Ishmael's thirteen *middôṯ*, which is consonant with his examples and intuitions, is consistent but incomplete. This allows us to consider the system of these rules as a kind of logic in a broad sense.

Keywords: Judaic logic, Judaic hermeneutics, Talmud, Hillel, Rabbi Ishmael.

1 Introduction

The Judaic hermeneutics contains some logical inference rules called *middôṯ* in Hebrew. It is difficult to date them, but it is known that Hillel the Elder (Hīllēl; ca. 110 B.C.–10 A.D.) first presented seven *middôṯ* for the interpretation of the Torah before the wise men from the Bathyra family. So, in the *'Āḇôṯ Dərabbî Nāṯān* 35: 10, we find the following reference to these rules listed by Hillel:

> ז׳ מדות דרש הילל הזקן לפני בני בתירה אלו הן ק״ו וגזרה שוה בנין אב מכלל ופרט ומפרט וכלל כיוצא בו במקום אחד דבר הלמד מענינו אלו שבע מדות שדרש הילל הזקן לפני בני בתירה

> Hillel the Elder presented seven hermeneutic rules before the House of Bathyra: (a) an *a fortiori* argument [*qal wāḥōmer*], (b) an analogy [*gəzērāh šāwāh*], (c) a prototype [*binyan 'āḇ*], (d) a general statement and a specific statement [*kəlāl ûpərāṭ*], (e) a specific statement and a general statement [*pərāṭ ûḵəlāl*], (f) a similar case in another place [*kayyôṣē' bô bəmāqôm 'aḥēr*], (g) a principle learned out from context [*dāḇār hallāmēḏ mē'inyānô*]; these are the seven hermeneutic rules that Hillel the Elder presented before the House of Bathyra.

Rabbi Ishmael (Rabbî Yišmā'ē'l; ca. 70–ca. 135 A.D.) developed this corpus of rules and added more rules to them. He only changed the names of rules (c) and (f) – according to him, (c) is called *binyan 'āḇ mikkāṯûḇ 'eḥāḏ* ("a prototype based on one passage"), and (f) is called *binyan 'āḇ miššənê ḵəṯûḇîm* ("a prototype based on two passages").

The purpose of this paper is to determine whether the thirteen hermeneutic rules of Rabbi Ishmael may be understood as a kind of logic in the modern sense.

We know that Judaic authorities have evaluated the *middôṯ* as a logic since the early medieval period. The Karaite scholars, such as Ya'qūb al-Qirqisānī (first half of the 10th century), who wrote the *Kitāb al-Anwār wa-al-Marāqib* (*Book of Luminaries and Observatories*), stated that instead of the *middôṯ*, the Aristotelian syllogisms should be involved in inferring laws from the Pentateuch. For instance, in the *Leviticus* 3: 17 there is the following general statement:

> It shall be a perpetual statute for your generations throughout all your dwellings, that ye eat neither fat nor blood.

It is the major premise, consisting of the universal affirmative proposition. Then Ya'qūb al-Qirqisānī noted in his book that the verse about the entire tail fat in the *Leviticus* 3: 9 is the minor premise, also consisting of the universal affirmative proposition. Consequently, we deal with the Aristotelian syllogism, called *Barbara*, and conclude that the entire tail fat is prohibited, too [16, p. 151-152].

In the dispute about the logic of hermeneutics, Maimonides (Mōšeh ben Maimôn; 1135–1204), the greatest Judaic philosopher and scholar of the medieval time, in his *Treatise on the Art of Logic* (*Maqālah fi ṣinā' at al-mantiq*), defended the *middôṯ* of Judaic hermeneutics and called them "dialectical arguments" (*maqāyis jadaliyyah*), see [14, p. 32], also see [13].

Different Judaic commentaries to Rabbi Ishmael's thirteen *middôṯ* are collected in [6], also see [15]. In the 19th and early 20th centuries, many Judaic scholars argued that the *middôṯ* is a logic that may even be formalized [8], [7], [20], [21],

[22], [23], [24], [25]. A more modern philosophical interpretation of the *middôṯ* is proposed in [9] and [10].

In the 21st century, the book series *Studies in Talmudic Logic* (College Publications, London) was organized in which the authors aim to formalize different fragments of Judaic reasoning including the *middôṯ*, from the point of view of modern symbolic logic, e.g., see [1], [2], [3] about a formalization of some of the hermeneutic rules.

The main subject of the paper is to prove that the thirteen hermeneutic rules provided by Rabbi Ishmael can be formalized as a consistent logical system. It is a good argument that the project of the book series *Studies in Talmudic Logic* really makes sense not only from the standpoint of modern logic, but also from the standpoint of historical studies. The matter is that this means that even the founders of Judaic logical hermeneutics – Hillel and Rabbi Ishmael – explicitly formulated the principles of inference to avoid any contradictions in commentary reasoning and did not make logical fallacies thereby.

2 Logical Analysis of the *Ḇāraytā’ Dərabbî Yišmā‘ē’l*

The earliest logical treatise in Hebrew, dated to the 2nd century A.D., is presented by the *Ḇāraytā’ Dərabbî Yišmā‘ē’l* that constitutes the Introduction to the *Sip̱rā’* – that is, to the *midrāš* to the Book of *Leviticus* [26]. It contains a compendium of thirteen hermeneutic rules, collected (but not invented) by Rabbi Ishmael to infer new laws from the Torah just logically. This short treatise begins as follows:

> רבי ישמעאל אומ׳ : בשלש עשרה מדות התורה נדרשת (א) מקל וחומר (ב) מגזירה שוה (ג) מבנין אב מכתוב אחד מבנין אב משני כתובים (ד) מכלל ופרט (ה) מפרט וכלל (ו) מכלל ופרט וכלל אי אתה דן אלא כעין הפרט (ז) מכלל שהוא צריך לפרט ומפרט שהוא צריך לכלל (ח) כל דבר שהיה בכלל ויצא מן הכלל ללמד לא ללמד על עצמו יצא אלא ללמד על הכלל כולו יצא (ט) כל דבר שהיה בכלל ויצא מן הכלל ליטען טען אחר שהוא כענינו יצא להקל ולא להחמיר (י) כל דבר שהיה בכלל ויצא מן הכלל ליטען טען אחר שלא כענינו יצא להקל ולהחמיר (יא) כל דבר שהיה בכלל ויצא מן הכלל לידון בדבר חדש אי אתה יכול להחזירו לכללו עד שיחזירנו הכתוב לכללו בפירוש (יב) דבר הלמד מעניינו ודבר הלמד מסופו (יג) וכן שני כתובין המכחישין זה את זה עד שיבא הכתוב השלישי ויכריע ביניהן
>
> *rabbî yišmā‘ē’l ’ômēr: bišlōš ‘ešōrēh middôṯ hattôrāh niḏrāšeṯ. miqqal wāḥōmer. ûmiggəzērāh šāwāh. mibbinyan ’āḇ mikkāṯûḇ ’eḥāḏ, ûmib-binyan ’āḇ miššənê ḵəṯûḇîm. mikkəlāl ûpərāṭ. ûmippərāṭ ûḵəlāl. kəlāl*

ûpərāṭ ûḵəlāl, 'î 'attāh ḏān 'ellā' kə'ên happərāṭ. mikkəlāl šehû' ṣārîḵ lipràṭ, ûmippərāṭ šehû' ṣārîḵ liḵlāl. kol dāḇār šehāyāh biḵlāl wəyāṣā' min hakkəlāl ləlammēḏ, lō' ləlammēḏ 'al 'aṣmô yāṣā', 'ellā' ləlammēḏ 'al hakkəlāl kullô yāṣā'. kol dāḇār šehāyāh biḵlāl, wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šehû' ḵə'inyānô, yāṣā' ləhāqēl wəlō' ləhaḥămîr. kol dāḇār šehāyāh biḵlāl wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šellō' ḵə'inyānô, yāṣā' ləhāqēl ûləhaḥămîr. kol dāḇār šehāyāh biḵlāl wəyāṣā' liddôn baddāḇār heḥāḏāš, 'î 'attāh yāḵôl ləhaḥăzîrô liḵlālô, 'aḏ šeyyaḥăzîrennû hakkāṯûḇ liḵlālô bəpērûš. dāḇār hallāmēḏ mē'inyānô, wəḏāḇār hallāmēḏ missôpô. wəḵēn šənê ḵəṯûḇîm hammaḵḥîšîn zeh 'eṯ zeh, 'aḏ šeyyāḇō' hakkāṯûḇ haššəlîšî wəyaḵrîa' bênêhen.

Rabbi Ishmael says: By thirteen methods the Torah is interpreted. [It is interpreted] (i) by means of *a fortiori* argument [*miqqal wāḥōmer*]; (ii) by means of an analogy [*ûmiggəzērāh šāwāh*]; (iii) by means of a prototype based on one passage [*mibbinyan 'āḇ mikkāṯûḇ 'eḥāḏ*] and by means of a prototype based on two passages [*ûmibbinyan 'āḇ miššənê ḵəṯûḇîm*]; (iv) by means of a general statement and a specific statement [*mikkəlāl ûpərāṭ*]; (v) by means of a specific statement and a general statement [*ûmippərāṭ ûḵəlāl*]; (vi) by means of a general statement and a specific statement and a general statement [*kəlāl ûpərāṭ ûḵəlāl*] – you decide only according to the subject of the specific statement [*'î 'attāh ḏān 'ellā' kə'ên happərāṭ*]; (vii) by means of a general statement which requires the specific statement, and by means of a specific statement which requires the general statement [*mikkəlāl šehû' ṣārîḵ liprāṭ, ûmippərāṭ šehû' ṣārîḵ liḵlāl*]; (viii) anything which is included in the general statement and which is specified in order to teach [something], teaches not only about itself but also teaches about everything included in the general statement [*kol dāḇār šehāyāh biḵlāl wəyāṣā' min hakkəlāl ləlammēḏ, lō' ləlammēḏ 'al 'aṣmô yāṣā', 'ellā' ləlammēḏ 'al hakkəlāl kullô yāṣā'*]; (ix) anything that is included in the general statement and which is specified as a requirement concerning another requirement which is inkeeping with the general statement, is specified in order to make [the second requirement] less stringent and not more stringent [*kol dāḇār šehāyāh biḵlāl, wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šehû' ḵə'inyānô, yāṣā' ləhāqēl wəlō' ləhaḥămîr*]; (x) anything that is included in the general statement and which is specified as a requirement in the general statement and which is specified as a requirement concerning another requirement which is not inkeeping with the general statement, is specified either to make less or more stringent [*kol dāḇār šehāyāh biḵlāl wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šellō' kə'inyānô, yāṣā' ləhāqēl ûləhaḥămîr*]; (xi) anything that is included in the general statement and which is excerpted from it by an entirely new [provision], you may not return it to [the provision] of its [original] statement unless Scripture expressly indicates that you may do so [*kol dāḇār šehāyāh biḵlāl wəyāṣā' liddôn baddāḇār heḥāḏāš, 'î 'attāh yāḵôl ləhaḥăzîrô liḵlālô, 'aḏ šeyyaḥăzîrennû hakkāṯûḇ liḵlālô bəpērûš*]; (xii) a thing is to be explained from its context, and a thing is to be explained from what follows it [*dāḇār hallāmēḏ mē'inyānô, wəḏāḇār hallāmēḏ missôpô*]; (xiii) and thus two passages which contradict each other [cannot be reconciled] unless a third passage comes and decides between them [*wəḵēn šənê kəṯûḇîm hammaḵḥîšîn zeh 'eṯ zeh, 'aḏ šeyyāḇō' hakkāṯûḇ haššəlîšî wəyaḵrîa' bênêhen*] [11, p. 63–64].

Hence, we face an enumeration of the hermeneutic rules/methods (*middôṯ*) for treating the Torah without demonstrating a criterion of their logical division. But then the author properly defines these rules and exemplifies each of them by showing how to interpret different passages from the Pentateuch in terms of each method. These examples allow us to understand to what extent the thirteen *middôṯ* may be evaluated by us as a logical system in the narrow sense.

We know that logical systems of antiquity, such as those of Aristotle and Chrysippus, were based on the idea that we should introduce some inference rules to avoid any inconsistency in our reasoning. So, the logical system is an inference process without possible contradictions. Meanwhile, there were no ideas of axiomatizations and subsequent verification of their completeness, as is the case of modern logic. The consistency, well defined and postulated for reasoning, was sufficient. And we see that Rabbi Ishmael also introduced the principle of non-contradiction as his thirteenth hermeneutic method: *wəḵēn šənê kəṯûḇîm hammaḵḥîšîn zeh 'eṯ zeh, 'aḏ šeyyāḇō' hakkāṯûḇ haššəlîšî wəyaḵrîa' bênêhen*.

This rule is illustrated by two examples in the *Ḇāraytā' Dərabbî Yišmā'ē'l*. Let us assume that we find two contradictory passages in the Bible: 'A', 'not-A'. According to the rule of introducing conjunction, we may then infer that "A and not-A", that is, obtain a contradiction. And Rabbi Ishmael proposes two ways to reject this contradiction. First, we can find a new passage B from the Bible such that we have: "If A, then B"; "If not-A, then not-B". Then, instead of two statements 'A', 'not-A', we formulate the following one axiom which is a tautology in propositional logic: "If A, then B or if not-A, then not-B." Rabbi Ishmael's own example is as follows. One verse (*Exodus* 19: 20) states: 'A' – "And the Lord came down upon Mount Sinai, on the top of the mount." Another verse (*Exodus* 20: 22): 'not-A' – "I have talked with you from heaven." So, the Lord spoke "from the top of the mount" ('A') and He spoke "from heaven" ('not-A'). A third verse resolves the contradiction (*Deuteronomy* 4: 36): "Out of heaven he made thee to hear his voice, that he might instruct thee: and upon earth he shewed thee his great fire; and thou heardest his words out of the midst of the fire." Thus, the voice is either not from fire if from heaven or from fire if it is from the top of the mount ("If not-A, then not-B **or** if A, then B", where 'B' is fire). The later statement is a tautology of propositional logic, where 'or' is a standard disjunction.

The second way to reject the contradiction of 'A' and 'not-A' is as follows. We should find 'B' such that "If B, then not-A **or** if not-B, then A". Rabbi Ishmael takes the following example to first show a contradiction: 'A' – "And when Moses was gone into the tabernacle of the congregation to speak with him" (*Numbers* 7: 89), and 'not-A' – "And Moses was not able to enter into the tent of the congregation" (*Exodus* 40: 35). This is resolved by the new verse (*Exodus* 40: 35): 'B' – "the cloud abode thereon, and the glory of the Lord filled the tabernacle". Indeed, if the cloud was there, Moses did not enter ("If B, then not-A"). When the cloud departed, he entered and spoke with Him ("If not-B, then A"). Consequently, we have the tautology: "If B, then not-A **or** if not-B, then A." Hence, if we face a contradiction "A and not-A" in the Torah, we replace it by one of the two axioms:

$(xiii')$ "A and not-A" $\rightarrowtail$ "If not-A, then not-B **or** if A, then B;"

$(xiii'')$ "A and not-A" $\rightarrowtail$ "If B, then not-A **or** if not-B, then A."

Here the sign $\rightarrowtail$ means a replacement of one passage by another, according to an appropriate hermeneutic rule.

To sum up, Rabbi Ishmael considers the *middôṯ* on the assumption that in the Torah there are no contradictions at all, therefore some logical inference rules are always applicable to this text.

According to the twelfth rule (*wəḏāḇār hallāmēḏ missôpô*), each obscure statement of the Pentateuch is clarified later. Let A be a statement which seems to be unclear, then, instead of A, we should take two strings simultaneously: AB, where B is a later clarification of A. In this way, the verse A from the *Leviticus* 14: 34 is clarified by the verse B from the *Leviticus* 14: 45. Therefore, both verses AB should be regarded together, instead of only one A:

$$(xii')\ \ A \rightarrowtail AB,$$

where B is a later statement clarifying A.

Let us assume that we have two strings in the Torah, occurring together: AB. In accordance with the second part of the twelfth rule (*dāḇār hallāmēḏ mēʻinyānô*),

B may be considered through its context as a particular case of B restricted to the situation of A. Formally: $p(B/A)$, where $p(B/A) =$ "A and $p(B)$." Then, instead of AB, we deal with $Ap(B/A)$, i.e., we replace B by $p(B/A)$. For instance, in the *Leviticus* 13: 40, "he is clean" is not understood as "clean of all plague-spot uncleanliness", but it concerns scalls alone:

$$(xii'')\ \ AB \rightarrowtail Ap(B/A).$$

Several rules from the *middôṯ* are grounded on the difference between particular and general notions. Let A be a statement. Then $p(A)$ means a particularization of A and $g(A)$ means a generalization of A. Subsequently, the following implications hold true: "If $p(A)$, then A"; "If A, then $g(A)$"; "If $p(A)$, then $g(A)$". By the seventh rule (*mikkəlāl šehû' ṣārîḵ liprāṭ, ûmippərāṭ šehû' ṣārîḵ liḵlāl*), some particulars are obtained by a set-theoretic intersection of their generalizations. Let us examine a string $g(A)g(B)g(C)$ consisting of three general notions: $g(A)$ – "all the firstborn"; $g(B)$ – "whatsoever openeth the womb"; $g(C)$ – "all the firstling males". They are taken from the following two verses: "Sanctify unto me all the firstborn, whatsoever openeth the womb among the children of Israel, both of man and of beast: it is mine" (*Exodus* 13: 2); "All the firstling males that come of thy herd and of thy flock thou shalt sanctify unto the Lord thy God" (*Deuteronomy* 15: 19). Each member (atomic string) from $g(A)g(B)g(C)$ is a particular case for another and, at the same time, it's a general case. This means that we have an intersection "$g(A)$ and $g(B)$ and $g(C)$". Therefore, we should replace the string $g(A)g(B)g(C)$ by this intersection. It gives all the firstborn who are male and who did not have the Caesarian section (i.e., "openeth the womb"):

$$(vii)\ \ g(A)g(B)g(C) \rightarrowtail \text{“}g(A) \text{ and } g(B) \text{ and } g(C)\text{”}.$$

According to the fourth, fifth, and sixth rules, the order of particular and general notions of the same genus in one and the same complex string changes its meaning. Assume that we find a specific statement after a general statement (*kəlāl ûpərāṭ*) of the same genus: $g(A)p(A)$. For example, "If any man of you bring an offering unto the Lord, ye shall bring your offering of the beasts, of the cattle, and of the sheep" (*Leviticus* 1: 2, the modified translation of KJV according to the Judaic commentaries). Here "of the beasts" is general (that is, all animals) and "of the cattle and of the sheep" is specific (that is, domesticated animals). Then, in accordance with the fourth rule, we consider only the specific (i.e., only the domesticated animals):

$$(iv)\ \ g(A)p(A) \rightarrowtail p(A).$$

Now, let us suppose that we see a general statement after a specific statement (*pərāṭ ûḵəlāl*). For instance, in the *Exodus* 22: 9, "for ox, for ass, for sheep, for raiment" is specific (let it be denoted by $p(A)$) and immediately after that we find the phrase "for any manner of lost thing which another challengeth to be his" that is general (let it be denoted by $g(A)$). Then, according to the fifth rule, we accept only the general:

$$(v)\;\; p(A)g(A) \rightarrowtail g(A).$$

According to the sixth rule (*kəlāl ûpərāṭ ûḵəlāl*), if we find the triple of general, specific, and general: $g'(A)p(A)g''(A)$, then we should accept the latter general as a restricted to the previous specific $g'''(p(A))$. For instance, "and thou shalt bestow that money for whatsoever thy soul lusteth after, for oxen, or for sheep, or for wine, or for strong drink, or for whatsoever thy soul desireth" (*Deuteronomy* 14: 26). Here "thou shalt bestow that money for whatsoever thy soul lusteth after" is general. "For oxen, or for sheep, or for wine, or for strong drink" is specific. "For whatsoever thy soul desireth" is again general, but it is a generalization of the previous specific. In this generalization, we accept something that is one nature from another (e.g., wine from grapes) to exclude mushrooms and truffles (which, although they are fruits, do not come from another fruit):

$$(vi)\;\; g'(A)p(A)g''(A) \rightarrowtail g'''(p(A)),$$

where $g'''(p(A))$ is a subset of $g''(A)$, but $g'''(p(A))$ is not equal to $g''(A)$.

According to the eighth rule (*kol dāḇār šehāyāh biḵlāl wəyāṣā' min hakkəlāl ləlammēḏ, lō' ləlammēḏ 'al 'aṣmô yāṣā', 'ellā' ləlammēḏ 'al hakkəlāl kullô yāṣā'*), if a particular case $p(A)$ already covered in a generalization $g(A)$, but it is nevertheless treated separately: $p(A)B$, then the same particularized treatment $p(A)B$ is applied to all other cases which are covered in that generalization: $g(A)B$. For example, in the following verse of the *Leviticus* 7: 20, we find $p(A)B$: "But the soul that eateth of the flesh of the sacrifice of peace offerings, that pertain unto the Lord, having his uncleanness upon him, even that soul shall be cut off from his people". So, we have a specific case presented by "the sacrifice of peace offerings" (i.e., $p(A)$) in order to teach about "cutting off from his people" (i.e., about B). Let the category of sacrifice, generalizing burnt-offering, meal-offering, sin-offering, guilt-offering, offering of investiture, and peace-offering, be denoted by $g(A)$. Then the following passage is to teach about "cutting off from his people" (i.e., about B) on the basis of all offerings (i.e., of $g(A)$): "Whosoever he be of all your seed among your generations, that goeth unto the holy things, which the children of Israel hallow unto the Lord, having his uncleanness upon him, that soul shall be cut off from my presence" (*Leviticus* 22: 3). Symbolically:

$$(viii)\ \ p(A)B \rightarrowtail g(A)B.$$

Quite close to this rule are also the ninth and tenth *middôṯ*. So, according to the ninth method (*kol dāḇār šehāyāh biḵlāl, wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šehû' ḵə'inyānô, yāṣā' ləhāqēl wəlō' ləhaḥămîr*), if a specific statement $p(A)$ is subsumed in a general category $g(A)$ and departed from that category for treating B, then it is departed for leniency and not for stringency – let us denote it by $p^l(B)$. For instance, in the verse: "Or if there be any flesh, in the skin whereof there is a hot burning, and the quick flesh that burneth have a white bright spot, somewhat reddish, or white" (*Leviticus* 13: 24), we see "a white bright spot", i.e., $p(A)$, departed from the category of all plague-spots (*Leviticus* 13: 2), to teach about quarantine, i.e., B. Then it is departed for leniency and not for stringency: $p^l(B)$. Therefore, one week of quarantine suffices as opposed to the two-week requirement of the general category:

$$(ix)\ \ p(A)B \rightarrowtail p(A)p^l(B).$$

By the tenth method (*kol dāḇār šehāyāh biḵlāl wəyāṣā' liṭ'ôn ṭō'an 'aḥēr šellō' ḵə'inyānô, yāṣā' ləhāqēl ûləhaḥămîr*), if a specific statement $p(A)$ is subsumed in a general category $g(A)$ and departed from that category for treating B, then it is departed both for leniency and for stringency – let us denote it by $p^{ls}(B)$. For example, "If a man or woman have a plague upon the head or the beard" (*Leviticus* 3: 29), this verse separately mentions "head and beard", which are in the general category of skin and flesh, to show that they are not affected by white hair (leniency) and that they are affected by yellow hair (stringency) – $p^{ls}(B)$:

$$(x)\ \ p(A)B \rightarrowtail p(A)p^{ls}(B).$$

According to the eleventh rule (*kol dāḇār šehāyāh biḵlāl wəyāṣā' liddôn baddāḇār heḥāḏāš, 'î 'attāh yāḵôl ləhaḥăzîrô liḵlālô, 'aḏ šeyyaḥăzîrennû hakkāṯûḇ liḵlālô bəpērûš*), if something specific $p(A)$ is subsumed in a general category $g(A)$, and departed from that category for a new learning (i.e., $p(A)B$), but after that the Holy Scripture shows another learning from the general category (i.e., $g(A)C$), then we accept it as a learning from this specific category, too (i.e., $p(A)C$). For example, "for as the sin offering is the priest's…" (*Leviticus* 14: 13). This is departed from the category for a new learning about the placing of the blood on the thumb of priest's right hand and of his right foot and on his right ear – $p(A)B$. Then the

Scripture explicitly restores it to its general category $g(A)$ to tell us that just as a sin-offering requires the placing of blood on the altar, so does this guilt-offering require it – $g(A)C$. Then we accept this learning for the specific category, too:

$$(xi)\ \ p(A)Bg(A)C \rightarrowtail p(A)Bp(A)C.$$

We see that the *middôṯ* from (viii) to (x) are competing – based on the same premise they draw different conclusions. This demonstrates the specifics of logical inference in the hermeneutic tradition of Rabbi Ishmael: in one logical conclusion, the inference rules numbered by (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii"), may only be applied once. So, these inference rules assume only one-step entailing. And if we see a premise $p(A)B$, then we can obtain either $g(A)B$ (by (viii)) or $p(A)p^l(B)$ (according to (ix)) or $p(A)p^{ls}(B)$ (by (x)), and we cannot continue inferring any further.

The first and second rules listed by Rabbi Ishmael are two types of inference by analogy. The first one (*qal wāḥōmer*) is perhaps one of the oldest inference rules in the West Semitic culture. Its early occurrences are found in the Amarna archive dated between ca. 1360–1332 B.C. This archive, mainly written in the Canaanite dialect of Akkadian, contains diplomatic correspondence between the Egyptian administration and its representatives in Canaan and Amurru (today's Israel and Syria). And we find many expressions containing the word *appūnamma* with the meaning 'even', 'moreover', 'furthermore' in a clause relating to previous statements to draw a conclusion by analogy corresponding to the *qal wāḥōmer*. This *appūnamma* (other spellings: *appūna*, *appunnāma*, *appunāna*) is close to the Hebrew *'ap* ('even'). For example, in the following conditional statement, its author claims that "if the king should come forth and all the lands are hostile to him", then moreover (*ap-pu-na-ma*) this king will even more destroy the author who is more hostile to him:

> *ù šum-ma ap-pu-na-ma yu-ṣa-na šàr-ru / ù ka-li* KUR.KUR.KI *nu-kúr-tu*$_4$ *a-na ša-šu / ù mi-na yi-pu-šu a-na ia-ši-nu.*
>
> But if, moreover, the king should come forth and all the lands are hostile to him, then what can he do to us? (EA, 74: 39–41); [12, p. 230].

In the next example, it is stated that if the pharaoh does not speak, then the author "will abandon the city and depart", but the author will even more do so it if the pharaoh even does not respond:

> *šum-ma ki-a-ma la-a ti-iq-*⌈*bu*⌉ (?) / *ù i-te*$_9$*-zi-ib* UR[U] *ù / pa-aṭ-ra-ti ša-ni-tam šum-ma la-a / tu-te-ru-na a-wa-ta*$_5$ *a-na ia-ši / ù i-te*$_9$*-zi-ib*

URU *ù* / *pa-aṭ-ra-ti qa-du* LÚ.MEŠ / *ša i-ra-a-mu-ni.*

> If thus you do not speak, then I will abandon the ci[ty] and depart; moreover, if you do not send word back to me, then I will abandon the city and depart (EA, 83: 45–51); [12, p. 360].

To the same extent as the Akkadian *appūnamma*, the Hebrew *'ap* in the Torah is sometimes used to express the inference by analogy, called *qal wāḥōmer*. For instance, in the following verse this conclusion is formulated by means of the particles *wə'ap kî* ("so then how much more"):

> *kî 'ānōḵî yāḏa'tî 'eṯ-meryəḵā wə'eṯ-'orpəḵā haqqāšeh hēn bə'ōwḏennî ḥai 'immāḵem hayyōwm mamrîm hĕyiṯem 'im-'ăḏōnāi* ***wə'ap kî****-'aḥărê mōwṯî.*

> For I know thy rebellion, and thy stiff neck: behold, while I am yet alive with you this day, ye have been rebellious against the Lord; and how much more after my death? (*Deuteronomy* 31: 27).

It means that if they are rebellious against the Lord when the author is alive, then moreover they will be even more rebellious after his death.

Other Hebrew particles to express the *qal wāḥōmer* (analogy through *a fortiori*) in the Pentateuch is presented by *wə'êḵ* ("then how"):

> *hēn kesep̱ 'ăšer māṣā'nû bəpî 'amtəḥōṯênû hĕšîḇōnû 'ēlêḵā mē'ereṣ kənā'an* ***wə'êḵ*** *nignōḇ mibbêṯ 'ăḏōnêḵā kesep̱ 'ōw zāhāḇ.*

> Behold, the money, which we found in our sacks' mouths, we brought again unto thee out of the land of Canaan: how then should we steal out of thy lord's house silver or gold? (*Genesis* 44: 8).

According to this passage, if they brought the money back, then they cannot steal out silver or gold from the lord's house (they will even bring this silver or gold back).

Since the *qal wāḥōmer* was very popular manner of thinking in the West Semitic culture, we also find many examples of this reasoning in the New Testament. First, in Jesus' sermons, we see particles *kəmā hāḵīl* ("how much more"), used to demonstrate the *qal wāḥōmer* in Aramaic (Syriac), for instance:

*’et̠baqaw bəna‘b̠e dəlā zār‘īn wəlā ḥāṣdīn wəlayt ləhon tawāne wəb̠ēt̠ qəpāse wălāhā mət̠arse ləhon **kəmā hāk̠īl** ’atton yattīrīn ’ətton men pārḥāt̠ā.*

Observe the ravens; for they do not sow nor reap, and they have no storerooms and barns; and yet God feeds them; how much more important are you than the fowls? (*Peshitta*, *Luke* 12: 24; translation by George Lamsa).

According to this reasoning, if the Lord feeds the ravens, then moreover He will even more feed the humans.

The conclusion by *qal wāḥōmer* was often drawn by apostles, too. Let us see the following inference composed by Paul the Apostle to underline the high importance of restoration:

wen tūqlat̠hon həwāt̠ ‘ūt̠rā lə‘ālmā wəḥayyāb̠ūt̠hon ‘ūt̠rā lə‘amme kəmā hāk̠īl šūmlāyhon.

Now if their stumbling has resulted in riches to the world, and their condemnation in riches to the Gentiles; how much more is their restoration? (*Peshitta*, *Romans* 11: 12; translation by George Lamsa).

Hence, the *qal wāḥōmer* rule was pointed out by Rabbi Ishmael as the first hermeneutic method, due to its extreme popularity in Hebrew and Aramaic discourses, which inherited some thought patterns of reasoning by analogy that took place in Akkadian using *appūnamma* (then in Hebrew *’ap*). Rabbi Ishmael’s own example of *qal wāḥōmer* refers to the following reasoning from the Pentateuch:

And the Lord said unto Moses, if her father had but spit in her face, should she not be ashamed seven days? (*Numbers* 12: 14).

According to this verse, if Miriam’s father had spat in her face (this means, according to Judaic commentaries, that he was just offended with her), then she would be in shame for seven days. It is a premise from the Torah. Then we know from the Torah that *Šək̠înāh* (the divine presence of God) rebukes her. We can then assume that she ought to be put to shame even for fourteen days. Nevertheless, according to the principle of sufficiency (*dayyô*), we cannot sentence more than what is stated in the premise (“seven days”). Then the ultimate sentence for Miriam is as follows:

> Let her be shut out from the camp seven days, and after that let her be received in again (*Numbers* 12: 14).

Rabbi Ishmael explains the *qal wāḥōmer* as containing only two premises, namely let $p'(A)$ and $p''(A)$ be two particulars of the same genus $g(A)$. Suppose that one $p'(A)$ teaches for B, i.e., $p'(A)B$. But we do not know what is taught by $p''(A)$. Then by the *dayyô*, we infer $p''(A)B$ after $p'(A)B$:

$$(i)\;\; p'(A)Bp''(A) \rightarrowtail p'(A)Bp''(A)B,$$

where either "If $p'(A)$, then $p''(A)$" or "If $p''(A)$, then $p'(A)$" holds true. In Rabbi Ishmael's example, "If her father rebukes her, then the Lord [any the more] rebukes her" holds true. It is a kind of a fortiori argument – we apply 'moreover' (*appūnamma*, *'ap*), connecting both particulars $p'(A)$ and $p''(A)$ through an implication in one of the two possible directions. It allows us to extrapolate a property ('teaching') B from $p'(A)$ to $p''(A)$.

More complicated examples discussing the *qal wāḥōmer* with more than two premises are given, for example, in the *Bāḇā' Qammā'* 25a of the Babylonian Talmud (as well as in the *Ṯôsāp̄ôṯ* to this folio). A logical formalization of these complex patterns of *qal wāḥōmer* is suggested in [4].

The second rule by analogy is called *gəzērāh šāwāh*. Let us look at Rabbi Ishmael's own example. We find in the *Exodus* 22: 11 in respect to a hired watchman that an oath shall be between him and his neighbor that the watchman did not put his hand against the deposit of the neighbor:

> Then shall an oath of the Lord be between them both, that he hath not put his hand unto his neighbour's goods; and the owner of it shall accept thereof, and he shall not make it good (*Exodus* 22: 11; *Šəmôṯ* 22: 10).

Let it be $p'(A)BC$, where $p'(A)$ is the hired watchman, B is his oath that he did not put his hand, and C is "between both", that is, except for the watchman's heirs and the owner's heirs. Then we read about someone who watches gratis (*Exodus* 22: 8; *Šəmôṯ* 22: 7): "[oath], whether he have put his hand unto his neighbour's goods". Let it be $p''(A)B$, where $p''(A)$ is the watchman for gratis and B is his oath that he did not put his hand. Then we conclude that in this oath his heirs as well as the owner's heirs are also excluded. Hence, we add "between both" (C) to him:

$$(ii)\;\; p'(A)BCp''(A)B \rightarrowtail p'(A)Bp''(A)BC.$$

The third hermeneutic rule of Rabbi Ishmael is called *binyan 'āḇ* ('prototype'). First, it may proceed by means of a prototype based on one passage (*binyan 'āḇ mikkāṯûḇ 'eḥāḏ*). For instance, *miškāḇ* ("what is lain upon"), A, and *môšāḇ* ("what is sat upon"), B, are different categories. We know that *zāḇ* ("one whose body flows") defiles them – $ADBD$, where D is defiling through *zāḇ*. This *zāḇ* also defiles clothing – CD, where C is clothing. Their possible joint general attribute "$g(A)$ and $g(B)$ and $g(C)$" is that they are articles only for our bodily comfort. Then, by induction, we conclude that all articles which are designed for the bodily comfort (that is, "$g(A)$ and $g(B)$ and $g(C)$") are defiled by *zāḇ*:

$$(iii')\ \ ADBDCD \rightarrowtail \text{“}g(A)\ and\ g(B)\ and\ g(C)\text{”}D.$$

Second, it may proceed by means of a prototype based on two passages (*binyan 'āḇ miššənê kəṯûḇîm*). The example provided by Rabbi Ishmael is as follows. The subject of the lamps on the *mənôrāh* (let it be A) is introduced through the notion *ṣaw* ('command') – let it be C, that is, we have CA:

> *ṣaw 'eṯ-bənê yiśrā'ēl wəyiqḥû 'ēlêḵā šemen zayiṯ zāḵ kāṯîṯ lammā'ōwr ləha'ălōṯ nēr tāmîḏ.*
>
> Command [*ṣaw*] the children of Israel, that they bring unto thee pure oil olive beaten for the light, to cause the lamps to burn continually (*Leviticus* 24: 2).

The subject of sending the unclean outside of the encampment (B) is introduced by the notion *ṣaw* (C), too, that is, CB:

> *ṣaw 'eṯ-bənê yiśrā'ēl wîšalləḥû min-hammaḥăneh kol-ṣārûa' wəḵol-zāḇ wəḵōl ṭāmē' lānāp̱eš.*
>
> Command [*ṣaw*] the children of Israel, that they put out of the camp every leper, and every one that hath an issue, and whosoever is defiled by the dead (*Numbers* 5: 2).

Then we know that CA and CB should be applied both immediately (D) and for future generations (E). The immediate (D) performing of the lamps as a command (CA) is discussed in the *Leviticus* 8: 2 and this (CA) for future generations (E) in the *Leviticus* 24: 3. So, we have $CADE$. The *Numbers* 5:4 speaks of the immediate (D) departure of the unclean outside the camp as a command (CB) and the *Numbers* 5: 4 speaks of that (CB) for future generations (E). Consequently, $CBDE$. From this reasoning, we conclude by induction that all commandments introduced by *ṣaw* are applicable both immediately and for future generations: C"$g(A)$ or $g(B)$"DE.

$$(iii'')\ \ CADECBDE \rightarrowtail C\text{“}g(A) \text{ or } g(B)\text{”}DE.$$

3 The Thirteen *middôṯ* of Rabbi Ishmael as a Consistent Logical System

We see that Rabbi Ishmael presents hermeneutic rules (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii") rather as *topoi* (*τόποι*, *loci*) from argumentation theory (evidently, he knew nothing about the Old Greek argumentation theory formed since Aristotle, because he belonged to the Aramaic legal tradition rooted in Babylonia). This means that these argumentation schemes can be applied to the same premises only once, unlike syllogisms, which can be repeatedly applied to the same set of premises, building a chain of inference.

Nevertheless, in the book *Zəḇāḥîm* (49b–51a) of the Babylonian Talmud, there is a hot discussion about chains of conclusions, applying several inference rules one after another. So, some limits of inference chains are considered for the following four rules: *qal wāḥōmer*, *gəzērāh šāwāh*, *binyan 'āḇ*, and *heqqēš*. Let us note that the *heqqēš* is not listed among Rabbi Ishmael's thirteen hermeneutic rules, but it means a kind of inference by analogy. Some limits discussed in the treatise are as follows:

- what is 'learnt' (inferred) through a *heqqēš* does not in turn 'teach' through a *heqqēš* (i.e., this *heqqēš* cannot be applied as next inference rule);
- what is 'learnt' by a *heqqēš* does not in turn 'teach' through a *gəzērāh šāwāh*;
- what is 'learnt' through a *heqqēš* cannot in turn 'teach' through a *binyan 'āḇ*;
- what is 'learnt' through a *heqqēš* 'teaches' in turn by a *qal wāḥōmer*.

Some attempts of a logical formalization of these limits in drawing inference chains are proposed in [5] and [18].

The thirteen hermeneutic rules (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii") have been symbolically written in the following formal system: $\mathbf{F} = (\mathbf{K}, \mathbf{V}, \mathbf{P}, \mathbf{I}, \mathbf{L})$, where $\mathbf{K} = \{A, B, C, \dots\}$ is an alphabet consisting of signs, $\mathbf{V} = \{X, Y, Z, \dots\}$ is a set of variables, $\mathbf{P} = \{p, g, p', g', p'', g'', \dots\}$ is a set of one-place predicates defined on signs or variables (where p is read as "a particular of" and g as "a general of" so that for any $x \in \mathbf{K} \cup \mathbf{V}$, "If $p(x)$, then

x" and "If x, then $g(x)$" hold true), **I** consists of comma (,) and the symbol of replacement ($\rightarrowtail$), **L** is a set of axioms defining strings and their replacements. Each sign, variable, or expression $h(x)$, where $x \in \mathbf{K} \cup \mathbf{V}$ and $h \in \mathbf{P}$, is called an atomic string. In **F**, we also construct the second-order predicates of the form $h'(h''(x))$, where $x \in \mathbf{K} \cup \mathbf{V}$ and $h', h'' \in \mathbf{P}$. Then $h'(h''(x))$ is an atomic string, too. Let us suppose that atomic strings are closed under the set of logical operations. It means that an atomic string can also be a compound proposition "$f(x, y, \ldots, z)$", where f is a propositional function of two-valued logic (built by using some logical connectives such as negation, conjunction, disjunction, or implication) and defined on atomic strings x, y, ..., z. The molecular string is of the form $x_1 \ldots x_n$, where $x_1, \ldots, x_n \in \mathbf{K} \cup \mathbf{V}$ or for some i, $x_i := h(y)$ or $h'(h''(y))$, where $y \in \mathbf{K} \cup \mathbf{V}$ and $h, h', h'' \in \mathbf{P}$, or for some i, $x_i :=$ "$f(y, \ldots, z)$", where f is a propositional function defined on atomic strings y, ..., z. It is a concatenation of atomic strings. Thus, each atomic or molecular string is a string. Rabbi Ishmael's intuition is that each string $x_1 \ldots x_n$ means that the sequence of symbols $x_1 \ldots x_n$ expresses one of the Judaic laws, contained in or derived from the Pentateuch. For any two strings x, y, we may construct their replacement: $x \rightarrowtail y$. Some strings and their replacements are collected as axioms in **L**.

This system **F** is to generate strings over signs, variables, predicates, and compound propositions, using axioms and the following two inference rules: (i) the substitution rule – we can replace the same variable by the same sequence of signs; (ii) *modus ponens* – if there are axioms x and $x \rightarrowtail y$, then we deduce a string y, where x and y are strings over **F**.

Proposition 1. *Let* $\mathbf{L}'$ *be a set of axioms of the first-order logic without quantifiers, formulated in the language of* **F**, *and* $\mathbf{S}'$ *be a logical system closed under applications of substitution rule and* modus ponens *to* $\mathbf{L}'$. *Then* $\mathbf{S}'$ *is consistent and complete.*

Proof. It is trivial. The concatenation operation $xy \ldots z$, coupling strings x, y, ..., z from **F**, can be treated as the implication "If x, then if y, then $\ldots z$". The Judaic inference $x \rightarrowtail y$ for replacing the string x by the string y can be treated as the implication "If x, then y", too. In this way, we can use any axioms $\mathbf{L}'$ of propositional logic, formulated in **F**. Then $\mathbf{S}'$ is the standard first-order logic without quantifiers that is consistent and complete, as we well know.

All the laws of the Pentateuch along with the thirteen *middôṯ* of Rabbi Ishmael (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii") may be understood as some non-logical axioms $\mathbf{L}''$ added to $\mathbf{L}'$,[1] where $\mathbf{L}'$ is

[1] Let us note that rules (xiii') and (xiii") are metarules in fact in order to present a set of strings from the Torah, $\mathbf{L}''$, as consistent to avoid visible contradictions of some verses from the Pentateuch. This is the first step before constructing logic itself.

taken from Proposition 1. Then the inference, based on (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii"), is directly drawn by *modus ponens* from $\mathbf{L}''$. For example, let us take (iv). If we find the string $g(A)p(A)$ in the Torah (it means that this string is contained in $\mathbf{L}''$), then from this string and (iv) (but this (iv) is also contained in $\mathbf{L}''$), we infer by *modus ponens* that $p(A)$.

Let $\mathbf{S}''$ be a logical system with axioms $\mathbf{L}'' \cup \mathbf{L}'$. We know that the inference rules of Rabbi Ishmael's hermeneutics from (viii) to (x) compete. To avoid possible contradictions because of drawing two competing propositions 'A' and 'not-A', applying these rules simultaneously, we must forbid the use of the substitution rule. In other words, we must delete the set $\mathbf{V}$ from $\mathbf{F}$. In this way, we obtain $\mathbf{F}'' = (\mathbf{K}, \mathbf{P}, \mathbf{I}, \mathbf{L}'' \cup \mathbf{L}')$ – the formal system for the logic $\mathbf{S}''$. It is natural since we only deal with signs and not with variables in the Pentateuch. Furthermore, we suppose that in the *middôṯ* of (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii"), all the signs are different. In this manner, we cannot apply these *middôṯ* more than once. Each application of them is fixed as an appropriate non-logical axiom (i), (ii), (iii'), (iii"), (iv), (v), (vi), (vii), (viii), (ix), (x), (xi), (xii'), (xii"), (xiii'), (xiii") from $\mathbf{L}''$. This immediately proves the following statement:

Proposition 2. *The logic* $\mathbf{S}''$ *is consistent.*

4 Conclusion

Thus, the thirteen *middôṯ* collected by Rabbi Ishmael should be considered as axioms of $\mathbf{L}''$ and not as inference rules in the narrow sense. However, they assume that *modus ponens* will be immediately applied to them in order to draw some new conclusions. From the examples of his *Ḇāraytā'* we see that Rabbi Ishmael clearly possessed such an intuition. It is a logic but in its wide meaning – a system of consistent reasoning, based on *modus ponens*. It is sufficient to be called logic.

In his *Ḇāraytā'*, Rabbi Ishmael explicitly states that the entire Pentateuch can be interpreted using these thirteen *middôṯ* (inference rules of Judaic hermeneutics) to draw new legal norms as logical conclusions: "Thirteen methods are used to interpret the Torah [*bišlōš 'ešōrēh middôṯ hattôrāh niḏrāšeṯ*]". His system is consistent, as we have demonstrated (Proposition 2), but it is not complete with respect to all the norms that are logically derived from the Pentateuch in Judaic hermeneutics. This means that the list of his *middôṯ* is not complete. There are sentences of the Torah to which the thirteen rules of Rabbi Ishmael cannot be applied, and one must resort to other rules. For instance, in the *Bāḇā' Qammā'* 64b: 4, a new rule was introduced called *šənê kəlālôṯ* ("two generals"):

אמרי והא שני כללות דסמיכי אהדדי נינהו אמר רבינא כדאמרי במערבא כל מקום שאתה מוצא שני כללות הסמוכים זה לזה הטל פרט ביניהם ודונם בכלל ופרט

> [The Sages] say: But these [repeated verbs] are two generals that are adjacent to each other. Ravina states: They say in the West [i.e., in Israel] that any place [in the Pentateuch] where you find two generals adjacent to each other, you should place the detail between them and then treat them as a general, and a specification, [and a general].

Symbolically:

$$g'(A)g''(A) \rightarrowtail g'(A)p(A)g''(A).$$

And then, according to Ravina (Ra<u>b</u>înā'; 3rd–4th century A.D.), we apply rule (vi) of Rabbi Ishmael's *<u>B</u>āraytā'*.

In addition to the logical treatise of Rabbi Ishmael, another treatise was written almost at the same time by a student of Rabbi Akiva (Rabbî 'Ăqî<u>b</u>ā'; ca. 40–137 A.D.), named 'Ĕlî'ezer ben Hôrəqənûs (1st–2nd centuries A.D.). The title of his work: *Bāraytā' Dərabbî 'Ĕlî'ezer* or *Pirqê Dərabbî 'Ĕlî'ezer*. Here he proposed thirty-two hermeneutical rules (*middô<u>t</u>*), see [17].

Hence, there are many different *middô<u>t</u>*, not listed in the *<u>B</u>āraytā' Dərabbî Yišmā'ē'l*. But all of them assume the application of *modus ponens*. This feature of Rabbi Ishmael's logic corresponds to the logical tradition of Babylonia [19], in which *modus ponens* was the main inference rule. This suggests that this logic is definitely non-Aristotelian, although it also deals with the conclusions from the fact that some concepts are special cases (species, *pərāṭ*), while others are general cases (genera, *kəlāl*). Even the logical terms used in Judaic hermeneutics, such as *pərāṭ* and *kəlāl*, are taken from the Babylonian jurisprudence [19]. For instance, the *pərāṭ* corresponds to the Akkadian word *parāsu*, while the *kəlāl* to the Akkadian word *kalû*. A (court) decision on the basis of *modus ponens* was called *dīna parāsu* in Akkadian.

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On the Completeness of Some First-order Extensions of C

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Abstract

We show the completeness of several Hilbert-style systems resulting from extending the propositional connexive logics C and C3 by the set of Nelsonian quantifiers, both in the varying domain and in the constant domain setting. In doing so, we focus on countable signatures and proceed by variations of the Henkin construction. We compare our work on the first-order extensions of C3 with the results of [10] and answer several open questions naturally arising in this respect. In addition, we consider possible extensions of C and C3 with a non-Nelsonian universal quantifier preserving a specific rapport between the interpretation of conditionals and the interpretation of the universal quantification which is visible in both intuitionistic logic and Nelson's logic but is lost if one adds the Nelsonian quantifiers on top of the propositional basis provided by C and C3. We briefly explore the completeness of systems resulting from adding this non-Nelsonian quantifier either together with the Nelsonian ones or separately to the two propositional connexive logics.

First-order logic, completeness, Nelson's logic, paraconsistent logic

1 Introduction

The present paper contains some completeness results concerning a family of first-order extensions of two propositional connexive logics, C and C3.[1] Among

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[1] More sources on connexivity in logic and on different systems of connexive logics can be found in [9] and [15].

connexive logics, C holds a special place in being, on the one hand, "one of the simplest and most natural"[12, p.178] connexive systems, and, on the other hand, being negation-inconsistent.

The propositional logic C was introduced in [14] as a connexive modification of the paraconsistent version N4 of Nelson's constructive logic of strong negation.[2] The logic C3, a variant of C which excludes the truth-value gaps, was then introduced in [10]. The connectives of C are defined in such a way that, as long as truth-value gaps are eradicated at the level of atoms, they also cannot occur at the level of compound sentences. As a result, the only difference between Hilbert-style axiomatizations of C and C3, respectively, is the presence in the latter of the axiom $\phi \vee \sim \phi$, corresponding to the law of excluded middle for the strong negation.

The quantified version of C, which we will call QC in this paper, was obtained by borrowing the semantics of $\forall$ and $\exists$ from QN4 (in the present paper, we will call them the Nelsonian quantifiers) and adding them on top of the propositional basis of C. A Hilbert-style axiomatization of QC was also proposed in [14] and was immediately shown to be complete via an embedding of the set of formulas of QC into positive intuitionistic logic. However, this work has not been extended yet to the extension of C3 with the Nelsonian quantifiers, although some proof-theoretic results about some extensions of this kind were reported already in [10]. A peculiar complication that arose relative to this type of extensions, consisted in the fact that the simple addition of Nelsonian quantifiers to C3 led to the reinstatement of truth-value gaps. This problem afflicts one of the first-order extensions of C3 introduced (in a purely proof-theoretic manner) in [10], namely, $\mathsf{QC3}_{At}$. The other system introduced in [10], QC3 eliminates them in a somewhat too direct manner. As a result, the set of admissible models is no longer closed for the models based on the same underlying Kripke frame so that the Kripke semantics of QC3 assumes a decidedly non-standard flavor.

One natural remedy to this adverse effect would have been to require the constancy of object domains associated to the nodes in Kripke models; but, in case this

[2]Nelson's original logic QN3 was introduced in [7]. It was from the very beginning a first-order logic, a first-order arithmetic even, with a semantics inspired by the Kleene's realizability semantics. However, the guiding idea behind Nelson's realizability clauses was clear enough so that their translation into Kripke semantics was completely unproblematic. See one of the early examples of such a translation — however, assuming the constancy of domains, – in [13]. QN4, on the other hand, was only introduced explicitly in a relatively recent [8, Section 4.1]; its only difference from QN3 is that the gluts, that is to say, the sentences that are both true and false at the same node of a Kripke model, are now allowed. The propositonal fragments of these logics, which we will denote by N3 and N4, respectively, also have been objects of separate study for many years now; in particular, N4 was introduced for the first time, to the best of our knowledge, in [6], and, independently, in [1].

move is taken, and the system $\mathsf{QC3}_{CD}$ is understood as the extension of C3 with the Nelsonian quantifiers under the assumption of constant domains, our attention is also inevitably drawn to the system which is now seen as a natural intermediary between QC and $\mathsf{QC3}_{CD}$. This third system, which we will denote by QC_{CD}, results from the addition of the Nelsonian $\forall$ and $\exists$ to C under the same assumption of constant domains which we had to impose on $\mathsf{QC3}_{CD}$.

The main goal of the present paper is then to spell out what happens with the completeness proofs in the family of the logics outlined in the previous paragraph, namely $\{\mathsf{QC}, \mathsf{QC}_{CD}, \mathsf{QC3}_{At}, \mathsf{QC3}, \mathsf{QC3}_{CD}\}$. Given that the first-order extensions of the propositional connexive logics remain largely unexplored, our plan for the paper is to provide a firm basis for further advancement by showing how fairly standard Henkin-style constructions can be produced for these logics, rather than to surprise the reader with new findings. That is why we also treat the completeness of QC even though it was already proven in [14] by an indirect argument; our aim is to spell out a direct proof by the usual Henkin technique that allows for further modifications aimed at getting the completeness results also for the other systems.

In achieving this goal, we adapt a mix of traditional techniques for proving completeness of intuitionistic and intermediate first-order logics; a knowledgeable reader will not fail to notice that we are influenced by the presentation of the completeness proofs given in [3, Ch. 4–5] and [5, Ch. 6–7].

However, given that C departs from N3 and N4 in its understanding of the propositonal connectives, the extension of C with the Nelsonian quantifiers cannot be viewed as the only acceptable choice, not without an additional argument that takes into account the range of other objectively existing options for such an extension. Although in this paper we mainly confine ourselves to preparing the ground for a comprehensive discussion of relative pros and cons of adopting the Nelsonian quantifiers in C, we also find it important to define and motivate, already at this point, at least one non-Nelsonian version of the universal quantifier. It turns out that it is relatively easy to take this new quantifier on board, both as an addition to the set of Nelsonian quantifiers and as the only quantifier extending the connexive propositional base — as long as one does not insist on eradicating the truth-value gaps in the style of C3. On the other hand, for the first-order extensions of C3 the non-Nelsonian universal quantifier exacerbates the problem of truth-value gaps to the point where even the assumption of constant domains is now no longer sufficient to eliminate them.

The corresponding completeness results for the extensions of C featuring the non-Nelsonian universal quantifier are then obtainable by repeating, with some minimal variations, the respective completeness proofs for the Nelsonian extensions of C and C3, which constitutes another reason for the inclusion of this whole discussion into the current preparatory work on first-order connexive logics.

The layout of the remaining part of present paper is then as follows. In Section 2, we define our notation and introduce the Kripke semantics for the first-order connexive logics with the Nelsonian quantifiers. In Section 3, we recall the axiomatization of QC given in [14], develop the basics of the Hilbertian proof theory for this system, and then prove both the general soundness theorem and its converse for the case of countable signatures. In Section 4, we introduce, for the first time in the existing literature, the axiomatizations for the Nelsonian systems, QC_{CD}, $\mathsf{QC3}_{CD}$, and show how to modify the proofs given in Section 3, so that they extend to these logics. In Section 5, we extend our completeness proofs to $\mathsf{QC3}_{At}$ and $\mathsf{QC3}$ and look at the results reported about these systems in [10] in the light of the notions and techniques developed in the previous sections. In particular, we address the question of Existence Property in the first-order extensions of $\mathsf{C3}$. Section 6 is devoted to the discussion of one possible definition of a non-Nelsonian universal quantifier which we denote by Æ, and of the axiomatizations of some logics featuring this quantifier.

Finally, in Section 7, we draw conclusions and try to map out some of the avenues for the future research.

2 Preliminaries and Notations

2.1 The First-order Language

We start by fixing some general notational conventions. In this paper, we identify the natural numbers with finite ordinals. We denote by ω the smallest infinite ordinal. For any $n \in \omega$, we will denote by $\bar{o}_n$ the sequence $(o_1, \ldots, o_n)$ of objects of any kind; moreover, somewhat abusing the notation, we will sometimes denote $\{o_1, \ldots, o_n\}$ by $\{\bar{o}_n\}$. The ordered 1-tuple will be identified with its only member. For any given $m, n \in \omega$, the notation $(\bar{p}_m)^\frown(\bar{q}_n)$ denotes the concatenation of $\bar{p}_m$ and $\bar{q}_n$.

Given a set X and a $k \in \omega$, the notation X^k (resp. $X^{\neq k}$) will denote the k-th Cartesian power of X (resp. the set of all k-tuples from X^k such that their elements are pairwise distinct). We also define that $X^\infty := \bigcup_{n \geq 0} X^n$. The *powerset* of X, that is to say, the set of its subsets, will be denoted by $\mathcal{P}(X)$; on the other hand, the *power of* X will be referred to by $|X|$, so that, for example, $|X| = \omega$ will mean that X is countably infinite. Finally, if X, Y are sets, then we will write $X \Subset Y$, if $X \subseteq Y$ and X is finite.

Given any relation R and a set X, we will denote by $R[X]$ the set $\{b \mid (\exists a \in X)(R(a, b))\}$; this notation naturally extends to cases when R is a function f or its inverse f^{-1}. In case $X = \{a\}$, we will also write $R[a]$ (resp. $f^{-1}[a]$) instead of $R[\{a\}]$ (resp. $f^{-1}[\{a\}]$). In case $\bar{a}_n \in X^n$, we will denote by $R\langle\bar{a}_n\rangle$ the set $\{\bar{b}_n \mid (\forall 1 \leq i \leq n)(b_i \in R[a_i])\}$. Similarly, by $f\langle\bar{a}_n\rangle$ we will denote the tuple $(f(a_1), \dots, f(a_n))$.

Given any function f, we will denote its domain by $dom(f)$; the *range* of f, denoted by $rang(f)$ is just $f[dom(f)]$. In case $rang(f) \subseteq X$, we will also write $f : dom(f) \to X$. Finally, for any set X, we will denote by id_X the identity function on X.

The notations introduced above for sets and functions are also freely applied in this paper to proper classes and class functions.

In this paper, we consider the first-order language without equality based on any set of predicate letters of any arity $k \in \omega$. In particular, 0-ary predicates, or propositional letters, are allowed in our language. We do not allow functions and constants[3], though.

We fix a proper class $Pred$ of possible predicate letters. Elements of $Pred$ will be normally denoted by capital Latin letters like P and Q. If $\Omega \subseteq Pred$ is a set, then any function $\Sigma : \Omega \to \omega$ is called a *signature*. Signatures will be denoted by letters Σ and Θ; moreover, we set that $|\Sigma| = |dom(\Sigma)|$. If $n \in \omega$ and $P \in \Sigma^{-1}[n]$, then we will also write $P^n \in \Sigma$.

Since signatures are functions, we can take their unions and intersections, in case the former make sense according to the general restrictions existing for such operations.

All these notations and all of the other notations introduced in this section can be decorated by all types of sub- and superscripts.

We are going to allow parameters in our formulas, therefore, we also fix a proper set Par which we assume to be disjoint from $Pred$. The elements of Par will be denoted by small Latin letters like a, b, c, and d.

Having fixed a signature Σ, and a set $\Pi \subseteq Par$ we generate a language out of it in the following way. We use $Log := \{\sim, \wedge, \vee, \to, \forall, \exists\}$ as the set of logical symbols and $Var := \{v_i \mid i < \omega\}$ as the set of (individual) variables. Both of these sets are assumed to be disjoint from $Pred \cup Par$. The set $L(\Sigma, \Pi)$ of Σ-formulas with parameters in Π can be then defined by the usual induction on the construction of a formula; in other words, $L(\Sigma, \Pi)$ is the smallest set such that:

1. $P(\bar{\alpha}_n) \in L(\Sigma, \Pi)$ for any $n \in \omega$, $P^n \in \Sigma$, and $\bar{\alpha}_n \in (Var \cup \Pi)^n$.
2. $\{\sim \phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \to \psi), \forall x\phi, \exists x\phi\} \subseteq L(\Sigma, \Pi)$ for all $\phi, \psi \in L(\Sigma, \Pi)$ and $x \in Var$.

[3]The parameters that we speak about are not proper constants since they are not required to be defined at every node of an appropriate model.

As per usual, we get that $|L(\Sigma, \Pi)| = \max(|\Sigma|, |\Pi|, \omega)$.

The elements of Var will be also denoted by x, y, z, w, and the elements of $L(\Sigma, \Pi)$ by Greek letters like ϕ, ψ and θ. In what follows, we will also freely use $\leftrightarrow$, understanding $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \to \psi) \wedge (\psi \to \phi)$. Moreover, given an $n \in \omega$ and a $\bar{\phi}_n \in L(\Sigma, \Pi)^n$ we define that $\bigwedge \bar{\phi}_n := \phi_1 \wedge \ldots \wedge \phi_n$ with the parentheses grouped to the left, and similarly for $\bigvee \bar{\phi}_n$.

Now we can also define the (always finite) sets of parameters, free and bound variables occurring in a given $\phi \in L(\Sigma, \Pi)$ as well as the smallest (finite) signature associated with ϕ (to be denoted by $Par(\phi)$, $FV(\phi)$, $BV(\phi)$, and $Sign(\phi)$, respectively). The definition is by induction on the construction of ϕ:

1. $Par(P(\bar{\alpha}_n)) = \{\bar{\alpha}_n\} \cap Par$, $FV(P(\bar{\alpha}_n)) = \{\bar{\alpha}_n\} \cap Var$, $BV(P(\bar{\alpha}_n)) = \emptyset$, and $Sign(\phi) = \{(P, \Sigma(P))\}$.

2. $\alpha(\sim \phi) = \alpha(\phi)$ and $\alpha(\phi \circ \psi) = \alpha(\phi) \cup \alpha(\psi)$ for all $\alpha \in \{Par, FV, BV, Sign\}$ and $\circ \in \{\wedge, \vee, \to\}$.

3. $Par(Qx\phi) = Par(\phi)$, $FV(Qx\phi) = FV(\phi) \setminus \{x\}$, $BV(Qx\phi) = BV(\phi) \cup \{x\}$, and $Sign(Qx\phi) = Sign(\phi)$ for all $Q \in \{\forall, \exists\}$.

For any $\Gamma \subseteq L(\Sigma, \Pi)$ and any $\alpha \in \{Par, FV, BV, Sign\}$, we define that $\alpha(\Gamma) := \bigcup\{\alpha(\phi) \mid \phi \in \Gamma\}$. Note that, for an infinite Γ, $FV(\Gamma)$ and $BV(\Gamma)$ can be countably infinte; as for the parameter sets and the associated signatures, we clearly have $|Par(\Gamma)|, |Sign(\Gamma)| \leq \max(|\Gamma|, \omega)$ for all $\Gamma \subseteq L(\Sigma, \Pi)$.

It is also clear that for any given $\phi \in L(\Sigma, \Pi)$, any signature Θ, and any set $\Xi \subseteq Par$, we have $\phi \in L(\Theta, \Xi)$ iff $Par(\phi) \subseteq \Xi$ and $Sign(\phi) \subseteq \Theta$.

We will denote the set of $L(\Sigma, \Pi)$-formulas with free variables among the elements of $\bar{x}_n \in Var^{\neq n}$ by $L_{\bar{x}_n}(\Sigma, \Pi)$; in case $\Pi = \emptyset$, we simply write $L_{\bar{x}_n}(\Sigma)$ instead of $L_{\bar{x}_n}(\Sigma, \emptyset)$. In particular, $L_{\emptyset}(\Sigma, \Pi)$ will stand for the set of Σ-sentences with parameters in Π. If $\phi \in L_{\bar{x}_n}(\Sigma, \Pi)$ (resp. $\Gamma \subseteq L_{\bar{x}_n}(\Sigma, \Pi)$), then we will also express this by writing $\phi(\bar{x}_n)$ (resp. $\Gamma(\bar{x}_n)$).

The formulas in $L(\Sigma)$ (resp. sentences in $L_{\emptyset}(\Sigma)$) will be called *pure* Σ-formulas (resp. *pure* Σ-sentences). It is $L_{\emptyset}(\Sigma)$ that can be called a language (over Σ, which in this case serves as a vocabulary) in the most direct and complete sense: every pure Σ-sentence says something in every possible Σ-model. Pure formulas with free variables are mainly of interest as possible constituent parts of pure sentences. Parametrized formulas, including parametrized sentences, are strange hybrid entities arising from pure sentences and formulas after these latter get (partially) interpreted in some particular model, which leads to a replacement of some variables in a formula by their denotations. The parametrized formulas are, therefore, always a mixture of linguistic entities like logical symbols or variables, and the objects in the world referred to by these linguistic entities *in a given interpretation attempt*; as such they are "neither here nor there".

However, the admission of these logical chimeras turns out to be very helpful both in defining the semantics and in formulating the calculi which are also complete for the sets of pure sentences over a given vocabulary, which is the reason for their introduction in this paper.

Given any $\phi \in L(\Sigma, \Pi)$, $\alpha \in Var \cup \Pi$, and $\beta \in \Pi \cup (Var \setminus BV(\phi))$, we denote by $\phi[\beta/\alpha] \in L(\Sigma, \Pi)$ the result of simultaneously replacing every occurrence of α by β (resp. every free occurrence in case $\alpha \in Var$). The precise definition of this operation proceeds by induction on the construction of $\phi \in L(\Sigma, \Pi)$ and runs as follows:

- $P(\bar{t}_n)[\beta/\alpha] := P(\bar{s}_n)$, where $P^n \in \Sigma$, and $\bar{t}_n, \bar{s}_n \in (Var \cup \Pi)^n$ are such that, for all $1 \leq i \leq n$ we have:

$$s_i := \begin{cases} \beta, & \text{if } t_i = \alpha; \\ t_i, & \text{otherwise.} \end{cases}$$

- $(\sim \phi)[\beta/\alpha] := \sim (\phi[\beta/\alpha])$.
- $(\phi \circ \psi)[\beta/\alpha] := \phi[\beta/\alpha] \circ \psi[\beta/\alpha]$, for $\circ \in \{\wedge, \vee, \rightarrow\}$.
- For every $x \in Var$ and $Q \in \{\forall, \exists\}$, we set:

$$(Qx\phi)[\beta/\alpha] := \begin{cases} Qx\phi, & \text{if } x = \alpha; \\ Qx(\phi[\beta/\alpha]), & \text{otherwise.} \end{cases}$$

The following lemma states that our substitution operations work as expected. We (mostly) omit the straightforward but tedious inductive proof.

Lemma 1. *Let Σ be a signature, let Π be a set of parameters, let $\phi \in L(\Sigma, \Pi)$, let $s, s' \in (Var \cup Par)$, and let $t, t' \in Par \cup (Var \setminus BV(\phi))$. Then the following statements hold:*

1. *$BV(\phi[t/s]) = BV(\phi)$, $FV(\phi[t/s]) \subseteq (FV(\phi) \setminus \{s\}) \cup \{t\}$, and $Par(\phi[t/s]) \subseteq (Par(\phi) \setminus \{s\}) \cup \{t\}$.*
2. *If $s \notin FV(\phi) \cup Par(\phi)$, then $\phi[t/s] = \phi$.*
3. *$\phi[t/t] = \phi$.*

4. We have

$$\phi[t/s][t'/s'] := \begin{cases} \phi[t'/s'], \text{ if } s' = s \text{ and } s' = t; \\ \phi[t/s], \text{ if } s' = s \text{ and } s' \neq t; \\ \phi[t'/s'][t'/s], \text{ if } s' \neq s \text{ and } s' = t; \\ \phi[t'/s'][t/s], \text{ if } s' \neq s, s \neq t', \text{ and } s' \neq t \end{cases}$$

Proof. We only sketch the proof for Part 4. If both $s' = s$ and $s' = t$, then also $s = t$. Thus we have $\phi[t/s][t'/s'] = \phi[s'/s'][t'/s'] = \phi[t'/s']$ by Part 3. Next, if both $s' = s$ and $s' \neq t$, then we have $s \neq t$. By Part 1, $FV(\phi[t/s]) \subseteq (FV(\phi) \setminus \{s\}) \cup \{t\}$ and $Par(\phi[t/s]) \subseteq (Par(\phi) \setminus \{s\}) \cup \{t\}$, therefore, $s \notin Par(\phi[t/s]) \cup FV(\phi[t/s])$. But then, Part 2 implies that $\phi[t/s][t'/s'] = \phi[t/s][t'/s] = \phi[t/s]$.

The proof for the remaining two cases proceeds by induction on the construction of ϕ. The basis and the induction step for the connectives are straightforward. As for the quantifiers, let $x \in Var$ and $Q \in \{\forall, \exists\}$ be such that $\phi = Qx\psi$. We may also assume that $s \neq s'$. The following cases arise:

Case 1. Assume that $s' = t$. Then, since $t \notin BV(\phi)$, we must also have $s' \neq x$. We have to consider the following subcases:

Case 1.1. $x = s$. By definition of substitution, we get that:

$$(Qx\psi)[t/s][t'/s'] = (Qx\psi)[t'/s'] = Qx(\psi[t'/s']) = Qx(\psi[t'/s'])[t'/s] = \\ = (Qx\psi)[t'/s'][t'/s].$$

Case 1.2. $x \neq s$. Then we argue by the Induction Hypothesis:

$$(Qx\psi)[t/s][t'/s'] = Qx(\psi[t/s])[t'/s'] = Qx(\psi[t/s][t'/s']) = \\ = Qx(\psi[t'/s'][t'/s]) = (Qx\psi)[t'/s'][t'/s].$$

Case 2. Assume that $s' \neq t$ and $t' \neq s$. The following subcases are possible:

Case 2.1. $x = s$. It follows then from $s \neq s'$ that also $s' \neq x$. The rest of the argument is as in Case 1.1.

Case 2.2. $x \neq s$. Again, two further subcases are possible. If $x = s'$ then we argue similarly to Cases 1.1 and 2.1. Otherwise, we argue by the Induction Hypothesis. □

The cases given in Lemma 1.4 are not exhaustive in that the case when $s \neq s'$, $s' \neq t$, and $s = t'$ is not solved. The following example shows that this is not a coincidence since under these assumptions one cannot, in general, push $[t'/s']$ inside the substitution cascade:

Example 1. *Let* $\Sigma = \{(P, 2)\}$*, let* $a, b, c \in Par$ *be pairwise distinct. Then we have* $P(a, c)[b/a][a/c] = P(b, c)[a/c] = P(b, a)$*, but* $P(a, c)[a/c] = P(a, a)$*, and any further substitutions can only lead to formulas of the form* $P(d, d)$*. Therefore,* $P(a, c)[b/a][a/c] \neq P(a, c)[a/c][t_1/s_1] \dots [t_n/s_n]$ *for any* $n \in \omega$ *and any* $\bar{s}_n, \bar{t}_n \in (Var \cup Par)^n$*.*

The final case in Lemma 1.4 is sufficiently well-behaved to allow for a (restricted) introduction of the operation of simultaneous substitution of variables/parameters by parameters. We formulate this fact as a separate corollary:

Corollary 1. *Let* Σ *be a signature, let* Π *be a set of parameters, let* $\phi \in L(\Sigma, \Pi)$*, let* $n \in \omega$*, let* $\bar{s}_n \in (Var \cup Par)^{\neq n}$ *and* $\bar{t}_n \in (Par \setminus \{\bar{s}_n\})^n$*. We let* $\phi[\bar{t}_n/\bar{s}_n]$ *denote* $\phi[t_1/s_1] \dots [t_n/s_n]$*. Then the following statements hold:*

1. *If* $(i_1, \dots, i_n)$ *is a permutation of* $(1, \dots, n)$*, then we have*
$$\phi[\bar{t}_n/\bar{s}_n] = \phi[t_{i_1}/s_{i_1}, \dots, t_{i_n}/s_{i_n}].$$
2. *If* $\{s_{i_1}, \dots, s_{i_k}\} = \{\bar{s}_n\} \cap (FV(\phi) \cup Par(\phi))$*, then*
$$\phi[\bar{t}_n/\bar{s}_n] = \phi[t_{i_1}/s_{i_1}, \dots, t_{i_k}/s_{i_k}].$$

In the special case when $t_1 = \dots = t_n = a \in Par$, we will write $\phi[a/\bar{s}_n]$ instead of $\phi[\bar{t}_n/\bar{s}_n]$.

The notion of substitution is necessary for the right inductive definition of a sentence that is independent from the inductive definition of an arbitrary formula. More precisely, let Σ be a signature, let Π be a subset of Par and let $c \in Par$, perhaps outside Π. Then $L_\emptyset(\Sigma, \Pi)$ is the smallest subset of $L(\Sigma, \Pi)$ satisfying the following conditions:

- $P(\bar{c}_n) \in L_\emptyset(\Sigma, \Pi)$ for all $n \geq 1$, $P^n \in \Sigma$, and $\bar{c}_n \in \Pi^n$.
- If $\phi, \psi \in L_\emptyset(\Sigma, \Pi)$, then $\sim \phi \in L_\emptyset(\Sigma, \Pi)$ and $(\phi \circ \psi) \in L_\emptyset(\Sigma, \Pi)$ for all $\circ \in \{\wedge, \vee, \rightarrow\}$.
- If $x \in Var$ and $\phi[c/x] \in L_\emptyset(\Sigma, \Pi \cup \{c\})$, then $\forall x\phi, \exists x\phi \in L_\emptyset(\Sigma, \Pi)$.

2.2 Semantics

In order to define our semantics we first fix yet another proper class *State* which is disjoint from $Log \cup Var \cup Pred \cup Par$.

For any given signature Σ, a Σ-model is a structure of the form $\mathcal{M} = (W, \leq, U, D, V^+, V^-)$ such that:

- $\emptyset \neq W \subseteq State$ is a non-empty set of *states*, or *nodes*.
- $\leq \subseteq W \times W$ is reflexive and transitive (i.e. a *pre-order*).
- $\emptyset \neq U \subseteq Par$ is a non-empty set of parameters serving, in this context, as the *universe of objects*.
- $D : W \to (\mathcal{P}(U) \setminus \{\emptyset\})$ is such that, for all $w, v \in W$ we have:

  $$w \leq v \Rightarrow D(w) \subseteq D(v).$$

  Given a $w \in W$, we will sometimes write D_w to denote $D(w)$.
- For all $\circ \in \{+, -\}$, we have that $V^\circ : dom(\Sigma) \times W \to \mathcal{P}(U^\infty)$ such that, for every $P^n \in \Sigma$, and all $w, v \in W$, it is true that:
  - $V^\circ(P, w) \subseteq (D_w)^n$.
  - $w \leq v \Rightarrow V^\circ(P, w) \subseteq V^\circ(P, v)$.

  Given a $w \in W$ and a $P^n \in \Sigma$, we will often write $V^\circ_w(P)$ in place of $V^\circ(P, w)$.

When we use subscripts and other decorated model notations, we strive for consistency in this respect. Some examples of this notational principle are given below:

$$\mathcal{M} = (W, \leq, U, D, V^+, V^-),\ \mathcal{M}' = (W', \leq', U', D', (V')^+, (V')^-),$$
$$\mathcal{M}_n = (W_n, \leq_n, U_n, D_n, (V_n)^+, (V_n)^-).$$

For a given model $\mathcal{M}$, its substructure $(W, \leq, U, D)$ is called the *underlying frame* of $\mathcal{M}$, and $\mathcal{M}$ is said to be based on $(W, \leq, U, D)$.

A model $\mathcal{M}$ is called a constant-domain model iff for all $w \in W$ we have $D_w = U$. A model $\mathcal{M}$ is called a $\mathsf{C3}$-model iff for all $w \in W$ and for every $P^n \in \Sigma$, we have $V^+_w(P) \cup V^-_w(P) = (D_w)^n$. We will denote the classes of constant domain and $\mathsf{C3}$-models by $\mathbb{CD}$ and $\mathbb{C}3$, respectively. In particular, if $\mathcal{M} \in \mathbb{CD} \cap \mathbb{C}3$, then we get that $V^+_w(P) \cup V^-_w(P) = U^n$ for all $w \in W$ and all $P^n \in \Sigma$.

We would like to say that a class $\mathbb{K}$ of models is *good* (that is to say, as a basis for a possible first-order extension of C) iff it is closed for the models based on the same underlying frame. Similarly, we will say that a class $\mathbb{K} \subseteq \mathbb{C}3$ is $\mathsf{C3}$-*good* (that is to say, as a basis for a possible first-order extension of $\mathsf{C3}$), iff whenever a Σ-model $\mathcal{M}$ is in $\mathbb{K}$ and a Σ-model $\mathcal{N} \in \mathbb{C}3$ is based on $(W, \leq, U, D)$, then $\mathcal{N} \in \mathbb{K}$. The goodness here is supposed to mean, somewhat loosely, a naturality of the resulting Kripke semantics, including (but not necessarily limited to) the possibility of a standard-looking frame correspondence theory.

It is easy to see that the class of all models and $\mathbb{CD}$ are good, whereas $\mathbb{C}3$ and $\mathbb{C}3 \cap \mathbb{CD}$ are $\mathsf{C3}$-good. Interestingly enough, $\mathbb{C}3$ itself is not good, which raises the question whether any first-order extension of $\mathsf{C3}$ can be also seen as a natural extension of C. We will not attempt to answer it in this paper. But, even if the question is to be answered negatively, the connection of the first-order extensions of $\mathsf{C3}$ with their propositional base is already sufficient to make them interesting to look at.

The semantics of QC, our main system, is given by the pair of ternary (class-)relations, $\models^+$ and $\models^-$ which are only defined for a triple (α, β, γ) in case α is a Σ-model $\mathcal{M}$ for some signature Σ, $\beta = w \in W$, and $\gamma \in L_\emptyset(\Sigma, D_w)$. The definition of these relations is then given by the following induction on the construction of γ for any Σ-model $\mathcal{M}$ and any $w \in W$:

$$\mathcal{M}, w \models^\circ P(\bar{c}_n) \Leftrightarrow \bar{c}_n \in V_w^\circ(P) \qquad \circ \in \{+,-\},\ P \in \Sigma_n,\ \bar{c}_n \in (D_w)^n$$
$$\mathcal{M}, w \models^+ \sim \phi \Leftrightarrow \mathcal{M}, w \models^- \phi$$
$$\mathcal{M}, w \models^- \sim \phi \Leftrightarrow \mathcal{M}, w \models^+ \phi$$
$$\mathcal{M}, w \models^+ \phi \wedge \psi \Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ and } \mathcal{M}, w \models^+ \psi$$
$$\mathcal{M}, w \models^- \phi \wedge \psi \Leftrightarrow \mathcal{M}, w \models^- \phi \text{ or } \mathcal{M}, w \models^- \psi$$
$$\mathcal{M}, w \models^+ \phi \vee \psi \Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ or } \mathcal{M}, w \models^+ \psi$$
$$\mathcal{M}, w \models^- \phi \vee \psi \Leftrightarrow \mathcal{M}, w \models^- \phi \text{ and } \mathcal{M}, w \models^- \psi$$
$$\mathcal{M}, w \models^+ \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \not\models^+ \phi \text{ or } \mathcal{M}, w \models^+ \psi)$$
$$\mathcal{M}, w \models^- \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \not\models^+ \phi \text{ or } \mathcal{M}, w \models^- \psi)$$
$$\mathcal{M}, w \models^+ \forall x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^+ \phi[a/x])$$
$$\mathcal{M}, w \models^- \forall x\phi \Leftrightarrow (\exists a \in D_w)(\mathcal{M}, w \models^- \phi[a/x])$$
$$\mathcal{M}, w \models^+ \exists x\phi \Leftrightarrow (\exists a \in D_w)(\mathcal{M}, w \models^+ \phi[a/x])$$
$$\mathcal{M}, w \models^- \exists x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^- \phi[a/x])$$

Given a pair $(\Gamma, \Delta) \subseteq \mathcal{P}(L_\emptyset(\Sigma, \Pi)) \times \mathcal{P}(L_\emptyset(\Sigma, \Pi))$, a Σ-model $\mathcal{M}$, and a $w \in W$, we say that $(\mathcal{M}, w)$ satisfies (Γ, Δ), and write $\mathcal{M}, w \models^+ (\Gamma, \Delta)$ iff $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, and we have $\mathcal{M}, w \models^+ \phi$ for every $\phi \in \Gamma$ and $\mathcal{M}, w \not\models^+ \psi$ for every $\psi \in \Delta$.[4] In case $\Delta = \emptyset$, we simply write $\mathcal{M}, w \models^+ \Gamma$. We say that (Γ, Δ) is satisfiable iff $\mathcal{M}, w \models^+ (\Gamma, \Delta)$ for some Σ-model $\mathcal{M}$, and some $w \in W$. Otherwise we say that

[4]The provision requiring inclusion of parameter sets into D_w is necessary to exclude the cases where the parametrized sentences from Δ fail to hold due to the absence of the corresponding parameters in the domain of the respective node.

Δ follows from Γ and write $\Gamma \models \Delta$; in other words, Δ follows from Γ iff for every Σ-model $\mathcal{M}$, and every $w \in W$ such that $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, $\mathcal{M}, w \models^+ \Gamma$ implies $\mathcal{M}, w \models^+ \phi$ for some $\phi \in \Delta$. As usual, we will suppress the brackets when Γ is a singleton. Given a $\psi \in L_\emptyset(\Sigma, \Pi)$, we say that ψ is satisfiable iff $\{\psi\}$ is, and that ψ is valid iff $\emptyset \models \psi$.

These notions can be easily relativized to any given subclass $\mathbb{K}$ of the class of models. Thus, we will say that Δ follows from Γ *over* $\mathbb{K}$ (and write $\Gamma \models_{\mathbb{K}} \Delta$) iff for every Σ-model $\mathcal{M} \in \mathbb{K}$, and every $w \in W$ such that $Par(\Gamma) \cup Par(\Delta) \subseteq D_w$, $\mathcal{M}, w \models^+ \Gamma$ implies $\mathcal{M}, w \models^+ \phi$ for some $\phi \in \Delta$; and similarly for the other notions introduced in the previous paragraph. In this sense we will speak of, e.g., $\mathbb{C}3$-consequence, $\mathbb{CD}$-consequence, and so on, and will write $\Gamma \models_{\mathbb{C}3} \Delta$, $\Gamma \models_{\mathbb{CD}} \Delta$, etc.

When handling the pairs of parametrized sentences (we will often call pairs of sets also bi-sets), we will assume that the usual set-theoretic relations and operations on them are defined componentwise. Thus, for example, we will write $(\Gamma, \Delta) \subseteq (\Gamma', \Delta')$ iff both $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$; we will understand $(\Gamma, \Delta) \Subset (\Gamma', \Delta')$, $(\Gamma, \Delta) \cup (\Gamma', \Delta')$ and so forth in a similar way.

The following lemma is a standard consequence of the definitions given in this subsection

Lemma 2. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $w, v \in W$ be such that $w \leq v$, and let $\phi \in L_\emptyset(\Sigma, D_w)$. Then we have $\mathcal{M}, w \models^\circ \phi \Rightarrow \mathcal{M}, v \models^\circ \phi$ for all $\circ \in \{+, -\}$.*

Proof (a sketch). The proof proceeds by induction on the construction of a parametrized sentence. We look into the following two cases:

Case 1. $\phi = \psi \to \theta$. If $\circ \in \{+, -\}$ and $w, v \in W$ are such that $w \leq v$ and $\mathcal{M}, w \models^\circ \psi \to \theta$, then let $v' \in W$ be such that $v' \geq v$. By transitivity, $v' \geq w$. Therefore, if $\mathcal{M}, v' \models^+ \psi$, then, by $\mathcal{M}, w \models^\circ \psi \to \theta$, we also have $\mathcal{M}, v' \models^\circ \theta$. But then, since $v' \in W$ was chosen arbitrarily under the condition that $v' \geq v$, we must also have $\mathcal{M}, v \models^\circ \psi \to \theta$.

Case 2. $\phi = \forall x\psi$. If $w, v \in W$ are such that $w \leq v$ and $\mathcal{M}, w \models^+ \forall x\psi$, then let $v' \in W$ and $a \in D_{v'}$ be such that $v' \geq v$. By transitivity, $v' \geq w$, therefore, $\mathcal{M}, w \models^+ \forall x\psi$ implies that $\mathcal{M}, v' \models^+ \psi[a/x]$. But then, since $v' \in W$ and $a \in D_{v'}$ were chosen arbitrarily under the condition that $v' \geq v$, we must also have $\mathcal{M}, v \models^+ \forall x\psi$.

On the other hand, assume that $\mathcal{M}, w \models^- \forall x\psi$, and choose an $a \in D_w$ such that $\mathcal{M}, w \models^- \psi[a/x]$. By definition, $a \in D_v$, and, by the Induction Hypothesis, $\mathcal{M}, v \models^- \psi[a/x]$, whence $\mathcal{M}, v \models^- \forall x\psi$ follows. $\square$

We observe that it follows from Lemma 2, that if $\mathcal{M}$ happens to be a constant-domain model, the quantifier clauses can be simplified:

Corollary 2. *Let Σ be a signature, let $\mathcal{M} \in \mathbb{CD}$ be a Σ-model, let $w \in W$, and let $\phi \in L_\emptyset(\Sigma, D_w)$. Then we have:*

$$\begin{aligned}
\mathcal{M}, w &\models^+ \forall x\phi \Leftrightarrow (\forall a \in U)(\mathcal{M}, w \models^+ \phi[a/x]) \\
\mathcal{M}, w &\models^- \forall x\phi \Leftrightarrow (\exists a \in U)(\mathcal{M}, w \models^- \phi[a/x]) \\
\mathcal{M}, w &\models^+ \exists x\phi \Leftrightarrow (\exists a \in U)(\mathcal{M}, w \models^+ \phi[a/x]) \\
\mathcal{M}, w &\models^- \exists x\phi \Leftrightarrow (\forall a \in U)(\mathcal{M}, w \models^- \phi[a/x])
\end{aligned}$$

Turning now to relativizations of the consequence relation w.r.t. the model subclasses introduced in this subsection, we observe that on the first-order level, in contrast with the propositional C3, restricting the class of admissible models to $\mathbb{C}3$ does not ensure the absence of truth-value gaps for arbitrary (pure) sentences. Indeed, consider the following example.

Example 2. *Let $\Sigma := \{(P, 1)\}$ and let the Σ-model $\mathcal{M}$ be defined as follows:*

$$W := \{w, v\},\ \leq := \{(w, v)\} \cup id_{\{w,v\}},\ U := \{a, b\},\ D := \{(w, \{a\}), (v, \{a, b\})\}.$$

Finally, set $V^+_\alpha(P) := \{a\}$ for all $\alpha \in W$, $V^-_w(P) := \emptyset$, and $V^-_v(P) := \{b\}$. Then $\mathcal{M} \in \mathbb{C}3$, but we have both $\mathcal{M}, w \not\models^+ \forall x P(x)$ and $\mathcal{M}, w \not\models^- \forall x P(x)$

However, the phenomena, illustrated by the above example, do not arise for the models in $\mathbb{CD} \cap \mathbb{C}3$, as the following lemma shows:

Lemma 3. *Let Σ be a signature, let $\mathcal{M} \in \mathbb{CD} \cap \mathbb{C}3$ be a Σ-model, and let $w \in W$. Then for any $\phi \in L_\emptyset(\Sigma, D_w)$ we have $\mathcal{M}, w \models^\circ \phi$ for some $\circ \in \{+, -\}$.*

Proof. By induction on the construction of ϕ. The atomic case, providing the basis for our induction, is obvious. We consider the induction steps where we have to deal with the following cases:

Case 1. $\phi = \psi \wedge \chi$. Then, by the Induction Hypothesis, we either have both $\mathcal{M}, w \models^+ \psi$ and $\mathcal{M}, w \models^+ \chi$, and thus also $\mathcal{M}, w \models^+ \psi \wedge \chi$, or else at least one of $\mathcal{M}, w \models^- \psi$, $\mathcal{M}, w \models^- \chi$ holds, implying that $\mathcal{M}, w \models^- \psi \wedge \chi$.

Case 2. $\phi = \psi \vee \chi$. Similar to Case 1.

Case 3. $\phi = \sim \psi$. Straightforward.

Case 4. $\phi = \psi \to \chi$. Note that the Induction Hypothesis implies that we have $\mathcal{M}, w \models^\circ \chi$ for some $\circ \in \{+, -\}$. If now $v \in W$ is such that both $v \geq w$ and $\mathcal{M}, v \models^+ \psi$, then we will also have $\mathcal{M}, v \models^\circ \chi$ by Lemma 2. But, since v was chosen arbitrarily, this also means that $\mathcal{M}, w \models^\circ \psi \to \chi$.

Case 5. $\phi = \forall x\psi$. Then, by the Induction Hypothesis, two subcases are possible: either we have $\mathcal{M}, w \models^- \psi[a/x]$ for some $a \in U$, and hence also $\mathcal{M}, w \models^- \forall x\psi$, or we have $\mathcal{M}, w \models^+ \psi[a/x]$ for all $a \in U$, and hence also $\mathcal{M}, w \models^+ \forall x\psi$ by the fact that $\mathcal{M}$ is a constant-domain model.

Case 6. $\phi = \exists x\psi$. Similar to Case 4. □

Lemma 3, together with Example 2, jointly explain why we consider $\mathbb{CD} \cap \mathbb{C}3$ a better setting for the first-order version of $\mathsf{C3}$ than the wider class $\mathbb{C}3$, even though this choice leads to a system which is different from both $\mathsf{C3}$-based first-order systems introduced in [10]. This discussion is taken up in more detail in Section 5 below. Of course, alternative natural settings for the first-order $\mathsf{C3}$ appear to be possible as well, but we leave their consideration to a future research.

We may understand a logic as a class-function, that, for any given signature Σ, returns the set of all pairs (Γ, ϕ) such that $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma)$ and ϕ is a consequence of Γ. If we use the (Nelsonian) semantics of quantifiers given in this section and interpret ϕ being a consequence of Γ by $\Gamma \models \phi$ (resp. $\Gamma \models_{\mathbb{CD}} \phi$, $\Gamma \models_{\mathbb{CD}\cap\mathbb{C}3} \phi$), then we get the definition of QC (resp. QC_{CD}, $\mathsf{QC3}_{CD}$).

Before we move on to the next section, we need to consider several important operations on models, that will be used later in the paper. The first one is a parameter substitution operation, very similar to the one we used for the formulas. More precisely, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. Consider the function $f_{[b/a]} : U \to (U \setminus \{a\}) \cup \{b\}$ such that, for every $c \in U$ we have:

$$f_{[b/a]}(c) := \begin{cases} b, & \text{if } c = a; \\ c, & \text{otherwise.} \end{cases}$$

Then we can define the model $\mathcal{M}_{[b/a]}$ resulting from the substitution of b for a as the tuple $(W, \leq, U_{[b/a]}, D_{[b/a]}, (V_{[b/a]})^+, (V_{[b/a]})^-)$, where:

- $U_{[b/a]} := f_{[b/a]}[U] = (U \setminus \{a\}) \cup \{b\}$.
- For every $w \in W$:

$$D_{[b/a]}(w) := f_{[b/a]}[D_w] = \begin{cases} (D_w \setminus \{a\}) \cup \{b\}, & \text{if } a \in D_w; \\ D_w, & \text{otherwise.} \end{cases}$$

- $(V_{[b/a]})^\circ(P, w) := \{f_{[b/a]}\langle \bar{a}_n \rangle \mid \bar{a}_n \in V^\circ(P, w)\}$ for all $\circ \in \{+, -\}$, $P^n \in \Sigma$, and $w \in W$.

The parameter substitutions in models are closely related to the parameter substitutions in formulas, so that the following lemma holds:

Lemma 4. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_{\emptyset}(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^{\circ} \phi \Leftrightarrow \mathcal{M}_{[b/a]}, w \models^{\circ} \phi[b/a].$$

Proof. See Appendix A for details. □

We immediately state a useful corollary to Lemma 4, namely that model substitutions do not affect the satisfaction of certain formulas:

Corollary 3. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_{\emptyset}(\Sigma, D_w \setminus \{a\})$, it is true that:*

$$\mathcal{M}, w \models^{\circ} \phi \Leftrightarrow \mathcal{M}_{[b/a]}, w \models^{\circ} \phi.$$

Proof. Note that, by Lemma 1.2, we must have $\phi[b/a] = \phi$. The corollary now follows from Lemma 4. □

Another useful operation on models allows us to add a new object to the domain of a model, as long as we make it indistinguishable from some already existing object. More precisely, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. Consider the relation $\rho_{[b:=a]} := id_{(U \cup \{b\})} \cup \{(a, b)\}$. Then we can define the model $\mathcal{M}_{[b:=a]}$ resulting from the addition of b as a copy a, setting it to the following tuple $(W, \leq, U_{[b:=a]}, D_{[b:=a]}, (V_{[b:=a]})^{+}, (V_{[b:=a]})^{-})$, where:

- $U_{[b:=a]} := \rho_{[b:=a]}[U] = U \cup \{b\}$.
- For every $w \in W$:

$$D_{[b:=a]}(w) := \rho_{[b:=a]}[D_w] = \begin{cases} D_w \cup \{b\}, & \text{if } a \in D_w; \\ D_w, & \text{otherwise.} \end{cases}$$

- $(V_{[b:=a]})^{\circ}(P, w) := \bigcup\{\rho_{[b:=a]}\langle \bar{a}_n \rangle \mid \bar{a}_n \in V^{\circ}(P, w)\}$ for all $\circ \in \{+, -\}$, $P^n \in \Sigma$, and $w \in W$.

Just as in the case of model substitution, the operation of adding a new copy of an existing object displays a close relation to a certain kind of parameter substitutions in formulas. As a result, the following lemma holds:

Lemma 5. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $n \in \omega$, every tuple $\bar{x}_n \in Var^{\neq n}$, every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^{\circ} \phi[a/\bar{x}_n] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^{\circ} \phi[b/\bar{x}_n].$$

Proof. See Appendix B for details. □

Lemma 5 also implies a useful corollary which we would like to state before we move on to axiomatizations of our logics.

Corollary 4. *Let Σ be a signature, let $\mathcal{M}$ be a Σ-model, let $a \in U$, and let $b \in Par \setminus U$. For every $\circ \in \{+, -\}$, every $w \in W$, and every $\phi \in L_\emptyset(\Sigma, D_w)$, it is true that:*

$$\mathcal{M}, w \models^\circ \phi \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^\circ \phi.$$

Proof. Note that, by Lemma 1.2, we must have $\phi[a/\bar{x}_n] = \phi = \phi[b/\bar{x}_n]$ for every $n \in \omega$ and every tuple $\bar{x}_n \in Var^{\neq n}$. The corollary now follows from Lemma 5. □

We note, in passing, that the subclasses of models that we have considered so far, like $\mathbb{CD}$, $\mathbb{C}3$, and $\mathbb{CD} \cap \mathbb{C}3$, are clearly closed for both operations on models.

3 A Hilbert-style Axiomatization of QC

We now start with the axiomatization work for QC, the first of the three logics introduced in the previous section. We will give a direct argument showing that the axiomatization of QC, as it is given in [14], is in general sound relative to the semantics defined above; in case the signature is assumed to be at most countable, we will also show completeness.

In this way, we will show that, for a countable signature Σ, the set of all pairs (Γ, ϕ) such that $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma)$, and ϕ follows from Γ, is recursively enumerable; in fact, our results will show that, even if we allow $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$ for an at most countable $\Pi \subseteq Par$, the respective set of pairs of the form (Γ, ϕ) remains enumerable. This is due to the fact that our axiomatization is given in a form that makes a generation of parametrized sentences from other sentences (parametrized or pure) an indispensable by-product in the process of the generation of pure sentences following from other pure sentences. The readers can easily convince themselves of this indispensability by paying attention to the form of axioms like (A15) below, as well as to their possible interaction with the rules like (MP).

Similar remarks apply to the axiomatizations of the other logical systems considered in this paper.

Given a signature Σ and an infinite set Π of parameters, the (Σ, Π)-instantiation of Hilbert-style axiomatization presented in [14] includes all parametrized sentences that are instances of the following schemes (for all $\phi, \psi, \chi \in L_\emptyset(\Sigma, \Pi)$, all $c \in \Pi$, all

$x \in Var$, and all $\theta \in L_x(\Sigma, \Pi)$):

$$\phi \to (\psi \to \phi) \tag{A1}$$
$$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)) \tag{A2}$$
$$(\phi \wedge \psi) \to \phi \tag{A3}$$
$$(\phi \wedge \psi) \to \psi \tag{A4}$$
$$(\chi \to \phi) \to ((\chi \to \psi) \to (\chi \to (\phi \wedge \psi))) \tag{A5}$$
$$\phi \to (\phi \vee \psi) \tag{A6}$$
$$\psi \to (\phi \vee \psi) \tag{A7}$$
$$(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi)) \tag{A8}$$
$$\sim\sim \phi \leftrightarrow \phi \tag{A9}$$
$$\sim (\phi \wedge \psi) \leftrightarrow (\sim \phi \vee \sim \psi) \tag{A10}$$
$$\sim (\phi \vee \psi) \leftrightarrow (\sim \phi \wedge \sim \psi) \tag{A11}$$
$$\sim (\phi \to \psi) \leftrightarrow (\phi \to \sim \psi) \tag{A12}$$
$$\sim \exists x\theta \leftrightarrow \forall x \sim \theta \tag{A13}$$
$$\sim \forall x\theta \leftrightarrow \exists x \sim \theta \tag{A14}$$
$$\forall x\theta \to \theta[c/x] \tag{A15}$$
$$\theta[c/x] \to \exists x\theta \tag{A16}$$

The rules of inference are then as follows:

$$\text{From } \phi, \phi \to \psi \text{ infer } \psi \tag{MP}$$
$$\text{From } \phi \to \theta[c/x] \text{ infer } \phi \to \forall x\theta \tag{R$\forall$}$$
$$\text{From } \theta[c/x] \to \psi \text{ infer } \exists x\theta \to \psi \tag{R$\exists$}$$

Given any particular application of the rules (R∀) and (R∃) the parameter c is called *the main parameter of the rule application* and must have no occurrences in $\phi \to \psi$.[5]

For any $\Delta \Subset L_\emptyset(\Sigma, \Pi)$ and any $\bar{\phi}_n \in L_\emptyset(\Sigma, \Pi)^n$, such that $\Delta \subseteq \{\bar{\phi}_n\}$, we say that $\bar{\phi}_n$ is a (Σ, Π)-deduction in QC of ϕ_n from the premises Δ iff, for every $1 \leq i \leq n$, ϕ_i is either (1) an instance of (A1)–(A16), or (2) $\phi_i \in \Delta$, or (3) ϕ_i is obtained from some ϕ_j, ϕ_k such that $1 \leq j, k < i$ by an application of (MP), or else (4) is obtained from some ϕ_j such that $1 \leq j < i$ by an application of either (R∀) or (R∃) and

[5] We do not need to require that $x \notin FV(\phi)$ since we assume that our deductions consist of parametrized sentences. Moreover, note that a parameter is always substitutable for a variable, hence the usual provisions associated to axiom schemas like (A15) and (A16) can be omitted in our case.

the main parameter of this application is outside $Par(\Delta)$. Moreover, $\bar{\phi}_n$ is called a proof iff it is a deduction from the empty set of premises. For any $\Gamma \subseteq L_\emptyset(\Sigma, \Pi)$, we say that $\phi \in L_\emptyset(\Sigma, \Pi)$ is (Σ, Π)-*deducible* from Γ (and write $\Gamma \vdash_{(\Sigma,\Pi)} \phi$) iff there exists a (Σ, Π)-deduction $\bar{\phi}_n$ from the premises Δ for some $\Delta \Subset \Gamma$ such that $\phi_n = \phi$. We say that $\phi \in L_\emptyset(\Sigma, \Pi)$ is *deducible* from Γ (and write $\Gamma \vdash \phi$) iff for every infinite set $Par(\Gamma \cup \{\phi\}) \subseteq \Xi \subseteq Par$, we have $\Gamma \vdash_{(Sign(\Gamma\cup\{\phi\}),\Xi)} \phi$.

We now take a brief look at some properties of deducibility. We establish, first, that certain renamings of parameters in deductions by "fresh" parameters are always possible:

Lemma 6. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Delta \cup \{\phi\} \Subset L_\emptyset(\Sigma, \Pi)$, let $\bar{\phi}_n$ be a (Σ, Π)-deduction of ϕ from the premises in Δ, and let $\bar{a}_m \in Par^{\neq m}$ be a non-repeating listing of $Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \{\phi\})$. Assume, moreover, that $\bar{b}_m \in (Par \setminus Par(\{\bar{\phi}_n\}))^{\neq m}$. Then $\phi_1[\bar{b}_m/\bar{a}_m], \dots, \phi_n[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi = \phi_n$ from the premises in Δ.*

Proof. By Lemma 1.2 and the choice of $\bar{a}_m$, we know that the formulas from $\Delta \cup \{\phi\}$ are not affected by the substitution of $\bar{b}_m$ for $\bar{a}_m$; on the other hand, for every $1 \leq i \leq n$ we have that:

$$\begin{aligned} Par(\phi_i[\bar{b}_m/\bar{a}_m]) &\subseteq (Par(\phi_i) \setminus \{\bar{a}_m\}) \cup \{\bar{b}_m\} && \text{(by Lemma 1.1)} \\ &\subseteq (Par(\{\bar{\phi}_n\}) \setminus \{\bar{a}_m\}) \cup \{\bar{b}_m\} \\ &= (Par(\{\bar{\phi}_n\}) \setminus (Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \{\phi\}))) \cup \{\bar{b}_m\} \\ &= Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\} && \text{(by } (\Delta \cup \{\phi\}) \subseteq \{\bar{\phi}_n\}) \end{aligned}$$

It remains to show that $\phi_1[\bar{b}_m/\bar{a}_m], \dots, \phi_n[\bar{b}_m/\bar{a}_m]$ is indeed a deduction; in doing so, we proceed by induction on $r \leq n$. More precisely, we show that, for every such r, $\phi_1[\bar{b}_m/\bar{a}_m], \dots, \phi_r[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi_r[\bar{b}_m/\bar{a}_m]$ from the premises in $\Delta \cap \{\bar{\phi}_k\}$.

Basis. $r = 1$. The following cases are then possible:

Case 1. $\phi_1 \in \Delta$. Then, by Lemma 1.2 and the fact that $\{\bar{a}_m\} \cap Par(\Delta) = \emptyset$, we must have $\phi_1[\bar{b}_m/\bar{a}_m] = \phi_1 \in \Delta$.

Case 2. ϕ_1 is an instance of an axiom schema. Then $\phi_1[\bar{b}_m/\bar{a}_m]$ is clearly an instance of the same axiom schema.

Step. $r = k+1$. Then, by IH, $\phi_1[\bar{b}_m/\bar{a}_m], \dots, \phi_k[\bar{b}_m/\bar{a}_m]$ is a $(\Sigma, Par(\Delta \cup \{\phi\}) \cup \{\bar{b}_m\})$-deduction of $\phi_k[\bar{b}_m/\bar{a}_m]$ from the premises in $\Delta \cap \{\bar{\phi}_k\}$. If now ϕ_r is in Δ or an instance of an axiom schema, then we reason as in the Basis. Otherwise, the following cases are possible:

Case 1. For some i, j such that $1 \leq i, j \leq k$ it is true that $\phi_j = \phi_i \to \phi_r$. But then $\phi_j[\bar{b}_m/\bar{a}_m] = \phi_i[\bar{b}_m/\bar{a}_m] \to \phi_r[\bar{b}_m/\bar{a}_m]$ so that $\phi_r[\bar{b}_m/\bar{a}_m]$ is obtained by an application of (MP) from $\phi_j[\bar{b}_m/\bar{a}_m]$ and $\phi_i[\bar{b}_m/\bar{a}_m]$.

Case 2. For some $1 \leq i \leq k$ we have $\phi_i = \psi \to \chi[c/x]$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \psi \to \forall x \chi$.

Two subcases are then possible.

Case 2.1. $c \in Par(\phi)$. Then we get that $c \notin \{\bar{a}_m\} \cup \{\bar{b}_m\}$, and we reason as follows:

$$\begin{aligned}
\phi_i[\bar{b}_m/\bar{a}_m] &= (\psi \to \chi[c/x])[\bar{b}_m/\bar{a}_m] \\
&= \psi[\bar{b}_m/\bar{a}_m] \to \chi[c/x][\bar{b}_m/\bar{a}_m] \\
&= \psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m][c/x] \qquad \text{(by Lemma 1.4)}
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\phi_n[\bar{b}_m/\bar{a}_m] &= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m]) \\
&= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m])
\end{aligned}$$

It remains to notice that $c \notin Par(\Delta)$, and that:

$$Par(\psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m]) \subseteq Par(\psi \to \chi) \cup \{\bar{b}_m\},$$

therefore, by the choice of $\bar{b}_m$, we must have

$$c \notin Par(\psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m]).$$

Case 2.2. $c \notin Par(\phi)$. Then $c \notin Par(\Delta) \cup Par(\phi)$ and yet c occurs in our deduction, therefore, $c = a_j$ for some $1 \leq j \leq m$. We may assume, wlog, that $j = m$ (otherwise, we can just re-shuffle our listing $\bar{a}_m$).

But then we get that:

$$\begin{aligned}
\phi_i[\bar{b}_m/\bar{a}_m] &= (\psi \to \chi[a_m/x])[\bar{b}_m/\bar{a}_m] \\
&= \psi[\bar{b}_m/\bar{a}_m] \to \chi[a_m/x][\bar{b}_m/\bar{a}_m] \\
&= \psi[\bar{b}_m/\bar{a}_m] \to \chi[\bar{b}_m/\bar{a}_m][b_m/x] \qquad \text{(by Lemma 1.4)} \\
&= \psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}][b_m/x] \\
&\qquad \text{(by Corollary 1.2 and } a_m = c \notin Par(\psi \to \chi))
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\phi_n[\bar{b}_m/\bar{a}_m] &= \psi[\bar{b}_m/\bar{a}_m] \to \forall x(\chi[\bar{b}_m/\bar{a}_m]) \\
&= \psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \forall x(\chi[\bar{b}_{m-1}/\bar{a}_{m-1}])
\end{aligned}$$

again, by Corollary 1.2 and $a_m = c \notin Par(\psi \to \chi)$. It remains to notice that $b_m \notin Par(\Delta)$, and that:

$$\begin{aligned} Par(\psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}]) &\subseteq Par(\psi \to \chi) \cup \{\bar{b}_{m-1}\} \\ &\subseteq Par(\{\bar{\phi}_n\}) \cup \{\bar{b}_{m-1}\}, \end{aligned}$$

therefore, by the choice of $\bar{b}_m$, we must have

$$b_m \notin Par(\psi[\bar{b}_{m-1}/\bar{a}_{m-1}] \to \chi[\bar{b}_{m-1}/\bar{a}_{m-1}]).$$

Therefore, $\phi_n[\bar{b}_m/\bar{a}_m]$ is obtained from $\phi_i[\bar{b}_m/\bar{a}_m]$ by a correct application of (R$\forall$).

Case 3. For some $1 \leq i \leq k$ we have $\phi_i = \psi[c/x] \to \chi$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \exists\psi \to \chi$. This case is dual to Case 2. □

Lemma 7. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. Then $\Gamma \vdash \phi$ iff $\Gamma \vdash_{(Sign(\Gamma\cup\{\phi\}),\Xi)} \phi$ for some infinite set $Par(\Gamma \cup \{\phi\}) \subseteq \Xi \subseteq Par$.*

Proof. The left-to-right direction is trivial. As for the right-to-left direction, assume that the set $Par(\Gamma\cup\{\phi\}) \subseteq \Xi \subseteq Par$ is infinite, and that we have $\Gamma \vdash_{(Sign(\Gamma\cup\{\phi\}),\Xi)} \phi$. Then for some $\Delta \Subset \Gamma$ and for some $\bar{\phi}_n \in L_\emptyset(Sign(\Gamma \cup \{\phi\}), \Xi)^n$ it is true that $\bar{\phi}_n$ is a $(Sign(\Gamma \cup \{\phi\}), \Xi)$-deduction of $\phi = \phi_n$ from the premises in Δ. Let $m \in \omega$ and let $\bar{a}_m \in \Xi^{\neq m}$ be a non-repeating listing of $Par(\{\bar{\phi}_n\}) \setminus Par(\Delta \cup \phi)$.

Choose any infinite parameter set $\Xi' \supseteq Par(\Gamma \cup \{\phi\})$. Since $Par(\{\bar{\phi}_n\})$ is finite, we can choose a non-repeating tuple $\bar{b}_m \in Par^{\neq m}$ such that $\{\bar{b}_m\} \subseteq \Xi' \setminus Par(\{\bar{\phi}_n\})$. Now Lemma 6 implies that $\phi_1[\bar{b}_m/\bar{a}_m], \ldots, \phi_n[\bar{b}_m/\bar{a}_m]$ is a $(Sign(\Gamma \cup \{\phi\}), Par(\Delta \cup \phi) \cup \{\bar{b}_m\})$-deduction (and hence also a $(Sign(\Gamma \cup \{\phi\}), \Xi')$-deduction) of $\phi = \phi_n$ from the premises in Δ. □

Next, we need to establish some particular deducibility relations to be used later:

Lemma 8. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, let $\phi, \psi, \chi \in L_\emptyset(\Sigma, \Pi)$, let $\Gamma \subseteq L_\emptyset(\Sigma, \Pi)$, and let $a \in Par \setminus Par(\Gamma)$. Next, let $x \in Var$, and let $\theta \in L_x(\Sigma, \Pi)$. Moreover, let $m, n \in \omega$, $\bar{\phi}_n \in L_\emptyset(\Sigma, \Pi)^n$, and $\bar{\psi}_m \in L_\emptyset(\Sigma, \Pi)^m$ be such that $\{\bar{\phi}_n\} \subseteq$*

$\{\bar{\psi}_m\}$. *Then the following deducibility relations hold:*

$$\vdash \phi \to \phi \tag{T1}$$
$$\vdash (\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \wedge \psi) \to \chi) \leftrightarrow (\psi \to (\phi \to \chi)) \tag{T2}$$
$$\vdash ((\phi \to \psi) \wedge (\phi \vee \psi \vee \chi)) \to (\psi \vee \chi) \tag{T3}$$
$$\vdash ((\phi \to \psi) \wedge (\phi \wedge \chi)) \to (\phi \wedge \psi \wedge \chi) \tag{T4}$$
$$\vdash \bigwedge \bar{\psi}_m \to \bigwedge \bar{\phi}_n \tag{T5}$$
$$\vdash \bigvee \bar{\phi}_n \to \bigvee \bar{\psi}_m \tag{T6}$$
$$\vdash \forall x(\theta \to \phi) \leftrightarrow (\exists x\theta \to \phi) \tag{T7}$$
$$\vdash \forall x(\phi \to \theta) \leftrightarrow (\phi \to \forall x\theta) \tag{T8}$$
$$(\Gamma \vdash \phi \to \psi \,\&\, \Gamma \vdash \psi \to \chi) \Rightarrow \Gamma \vdash \phi \to \chi \tag{DR1}$$

The proof is as in the intuitionistic (and classical) case. Using (T5) and (T6), we may extend our notational conventions and write $\bigwedge \Gamma$ and $\bigvee \Gamma$ for an arbitrary $\Gamma \Subset L_\emptyset(\Sigma, \Pi)$.

Lemma 9 (Deduction Theorem)**.** *Let* Σ *be a signature, let* $\Pi \subseteq Par$ *be a set, and let* $\Gamma \cup \{\phi, \psi\} \subseteq L_\emptyset(\Sigma, \Pi)$. *Then* $\Gamma, \phi \vdash \psi$ *iff* $\Gamma \vdash \phi \to \psi$.

Proof. The right-to-left direction is straightforward due to the presence of (MP) in our system. As for the other direction, assume that, for some infinite $\Xi \subseteq Par$ such that $Par(\Gamma \cup \{\phi, \psi\}) \subseteq \Xi$, and for some $n \in \omega$, the sequence $\bar{\phi}_n \in L_\emptyset(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)^n$ is a $(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)$-deduction of $\psi = \phi_n$ from the premises in $\Delta \Subset \Gamma \cup \{\phi\}$.

Now, if $\phi \notin \Delta$, then we must also have $\Gamma \vdash \psi$. But then we can append to $\bar{\phi}_n$ the sentence $\psi \to (\phi \to \psi)$ as an instance of (A1) followed by $\phi \to \psi$ as the result of applying (MP) to the previous sentence and ψ. The resulting sequence is clearly a deduction of $\phi \to \psi$ from the premises in $\Delta \subseteq \Gamma$ so that $\Gamma \vdash \phi \to \psi$.

On the other hand, if $\phi \in \Delta$, then consider the sequence $\phi \to \phi_1, \ldots, \phi \to \phi_n$, and show, by induction on n, that, for every $1 \leq k \leq n$, we can add enough elements to it so that its initial fragment $\phi \to \phi_1, \ldots, \phi \to \phi_k$ turns into a deduction of $\phi \to \phi_k$ from the premises in $(\Delta \setminus \{\phi\}) \cap \{\bar{\phi}_k\}$.

Basis. $k = 1$. We reason as in the intuitionistic (and classical) case.

Step. $k = r + 1$ for some $r \geq 1$. In case ϕ_k is in $\Delta \cup \{\phi\}$, or is an instance of an axiom schema, or is obtained from earlier formulas by an application of (MP), we again reason as in the intuitionistic (and classical) case. There remain two cases connected with the use of the quantifier rules:

Case 1. For some $1 \leq i \leq r$ we have $\phi_i = \theta \to \chi[c/x]$ for corresponding x, c, θ, and χ, such that $c \notin Par(\Delta \cup \{\phi\}) \cup Par(\theta \to \chi)$, whereas $\phi_k = \theta \to \forall x\chi$. The Induction Hypothesis then implies that for some $s \in \omega$ we have transformed the sequence $\phi \to \phi_1, \ldots, \phi \to \phi_r$ into some $(Sign(\Gamma \cup \{\phi, \psi\}), \Xi)$-deduction $\chi_1, \ldots, \chi_s$ of $\phi \to \phi_r = \chi_s$ from the premises in $(\Delta \setminus \{\phi\}) \cap \{\bar{\phi}_r\}$. We now extend $\chi_1, \ldots, \chi_s$ by adding the proof of $(\phi \to (\theta \to \chi[c/x])) \to ((\phi \wedge \theta) \to \chi[c/x])$ as an instance of (T2) followed by an occurrence of $(\phi \wedge \theta) \to \chi[c/x]$ resulting from an application of (MP) to this instance of (T2) and the formula $\phi \to (\theta \to \chi[c/x]) = \phi \to \phi_i = \chi_j$ for some $1 \leq j \leq s$. Immediately after that, we add the formula $(\phi \wedge \theta) \to \forall x\chi$. Since $c \notin Par(\Delta \cup \{\theta, \phi, \chi\})$, the latter formula is obtained from $(\phi \wedge \theta) \to \chi[c/x]$ by an application of (R$\forall$). We insert, next, the proof of $((\phi \wedge \theta) \to \forall x\chi) \to (\phi \to (\theta \to \forall x\chi))$ as an instance of (T2). The sentence $\phi \to (\theta \to \forall x\chi) = \phi \to \phi_k$ now follows from the latter sentence and from $(\phi \wedge \theta) \to \forall x\chi$ by an application of (MP).

Case 2. For some $1 \leq i \leq r$ we have $\phi_i = \theta[c/x] \to \chi$ for corresponding x, c, θ, and χ, such that $c \notin Par(\Delta \cup \{\phi\}) \cup Par(\theta \to \chi)$, whereas $\phi_k = \exists x\theta \to \chi$. The reasoning here is parallel to the argument for Case 1. □

We immediately state a useful corollary:

Corollary 5. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$, let $\Delta \Subset L_\emptyset(\Sigma, \Pi)$, let $x \in Var$, let $\psi \in L_x(\Sigma, \Pi)$, and let $a \in Par \setminus Par(\Gamma \cup \{\psi\})$. Then the following statements hold:*

1. $\Gamma \cup \Delta \vdash \phi \Leftrightarrow \Gamma \vdash \bigwedge \Delta \to \phi$.

2. $\Gamma \vdash \psi[a/x] \Leftrightarrow \Gamma \vdash \forall x\psi$.

Proof. (Part 1). By Lemma 9 and (T2).

(Part 2). The right-to-left direction follows by (A15). For the left-to-right direction, note that $Sign(\psi)$ must be non-empty, so choose any $k \in \omega$ and any P such that $P^k \in Sign(\psi)$. By (T1) and (A16), we must have $\vdash \chi$ for $\chi := \exists \bar{x}_k(P(\bar{x}_k) \to P(\bar{x}_k))$, hence also $\Gamma \vdash \chi$. On the other hand, since $Par(\chi) = \emptyset$, we must have $\Gamma \cup \{\chi\} \vdash \psi[a/x]$, so that, by Lemma 9, also $\Gamma \vdash \chi \to \psi[a/x]$. Since $a \notin Par(\Gamma \cup \{\chi \to \psi\})$, the rule (R$\forall$) is applicable, and we get that $\Gamma \vdash \chi \to \forall x\psi$. One further application of (MP) gives us that $\Gamma \vdash \forall x\psi$. □

Our proof system is sound relative to the semantics of QC introduced in the previous section; more precisely, the following theorem holds:

Theorem 1. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \vdash \phi$, then $\Gamma \models \phi$.*

Proof. Assume that $\Gamma \vdash \phi$. Fix any infinite parameter set $\Xi \subseteq Par(\Gamma \cup \{\phi\})$ and any $(Sign(\Gamma \cup \{\phi\}), \Xi)$-deduction $\bar{\phi}_n$ of $\phi = \phi_n$ from the premises in $\Delta \Subset \Gamma$. We will show that, for every $r \leq n$, we have $\Delta \models \phi_r$ by induction on r, whence $\Gamma \models \phi = \phi_n$ obviously follows.

Basis. $r = 1$. Let $\Sigma \supseteq Sign(\Gamma \cup \{\phi_1\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_1\})$ and $\mathcal{M}, w \models^+ \Delta$. Two cases are possible. If ϕ_1 is an instance of an axiom schema, then clearly $\mathcal{M}, w \models^+ \phi_1$. Otherwise, we must have $\phi_1 \in \Delta$, and then $\mathcal{M}, w \models^+ \phi_1$ follows from $\mathcal{M}, w \models^+ \Delta$.

Step. $r = k + 1$. Then the Induction Hypothesis implies that $\Delta \models^+ \phi_i$ for any $1 \leq i \leq k$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. If ϕ_r is an instance of an axiom schema or a premise, then we reason as in the Basis. Otherwise the following cases are possible:

Case 1. For some i, j such that $1 \leq i, j \leq k$ it is true that $\phi_j = \phi_i \to \phi_r$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. Assume, for contradiction, that $\mathcal{M}, w \not\models^+ \phi_r$. Then we choose a tuple $\bar{a}_m \in Par^{\neq m}$ such that $\{\bar{a}_m\} = Par(\phi_i) \setminus D_w$ and choose any $b \in D_w$. Next, we set $\mathcal{M}' := \mathcal{M}_{[a_1:=b]\ldots[a_m:=b]}$. By Corollary 4, we have both $\mathcal{M}', w \models^+ \Delta$ and $\mathcal{M}', w \not\models^+ \phi_r$. On the other hand, $Par(\phi_i) \cup Par(\phi_r) \subseteq D'_w$ so that the Induction Hypothesis implies that $\mathcal{M}', w \models^+ \phi_i \wedge (\phi_i \to \phi_r)$, which is in contradiction with $\mathcal{M}', w \not\models^+ \phi_r$.

Case 2. For some $1 \leq i \leq k$ we have $\phi_i = \psi \to \chi[c/x]$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \psi \to \forall x\chi$. Again, let $\Sigma \supseteq Sign(\Delta \cup \{\phi_r\})$, let $\mathcal{M}$ be a Σ-model and let $w \in W$ be such that $D_w \supseteq Par(\Delta \cup \{\phi_r\})$ and $\mathcal{M}, w \models^+ \Delta$. Observe that we have then that $\Delta \vdash \psi \to \chi[c/x]$. Assume, for contradiction, that $\mathcal{M}, w \not\models^+ \psi \to \forall x\chi$. Then there must be a $v \in W$ such that $w \leq v$ and we have both $\mathcal{M}, v \models^+ \psi$ and $\mathcal{M}, v \not\models^+ \forall x\chi$; the latter means that, for some $u \in W$ such that $v \leq u$ and for some $a \in D_u$ we must have $\mathcal{M}, u \not\models^+ \chi[a/x]$. By transitivity of $\leq$ and Lemma 2, we get that $\mathcal{M}, u \models^+ (\Delta \cup \{\psi\}, \{\chi[a/x]\})$. If $a = c$, then we are done. Otherwise, we choose any $d \in Par \setminus U$ and set $\mathcal{M}' := \mathcal{M}_{[d/c][c:=a]}$. By Corollary 3 and Corollary 4, we get that $\mathcal{M}', u \models^+ \Delta \cup \{\psi\}$; on the other hand, we get, by Corollary 3, that $\mathcal{M}_{[d/c]}, u \not\models^+ \chi[a/x][d/c]$. However, by the choice of a, c we have $c \notin Par(\chi) \cup \{a\}$ and so Lemma 1.2 implies that $\chi[a/x][d/c] = \chi[a/x]$. Therefore, $\mathcal{M}_{[d/c]}, u \not\models^+ \chi[a/x]$, whence, by Lemma 5, it follows that $\mathcal{M}', u \not\models^+ \chi[c/x]$. Now, since $c \in D'_u$, and the Induction Hypothesis implies that $\Delta \models^+ \psi \to \chi[c/x]$, we must also have $\mathcal{M}', u \models^+ \psi \to \chi[c/x]$. The obtained contradiction shows that, in fact, we must have had $\mathcal{M}, w \models^+ \psi \to \forall x\chi$ all along.

Case 3. For some $1 \leq i \leq k$ we have $\phi_i = \psi[c/x] \to \chi$ for corresponding x, c, ψ, and χ, such that $c \notin Par(\Delta) \cup Par(\psi \to \chi)$, whereas $\phi_r = \exists x\psi \to \chi$. This case is dual to Case 2. $\square$

We now proceed to show the converse of Theorem 1. We will only show it for countable signatures and countable parameter sets. Again, we start with some definitions. A given bi-set $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, is called:

- *Non-trivial*, if $\Delta \neq \emptyset$ and $\Gamma \not\vdash \bigvee \Delta'$ for every $\emptyset \neq \Delta' \Subset \Delta$.
- *Complete*, if $\Gamma \cup \Delta = L_\emptyset(Sign(\Gamma \cup \Delta), Par(\Gamma \cup \Delta))$.
- $\exists$-*complete*, if for every $\exists x\phi \in L_\emptyset(Sign(\Gamma \cup \Delta), Par(\Gamma \cup \Delta))$ such that $\exists x\phi \in \Gamma$, there exists an $a \in Par(\Gamma \cup \Delta)$ such that $\phi[a/x] \in \Gamma$.

A given $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ is called (Σ, Π)-appropriate iff $Sign(\Gamma \cup \Delta) = \Sigma$, $Par(\Gamma \cup \Delta) = \Pi$, and (Γ, Δ) is non-trivial, complete, and $\exists$-complete. In the lemmas that follow below, we list some properties of non-trivial bi-sets and then, more specifically, some properties of the non-trivial bi-sets that also happen to be appropriate.

Lemma 10. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be non-trivial. Then the following statements hold:*

1. *If $(\Gamma', \Delta') \subseteq (\Gamma, \Delta)$ and $\Delta' \neq \emptyset$, then (Γ', Δ') is non-trivial.*
2. *If $\phi \in L_\emptyset(\Sigma, \Pi)$, then one of $(\Gamma \cup \{\phi\}, \Delta)$, $(\Gamma, \Delta \cup \{\phi\})$ is non-trivial.*
3. *If $\exists x\phi \in L_\emptyset(\Sigma, \Pi)$, and $a \in Par \setminus \Pi$, then one of $(\Gamma \cup \{\exists x\phi, \phi[a/x]\}, \Delta)$, $(\Gamma, \Delta \cup \{\exists x\phi\})$ is non-trivial.*
4. *If $\phi \to \psi \in \Delta$, then $(\Gamma \cup \{\phi\}, \{\psi\})$ is non-trivial.*
5. *If $\sim (\phi \to \psi) \in \Delta$, then $(\Gamma \cup \{\phi\}, \{\sim \psi\})$ is non-trivial.*
6. *If $\forall x\phi \in \Delta$, and $a \in Par \setminus \Pi$, then $(\Gamma, \{\phi[a/x]\})$ is non-trivial.*
7. *If $\sim \exists x\phi \in \Delta$, and $a \in Par \setminus \Pi$, then $(\Gamma, \{\sim \phi[a/x]\})$ is non-trivial.*

Proof. Part 1 is straightforward. As for Part 2, assume that both $(\Gamma \cup \{\phi\}, \Delta)$ and $(\Gamma, \Delta \cup \{\phi\})$ are trivial. Then there must be $\Delta', \Delta'' \Subset \Delta$ such that (wlog, due to (A6) and (MP)), both $\Gamma \cup \{\phi\} \vdash \bigvee \Delta'$ and $\Gamma \vdash \phi \vee \bigvee \Delta''$. By Lemma 9, the former deducibility relation implies that also $\Gamma \vdash \phi \to \bigvee \Delta'$. Applying (A6) and (T3), we infer that $\Gamma \vdash \bigvee \Delta' \vee \bigvee \Delta''$. Applying (T6) next, we can show that also

$\Gamma \vdash \bigvee(\Delta' \cup \Delta'')$ (we basically need to erase the repetitions in $\bigvee \Delta' \vee \bigvee \Delta''$). Since $\Delta' \cup \Delta'' \Subset \Delta$, this contradicts the non-triviality of (Γ, Δ).

(Part 3). Again, assume that both $(\Gamma \cup \{\exists x\phi, \phi[a/x]\}, \Delta)$ and $(\Gamma, \Delta \cup \{\exists x\phi\})$ are trivial. Then, by Part 2, $(\Gamma \cup \{\exists x\phi\}, \Delta)$ must be non-trivial. Let $\Delta' \Subset \Delta$ be such that $\Gamma \cup \{\exists x\phi, \phi[a/x]\} \vdash \bigvee \Delta'$. By Lemma 9, we must have then $\Gamma \cup \{\exists x\phi\} \vdash \phi[a/x] \to \bigvee \Delta'$. Since $a \in Par$, by its choice, is outside $Par(\Gamma \cup \Delta \cup \{\phi\})$, we get that:

$$\Gamma \cup \{\exists x\phi\} \vdash \exists x\phi \to \bigvee \Delta' \qquad \text{(by (R}\exists\text{))}$$

$$\Gamma \cup \{\exists x\phi\} \vdash \bigvee \Delta' \qquad \text{(by (MP))}$$

The latter deducibility clearly contradicts the non-triviality of $(\Gamma \cup \{\exists x\phi\}, \Delta)$.

(Part 4). Assume that $\phi \to \psi \in \Delta$, but $(\Gamma \cup \{\phi\}, \{\psi\})$ is trivial, that is to say, that we have $\Gamma \cup \{\phi\} \vdash \psi$. By Lemma 9, we have $\Gamma \vdash \phi \to \psi$, which contradicts the non-triviality of (Γ, Δ).

(Part 5). Assume that $\sim (\phi \to \psi) \in \Delta$. By Part 2, either $(\Gamma \cup \{\phi \to\sim \psi\}, \Delta)$ or $(\Gamma, \Delta \cup \{\phi \to\sim \psi\})$ must be non-trivial. The former case is in contradiction with (A12), therefore, $(\Gamma, \Delta \cup \{\phi \to\sim \psi\})$ must be non-trivial, and, by Part 4, $(\Gamma \cup \{\phi\}, \{\sim \psi\})$ must be non-trivial as well.

(Part 6). Assume that $\forall x\phi \in \Delta$, and that $a \in Par \setminus \Pi$, but $(\Gamma, \{\phi[a/x]\})$ is trivial, that is to say, that we have $\Gamma \vdash \phi[a/x]$. By Corollary 5.2, we must have then that $\Gamma \vdash \forall x\phi$ which is in contradiction with the non-triviality of (Γ, Δ).

(Part 7). Assume that $\sim \exists x\phi \in \Delta$, and that $a \in Par \setminus \Pi$. By Part 2, either $(\Gamma \cup \{\forall x \sim \phi\}, \Delta)$ or $(\Gamma, \Delta \cup \{\forall x \sim \phi\})$ must be non-trivial. The former case is in contradiction with (A13), therefore, $(\Gamma, \Delta \cup \{\forall x \sim \phi\})$ must be non-trivial, and, by Part 6, $(\Gamma, \{\sim \phi[a/x]\})$ must be non-trivial as well. □

Lemma 11. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be (Σ, Π)-appropriate. Let $\phi, \psi \in L_\emptyset(\Sigma, \Pi)$, let $x \in Var$, and let $\chi \in L_x(\Sigma, \Pi)$. Then the following statements hold:*

1. *If $\Gamma \vdash \phi$, then $\phi \in \Gamma$.*
2. *$\phi \wedge \psi \in \Gamma$ iff $\phi, \psi \in \Gamma$.*
3. *$\sim (\phi \wedge \psi) \in \Gamma$ iff $\sim \phi \in \Gamma$ or $\sim \psi \in \Gamma$.*
4. *$\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.*
5. *$\sim (\phi \vee \psi) \in \Gamma$ iff $\sim \phi, \sim \psi \in \Gamma$.*
6. *$\sim\sim \phi \in \Gamma$ iff $\phi \in \Gamma$.*

7. $\exists x\chi \in \Gamma$ *iff* $\chi[a/x] \in \Gamma$ *for some* $a \in Par(\Gamma \cup \Delta)$.

8. $\sim \forall x\chi \in \Gamma, \Delta$ *iff* $\sim \chi[a/x] \in \Gamma$ *for some* $a \in Par(\Gamma \cup \Delta)$.

9. *If* $\phi \to \psi \in \Gamma$ *and* $\phi \in \Gamma$, *then* $\psi \in \Gamma$.

10. *If* $\sim (\phi \to \psi) \in \Gamma$ *and* $\phi \in \Gamma$, *then* $\sim \psi \in \Gamma$.

11. *If* $\forall x\chi \in \Gamma$ *and* $a \in Par(\Gamma \cup \Delta)$, *then* $\chi[a/x] \in \Gamma$.

12. *If* $\sim \exists x\chi \in \Gamma$ *and* $a \in Par(\Gamma \cup \Delta)$, *then* $\sim \chi[a/x] \in \Gamma$.

Proof. Part 1 follows from the non-triviality and completeness of (Γ, Δ).

Most of the remaining parts are proven by a straightforward reference to Part 1 plus the corresponding part of our axiomatization sometimes combined with the reference to the earlier dual parts of the Lemma. Exceptions are Part 4 (reference to (T1)) and Part 7, where one must use the $\exists$-completeness of (Γ, Δ). □

Certain types of non-trivial bi-sets are in general extendable to certain types of appropriate bi-sets, which is the subject of the next lemma:

Lemma 12. *Let* Σ *be an at most countable signature, let* $\Pi \subseteq Par$ *be an at most countable set, and let* $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ *be non-trivial. Then, for every* $\Xi \subseteq Par$ *disjoint from* Π *and such that* $|\Xi| = \omega$, *there exists a* $(\Sigma, \Pi \cup \Xi)$*-appropriate bi-set* (Γ', Δ') *such that* $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$.

Proof. Let $\{a_n \mid n \in \omega\}$ be an enumeration of Ξ, and let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma, \Pi \cup \Xi)$. We now define a countably infinite increasing chain of non-trivial bi-sets

$$(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$$

by setting $(\Gamma, \Delta) := (\Gamma_0, \Delta_0)$, and for any $k \in \omega$, if ψ_k is not of the form $\exists x\phi$ we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\psi_k\}, \Delta_k), & \text{if } (\Gamma_k \cup \{\psi_k\}, \Delta_k) \text{ is non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\psi_k\}), & \text{otherwise.} \end{cases}$$

In case ψ_k has the form $\exists x\phi$, we set

$$\nu[\Gamma_k, \Delta_k, \psi_k] := \{n \in \omega \mid a_n \in \Xi \setminus Par(\Gamma_k \cup \Delta_k \cup \{\psi_k\})\}$$

and define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k), & \text{if } m = \min \nu[\Gamma_k, \Delta_k, \exists x\phi] \\ & \text{and } (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ is non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\exists x\phi\}), & \text{otherwise.} \end{cases}$$

We show that the chain $(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$ is well-defined and that, for every $k \in \omega$ the bi-set (Γ_k, Δ_k) is non-trivial and we have $|\nu[\Gamma_k, \Delta_k, \psi_k]| = \omega$.

This claim is obviously true when $k = 0$. If $k = r + 1$, and the claim is true for (Γ_r, Δ_r), then $(\Gamma_{r+1}, \Delta_{r+1})$ is well-defined by the Induction Hypothesis and is non-trivial by Lemma 10.2–3. Finally, we have $Par(\Gamma_{r+1} \cup \Delta_{r+1} \cup \{\psi_{r+1}\}) = Par(\Gamma_r \cup \Delta_r \cup \{\psi_r, \psi_{r+1}\})$ in case ψ_r is not of the form $\exists x\phi$ and $Par(\Gamma_{r+1} \cup \Delta_{r+1} \cup \{\psi_{r+1}\}) \subseteq Par(\Gamma_r \cup \Delta_r \cup \{\exists x\phi, \phi[a_m/x], \psi_{k+1}\})$ for certain fresh $a_m \in Par$ when $\psi_r = \exists x\phi$. In both cases the difference with $Par(\Gamma_r \cup \Delta_r \cup \{\psi_r\})$ is clearly finite so that $|\nu[\Gamma_{r+1}, \Delta_{r+1}, \psi_{r+1}]| = \omega$ obviously holds.

We now set $(\Gamma', \Delta') := (\bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n)$ and show that this bi-set satisfies the requirements of the Lemma. It is clear that $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$. Moreover, for every $k \in \omega$, we have $\psi_k \in \Gamma_{k+1} \cup \Delta_{k+1}$, therefore it is also clear that $Sign(\Gamma' \cup \Delta') = \Sigma$, that $Par(\Gamma' \cup \Delta') = \Pi \cup \Xi$, and that (Γ', Δ') is complete.

It remains to show non-triviality and $\exists$-completeness of (Γ', Δ'). If $\emptyset \neq \Delta^* \Subset \Delta'$ is such that $\Gamma' \vdash \bigvee \Delta^*$, then consider any deduction of $\bigvee \Delta^*$ from the premises in $\Gamma^* \Subset \Gamma$ and choose any $k \in \omega$ such that $(\Gamma^*, \Delta^*) \subseteq (\Gamma_k, \Delta_k)$. Then $\Gamma_k \vdash \bigvee \Delta^*$, which contradicts the non-triviality of (Γ_k, Δ_k).

As for the $\exists$-completeness, if $\exists x\phi \in \Gamma'$, then $\exists x\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$, therefore, for some $k \in \omega$, we must have $\exists x\phi = \psi_k$. Clearly, $\psi_k \in \Delta_{k+1} \subseteq \Delta'$ would contradict the non-triviality of (Γ', Δ'). Therefore, we must have $\exists x\phi, \phi[a_m/x] \in \Gamma_{k+1} \subseteq \Gamma'$ for an appropriate $a_m \in \Xi$. □

We are now in a position to prove the completeness of our axiomatization in the countable case. Given a signature Σ and a parameter set Π, we will call a bi-set (Γ, Δ) (Σ, Π)-nice iff (Γ, Δ) is (Σ, Ξ)-appropriate, for some $\Xi \subseteq \Pi$ such that $|\Pi \setminus \Xi| = \omega$. Given a (Σ, Π)-appropriate bi-set (Γ, Δ), and a countably infinite $\Xi \subseteq Par$ which is disjoint from Π, we can define the following Σ-model $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ by setting:

- $W_{(\Gamma,\Delta,\Xi)} := \{(\Gamma', \Delta') \mid (\Gamma', \Delta') \text{ is } (\Sigma, \Pi \cup \Xi)\text{-nice}\}$.
- For $(\Gamma_0, \Delta_0), (\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta,\Xi)}$, we have $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$ iff $\Gamma_0 \subseteq \Gamma_1$.
- $U_{(\Gamma,\Delta,\Xi)} := \Pi \cup \Xi$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, we have $D_{(\Gamma,\Delta,\Xi)}(\Gamma_0, \Delta_0) := Par(\Gamma_0 \cup \Delta_0)$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, $n \in \omega$, $P^n \in Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^+_{(\Gamma,\Delta,\Xi)}(P, (\Gamma_0, \Delta_0))$ iff $P(\bar{a}_n) \in \Gamma_0$.

- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$, $n \in \omega$, $P^n \in Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^-_{(\Gamma,\Delta,\Xi)}(P, (\Gamma_0, \Delta_0))$ iff $\sim P(\bar{a}_n) \in \Gamma_0$.

It is straightforward to show that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ is indeed a model of QC, and using the usual methods, a truth lemma can be shown for this model:

Lemma 13. *Let* (Γ, Δ) *be a* (Σ, Π)*-appropriate bi-set and let* $\Xi \subseteq Par$ *be countably infinite and disjoint from* Π*. Then, for every* $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta,\Xi)}$ *and every* $\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$ *it is true that:*

1. $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \phi$ *iff* $\phi \in \Gamma_0$.
2. $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \phi$ *iff* $\sim \phi \in \Gamma_0$.

Proof. We prove both parts of the Lemma simultaneously by induction on the construction of $\phi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$.

Basis. ϕ is atomic. Both parts of the Lemma hold by the definition of $V^+_{(\Gamma,\Delta,\Xi)}$ and $V^-_{(\Gamma,\Delta,\Xi)}$

Step. The following cases are possible.

Case 1. $\phi = \psi \wedge \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Then, for Part 1 of the Lemma we reason as follows:

$$\begin{aligned} \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \wedge \chi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \psi \\ &\qquad \text{and } \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \chi \\ &\Leftrightarrow \psi \in \Gamma_0 \text{ and } \chi \in \Gamma_0 && \text{(by IH)} \\ &\Leftrightarrow \psi \wedge \chi \in \Gamma_0 && \text{(by Lemma 11.2)} \end{aligned}$$

For Part 2, the reasoning is similar:

$$\begin{aligned} \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \wedge \chi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \\ &\qquad \text{or } \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \chi \\ &\Leftrightarrow \sim \psi \in \Gamma_0 \text{ or } \sim \chi \in \Gamma_0 && \text{(by IH)} \\ &\Leftrightarrow \sim (\psi \wedge \chi) \in \Gamma_0 && \text{(by Lemma 11.3)} \end{aligned}$$

Case 2. $\phi = \psi \vee \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Similar to Case 1.

Case 3. $\phi = \sim \psi$ or some $\psi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Then, for Part 1 of the Lemma we reason as follows:

$$\begin{aligned} \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^+ \sim \psi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi \\ &\Leftrightarrow \sim \psi \in \Gamma_0 && \text{(by IH)} \end{aligned}$$

As for Part 2, the argument is as follows:

$$\begin{aligned}\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^{-} \sim \psi &\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^{+} \psi \\ &\Leftrightarrow \psi \in \Gamma_0 && \text{(by IH)} \\ &\Leftrightarrow \sim\sim \psi \in \Gamma_0 && \text{(by Lemma 11.6)}\end{aligned}$$

Case 4. $\phi = \psi \to \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi \cup \Xi)$. Again, we consider Part 1 first:

($\Leftarrow$). If $\psi \to \chi \in \Gamma_0$, and $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$, then $\Gamma_0 \subseteq \Gamma_1$ so that $\psi \to \chi \in \Gamma_1$. Now, if $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^{+} \psi$, then, by the Induction Hypothesis, $\psi \in \Gamma_1$, and, by Lemma 11.9, $\chi \in \Gamma_1$. Applying the Induction Hypothesis one more time, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^{+} \chi$. Since $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$ was chosen arbitrarily, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^{+} \psi \to \chi$.

($\Rightarrow$). If $\psi \to \chi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\psi \to \chi \in \Delta_0$. But then, by Lemma 10.4, we get that $(\Gamma_0 \cup \{\psi\}, \{\chi\})$ must be non-trivial, and, clearly $Par(\Gamma_0 \cup \{\psi, \chi\}) \subseteq Par(\Gamma_0 \cup \Delta_0)$. Therefore, the parameter set $\Pi' := (\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \{\psi, \chi\})$ must be countably infinite. Now, we partition Π' into two further countably infinite sets, Π_0 and Π_1. By Lemma 12, we can find a $(\Sigma, Par(\Gamma_0 \cup \{\psi, \chi\}) \cup \Pi_0)$-appropriate bi-set $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0 \cup \{\psi\}, \{\chi\})$. For this latter bi-set, we have that $(\Pi \cup \Xi) \setminus Par(\Gamma_1 \cup \Delta_1) = \Pi_1$, so that (Γ_1, Δ_1) is also $(\Sigma, \Pi \cup \Xi)$-nice and thus in $W_{(\Gamma,\Delta,\Xi)}$. Moreover, we must have $\Gamma_1 \supseteq \Gamma_0$ so that $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$. Next, we have $\psi \in \Gamma_1$ so that the Induction Hypothesis implies that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^{+} \psi$. Finally, we have $\chi \in \Delta_1$, hence also $\chi \notin \Gamma_1$ by the non-triviality of (Γ_1, Δ_1), whence further $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \not\models^{+} \chi$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \not\models^{+} \psi \to \chi$.

Part 2 of the Lemma in this Case is similar to Part 1.

Case 5. $\phi = \forall x \psi$ for some $\psi \in L_x(\Sigma, \Pi \cup \Xi)$. We consider Part 1 first:

($\Leftarrow$). If $\forall x \psi \in \Gamma_0$, and $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta,\Xi)} (\Gamma_1, \Delta_1)$, then $\Gamma_0 \subseteq \Gamma_1$ so that $\forall x \psi \in \Gamma_1$. Now, if $a \in D_{(\Gamma,\Delta,\Xi)}(\Gamma_1, \Delta_1) = Par(\Gamma_1 \cup \Delta_1)$, then, by Lemma 11.11, $\psi[a/x] \in \Gamma_1$, and further, by the Induction Hypothesis, $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \models^{+} \psi[a/x]$. Since $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$ and $a \in D_{(\Gamma,\Delta,\Xi)}(\Gamma_1, \Delta_1)$ were chosen arbitrarily, we get that $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^{+} \forall x \psi$.

($\Rightarrow$). If $\forall x \psi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\forall x \psi \in \Delta_0$. We know that $(\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \Delta_0)$ is infinite, therefore, we can choose any parameter a in this set. Now Lemma 10.6 tells us that $(\Gamma_0, \{\psi[a/x]\})$ must be non-trivial, and, clearly $Par(\Gamma_0 \cup \{\psi[a/x]\}) \subseteq Par(\Gamma_0 \cup \Delta_0) \cup \{a\}$. Therefore, the parameter set $\Pi' := (\Pi \cup \Xi) \setminus Par(\Gamma_0 \cup \{\psi[a/x]\})$ must be countably infinite. We partition Π' into two further countably infinite sets, Π_0 and Π_1. By Lemma 12, we can find a $(\Sigma, Par(\Gamma_0 \cup \{\psi[a/x]\}) \cup \Pi_0)$-appropriate bi-set $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0, \{\psi[a/x]\})$.

For this latter bi-set, we have that $(\Pi \cup \Xi) \setminus Par(\Gamma_1 \cup \Delta_1) = \Pi_1$, so that (Γ_1, Δ_1) is also $(\Sigma, \Pi \cup \Xi)$-nice and thus in $W_{(\Gamma,\Delta,\Xi)}$. Moreover, we must have $\Gamma_1 \supseteq \Gamma_0$ so that $(\Gamma_1, \Delta_1) \geq_{(\Gamma,\Delta,\Xi)} (\Gamma_0, \Delta_0)$. Next, we have $\psi[a/x] \in \Delta_1$, hence also $\psi[a/x] \notin \Gamma_1$ by the non-triviality of (Γ_1, Δ_1), whence further $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_1, \Delta_1) \not\models^+ \psi[a/x]$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \not\models^+ \forall x\psi$.

We know turn to Part 2 of the Lemma, and reason as follows:

$$\begin{aligned}
\sim \forall x\psi \in \Gamma_0 &\Leftrightarrow (\exists a \in Par(\Gamma_0 \cup \Delta_0))(\sim \psi[a/x] \in \Gamma_0) && \text{(by Lemma 11.8)}\\
&\Leftrightarrow (\exists a \in Par(\Gamma_0 \cup \Delta_0)(\mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \psi[a/x]) && \text{(by IH)}\\
&\Leftrightarrow \mathcal{M}_{(\Gamma,\Delta,\Xi)}, (\Gamma_0, \Delta_0) \models^- \forall x\psi
\end{aligned}$$

Case 6. $\phi = \exists x\psi$ for some $\psi \in L_x(\Sigma, \Pi \cup \Xi)$. Similar to Case 5. □

Theorem 2. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.*

Proof. We argue by contraposition. If $\Gamma \nvdash \phi$, then the bi-set $(\Gamma, \{\phi\})$ must be non-trivial. But then, choose two infinitely countable parameter sets Ξ_0 and Ξ_1 such that $\{\Pi, \Xi_0, \Xi_1\}$ forms a pairwise disjoint family of sets. Then we can find, by Lemma 12, a $(\Pi \cup \Xi_0)$-appropriate bi-set $(\Gamma', \Delta') \supseteq (\Gamma, \{\phi\})$; (Γ', Δ') is also $(\Pi \cup \Xi_0 \cup \Xi_1)$-nice. We clearly have $\phi \in \Delta'$, so also $\phi \notin \Gamma'$ by the non-triviality of (Γ', Δ'). Now Lemma 13 implies that we have both $\mathcal{M}_{(\Gamma',\Delta',\Xi_1)}, (\Gamma', \Delta') \models^+ \Gamma' \supseteq \Gamma$ and $\mathcal{M}_{(\Gamma',\Delta',\Xi_1)}, (\Gamma', \Delta') \not\models^+ \phi$. Therefore, $\Gamma \not\models \phi$ as desired. □

4 Hilbert-style axiomatizations of QC_{CD} and $\mathsf{QC3}_{CD}$

In order to obtain the axiomatization of QC_{CD}, we extend the set of axioms with the parametrized sentences which are instances of the following scheme:

$$\forall x(\phi \vee \psi) \to (\phi \vee \forall x\psi) \tag{A17}$$

We do not need to require separately that $x \notin FV(\phi)$ since this already follows from the fact that $\forall x(\phi \vee \psi) \to (\phi \vee \forall x\psi)$ is a parametrized sentence.

We can then define the notion of $(\Sigma, \Pi)_{CD}$-deduction and the deducibility relation $\vdash_{CD}$ for this extended system. Lemmas 6–9 then extend to our amended deduction and deducibility notions and the only change in the proofs is that one needs to mention the extended set of axioms in place of the set of axioms for QC.

Similarly, we can now prove the following theorem in almost the same way as Theorem 1:

Theorem 3. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \vdash_{CD} \phi$, then $\Gamma \models_{\mathbb{CD}} \phi$.*

Turning now to the converse of Theorem 3 in the countable case, we observe, first, that we need to extend the notion of an appropriate bi-set. More precisely, given a bi-set $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, we say that (Γ, Δ) is $\forall$-complete iff for every $\forall x\phi \in L_\emptyset(Sign(\Gamma \cup \Delta), Par(\Gamma \cup \Delta))$ such that $\forall x\phi \in \Delta$, there exists an $a \in Par(\Gamma \cup \Delta)$ such that $\phi[a/x] \in \Delta$. A bi-set (Γ, Δ) is then called $(\Sigma, \Pi)_{CD}$-appropriate iff it is (Σ, Π)-appropriate (in the sense of the previous section, except that non-triviality is understood relative to $\vdash_{CD}$) and $\forall$-complete.

Next, we need to extend the lemma on non-trivial bi-sets:

Lemma 14. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be CD-non-trivial. Then all of the statements in Lemma 10 hold, and, in addition, it is true that, if $\forall x\phi \in L_\emptyset(\Sigma, \Pi)$, and $a \in Par \setminus \Pi$, then one of $(\Gamma \cup \{\forall x\phi\}, \Delta)$, $(\Gamma, \Delta \cup \{\forall x\phi, \phi[a/x]\})$ is CD-non-trivial.*

Proof. The proof of Lemma 10 can be simply repeated replacing the non-triviality everywhere with the CD-non-triviality. As for the additional part, assume that both $(\Gamma \cup \{\forall x\phi\}, \Delta)$ and $(\Gamma, \Delta \cup \{\forall x\phi, \phi[a/x]\})$ are CD-trivial. Then, by Lemma 10.2, $(\Gamma, \Delta \cup \{\forall x\phi\})$ must be CD-non-trivial. Let $\emptyset \neq \Delta' \Subset \Delta$ be such that, wlog, $\Gamma \vdash_{CD} \phi[a/x] \vee (\forall x\phi \vee \bigvee \Delta')$. Since $x \notin FV(\forall x\phi \vee \bigvee \Delta')$, Lemma 1.2 implies that $\Gamma \vdash_{CD} (\phi \vee (\forall x\phi \vee \bigvee \Delta'))[a/x]$. By Corollary 5.2, we must have then $\Gamma \vdash_{CD} \forall x(\phi \vee (\forall x\phi \vee \bigvee \Delta'))$, whence, by (A17) and (MP), $\Gamma \vdash_{CD} \forall x\phi \vee \forall x\phi \vee \bigvee \Delta'$. By (T6), we get, next, that $\Gamma \vdash_{CD} \forall x\phi \vee \bigvee \Delta'$, which contradicts the CD-non-triviality of $(\Gamma, \Delta \cup \{\forall x\phi\})$. □

We note, furthermore, that Lemma 11 (on appropriate bi-sets) carries over to CD-appropriate bi-sets without any non-trivial change in the proof. Next, we show that in the countable case any CD-non-trivial bi-set can be extended to a CD-appropriate one over an extended set of parameters.

Lemma 15. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be CD-non-trivial. Then, for every $\Xi \subseteq Par$ disjoint from Π and such that $|\Xi| = \omega$, there exists a $(\Sigma, \Pi \cup \Xi)_{CD}$-appropriate bi-set (Γ', Δ') such that $(\Gamma', \Delta') \supseteq (\Gamma, \Delta)$.*

Proof. We adapt the proof of Lemma 12 to our current environment. Again, let $\{a_n \mid n \in \omega\}$ be an enumeration of Ξ, and let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma, \Pi \cup \Xi)$. We now define a countably infinite increasing chain of CD-non-trivial bi-sets

$$(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$$

by setting $(\Gamma, \Delta) := (\Gamma_0, \Delta_0)$, and for any $k \in \omega$, if ψ_k is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$, then we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\psi_k\}, \Delta_k), & \text{if } (\Gamma_k \cup \{\psi_k\}, \Delta_k) \text{ is } CD\text{-non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\psi_k\}), & \text{otherwise.} \end{cases}$$

For the remaining cases, we will use the subsets of ω of the form $\nu[\Gamma_k, \Delta_k, \psi_k]$ as defined in the proof of Lemma 12.

Namely, in case ψ_k has the form $\exists x\phi$, we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ if } m = \min \nu[\Gamma_k, \Delta_k, \exists x\phi] \\ \qquad\qquad \text{and } (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ is CD-non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\exists x\phi\}), \text{ otherwise.} \end{cases}$$

Finally, in case ψ_k has the form $\forall x\phi$, we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\forall x\phi\}, \Delta_k), & \text{if } (\Gamma_k \cup \{\forall x\phi\}, \Delta_k) \text{ is } CD\text{-non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}), & \text{if } m = \min \nu[\Gamma_k, \Delta_k, \forall x\phi], \text{ otherwise.} \end{cases}$$

The rest of the argument is exactly as in the proof of Lemma 12 except that we need to add the reference to Lemma 14 in order to show that in the latter case the bi-set remains CD-non-trivial. Another addition is the argument for $\forall$-completeness of the resulting set $(\Gamma', \Delta') := (\bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n)$ which is similar to the one for the $\exists$-completeness given in the proof of Lemma 12. □

Before we start with the construction of the canonical model, we need one final ingredient which was not necessary in the case of QC but which is normally required as long as the domains are assumed to be constant. We formulate this additional argumentative ingredient in the following lemma:

Lemma 16. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, let $(\Gamma, \Delta) \subseteq (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be $(\Sigma, \Pi)_{CD}$-appropriate, and let $(\Gamma_0, \Delta_0) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ be such that $(\Gamma \cup \Gamma_0, \Delta_0)$ is CD-non-trivial. Then there exists a $(\Sigma, \Pi)_{CD}$-appropriate bi-set (Γ', Δ') such that $(\Gamma', \Delta') \supseteq (\Gamma \cup \Gamma_0, \Delta_0)$.*

Proof. Once again we re-use the construction from Lemma 15 with a further additional twist. Namely, we let $\{a_n \mid n \in \omega\}$ be an enumeration of Π and we let $\{\psi_n \mid n \in \omega\}$ be an enumeration of $L_\emptyset(\Sigma, \Pi)$. But this time we define a countably infinite increasing chain of finite bi-sets

$$(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$$

such that, for every $k \in \omega$, the bi-set $(\Gamma \cup \Gamma_k, \Delta_k)$ is CD-non-trivial. In this chain, (Γ_0, Δ_0) is given in the formulation of the lemma and for any $k \in \omega$, if ψ_k is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$, then we set:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\psi_k\}, \Delta_k), & \text{if } (\Gamma \cup \Gamma_k \cup \{\psi_k\}, \Delta_k) \text{ is } CD\text{-non-trivial} \\ (\Gamma_k, \Delta_k \cup \{\psi_k\}), & \text{otherwise.} \end{cases}$$

In case ψ_k has the form $\exists x\phi$, we set:

$$\mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] := \{n \in \omega \mid a_n \in \Pi \mid (\Gamma \cup \Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k) \text{ is } CD\text{-non-trivial}\}$$

and we define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\exists x\phi, \phi[a_m/x]\}, \Delta_k), \\ \qquad\qquad \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] \neq \emptyset \text{ and } m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] \\ (\Gamma_k, \Delta_k \cup \{\exists x\phi\}), \text{ if } \mu_\Gamma[\Gamma_k, \Delta_k, \exists x\phi] = \emptyset. \end{cases}$$

Finally, in case ψ_k has the form $\forall x\phi$, we set

$$\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] := \{n \in \omega \mid a_n \in \Pi \mid (\Gamma \cup \Gamma_k, \Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}) \text{ is } CD\text{-non-trivial}\}$$

and we define:

$$(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{\forall x\phi\}, \Delta_k), \text{ if } \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] = \emptyset \\ (\Gamma_k, \Delta_k \cup \{\forall x\phi, \phi[a_m/x]\}), \\ \qquad\qquad \text{if } \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] \neq \emptyset \text{ and } m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi]. \end{cases}$$

We show that the chain $(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_n, \Delta_n) \subseteq \ldots$ is well-defined and that, for every $k \in \omega$, we have $(\Gamma_k, \Delta_k) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$ and the bi-set $(\Gamma \cup \Gamma_k, \Delta_k)$ is CD-non-trivial.

This claim is obviously true when $k = 0$. If $k = r + 1$, and the claim is true for (Γ_r, Δ_r), then $(\Gamma_{r+1}, \Delta_{r+1})$ is well-defined by the Induction Hypothesis and we clearly have $(\Gamma_{r+1}, \Delta_{r+1}) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$. It remains to show the CD-non-triviality, and, in doing so, we have to consider the three cases in our definition:

Case 1. ψ_r is neither of the form $\exists x\phi$ nor of the form $\forall x\phi$. Then the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ follows from (the CD-version of) Lemma 10.2.

Case 2. ψ_r has the form $\exists x\phi$. If $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] \neq \emptyset$, then we are done. Otherwise, we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] = \emptyset$. If now $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\exists x\phi\})$ is CD-trivial, then by (the CD-version of) Lemma 10.2, the bi-set $(\Gamma \cup \Gamma_r \cup \{\exists x\phi\}, \Delta_r)$ must be CD-non-trivial. On the other hand, since $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] = \emptyset$, we must have, wlog,

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \cup \{\exists x\phi, \phi[a_m/x]\} \vdash_{CD} \bigvee \Delta_r) \tag{1}$$

We now reason as follows:

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\exists x\phi \wedge \phi[a_m/x] \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r) \text{ (by (1) and Cor. 5.1)} \quad (2)$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\phi[a_m/x] \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r) \text{ (by (2), (T4) and (DR1))} \quad (3)$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} \phi[a_m/x] \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \text{ (by (3), (T2) and (DR1))} \quad (4)$$

Now, since $(\Gamma_r, \Delta_r) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, Lemma 1.2 implies that

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x]) \quad (5)$$

Since we clearly have $(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \in L_\emptyset(\Sigma, \Pi)$ for every $m \in \omega$, the (CD-version of) Lemma 11.1 allows us to infer that:

$$(\forall m \in \omega)((\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \in \Gamma) \quad (6)$$

By the CD-non-triviality of (Γ, Δ), it follows, further, that:

$$(\forall m \in \omega)((\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r))[a_m/x] \notin \Delta) \quad (7)$$

Finally, since $\{a_n \mid n \in \omega\}$ is an enumeration of Π, $\forall$-completeness of (Γ, Δ) implies that:

$$\forall x(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \notin \Delta \quad (8)$$

Applying the completeness of (Γ, Δ), we get that:

$$\Gamma \vdash_{CD} \forall x(\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r)) \quad (9)$$

Now it remains to apply (T7) and (T2) to get, successively:

$$\Gamma \vdash_{CD} \exists x\phi \to (\bigwedge \Gamma_r \to \bigvee \Delta_r) \quad (10)$$

and:

$$\Gamma \vdash_{CD} (\exists x\phi \wedge \bigwedge \Gamma_r) \to \bigvee \Delta_r \quad (11)$$

but the latter equation implies, by Corollary 5.1, that $\Gamma \cup \Gamma_r \cup \{\exists x\phi\} \vdash_{CD} \bigvee \Delta_r$ which is in contradiction with the CD-non-triviality of $(\Gamma \cup \Gamma_r \cup \{\exists x\phi\}, \Delta_r)$. The obtained contradiction shows that we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \exists x\phi] \neq \emptyset$, whence the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ easily follows.

Case 3. ψ_r has the form $\forall x\phi$. If $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] \neq \emptyset$, then we are done. Otherwise, we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] = \emptyset$. If now $(\Gamma \cup \Gamma_r \cup \{\forall x\phi\}, \Delta_r)$ is CD-trivial, then by (the CD-version of) Lemma 10.2, the bi-set $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\forall x\phi\})$ must be CD-non-trivial. On the other hand, since $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] = \emptyset$, we must have, wlog,

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \vdash_{CD} \forall x\phi \vee \phi[a_m/x] \vee \bigvee \Delta_r) \tag{12}$$

$$(\forall m \in \omega)(\Gamma \cup \Gamma_r \vdash_{CD} \phi[a_m/x] \vee \bigvee \Delta_r) \quad \text{(by (12), (T3), and (A15))} \tag{13}$$

$$(\forall m \in \omega)(\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \phi[a_m/x] \vee \bigvee \Delta_r) \quad \text{(by (13) and Cor. 5.1)} \tag{14}$$

Now, since $(\Gamma_r, \Delta_r) \Subset (L_\emptyset(\Sigma, \Pi), L_\emptyset(\Sigma, \Pi))$, Lemma 1.2 implies that

$$(\forall m \in \omega)(\Gamma \vdash_{CD} (\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x]) \tag{15}$$

Since we clearly have $(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \in L_\emptyset(\Sigma, \Pi)$ for every $m \in \omega$, the (CD-version of) Lemma 11.1 allows us to infer that:

$$(\forall m \in \omega)((\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \in \Gamma) \tag{16}$$

By the CD-non-triviality of (Γ, Δ), it follows, further, that:

$$(\forall m \in \omega)((\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r)[a_m/x] \notin \Delta) \tag{17}$$

Finally, since $\{a_n \mid n \in \omega\}$ is an enumeration of Π, $\forall$-completeness of (Γ, Δ) implies that:

$$\forall x(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r) \notin \Delta \tag{18}$$

Applying again the completeness of (Γ, Δ), we get that:

$$\Gamma \vdash_{CD} \forall x(\bigwedge \Gamma_r \to \phi \vee \bigvee \Delta_r) \tag{19}$$

Now it remains to apply (T8), (DR1), and (A17) to get, successively:

$$\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \forall x(\phi \vee \bigvee \Delta_r) \tag{20}$$

and:

$$\Gamma \vdash_{CD} \bigwedge \Gamma_r \to \forall x\phi \vee \bigvee \Delta_r \tag{21}$$

but the latter equation implies, by Corollary 5.1, that $\Gamma \cup \Gamma_r \vdash_{CD} \forall x\phi \vee \bigvee \Delta_r$ which is in contradiction with the CD-non-triviality of $(\Gamma \cup \Gamma_r, \Delta_r \cup \{\forall x\phi\})$. The obtained contradiction shows that we must have $\mu_\Gamma[\Gamma_r, \Delta_r, \forall x\phi] \neq \emptyset$, whence the CD-non-triviality of $(\Gamma \cup \Gamma_{r+1}, \Delta_{r+1})$ easily follows.

Having defined our chain of bi-sets, we set:

$$(\Gamma', \Delta') = (\Gamma \cup \bigcup_{n\in\omega} \Gamma_n, \bigcup_{n\in\omega} \Delta_n),$$

and we show that this latter bi-set satisfies the conditions of the Lemma arguing as in the proofs of Lemmas 12 and 15. For example, to show that (Γ', Δ') is $\forall$-complete, assume that $\forall x\phi \in \Delta' \subseteq L_\emptyset(\Sigma, \Pi)$. Then, for some $k \in \omega$, we must have $\forall x\phi = \psi_k$. Consider $(\Gamma_{k+1}, \Delta_{k+1})$. If $\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] = \emptyset$, then we must have $\forall x\phi \in \Gamma_{k+1} \subseteq \Gamma'$, which would contradict the non-triviality of (Γ', Δ'). Therefore, we must have $\mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi] \neq \emptyset$, but then also $\phi[a_m/x] \in \Delta_{k+1} \subseteq \Delta'$ for $m = \min \mu_\Gamma[\Gamma_k, \Delta_k, \forall x\phi]$. □

Our canonical model construction for QC_{CD} now looks as follows. Given a signature Σ, a parameter set Π, and a $(\Sigma, \Pi)_{CD}$-appropriate bi-set (Γ, Δ), we can define the following constant domain Σ-model $\mathcal{M}_{(\Gamma,\Delta)}$ by setting:

- $W_{(\Gamma,\Delta)} := \{(\Gamma', \Delta') \mid (\Gamma', \Delta') \text{ is } (\Sigma, \Pi)_{CD}\text{-appropriate}\}$.
- For $(\Gamma_0, \Delta_0), (\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta)}$, we have $(\Gamma_0, \Delta_0) \leq_{(\Gamma,\Delta)} (\Gamma_1, \Delta_1)$ iff $\Gamma_0 \subseteq \Gamma_1$.
- $U_{(\Gamma,\Delta)} := \Pi = D_{(\Gamma,\Delta)}(\Gamma_0, \Delta_0) = Par(\Gamma_0 \cup \Delta_0)$ for every $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$, $n \in \omega$, $P^n \in \Sigma = Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in \Pi^n = Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^+_{(\Gamma,\Delta)}(P, (\Gamma_0, \Delta_0))$ iff $P(\bar{a}_n) \in \Gamma_0$.
- For $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$, $n \in \omega$, $P^n \in \Sigma = Sign(\Gamma_0 \cup \Delta_0)$, and $\bar{a}_n \in \Pi^n = Par(\Gamma_0 \cup \Delta_0)^n$ we have $\bar{a}_n \in V^-_{(\Gamma,\Delta)}(P, (\Gamma_0, \Delta_0))$ iff $\sim P(\bar{a}_n) \in \Gamma_0$.

It is straightforward to show that $\mathcal{M}_{(\Gamma,\Delta)}$ is indeed a constant domain model of QC, and using the usual methods, a truth lemma can be shown for this model:

Lemma 17. *Let* (Γ, Δ) *be a* $(\Sigma, \Pi)_{CD}$*-appropriate bi-set. Then, for every* $(\Gamma_0, \Delta_0) \in W_{(\Gamma,\Delta)}$ *and every* $\phi \in L_\emptyset(\Sigma, \Pi)$ *it is true that:*

1. $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \models^+ \phi$ *iff* $\phi \in \Gamma_0$.
2. $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \models^- \phi$ *iff* $\sim \phi \in \Gamma_0$.

Proof. We prove both parts of the Lemma simultaneously by induction on the construction of $\phi \in L_\emptyset(\Sigma, \Pi)$. The proof for the induction basis and for the induction steps associated with $\wedge$, $\vee$, and $\sim$ are exactly as in the proof of Lemma 13. We consider the remaining cases:

Step. The following cases are possible.

Case 4. $\phi = \psi \to \chi$ for some $\psi, \chi \in L_\emptyset(\Sigma, \Pi)$. We consider Part 1 first:

($\Rightarrow$). If $\psi \to \chi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\psi \to \chi \in \Delta_0$. But then, by (the CD-version of) Lemma 10.4, we get that $(\Gamma_0 \cup \{\psi\}, \{\chi\})$ must be CD-non-trivial. Next, by Lemma 16, there must be a $(\Gamma_1, \Delta_1) \in W_{(\Gamma,\Delta)}$ such that $(\Gamma_1, \Delta_1) \supseteq (\Gamma_0 \cup \{\psi\}, \{\chi\})$. Now, $\psi \in \Gamma_1$ implies, by the Induction Hypothesis, that $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_1, \Delta_1) \models^+ \psi$. On the other hand, we have $\chi \in \Delta_1$, hence also $\chi \notin \Gamma_1$ by the CD-non-triviality of (Γ_1, Δ_1), whence, further, $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_1, \Delta_1) \not\models^+ \chi$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \psi \to \chi$.

The proofs for the $(\Leftarrow)$-part and for Part 2 are as in Lemma 13.

Case 5. $\phi = \forall x \psi$ for some $\psi \in L_x(\Sigma, \Pi)$. We consider Part 1 first:

$(\Rightarrow)$. If $\forall x\psi \notin \Gamma_0$, then, by the completeness of (Γ_0, Δ_0), we get that $\forall x\psi \in \Delta_0$. Therefore, by $\forall$-completeness of (Γ_0, Δ_0), we must have $\psi[a/x] \in \Delta$ for some $a \in \Pi$. By the CD-non-triviality of (Γ_0, Δ_0), it follows that $\psi[a/x] \notin \Gamma_0$, whence further $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \psi[a/x]$ by the Induction Hypothesis. But then $\mathcal{M}_{(\Gamma,\Delta)}, (\Gamma_0, \Delta_0) \not\models^+ \forall x\psi$.

Again, the proofs for the $(\Leftarrow)$-part and for Part 2 are as in Lemma 13, and the case of the existential quantifier is parallel to Case 5. □

We now formulate and prove the converse of Theorem 3 for the countable case:

Theorem 4. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \models_{\mathbb{CD}} \phi$ then $\Gamma \vdash_{CD} \phi$.*

Proof. Again, we argue by contraposition. If $\Gamma \nvdash_{CD} \phi$, then the bi-set $(\Gamma, \{\phi\})$ must be CD-non-trivial. But then, choose an infinitely countable parameter set Ξ disjoint from Π. We can find, by Lemma 15, a $(\Sigma, \Pi \cup \Xi)_{CD}$-appropriate bi-set $(\Gamma', \Delta') \supseteq (\Gamma, \{\phi\})$. We clearly have $\phi \in \Delta'$, so also $\phi \notin \Gamma'$ by the CD-non-triviality of (Γ', Δ'). Now Lemma 17 implies that we have both $\mathcal{M}_{(\Gamma',\Delta')}, (\Gamma', \Delta') \models^+ \Gamma' \supseteq \Gamma$ and $\mathcal{M}_{(\Gamma',\Delta')}, (\Gamma', \Delta') \not\models^+ \phi$. Therefore, $\Gamma \not\models_{\mathbb{CD}} \phi$ as desired. □

It is now easy to see that one can obtain a complete axiomatization for $\mathsf{QC3}_{CD}$ by extending the axiomatization for QC_{CD} with the following additional axiom schema:

$$\phi \vee \sim \phi \tag{A18}$$

Re-using, with a slight modification, the previous definitions of this sort, one can define the deducibility relation $\vdash_{C3CD}$ and prove the following theorem:

Theorem 5. *Let Σ be a signature, let $\Pi \subseteq Par$ be a set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \vdash_{C3CD} \phi$, then $\Gamma \models_{\mathbb{C}3 \cap \mathbb{CD}} \phi$.*

Moreover, by repeating the series of constructions leading to Theorem 4 above, it is straigtforward to check that the presence of (A18) in our axiomatization guarantees that the respective canonical model is in $\mathbb{C}3$. Proceeding in this way, one also arrives at the corresponding completeness theorem for the countable case:

Theorem 6. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. If $\Gamma \models_{\mathbb{C}3\cap\mathbb{CD}} \phi$ then $\Gamma \vdash_{C3CD} \phi$.*

5 Comparison of $\mathsf{QC3}_{CD}$ with the systems $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$

The systems $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$ were introduced in [10], purely proof-theoretically, as the first-order extensions of $\mathsf{C3}$. Each of these two systems was given in two forms: first, in the form of a Hilbert-style calculus and then in the form of its (unlabelled) sequent counterpart. The two forms were shown in [10] to be equivalent in the sense that the derivability relations from a finite set of premises obtained in each of the two types of proof systems were shown to coincide for both $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$.

Since in the present paper we are focusing on the Hilbert-style axiomatizations of various first-order extensions of C, we will omit the discussion of sequent calculi introduced in [10]. As for the Hilbert-style calculi for $\mathsf{QC3}$ and $\mathsf{QC3}_{At}$, they are obtained by extending the axiomatization of QC by (A18) in the case of $\mathsf{QC3}$ and by the following restriction of (A18) in the case of $\mathsf{QC3}_{At}$:

$$\phi \vee \sim \phi \qquad\qquad \text{for } \phi \text{ atomic} \qquad\qquad (\text{A18}_{At})$$

It is clear that $\mathsf{QC3}_{At}$ can be shown to axiomatize the logic of $\mathsf{C3}$-models by a trivial modification of the completeness proof given for QC in Section 3 of the present paper. Denoting by $\vdash_{C3}$ the deducibility relation induced by $\mathsf{QC3}_{At}$, we get the following

Theorem 7. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_\emptyset(\Sigma, \Pi)$. Then $\Gamma \models_{\mathbb{C}3} \phi$ iff $\Gamma \vdash_{C3} \phi$.*

Proof (a sketch). We repeat the proofs of Theorem 1 and 2, noting that the canonical Σ-model $\mathcal{M}_{(\Gamma,\Delta,\Xi)}$ constructed for a given (Σ, Π)-appropriate bi-set (Γ, Δ), and a given countably infinite $\Xi \subseteq Par$ disjoint from Π must be in $\mathbb{C}3$, due to the presence of (A18_{At}) in our system and by Lemma 11.4. □

Incidentally, the observation made in the proof of Theorem 7 also implies that one can equivalently axiomatize $\mathsf{QC3}_{CD}$ by replacing (A18) with (A18_{At}). Of course, one could also directly infer the remaining cases of (A18) in the axiomatization of $\mathsf{QC3}_{CD}$ based on (A18_{At}) arguing by induction on the construction of a parametrized sentence, but the semantic argument provides us with a shortcut to this result as well.

The question of the right semantics for $\mathsf{QC3}$ is more tricky. Example 2 shows that $\mathsf{QC3}$ is strictly stronger than the logic of $\mathsf{C3}$-models which we just recognized as $\mathsf{QC3}_{At}$. On the other hand, Theorem 6 above shows that $\mathsf{QC3}$ must be a subsystem of $\mathsf{QC3}_{CD}$. There remains the question whether this subsystem is proper.

First of all, it is clear that, seen from a semantical point of view, QC3 must be the logic of QC3-complete models, where, for any given signature Σ, a Σ-model is QC3-complete iff for every $w \in W$ and every $\phi \in L_{\emptyset}(\Sigma, D_w)$ it is true that $\mathcal{M}, w \models^+ \phi$ or $\mathcal{M}, w \models^- \phi$. So let us denote by $\mathbb{QC}3$ the class of QC3-complete models and by $\vdash_{QC3}$ the deducibility relation induced by QC3. The corresponding completeness proof is obtained from the completeness proof for QC by a trivial modification very similar to the one required in the case of $\mathsf{QC3}_{At}$. In this way, we get that:

Theorem 8. *Let Σ be an at most countable signature, let $\Pi \subseteq Par$ be an at most countable set, and let $\Gamma \cup \{\phi\} \subseteq L_{\emptyset}(\Sigma, \Pi)$. Then $\Gamma \models_{\mathbb{QC}3} \phi$ iff $\Gamma \vdash_{QC3} \phi$.*

Proof (a sketch). Similar to Theorem 7. □

Now, Example 2 shows that $\mathbb{C}3 \not\subseteq \mathbb{QC}3$, and, on the other hand, Lemma 3 shows that $\mathbb{C}3 \cap \mathbb{CD} \subseteq \mathbb{QC}3$. The question is whether we also have $\mathbb{QC}3 \subseteq \mathbb{C}3 \cap \mathbb{CD}$. The following example can be used to show that this question must be answered in the negative:

Example 3. *Consider the signature $\Sigma = \{(p, 0), (Q, 1)\}$ and consider the following varying-domain Σ-model $\mathcal{M}$, where $W = \{1, 2\}$, $\leq$ is the natural order on W, $U = \{a, b\}$, $D(1) = \{a\}$, $D(2) = U$, $V^+(p, i) = 1$ iff $i = 2$, $V^-(p, i) = 1$ for all $i \in W$, and we have $V^+(Q, i) = \{a\}$ and $V^-(Q, i) = D(i)$ for all $i \in W$.*

The following lemma can then be shown to hold:

Lemma 18. *Let Σ and $\mathcal{M}$ be defined as in the Example 3. Then the following statements are true:*

1. *$\mathcal{M}$ is QC3-complete.*
2. *$\mathcal{M}, 1 \not\models^+ \forall x(p \vee Q(x)) \to (p \vee \forall x Q(x))$.*

Even though the model $\mathcal{M}$ of Example 3 is an obvious paraconsistent variant of a model often used to show that (A17) fails in intuitionistic logic, the proof of Lemma 18 requires a surprisingly careful and tiresome induction on the construction of the parametrized sentence. It is therefore relegated to Appendix C.

Lemma 18 shows that QC3 is a proper subsystem of $\mathsf{QC3}_{CD}$ (as long as the signature is not too small) since we must have $\mathcal{M}, 1 \models^+ \mathsf{QC3}$, yet $\mathcal{M}, 1 \not\models^+ \forall x(p \vee Q(x)) \to (p \vee \forall x Q(x)) \in \mathsf{QC3}_{CD}$.

It also shows that $\mathbb{C}3 \cap \mathbb{CD} \subsetneq \mathbb{QC}3$ (as long as the signature is not too small).

Finally, it shows that the frame correspondence theory in its usual form is not possible for QC3, since the class of QC3-complete models in the corresponding signature is neither good nor C3-good. Indeed, whereas $\mathcal{M}$ of Example 3 was shown to be $\mathsf{QC3}$-complete, this is not the case for the model $\mathcal{N} \in \mathbb{C}3$ which only differs from $\mathcal{M}$ in that $\mathcal{N}, 1 \not\models^- Q(a)$. That $\mathcal{N} \notin \mathbb{QC}3$ is evident from the fact that we have both $\mathcal{N}, 1 \not\models^+ \forall x Q(x)$ and $\mathcal{N}, 1 \not\models^- \forall x Q(x)$.

To sum up, we have shown that all the systems in the set $\{\mathsf{QC3}_{At}, \mathsf{QC3}, \mathsf{QC3}_{CD}\}$ are pairwise disjoint. Out of these three systems, $\mathsf{QC3}_{At}$ is complete relative to a $\mathsf{C3}$-good class of models, but suffers from the truth-value gap problem in that it fails to verify the general form of the law of excluded middle given by (A18); it is also inconvenient that the set of theorems of $\mathsf{QC3}_{At}$ is not closed for formula substitutions. The truth-value gap problem is avoided in $\mathsf{QC3}$, however, this system is complete relative to a class of models which, as is shown above, is not $\mathsf{C3}$-good and it is not clear how to supply $\mathsf{QC3}$ with a better semantics. Therefore, at least as long as a better candidate is not found and proposed, we are inclined to favor $\mathsf{QC3}_{CD}$ as the correct first-order version of the propositional logic $\mathsf{C3}$ since it is both complete relative to a $\mathsf{C3}$-good class of models and verifies the unrestricted version of the law of excluded middle which we take to be a distinctive mark of $\mathsf{C3}$-like systems.

We end this section with a brief discussion of constructive truth and constructible falsity properties in the first-order extensions of C, since this subject was also discussed in [10]. It is known that in the intuitionistic first-order logic, its characteristic constructive understanding of truth manifests itself in the following properties:

(DP) Disjunctive Property: for every signature Σ, if $\phi, \psi \in L_\emptyset(\Sigma)$ and $\phi \vee \psi$ is a theorem, then either ϕ or ψ is a theorem.

(EP) Existence Property: for every signature Σ, if $\exists x\phi \in L_\emptyset(\Sigma)$ and $\exists x\phi$ is a theorem then there exists an $a \in Par$ such that $\phi[a/x]$ is a theorem.

Whereas both (DP) and (EP) fail in classical logic, they are preserved in the first-order Nelson's logics, both $\mathsf{QN3}$ and $\mathsf{QN4}$; moreover, they are complemented in these logics by the following constructible falsity counterparts, showing that the treatment of falsehoods now also becomes constructive:

(DP_F) Negated Conjunction Property: for every signature Σ, if $\phi, \psi \in L_\emptyset(\Sigma)$ and $\sim (\phi \wedge \psi)$ is a theorem, then either $\sim \phi$ or $\sim \psi$ is a theorem.

(EP_F) Negated Universal Property: for every signature Σ, if $\sim \forall x\phi \in L_\emptyset(\Sigma)$ and $\sim \forall x\phi$ is a theorem then there exists an $a \in Par$ such that $\sim \phi[a/x]$ is a theorem.

The logic QC is known to have all the four properties in $\{DP, DP_F, EP, EP_F\}$.[6] It

[6] Apparently this was known at the time of writing [14], although, quite surprisingly, neither EP nor EP_F are mentioned there; on the other hand, the satisfaction of both DP and DP_F is established in [14, Proposition 2].

is easy to see that the same sort of arguments can be used to show that the said four properties of constructive truth and constructible falsity are still satisfied by QC_{CD}.

With the first-order extensions of C3 the situation is a little bit more tricky. Due to the presence of (A18$_{At}$) in all such systems, it is easy to see right away that both DP and DP_F must fail. However, a proof-theoretic argument given for [10, Theorem 6.5] shows that, surprisingly, both EP and EP_F are still satisfied by $\mathsf{QC3}_{At}$.

It remains to see whether this rather peculiar (although not completely unknown: see [11]) phenomenon persists when we extend $\mathsf{QC3}_{At}$ to QC3 and then further to $\mathsf{QC3}_{CD}$. The answer is, again, in the negative:

Proposition 1. *Both* QC3 *and* $\mathsf{QC3}_{CD}$ *fail every property in* $\{EP, EP_F\}$

Proof. Indeed, consider signature $\Sigma = \{(P, 1)\}$. Then the pure Σ-sentence $\exists x(\forall x P(x) \vee \sim P(x))$ is provable in both QC3 and $\mathsf{QC3}_{CD}$, as the following derivation in QC3 shows (where $a \in Par$ is chosen arbitrarily):

$$\sim P(a) \to (\forall x P(x) \vee \sim P(a)) \quad \text{by (A7)} \tag{22}$$
$$(\forall x P(x) \vee \sim P(a)) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{by (A16)} \tag{23}$$
$$\sim P(a) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (22)–(23) by (DR1)} \tag{24}$$
$$\exists x \sim P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (24) by (R}\exists\text{)} \tag{25}$$
$$\sim \forall x P(x) \to \exists x \sim P(x) \quad \text{by (A14)} \tag{26}$$
$$\sim \forall x P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (25)–(26) by (DR1)} \tag{27}$$
$$\forall x P(x) \to (\forall x P(x) \vee \sim P(a)) \quad \text{by (A6)} \tag{28}$$
$$\forall x P(x) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (23),(28) by (DR1)} \tag{29}$$
$$(\forall x P(x) \vee \sim \forall x P(x)) \to \exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (27),(29) by (A8)} \tag{30}$$
$$\forall x P(x) \vee \sim \forall x P(x) \quad \text{by (A18)} \tag{31}$$
$$\exists x(\forall x P(x) \vee \sim P(x)) \quad \text{from (30),(31) by (MP)} \tag{32}$$

However, $\forall x P(x) \vee \sim P(a)$ is not a theorem of $\mathsf{QC3}_{CD}$ for any $a \in Par$ (and hence also not a theorem of its proper subsystem QC3) as the following constant domain C3-model shows.

Indeed, let $\Sigma = \{(P, 1)\}$ and let Σ-model $\mathcal{M}$ be such that with $W = \{1\}$, $\leq = \{(1, 1)\}$, $U = D(1) = \{a, b\}$, $V^+(P, 1) = \{a\}$ and $V^-(P, 1) = \{b\}$. It is easy to see that we have both $\mathcal{M}, 1 \not\models^+ \sim P(a)$ and $\mathcal{M}, 1 \not\models^+ \forall x P(x)$. The preceding argument disproves EP for both QC3 and $\mathsf{QC3}_{CD}$; as for EP_F, it is enough to notice that the formula $\sim \forall x(\sim \forall x P(x) \wedge P(x))$ is provable by applying (A14), (A11) and (A9) to $\exists x(\forall x P(x) \vee \sim P(x))$. On the other hand, $\sim (P(a) \wedge \sim \forall x P(x))$ is not a theorem for any $a \in Par$ as is witnessed by the model $\mathcal{M}$ defined above. □

6 The peculiar quantifier Æ

Both intuitionistic logic and some of the logics inspired by it, display a very close parallelism between the interpretation of the implication connective and the interpretation of the universal quantifier. For example, in one typical description of the intuitionistic meaning of logical symbols (clearly paraphrasing the so-called Brouwer-Heyting-Kolmogorov interpretation) we can read that:

> The second group is composed of $\forall$, $\to$, and $\neg$. A proof of $\forall x A(x)$ is a construction of which we can recognize that, when applied to any number n, it yields a proof of $A(\bar{n})$. Such a proof is therefore an operation that carries natural numbers into proofs. A proof of $A \to B$ is a construction of which we can recognize that, applied to any proof of A, it yields a proof of B. Such a proof is therefore an operation carrying proofs into proofs.
>
> (M. Dummett — [2, p. 8])

We see that $\to$ and $\forall$ are grouped together in that they both refer to a general construction producing proofs, the one out of (other) proofs, the other out of objects in the domain of discourse, which, in the example at hand, are natural numbers. The difference between the two constructions consists, first of all, in the input allowed by each of them. And this difference is not that big, since both natural numbers and proofs are, according to intuitionism, just two varieties of constructions, and one of this varieties can serve as a representative of the other one as the goedelization technique has taught us.

The other obvious difference is of course that the implicational construction returns a proof of one and the same sentence for every possible input, whereas the universal quantifier construction each time returns a proof of a different substitution instance based on the input. This difference is much more serious and we are not going to downplay it, although it does not cancel the objectively existing close parallelism between the two constructions.

This close parallelism is also reflected in the Kripke semantics for intuitionistic logic by the coincidence of the quantifier patterns in the corresponding clauses in the definition of the satisfaction relation. These clauses can be given, in view of the notational conventions accepted in this paper, as follows:

$$\mathcal{M}, w \models \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \models \phi \Rightarrow \mathcal{M}, v \models \psi)$$
$$\mathcal{M}, w \models \forall x \psi \Leftrightarrow (\forall v \geq w)(\forall a)(a \in D_v \Rightarrow \mathcal{M}, v \models \psi[a/x])$$

The introduction of Nelson's logic made it necessary to conceive of the falsification conditions for connectives and quantifiers as something possibly different from a mere negation of verification conditions. Thus, although the clauses above were still accepted for the definition of the verification relation $\models^+$, the conditions for falsifying the implications and universally quantified sentences had to be given independently. But also in this extension of intuitionistic logic the parallelism between the implication and the universal quantifier remained untouched, as is evident from the formulation of these conditions used in both QN3 and QN4 (again, adapted to our notational conventions):

$$\mathcal{M}, w \models^- \phi \to \psi \Leftrightarrow \mathcal{M}, w \models^+ \phi \text{ and } \mathcal{M}, w \models^- \psi$$
$$\mathcal{M}, w \models^- \forall x\psi \Leftrightarrow (\exists a)(a \in D_w \text{ and } \mathcal{M}, w \models^- \psi[a/x])$$

One could rephrase the idea behind these stipulations along the lines of the BHK approach to Nelson's logic by saying that a falsification of a conditional sentence consists in the fact that a proof of an antecedent has been constructed, along with a refutation of a consequent. Similarly, a falsification of a quantified sentence means that an object has been constructed, along with a refutation of a substitution instance of the quantified formula induced by this object.

Now, in QC as well as in the other Nelsonian extensions of C considered in this paper, this parallelism of the falsification conditions between $\to$ and $\forall$ appears to be lost in that we have:

$$\mathcal{M}, w \models^- \phi \to \psi \Leftrightarrow (\forall v \geq w)(\mathcal{M}, v \models^+ \phi \Rightarrow \mathcal{M}, v \models^- \psi),$$

whereas the falsification clause for the universal quantifier remains Nelsonian. Through the BHK lens, the matter looks as if we are now saying that a proper refutation of a conditional sentence must be a general construction, which, given a proof of the antecedent, spits out a refutation of the consequent (and does that recognizably, as M. Dummett would probably insist). However, were we to think of the possible refutations of the universally quantified sentences along the same lines, we would probably have to say that a proper refutation of a universally quantified sentence must be a general construction, which, given a construction of a possible object in our domain, recognizably returns a refutation of the substitution instance of the quantified formula induced by this object. It is natural to think that a formal explication of this idea may have looked as something like this:

$$\mathcal{M}, w \models^- \forall x\phi \Leftrightarrow (\forall v \geq w)(\forall a)(a \in D_v \Rightarrow \mathcal{M}, v \models^- \phi[a/x]).$$

However, it is now evident that, in doing so, we are just ascribing to the universal quantifier the falsification condition borrowed from the Nelsonian existential quantifier (also used in the semantics of QC).

We leave it to the reader to judge whether the idea of keeping the interpretations of $\forall$ and $\to$ bound together also in the first-order extensions of C has any intuitive appeal.[7] In the present paper, we confine ourselves to pointing out some of the formal consequences of realizing this idea by having a quantifier with the verification clause borrowed from the Nelsonian $\forall$ and the verification clause borrowed from the Nelsonian $\exists$. We will denote this quantifier by Æ and will assign it the following semantics:

$$\mathcal{M}, w \models^{+} \text{Æ}x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models \phi[a/x])$$
$$\mathcal{M}, w \models^{-} \text{Æ}x\phi \Leftrightarrow (\forall v \geq w)(\forall a \in D_v)(\mathcal{M}, v \models^{-} \phi[a/x]).$$

One very interesting property of Æ is that it commutes with the strong negation, that is to say, the following principle becomes valid:

$$\sim \text{Æ}x\phi \leftrightarrow \text{Æ}x \sim \phi \tag{A19}$$

One may also express this property of Æ by saying that this quantifier is "self-dual". It is also clear that if we simply want to extend with Æ the language of any system in the set $\{\mathsf{QC}, \mathsf{QC}_{CD}\}$, then we can obtain a sound and complete (in the countable case) axiomatization for such an extension by simply adding the following two schemas to the list of its axioms:

$$\text{Æ}x\phi \leftrightarrow \forall x\phi \tag{A20}$$
$$\sim \text{Æ}x\phi \leftrightarrow \sim \exists x\phi \tag{A21}$$

The situation is somewhat different if we wish to have Æ as the only quantifier in our language. In this case, given an axiomatization for any system in $\{\mathsf{QC}, \mathsf{QC}_{CD}\}$, one has to omit the axioms (A13)–(A17) together with the rules (R$\forall$) and (R$\exists$), and replace them with (A19) and the following Æ-analogues of (A15), (A17) and (R$\forall$), respectively:

$$\text{Æ}x\theta \to \theta[c/x] \tag{A15$_{\text{Æ}}$}$$
$$\text{Æ}x(\phi \vee \psi) \to (\phi \vee \text{Æ}x\psi) \tag{A17$_{\text{Æ}}$}$$
$$\text{From } \phi \to \theta[c/x] \text{ infer } \phi \to \text{Æ}x\theta \tag{R$\forall_{\text{Æ}}$}$$

In this way, we get two additional systems $\mathsf{C}(\text{Æ})$ and $\mathsf{C}_{CD}(\text{Æ})$.

The soundness and completeness proofs for these new systems are simpler versions of the proofs given in the earlier sections of this paper for their Nelsonian

[7] The reader may usefully compare our discourse with the attempt at "connexivization" of other propositional connectives besides $\to$ and $\sim$ in [4].

analogues. For example, in the completeness proof of $\mathsf{C}(\text{Æ})$, we no longer need to require that appropriate (and nice) bi-sets are $\exists$-complete, moreover, we no longer need several auxiliary statements like Lemma 10.8 and the second case in the main construction given in the proof of Lemma 12 is no longer relevant. These simplifications also apply to $\mathsf{C}_{CD}(\text{Æ})$. However, in the case of $\mathsf{C}_{CD}(\text{Æ})$, we will still need both the $\forall$-completeness (more precisely, its Æ-analogue) and Case 3 in the main construction given in the proofs of the statements like Lemma 15 and 16. The rest of the argument is basically the same as for the corresponding Nelsonian systems.

The introduction of Æ into first-order extensions of $\mathsf{C3}$, however, can only be easily done in the case of $\mathsf{QC3}_{At}$, where Æ can function both as an addition to the set of Nelsonian quantifiers and as the only quantifier in the same fashion as for QC. In the case of $\mathsf{QC3}$ one needs to further amend its already non-standard semantics and speak of the ($\mathsf{QC3}$+Æ)-complete models and ($\mathsf{C3}$+Æ)-complete models depending on whether we add Æ together with the set of Nelsonian quantifiers or alone. In this case ($\mathsf{QC3}$+Æ)-complete (resp. ($\mathsf{C3}$+Æ)-complete) models are the models that never display truth-value gaps for the parametrized sentences in the language based on $\{\wedge, \vee, \sim, \to, \forall, \exists, \text{Æ}\}$ (resp. $\{\wedge, \vee, \sim, \to, \text{Æ}\}$) as the set of logical symbols.

We have seen in Section 5, that the class of $\mathsf{QC3}$-complete models is not closed for the models based on the same underlying frame; the same clearly holds for the classes of ($\mathsf{C3}$+Æ)-complete models and ($\mathsf{QC3}$+Æ)-complete models. Indeed, the model $\mathcal{M}$ constructed in the proof of Proposition 1 is neither ($\mathsf{C3}$+Æ)-complete nor ($\mathsf{QC3}$+Æ)-complete since we have both $\mathcal{M}, 1 \not\models^{+} \text{Æ}xP(x)$ and $\mathcal{M}, 1 \not\models^{-} \text{Æ}xP(x)$. However, the model $\mathcal{M}'$ which is only different from $\mathcal{M}$ in that we have $\mathcal{M}', 1 \models^{+} P(b)$ is easily shown to be both ($\mathsf{C3}$+Æ)-complete and ($\mathsf{QC3}$+Æ)-complete. The following lemma provides the main stepping stone to the latter claim:

Lemma 19. *Denote by $L^{\text{Æ}}$ the language based on $\{\wedge, \vee, \sim, \to, \forall, \exists, \text{Æ}\}$ as the set of logical symbols. Let $x \in Var$, let $\Sigma = \{(P, 1)\}$ and let Σ-model $\mathcal{M}'$ be such that with $W = \{1\}$, $\leq = \{(1, 1)\}$, $U = D(1) = \{a, b\}$, $V^{+}(P, 1) = \{a, b\}$ and $V^{-}(P, 1) = \emptyset$. Then, for all $\phi \in L^{\text{Æ}}_{x}(\Sigma, U)$ and for every $\circ \in \{+, -\}$ it is true that:*

$$\mathcal{M}', 1 \models^{\circ} \phi[a/x] \Leftrightarrow \mathcal{M}', 1 \models^{\circ} \phi[b/x].$$

Proof. By induction on the construction of $\phi[a/x]$. Both the basis and the induction step cases for the propositional connectives are straightforward (for the implication case, note that our model consists of a single state). We treat the quantifier cases.

Case 1. $\phi[a/x] = \forall y\psi[a/x]$. We may assume, wlog, that $y \neq x$, and we reason as follows:

(Part 1). We have $\mathcal{M}', 1 \models^{+} \phi[a/x]$ iff $\mathcal{M}', 1 \models^{+} \psi[a/x, a/y] \wedge \psi[a/x, b/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^{+} \psi[a/y, a/x] \wedge \psi[b/y, a/x]$, iff, by the Induction Hypothesis,

$\mathcal{M}', 1 \models^+ \psi[a/y, b/x] \wedge \psi[b/y, b/x]$, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^+ \psi[b/x, a/y] \wedge \psi[b/x, b/y]$ iff $\mathcal{M}', 1 \models^+ \phi[b/x]$.

(Part 2). We have $\mathcal{M}', 1 \models^- \phi[a/x]$ iff, for some $c \in \{a, b\}$, we have $\mathcal{M}', 1 \models^- \psi[a/x, c/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^- \psi[c/y, a/x]$ for this c, iff, by the Induction Hypothesis, $\mathcal{M}', 1 \models^- \psi[c/y, b/x]$ for the said c, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^- \psi[b/x, c/y]$ iff $\mathcal{M}', 1 \models^- \phi[b/x]$.

Case 2. $\phi[a/x] = \exists y\psi[a/x]$. Similar to Case 1.

Case 3. $\phi[a/x] = \text{Æ}y\psi[a/x]$. We argue similarly to Part 1 of Case 1, since we know that for any $\circ \in \{+, -\}$ we have $\mathcal{M}', 1 \models^\circ \phi[a/x]$ iff $\mathcal{M}', 1 \models^\circ \psi[a/x, a/y] \wedge \psi[a/x, b/y]$ iff, by Corollary 2.1, $\mathcal{M}', 1 \models^\circ \psi[a/y, a/x] \wedge \psi[b/y, a/x]$, iff, by the Induction Hypothesis, $\mathcal{M}', 1 \models^\circ \psi[a/y, b/x] \wedge \psi[b/y, b/x]$, iff, by Corollary 2.1, $\mathcal{M}', 1 \models^\circ \psi[b/x, a/y] \wedge \psi[b/x, b/y]$ iff $\mathcal{M}', 1 \models^\circ \phi[b/x]$. □

The following Proposition then makes our claim about $\mathcal{M}'$ more precise:

Proposition 2. *Let* Σ, $\mathcal{M}'$, *and* $L^{\text{Æ}}$ *be defined as in Lemma 19. Then for every* $\phi \in L^{\text{Æ}}_{\emptyset}(\Sigma, U)$ *it is true that* $\mathcal{M}', 1 \models^\circ \phi$ *for some* $\circ \in \{+, -\}$.

Proof. Again, we argue by induction on the construction of $\phi \in L^{\text{Æ}}_{\emptyset}(\Sigma, U)$. The basis and most of the induction cases are as in the proof of Lemma 3 since $\mathcal{M}' \in \mathbb{CD} \cap \mathbb{C}3$. As for the only new induction case, assume that $\phi = \text{Æ}x\psi$. Then, by the Induction Hypothesis, we must have either $\mathcal{M}', 1 \models^+ \psi[a/x]$ or $\mathcal{M}', 1 \models^- \psi[a/x]$. In the former case, Lemma 19 implies that also $\mathcal{M}', 1 \models^+ \psi[b/x]$ and hence $\mathcal{M}', 1 \models^+ \phi$. In the latter case, Lemma 19 implies that also $\mathcal{M}', 1 \models^- \psi[b/x]$ and hence $\mathcal{M}', 1 \models^- \phi$. □

The fact that $\mathcal{M}, \mathcal{M}' \in \mathbb{CD}$ is particularly important in that it shows that, as long as Æ is present in the language, even the imposition of constant domains does not return us to a standard type of semantics and thus cannot be considered as any sort of remedy for the truth-value gap problem. In other words, not only do the classes of (QC3+Æ)-complete models and (C3+Æ)-complete models fail to be C3-good themselves, but their intersections with $\mathbb{CD}$ also fail to be C3-good.

Due to this phenomenon, also the addition of Æ to $\mathsf{QC3}_{CD}$ inevitably leads to a system with a non-standard semantics and with poor prospects for any traditional forms of frame correspondence theory.

7 Conclusion and future work

In the main part of our paper, we were focused on the completeness for the three systems QC, QC_{CD}, and $\mathsf{QC3}_{CD}$. These systems naturally arise as the result of extension of the propositional paraconsistent logics C and C3 with the Nelsonian quantifiers. We have succeeded in proving the general version of the soundness theorem for all these logics, as well as its converse in the countable case.

The Henkin technique used in these proofs proved to be easily adaptable to the treatment of the systems $\mathsf{QC3}_{At}$ and QC3, introduced in [10], even though it turned out that QC3 is a somewhat inconvenient extension of C3 since its class of intended models is not closed for the models based on the same underlying frame even if we restrict our attention to C3-models only. Finally, we have answered in the negative the question about the existence properties in QC3 and $\mathsf{QC3}_{CD}$.

Moreover, we have considered a relatively novel and peculiar quantifier Æ which combines the verification and falsification conditions of the two Nelsonian quantifiers. The intuitive motivation for the introduction of Æ in place of the Nelsonian version of ∀ is that such an introduction would be parallel to the amendment of the Nelsonian interpretation of the implication connective in C. We have sketched the application of the techniques developed in the main part of our paper to the systems where Æ is either added to the Nelsonian quantifiers or replaces them, and found that, in each case, a modicum of an amendment allows to obtain a Hilbert-style proof system which is sound and (in the countable case) complete for the logic at hand. We have also observed how the presence of this novel quantifier tends to exacerbate the problem of truth-value gap reinstatement in first-order extensions of C3 which appeared earlier in relation to Nelsonian quantifiers in $\mathsf{QC3}_{At}$.

However, the more general issue of the possibility of extending C with a (partially) non-Nelsonian set of quantifiers is by no means exhausted by the sketchy discourse contained in Section 6 of our paper. It is our hope that we will be able to return to this topic in our future research and to consider other well-motivated examples of non-Nelsonian quantifiers which show a certain degree of harmony with the basic motivating intuitions of C.

Turning one more time to the family of logics extending C with the Nelsonian set of quantifiers, we would like to add that one can easily see that the argument for the completeness of QC given in our paper can be straightforwardly extended to the signatures of arbitrary power by replacing every induction on ω with a suitable transfinite induction and by increasing the power of the sets of "fresh parameters" used in Lemmas 12 and 13 accordingly.

Unfortunately, such an easy extension is not possible in the case of QC_{CD}, and $\mathsf{QC3}_{CD}$, since the proof of the respective version of Lemma 16 for any of the two systems requires essentially that every bi-set in the increasing chain obtained in its main construction is finite. However, a standard workaround for this difficulty is also well-known and boils down to giving an independent proof of the compactness theorem for the system at hand. Again, in our future work, we hope to provide a satisfactorily complete version of such a proof and thus to close the issue of completeness for the axiomatizations of QC_{CD}, and $\mathsf{QC3}_{CD}$ presented in this paper.

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A Proof of Lemma 4

The proof proceeds by induction on the construction of ϕ for all $\circ \in \{+, -\}$ and all $w \in W$ simultaneously.

Basis. Let $\phi = P(\bar{a}_n)$ for some $P^n \in \Sigma$ and some $\bar{a}_n \in (D_w)^n$. Then we have:

$$\begin{aligned}\mathcal{M}, w \models^\circ P(\bar{a}_n) &\Leftrightarrow \bar{a}_n \in V^\circ(P, w)\\ &\Leftrightarrow f_{[b/a]}\langle \bar{a}_n\rangle \in (V_{[b/a]})^\circ(P, w)\\ &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ P(f_{[b/a]}\langle \bar{a}_n\rangle)\\ &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^\circ P(\bar{a}_n)[b/a]\end{aligned}$$

Step. The cases for $\wedge$, $\vee$, and $\to$ are straightforward, given that the parameter substitutions in formulas commute with the connectives. We consider the quantifiers:

Case 1. We have $\circ = +$ and $\phi = \forall x\psi$ for some $\psi \in L_x(\Sigma, D_w)$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}\mathcal{M}, w \models^+ \forall x\psi &\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x])\\ &\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b/a]}, v \models^+ \psi[c/x][b/a]) \qquad \text{(by IH)}\end{aligned}$$

If now $v \geq w$ and $d \in D_{[b/a]}(v)$, then two cases are possible:

Case 1.1. $d \in D_v \setminus \{a\}$. Then we must have $\mathcal{M}_{[b/a]}, v \models^+ \psi[d/x][b/a] = \psi[b/a][d/x]$ by Lemma 1.4 and the fact that $d \neq a$ and $x \notin \{a, b\}$.

Case 1.2. $d = b$. Then $a \in D_v$ and we must have $\mathcal{M}_{[b/a]}, v \models^+ \psi[a/x][b/a] = \psi[b/a][b/x] = \psi[b/a][d/x]$ by Lemma 1.4.

Summing up, we get that $(\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[b/a][d/x])$, so that $\mathcal{M}_{[b/a]}, w \models^+ \forall x(\psi[b/a])$ and hence also $\mathcal{M}_{[b/a]}, w \models^+ (\forall x\psi)[b/a]$.

For the ($\Leftarrow$)-part, we have:

$$\begin{aligned}\mathcal{M}_{[b/a]}, w \models^+ (\forall x\psi)[b/a] &\Leftrightarrow \mathcal{M}_{[b/a]}, w \models^+ \forall x(\psi[b/a])\\ &\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[b/a][d/x])\\ &\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b/a]}(v))(\mathcal{M}_{[b/a]}, v \models^+ \psi[d/x][b/a]),\end{aligned}$$

where the latter equivalence holds by Lemma 1.4. and the fact that $d \neq a$ and $x \notin \{a, b\}$. But then the Induction Hypothesis implies that $(\forall v \geq w)(\forall d \in D_v \setminus \{a\})(\mathcal{M}, v \models^+ \psi[d/x])$. In case $a \notin D_v$, we also get that $\mathcal{M}, w \models^+ \forall x\psi$. Otherwise, we must have $a \in D_v$ and, therefore, $b \in D_{[b/a]}(v)$. Now, given any $v \geq w$, our chain of equivalences implies that $\mathcal{M}_{[b/a]}, v \models^+ \psi[b/x][b/a] = \psi[b/a][b/x] = \psi[a/x][b/a]$ by Lemma 1.4 and the fact that $a \neq b$ and $x \notin \{a, b\}$. Therefore, the Induction Hypothesis again implies that $\mathcal{M}, v \models^+ \psi[a/x]$, and we get that $\mathcal{M}, w \models^+ \forall x\psi$ also in this case.

Case 2. We have $\circ = -$ and $\phi = \forall x\psi$ for some $\psi \in L_x(\Sigma, D_w)$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}\mathcal{M}, w \models^- \forall x\psi &\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x])\\ &\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b/a]}, w \models^- \psi[c/x][b/a]) \qquad \text{(by IH)}\end{aligned}$$

We now choose the corresponding $c \in D_w$. If $c \in D_w \setminus \{a\}$, then also $c \in D_{[b/a]}(w)$, and we must have $\mathcal{M}_{[b/a]}, w \models^- \psi[c/x][b/a] = \psi[b/a][c/x]$ by Lemma 1.4. and the fact that $b \neq x$ and $a \notin \{c, x\}$, whence $\mathcal{M}_{[b/a]}, w \models^- \forall x\psi[b/a]$. Otherwise, we must have $c = a \in D_w$ so that also $b \in D_{[b/a]}(w)$. But then, $\mathcal{M}_{[b/a]}, w \models^- \psi[a/x][b/a] = \psi[b/a][b/x]$ by Lemma 1.4, and, again, $\mathcal{M}_{[b/a]}, w \models^- \forall x\psi[b/a]$ follows.

For the ($\Leftarrow$)-part, we have:

$$\begin{aligned}\mathcal{M}_{[b/a]}, w \models^- (\forall x\psi)[b/a] &\Leftrightarrow (\exists d \in D_{[b/a]}(w))(\mathcal{M}_{[b/a]}, w \models^- \psi[b/a][d/x])\\ &\Leftrightarrow (\exists d \in D_{[b/a]}(w))(\mathcal{M}_{[b/a]}, w \models^- \psi[d/x][b/a]),\end{aligned}$$

where the latter equivalence holds by Lemma 1.4. and the fact that $d \neq a$ and $x \notin \{a, b\}$. Now, if $d \in D_w \setminus \{a\}$, then also $\mathcal{M}, w \models^- \psi[d/x]$ by the Induction Hypothesis, and thus $\mathcal{M}, w \models^- \forall x\psi$. Otherwise, we must have $d = b$, but then also $a \in D_w$, and we get that $\mathcal{M}_{[b/a]}, w \models^- \psi[d/x][b/a] = \psi[b/a][b/x] = \psi[a/x][b/a]$ by Lemma 1.4 and $a \neq x$. Therefore, $\mathcal{M}, w \models^- \psi[a/x]$ by the Induction Hypothesis, and, again, $\mathcal{M}, w \models^- \forall x\psi$.

The case of the existential quantifier is parallel to the case of the universal quantifier.

B Proof of Lemma 5

Again, the proof is by induction on the construction of $\phi[a/\bar{x}_n]$ for all $\circ \in \{+, -\}$, all $\bar{x}_n \in Var^{\neq n}$, and all $w \in W$ simultaneously.

Basis. Let $\phi[a/\bar{x}_n] = P(\bar{a}_m)$ for some $P^m \in \Sigma$ and some $\bar{a}_m \in (D_w)^m$. If now $\mathcal{M}, w \models^\circ P(\bar{a}_m)$, then $\bar{a}_m \in V^\circ(P, w)$. Let $\phi[b/\bar{x}_n] = P(\bar{b}_m)$. We want to show that $\bar{b}_m \in \rho_{[b:=a]}\langle \bar{a}_m \rangle \subseteq (V_{[b:=a]})^\circ(P, w)$. Indeed, fix an $1 \leq i \leq m$. If $a_i \neq a$, then a_i does not replace an occurrence of x_j for any $1 \leq j \leq n$, and, therefore, also $b_i = a_i \in \rho_{[b:=a]}[a_i]$. Otherwise $a_i = a$, and then, depending on whether a replaces an occurrence of x_j for some $1 \leq j \leq n$ or not, we will have $b_i = a$ or $b_i = b$, so, in any case, $b_i \in \rho_{[b:=a]}[a_i]$. But then $\bar{b}_m \in \rho_{[b:=a]}\langle \bar{a}_m \rangle \subseteq (V_{[b:=a]})^\circ(P, w)$, and we must have $\mathcal{M}_{[b:=a]}, w \models^\circ P(\bar{b}_m)$.

In the other direction, if $\phi[b/\bar{x}_n] = P(\bar{b}_m)$ and $\mathcal{M}_{[b:=a]}, w \models^\circ P(\bar{b}_m)$, then we must have $\bar{b}_m \in \rho_{[b:=a]}\langle \bar{c}_m \rangle$ for some $\bar{c}_m \in V^\circ(P, w)$. Moreover, it is easy to see that there exists a unique $\bar{c}_m \in (D_w)^m$ such that $\bar{b}_m \in \rho_{[b:=a]}\langle \bar{c}_m \rangle$ (since we have to replace all b's in $\bar{b}_m$ with a's in order for $\bar{c}_m$ to end up in $(D_w)^m$), so we must have $\bar{c}_m \in V^\circ(P, w)$ for this unique tuple. Let $\phi[a/\bar{x}_n] = P(\bar{a}_m)$. We will show that $\bar{a}_m = \bar{c}_m$. Indeed, fix an $1 \leq i \leq m$. If $b_i \notin \{a, b\}$, then $c_i \ \rho_{[b:=a]} \ b_i$ implies that $c_i = b_i$; on the other hand, if $b_i \notin \{a, b\}$, then b_i does not replace an occurrence of x_j in $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$ for any $1 \leq j \leq n$, and we must have $a_i = b_i$. Therefore $a_i = c_i$. Next, if $b_i = b$, then $c_i = a$ since $\bar{c}_m \in (D_w)^m$ and $b \notin D_w$; on the other hand, if $b_i = b$ then b_i must replace an occurrence of x_j in $\phi \in L_{\bar{x}_n}(\Sigma, D_w)$ for some $1 \leq j \leq n$. This same occurrence will be replaced by a in $P(\bar{a}_m)$, hence $a_i = a$. Summing up, we get that $c_i = a = a_i$. Finally, if $b_i = a$, then, again $c_i = a$ and also the occurrence of b_i does not replace an occurrence of any x_j, so also $a_i = a$. Again we get that $c_i = a = a_i$.

In this way, we see that $\bar{a}_m = \bar{c}_m \in V^\circ(P, w)$ and thus $\mathcal{M}, w \models^\circ P(\bar{a}_m) = \phi[a/\bar{x}_n]$.

Step. The cases for $\wedge$, $\vee$, and $\to$ are straightforward, given that the parameter substitutions in formulas commute with the connectives. We consider the quantifiers:

Case 1. We have $\circ = +$ and $\phi = \forall y \psi$ for some $\psi \in L_{(\bar{x}_n)^\frown y}(\Sigma, D_w)$ and some $y \in Var \setminus \{\bar{x}_n\}$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}
\mathcal{M}, w \models^+ (\forall y \psi)[a/\bar{x}_n] &\Leftrightarrow \mathcal{M}, w \models^+ \forall y(\psi[a/\bar{x}_n]) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[a/\bar{x}_n][c/y]) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/y][a/\bar{x}_n]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}) \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/y][b/\bar{x}_n]) && \text{(by IH)} \\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y]) && \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\})
\end{aligned}$$

We now fix any $v \geq w$. In case $a \notin D_v$, we have $D_v = D_{[b:=a]}(v)$ so we can already conclude that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$. Otherwise, we have $a \in D_v$ and $D_{[b:=a]}(v) = D_v \cup \{b\}$, so, in particular, we have that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][a/y]$. By Lemma 1.4 and $y \notin \{\bar{x}_n\}$, we conclude that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[a/y][b/\bar{x}_n]$. Now, the Induction Hypothesis implies that $\mathcal{M}, v \models^+ \psi[a/y][a/\bar{x}_n] = \psi[a/\bar{x}_n][a/y]$, and, applying the Induction Hypothesis one more time, we get that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][b/y]$. Summing this up with the fact that

$(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$, we again arrive at the conclusion that

$(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$.

Since we thus get the latter conclusion for an arbitrary $v \geq w$, we infer that $(\forall v \geq w)(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][c/y])$, whence it follows that $\mathcal{M}_{[b:=a]}, w \models^+ \forall y(\psi[b/\bar{x}_n]) = (\forall y \psi)[b/\bar{x}_n]$.

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$\begin{aligned}
\mathcal{M}_{[b:=a]}, w \models^+ (\forall y\psi)[b/\bar{x}_n] &\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^+ \forall y(\psi[b/\bar{x}_n])\\
&\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/\bar{x}_n][d/y])\\
&\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/y][b/\bar{x}_n])\\
&\qquad\qquad \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\})\\
&\Rightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/y][b/\bar{x}_n])\\
&\qquad\qquad \text{(by } D_v \subseteq D_{[b:=a]}(v))\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/y][a/\bar{x}_n]) \qquad \text{(by IH)}\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[a/\bar{x}_n][c/y])\\
&\qquad\qquad \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\})\\
&\Leftrightarrow \mathcal{M}, w \models^+ \forall y(\psi[a/\bar{x}_n]) = (\forall y\psi)[a/\bar{x}_n]
\end{aligned}$$

Case 2. We have $\circ = +$ and $\phi = \forall y\psi$ for some $\psi \in L_{\bar{x}_n}(\Sigma, D_w)$, where $y = x_i$ for some $1 \leq i \leq n$. Then we must have $n \geq 1$. If now $n > 1$, then we have $\forall y\psi[a/\bar{x}_n] = \forall y\psi[a/(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)]$, and similarly for $\forall y\psi[b/\bar{x}_n]$, so we can reason as in Case 1 replacing $\bar{x}_n$ everywhere with $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. In case $n = 1$, we must have $y = x_1$. Then we have, for the ($\Rightarrow$)-part:

$$\begin{aligned}
\mathcal{M}, w \models^+ (\forall x_1\psi)[a/x_1] &\Leftrightarrow \mathcal{M}, w \models^+ \forall x_1\psi\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x_1])\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}, v \models^+ \psi[c/x_1][a/x_1]) \qquad \text{(by Lemma 1.4)}\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1][b/x_1]) \qquad \text{(by IH)}\\
&\Leftrightarrow (\forall v \geq w)(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1]) \qquad \text{(by Lemma 1.4)}
\end{aligned}$$

We now fix any $v \geq w$. In case $a \notin D_v$, we have $D_v = D_{[b:=a]}(v)$ so we can already conclude that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$. Otherwise, we have $a \in D_v$ and $D_{[b:=a]}(v) = D_v \cup \{b\}$, so, in particular, we have that $\mathcal{M}, v \models^+ \psi[a/x_1]$. Now, the Induction Hypothesis implies that $\mathcal{M}_{[b:=a]}, v \models^+ \psi[b/x_1]$. Summing this up with the fact that $(\forall c \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$, we again arrive at the conclusion that $(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$.

Since we thus get the latter conclusion for an arbitrary $v \geq w$, we infer that $(\forall v \geq w)(\forall c \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[c/x_1])$, whence it follows that $\mathcal{M}_{[b:=a]}, w \models^+ \forall x_1\psi = (\forall x_1\psi)[b/x_1]$.

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$
\begin{aligned}
&\mathcal{M}_{[b:=a]}, w \models^+ (\forall x_1\psi)[b/x_1] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^+ \forall x_1\psi \\
&\quad\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1]) \\
&\quad\Leftrightarrow (\forall v \geq w)(\forall d \in D_{[b:=a]}(v))(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1][b/x_1]) \quad \text{(by Lemma 1.4)} \\
&\quad\Rightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}_{[b:=a]}, v \models^+ \psi[d/x_1][b/x_1]) \quad \text{(by } D_v \subseteq D_{[b:=a]}(v)\text{)} \\
&\quad\Leftrightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}, v \models^+ \psi[d/x_1][a/x_1]) \quad \text{(by IH)} \\
&\quad\Leftrightarrow (\forall v \geq w)(\forall d \in D_v)(\mathcal{M}, v \models^+ \psi[d/x_1]) \quad \text{(by Lemma 1.4)} \\
&\quad\Leftrightarrow \mathcal{M}, w \models^+ \forall x_1\psi = (\forall x_1\psi)[a/x_1]
\end{aligned}
$$

Case 3. We have $\circ = -$ and $\phi = \forall y\psi$ for some $\psi \in L_{(\bar{x}_n)^\frown y}(\Sigma, D_w)$ and some $y \in Var \setminus \{\bar{x}_n\}$. Then we have, for the ($\Rightarrow$)-part:

$$
\begin{aligned}
&\mathcal{M}, w \models^- (\forall y\psi)[a/\bar{x}_n] \Leftrightarrow \mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) \\
&\quad\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[a/\bar{x}_n][c/y]) \\
&\quad\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/y][a/\bar{x}_n]) \quad \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}\text{)} \\
&\quad\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/y][b/\bar{x}_n]) \quad \text{(by IH)} \\
&\quad\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][c/y]) \quad \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}\text{)} \\
&\quad\Rightarrow (\exists c \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][c/y]) \quad \text{(by } D_w \subseteq D_{[b:=a]}(w)\text{)} \\
&\quad\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall y(\psi[b/\bar{x}_n]) = (\forall y\psi)[b/\bar{x}_n]
\end{aligned}
$$

Turning now to the ($\Leftarrow$)-part, we reason as follows:

$$
\begin{aligned}
&\mathcal{M}_{[b:=a]}, w \models^- (\forall y\psi)[b/\bar{x}_n] \Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall y(\psi[b/\bar{x}_n]) \\
&\quad\Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[b/\bar{x}_n][d/y]) \\
&\quad\Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[d/y][b/\bar{x}_n]) \quad \text{(by Lemma 1.4 and } y \notin \{\bar{x}_n\}\text{)}
\end{aligned}
$$

We now choose a corresponding $d \in D_{[b:=a]}(w)$. If $d \in D_w$, then, by IH, we get that $\mathcal{M}, w \models^- \psi[d/y][a/\bar{x}_n]$ whence $\mathcal{M}, w \models^- \psi[a/\bar{x}_n][d/y]$ by Lemma 1.4 and $y \notin \{\bar{x}_n\}$. Now $\mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) = (\forall y\psi)[a/\bar{x}_n]$ follows immediately.

Otherwise we have $d = b$, and we get that $\mathcal{M}_{[b:=a]}, w \models^- \psi[b/y][b/\bar{x}_n] = \psi[b/(\bar{x}_n)^\frown y]$, whence, by IH, $\mathcal{M}, w \models^- \psi[a/(\bar{x}_n)^\frown y] = \psi[a/\bar{x}_n][a/y]$. Hence also $\mathcal{M}, w \models^- \forall y(\psi[a/\bar{x}_n]) = (\forall y\psi)[a/\bar{x}_n]$ follows.

Case 4. We have $\circ = -$ and $\phi = \forall y\psi$ for some $\psi \in L_{\bar{x}_n}(\Sigma, D_w)$, where $y = x_i$ for some $1 \leq i \leq n$. Then we must have $n \geq 1$. Again, the subcase $n > 1$ can be reduced to Case 3 above. In case $n = 1$, we must have $y = x_1$. Then we have, for the (⇒)-part:

$$\begin{aligned}
\mathcal{M}, w \models^- (\forall x_1 \psi)[a/x_1] &\Leftrightarrow \mathcal{M}, w \models^- \forall x_1 \psi \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x_1]) \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}, w \models^- \psi[c/x_1][a/x_1]) && \text{(by Lemma 1.4)} \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/x_1][b/x_1]) && \text{(by IH)} \\
&\Leftrightarrow (\exists c \in D_w)(\mathcal{M}_{[b:=a]}, w \models^- \psi[c/x_1]) && \text{(by Lemma 1.4)} \\
&\Rightarrow (\exists c \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^\cdot \psi[c/x_1]) && \text{(by } D_v \subseteq D_{[b:=a]}(v)) \\
&\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall x_1 \psi = (\forall x_1 \psi)[b/x_1]
\end{aligned}$$

Turning now to the (⇐)-part, we reason as follows:

$$\begin{aligned}
\mathcal{M}_{[b:=a]}, w \models^- (\forall x_1 \psi)[b/x_1] &\Leftrightarrow \mathcal{M}_{[b:=a]}, w \models^- \forall x_1 \psi \\
&\Leftrightarrow (\exists d \in D_{[b:=a]}(w))(\mathcal{M}_{[b:=a]}, w \models^- \psi[d/x_1])
\end{aligned}$$

We now choose a corresponding $d \in D_{[b:=a]}(w)$. In the subcase $d \in D_w$ we are done by the Induction Hypothesis.

In the subcase $d = b$, we must have $a \in D_w$, and we get that $\mathcal{M}_{[b:=a]}, w \models^- \psi[b/x_1]$, whence, by the Induction Hypothesis, $\mathcal{M}, w \models^- \psi[a/x_1]$.

In this way, we get that $\mathcal{M}, w \models^- \forall x_1 \psi = (\forall x_1 \psi)[a/x_1]$ in both subcases.

The case of the existential quantifier is parallel to the case of the universal quantifier.

C Proof of Lemma 18

We assume that the signature Σ and the Σ-model $\mathcal{M}$ are defined as in Example 3. We prove a couple of auxiliary lemmas first:

Lemma 20. *Let $\phi \in L_\emptyset(\Sigma, U)$. Then, for some $\circ \in \{+, -\}$, we have $\mathcal{M}, 2 \models^\circ \phi$.*

Proof. 2 is the maximal state in $\mathcal{M}$. □

Lemma 21. *Let $x \in Var$, let $\phi \in L_x(\Sigma, U)$. Then the following statements hold:*

1. *If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then $\mathcal{M}, 2 \models^+ \phi[a/x]$.*

2. *If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then $\mathcal{M}, 2 \models^- \phi[a/x]$.*

Proof. By induction on the construction of $\phi[b/x]$.

Basis. If $\phi[b/x]$ is atomic, then we must have $\phi[b/x] \in \{Q(a), Q(b), p\}$.

(Part 1). The situation when both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$ is therefore impossible, so our statement holds vacuously.

(Part 2). If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then we must have $\phi[b/x] = Q(b)$. Two cases are possible:

Case 1. $\phi = Q(b)$. Then $\phi[a/x] = Q(b)$, and we have $\mathcal{M}, 2 \models^- \phi[a/x] = Q(b) = \phi[b/x]$ by our assumption.

Case 2. $\phi = Q(x)$. Then $\phi[a/x] = Q(a)$, and we have $\mathcal{M}, 2 \models^- \phi[a/x] = Q(a)$ by the definition of $\mathcal{M}$.

Step. The following cases are possible:

Case 1. $\phi[b/x] = \psi[b/x] \wedge \chi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then we must have, on the one hand, that both $\mathcal{M}, 2 \models^+ \psi[b/x]$ and $\mathcal{M}, 2 \models^+ \chi[b/x]$. On the other hand, we must have both $\mathcal{M}, 2 \not\models^- \psi[b/x]$ and $\mathcal{M}, 2 \not\models^- \chi[b/x]$. Therefore, by IHp1, we must have also that $\mathcal{M}, 2 \models^+ \psi[a/x] \wedge \chi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then we must have, on the one hand, that either $\mathcal{M}, 2 \models^- \psi[b/x]$ or $\mathcal{M}, 2 \models^- \chi[b/x]$. On the other hand, we must have either $\mathcal{M}, 2 \not\models^+ \psi[b/x]$ or $\mathcal{M}, 2 \not\models^+ \chi[b/x]$.

Assume, wlog, that $\mathcal{M}, 2 \models^- \psi[b/x]$. If also $\mathcal{M}, 2 \not\models^+ \psi[b/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^- \psi[a/x]$, whence $\mathcal{M}, 2 \models^- \psi[a/x] \wedge \chi[a/x]$. Otherwise, we must have $\mathcal{M}, 2 \models^+ \psi[b/x]$, but then we must have $\mathcal{M}, 2 \not\models^+ \chi[b/x]$, and, by Lemma 20, that $\mathcal{M}, 2 \models^- \chi[b/x]$. But now IHp2 is again applicable and yields that $\mathcal{M}, 2 \models^- \chi[a/x]$ whence also $\mathcal{M}, 2 \models^- \psi[a/x] \wedge \chi[a/x]$.

Case 2. $\phi[b/x] = \psi[b/x] \vee \chi[b/x]$. Similar to Case 1.

Case 3. $\phi[b/x] =\sim \psi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then we must have both $\mathcal{M}, 2 \models^- \psi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \psi[b/x]$. But then, also $\mathcal{M}, 2 \models^- \psi[a/x]$ follows by IHp2 and, further, $\mathcal{M}, 2 \models^+ \sim \psi[a/x]$.

(Part 2). Parallel to Part 1.

Case 4. $\phi[b/x] = \psi[b/x] \to \chi[b/x]$.

(Part 1). If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then we must have $\mathcal{M}, 2 \not\models^- \chi[b/x]$ since 2 is the maximal node; whence Lemma 20 implies that also $\mathcal{M}, 2 \models^+ \chi[b/x]$. Now, by IHp1, we also get that $\mathcal{M}, 2 \models^+ \chi[a/x]$, whence, further, $\mathcal{M}, 2 \models^+ \psi[a/x] \to \chi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then we must have $\mathcal{M}, 2 \not\models^+ \chi[b/x]$ since 2 is the maximal node; whence Lemma 20 implies that also $\mathcal{M}, 2 \models^- \chi[b/x]$. Now, by IHp2, we also get that $\mathcal{M}, 2 \models^- \chi[a/x]$, whence, further, $\mathcal{M}, 2 \models^- \psi[a/x] \to \chi[a/x]$.

Case 5. $\phi[b/x] = \forall y\psi[b/x]$. We may assume, wlog, that $y \neq x$. Note that we have, by Corollary 1, that $\psi[b/x][c/y] = \psi[c/y][b/x]$ for every $c \in U$ under this condition.

(Part 1). If both $\mathcal{M}, 2 \models^+ \phi[b/x]$ and $\mathcal{M}, 2 \not\models^- \phi[b/x]$, then we must have, on the one hand, both $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][b/x]$. On the other hand, we must have both $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$ and $\mathcal{M}, 2 \not\models^- \psi[b/y][b/x]$. But then IHp1 implies that both $\mathcal{M}, 2 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][a/x] = \psi[a/x][b/y]$, whence, given that 2 is a maximal node, it follows that $\mathcal{M}, 2 \models^+ \forall y\psi[a/x]$.

(Part 2). If both $\mathcal{M}, 2 \models^- \phi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \phi[b/x]$, then we must have, on the one hand, either $\mathcal{M}, 2 \models^- \psi[a/y][b/x]$ or $\mathcal{M}, 2 \models^- \psi[b/y][b/x]$. On the other hand, we must have either $\mathcal{M}, 2 \not\models^+ \psi[a/y][b/x]$ or $\mathcal{M}, 2 \not\models^+ \psi[b/y][b/x]$.

Assume, wlog, that $\mathcal{M}, 2 \models^- \psi[a/y][b/x]$. If also $\mathcal{M}, 2 \not\models^+ \psi[a/y][b/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^- \psi[a/y][a/x] = \psi[a/x][a/y]$, whence $\mathcal{M}, 2 \models^- \forall y\psi[a/x]$. Otherwise, we must have $\mathcal{M}, 2 \not\models^+ \psi[b/y][b/x]$, whence, by Lemma 20, it follows that $\mathcal{M}, 2 \models^- \psi[b/y][b/x]$. But then IHp2 is applicable and yields that $\mathcal{M}, 2 \models^- \psi[b/y][a/x] = \psi[a/x][b/y]$, whence again $\mathcal{M}, 2 \models^- \forall y\psi[a/x]$.

Case 6. $\phi[b/x] = \exists y\psi[b/x]$. Similar to Case 5. □

Lemma 22. *Let $x \in Var$, let $\phi \in L_x(\Sigma, \{a\})$. Then the following statements hold:*

1. *If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then $\mathcal{M}, 2 \models^+ \phi[b/x]$.*
2. *If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then $\mathcal{M}, 2 \models^- \phi[b/x]$.*
3. *For some $\circ \in \{+, -\}$, we have $\mathcal{M}, 1 \models^\circ \phi[a/x]$.*

Proof. By induction on the construction of $\phi[a/x]$.

Basis. If $\phi[a/x]$ is atomic, then we must have $\phi[a/x] \in \{Q(a), p\}$.

(Part 1). The situation when both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$ is therefore impossible, so our statement holds vacuously.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then we must have $\phi[a/x] = p$. Then $\phi[b/x] = p$ as well, and we have $\mathcal{M}, 2 \models^- \phi[b/x] = p$ by the definition of $\mathcal{M}$.

(Part 3). Trivial by the definition of $\mathcal{M}$.

Step. The following cases are possible:

Case 1. $\phi[a/x] = \psi[a/x] \wedge \chi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then we must have, on the one hand, that both $\mathcal{M}, 1 \models^+ \psi[a/x]$ and $\mathcal{M}, 1 \models^+ \chi[a/x]$. On the other hand, we must have both $\mathcal{M}, 1 \not\models^- \psi[a/x]$ and $\mathcal{M}, 1 \not\models^- \chi[a/x]$. Therefore, by IHp1, we must have also that $\mathcal{M}, 2 \models^+ \psi[b/x] \wedge \chi[b/x]$.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then we must have, on the one hand, that either $\mathcal{M}, 1 \models^- \psi[a/x]$ or $\mathcal{M}, 1 \models^- \chi[a/x]$. On the other hand, we must have either $\mathcal{M}, 1 \not\models^+ \psi[a/x]$ or $\mathcal{M}, 1 \not\models^+ \chi[a/x]$.

Assume, wlog, that $\mathcal{M}, 1 \models^- \psi[a/x]$. If also $\mathcal{M}, 1 \not\models^+ \psi[a/x]$, then, by IHp2, we must have $\mathcal{M}, 2 \models^- \psi[b/x]$, whence $\mathcal{M}, 2 \models^- \psi[b/x] \wedge \chi[b/x]$. Otherwise, we must have $\mathcal{M}, 1 \models^+ \psi[a/x]$, but then we must have $\mathcal{M}, 1 \not\models^+ \chi[a/x]$, and, by IHp3, that $\mathcal{M}, 1 \models^- \chi[a/x]$. But now IHp2 is again applicable and yields that $\mathcal{M}, 2 \models^- \chi[b/x]$ whence also $\mathcal{M}, 2 \models^- \psi[b/x] \wedge \chi[b/x]$.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 2. $\phi[a/x] = \psi[a/x] \vee \chi[a/x]$. Similar to Case 1.

Case 3. $\phi[a/x] = \sim \psi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then we must have both $\mathcal{M}, 1 \models^- \psi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \psi[a/x]$. But then, also $\mathcal{M}, 2 \models^- \psi[b/x]$ follows by IHp2 and, further, $\mathcal{M}, 2 \models^+ \sim \psi[b/x]$.

(Part 2). Parallel to Part 1.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 4. $\phi[a/x] = \psi[a/x] \to \chi[a/x]$.

(Part 1). If both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^+ \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \models^+ \psi[b/x]$ and $\mathcal{M}, 2 \not\models^+ \chi[b/x]$, whence it follows, by IHp1, that either $\mathcal{M}, 1 \not\models^+ \chi[a/x]$ or $\mathcal{M}, 1 \models^- \chi[a/x]$. By IHp3, we know that we must have $\mathcal{M}, 1 \models^- \chi[a/x]$ in both cases. But the latter means that we must have $\mathcal{M}, 1 \models^- \psi[a/x] \to \chi[a/x]$, which contradicts our assumption. Therefore, we must have $\mathcal{M}, 2 \models^+ \phi[b/x]$.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^- \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \models^+ \psi[b/x]$ and $\mathcal{M}, 2 \not\models^- \chi[b/x]$, whence it follows, by IHp2, that either $\mathcal{M}, 1 \not\models^- \chi[a/x]$ or $\mathcal{M}, 1 \models^+ \chi[a/x]$. By IHp3, we know that we must have $\mathcal{M}, 1 \models^+ \chi[a/x]$ in both cases. But the latter means that we must have $\mathcal{M}, 1 \models^+ \psi[a/x] \to \chi[a/x]$, which contradicts our assumption. Therefore, we must have $\mathcal{M}, 2 \models^- \phi[b/x]$.

(Part 3). Trivial (by application of the corresponding truth-table).

Case 5. $\phi[b/x] = \forall y \psi[b/x]$. We may assume, wlog, that $y \neq x$. Note that we have, by Corollary 1, that $\psi[b/x][c/y] = \psi[c/y][b/x]$ for every $c \in U$ under this condition.

(Part 1). Assume that both $\mathcal{M}, 1 \models^+ \phi[a/x]$ and $\mathcal{M}, 1 \not\models^- \phi[a/x]$. Now $\mathcal{M}, 1 \models^+ \phi[a/x]$ implies that $\mathcal{M}, 1 \models^+ \psi[a/x][a/y]$, whereas $\mathcal{M}, 1 \not\models^- \phi[a/x]$, by the definition of $\mathcal{M}$, implies that $\mathcal{M}, 1 \not\models^- \psi[a/x][a/y]$. Therefore, by IHp1, we must have $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$.

Next, choose a $z \in Var$ such that $z \notin FV(\psi[a/y][a/x]) \cup BV(\psi[a/y][a/x]) \cup \{x, y\}$

and consider $\psi[z/x][z/y]$. Then Lemma 1.4 and Lemma 1.2 imply that

$$\psi[z/x][z/y][b/z] = \psi[z/x][b/z][b/y] = \psi[b/z][b/x][b/y] = \psi[b/x][b/y].$$

A parallel argument shows that also $\psi[z/x][z/y][a/z] = \psi[a/x][a/y]$. Thus we have shown that both $\mathcal{M}, 1 \models^+ \psi[z/x][z/y][a/z]$ and $\mathcal{M}, 1 \not\models^- \psi[z/x][z/y][a/z]$, whence, by IHp1, $\mathcal{M}, 2 \models^+ \psi[z/x][z/y][b/z] = \psi[b/x][b/y]$.

Thus we have shown that

$$\mathcal{M}, 2 \models^+ \psi[b/x][a/y] \wedge \psi[b/x][b/y],$$

so that also $\mathcal{M}, 2 \models^+ \phi[b/x] = \forall y\psi[b/x]$ holds.

(Part 2). If both $\mathcal{M}, 1 \models^- \phi[a/x]$ and $\mathcal{M}, 1 \not\models^+ \phi[a/x]$, then assume that $\mathcal{M}, 2 \not\models^- \phi[b/x]$. The latter means that we have both $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$ and $\mathcal{M}, 2 \not\models^- \psi[b/y][b/x]$. Now Lemma 20 implies that we must also have both $\mathcal{M}, 2 \models^+ \psi[a/y][b/x]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][b/x]$, whence, by Lemma 21.1, we must have both $\mathcal{M}, 2 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ and $\mathcal{M}, 2 \models^+ \psi[b/y][a/x] = \psi[a/x][b/y]$.

Next, since we have $\mathcal{M}, 2 \not\models^- \psi[a/y][b/x]$, it also follows by IHp2 that either $\mathcal{M}, 1 \not\models^- \psi[a/y][a/x]$ or $\mathcal{M}, 1 \models^+ \psi[a/y][a/x]$. By IHp3, $\mathcal{M}, 1 \models^+ \psi[a/y][a/x] = \psi[a/x][a/y]$ holds in both cases.

Summing up, we have shown that all of the following holds:

$$\mathcal{M}, 1 \models^+ \psi[a/x][a/y],\ \mathcal{M}, 2 \models^+ \psi[a/x][a/y],\ \mathcal{M}, 2 \models^+ \psi[a/x][b/y],$$

which, by definition of $\mathcal{M}$, implies that $\mathcal{M}, 1 \models^+ \forall y\psi[a/x]$, contrary to our assumption. The obtained contradiction shows that we must have $\mathcal{M}, 2 \models^- \phi[b/x]$.

(Part 3). Assume that $\mathcal{M}, 1 \not\models^- \phi[a/x]$. Then we must have $\mathcal{M}, 1 \not\models^- \phi[a/x][a/y]$, whence IHp3 further implies that $\mathcal{M}, 1 \models^+ \phi[a/x][a/y]$. But then, by IHp1, it follows that $\mathcal{M}, 2 \models^+ \phi[a/x][b/y]$. Moreover, we must have $\mathcal{M}, 2 \models^+ \phi[a/x][a/y]$ by monotonicity. Summing up, we get that $\mathcal{M}, 1 \models^+ \forall y\psi[a/x] = \phi[a/x]$.

Case 6. $\phi[b/x] = \exists y\psi[b/x]$. Similar to Case 5. □

Proof of Lemma 18. (Part 1). By Lemma 20 and Lemma 22.3.

(Part 2). We have $\mathcal{M}, 1 \models^+ p \vee Q(a)$ as well as $\mathcal{M}, 2 \models^+ p \vee Q(a)$ and $\mathcal{M}, 2 \models^+ p \vee Q(b)$, so that $\mathcal{M}, 1 \models^+ \forall x(p \vee Q(x))$. However, we also have $\mathcal{M}, 1 \not\models^+ p$ and $\mathcal{M}, 2 \not\models^+ Q(b)$, whence $\mathcal{M}, 1 \not\models^+ \forall xQ(x)$, so that, finally, $\mathcal{M}, 1 \not\models^+ p \vee \forall xQ(x)$. □

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