Relevance and Importance Preferential Attachment

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Relevance and importance are the main factors when humans build network connections. We propose an evolutionary network model based on preferential attachment (PA) considering these factors. We analyze and compute several important features of the network class generated by this algorithm, including scale-free degree distribution, high clustering coefficient, small-world property and core-periphery structure. We then compare this model with other network models and empirical data such as intercity road transportation and air traffic networks.

Keywords: geometric preferential attachment; network growth; social networks; applied probability; traffic networks

1. Introduction

Preferential attachment (PA) is the key ingredient in the growth/evolution of many scale-free social, biological and ecological networks. The linking of new nodes to older nodes of highest degrees is essentially a topological, or more precisely, a graph-theoretical algorithm for network growth. It does not depend on the geographic/geometric properties of the nodes in the networks. At the other end of the spectrum of networks, there are geometric ones such as the random geographic graphs (RGG) and the online nearest-neighbor graphs (ONGs). In the case of the ONGs, the algorithm consists of randomly (by a Poisson process) locating a new node within a compact subset of some Euclidean space and then linking it to the nearest older node. Recently, the topological PA algorithm has been modified geometrically to yield a family of so-called geometric preferential attachment networks. The geometric or metric component of this algorithm forms a combination with varying weights to the topological PA component [1], where the location of a new node remains random Poisson-like in the ONGs, and the linkages to older nodes are chosen according to a probability distribution that depends on a weighted combination of the degrees of the older nodes and their distances from the new node [2]. The rigorous analysis of the geometric PA

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networks consisted of results on their expected degree sequence and their moments, departures from the ONGs as a function of the relative weighting of the metric component versus the topological PA component, ultimately leading to a phase transition. The introduction of the geometric PA networks by [2] and their proofs of convergence to the power-law degrees distribution under certain conditions on the strength of the metric component depended on a parameter $\alpha > 0$ that controlled the presence of self-loops. Their results on the small-world property, and the existence of small separators, were obtained under similar conditions. Jordan [1] provided proofs in the $\alpha = 0$ case. Partly due to the technical difficulty in the proofs of these results for the family of geometric PA networks, and also due to their dependence on certain unnatural conditions in parameters, we are motivated to seek a precise modification to that family of algorithms for generating random social networks, which would yield an algorithm (RIPA) that is substantially simpler to analyze and of equal if not greater utility to the understanding of social networks. The main aim of this paper is to show that such modifications exist and are simply prescribed by adding a metric component to the first stage of the algorithms—instead of the random Poisson process, the location x of the new node is chosen in a compact subset of a metric space according to a probability distribution that depends only on the centrality of the new site x relative to the older nodes. Details and a formal definition of the centrality measure in terms of the same metric as in the second stage of the algorithm for the linkages to the older nodes will be given in the following.

There is another reason for us to invent this modified model, in which the probability measure of the emergence of a new node is adaptive to the current network state. In many real-world networks, it is more reasonable to consider the new node as the offspring of the old nodes than as spontaneous growth in the blank space. For example, an influential research article may trigger many relevant articles, while the opposite case, a new article not motivated by any previous research, is very rare. Following this philosophy, we assume the probability measure of the emergence of a new node at a specific location is proportional to the superposition of the influences of all previous nodes at that point. This assumption leads to an equivalent presentation of our model described in Section 7, the RIIP model.

For many complex networks in society, it is arguable that *relevance* and *importance* are two of the main factors influencing how new network connections are formed in existing dynamic networks. One typical scenario is in scientific research and the publication process. In choosing references, authors are more likely to cite articles with high impact (importance) and also those using similar methods or discussing relevant issues (relevance). Another example is in the design

and organic growth of intercity transportation networks. Traffic engineers and city designers prefer to connect a given city to big cities with high connectivity (importance) but also want to reduce the expense by giving priority to the connections between nearby cities (relevance). Complex networks involving both relevance and importance also include aspects of the World Wide Web (WWW) and many social networks. An interesting and ironic point is that people are still striving to understand the properties of these complex networks, which are largely manmade. As a related point, we emphasize that the network evolutions studied here are governed by a distributed decision-making system rather than centrally organized. For each agent in the networks that makes local decisions, the rule of adding or deleting links may be simple and clear. Intuitively, the complexity of the networks arises from some other reasons, such as cooperative and bulk properties of large systems consisting of many similar subunits. While this complexity is not explicit in the local design rules and is often beyond the total control of the network designers, human society nonetheless seeks to understand and manage this complexity. Hence, the current scientific and technological interests in studying the origins and properties of these dynamic complex systems. In this paper, we study systematically one of the origins of this complexity—an underlying metric space defining the relevance structure, which we will introduce and discuss in detail later.

In the past few decades, several evolutionary network models have been proposed with respect to one or both of the two factors, importance and relevance. For importance alone, the most famous model was invented by Barabasi, known as the BA network model [3] or the "preferential attachment" (PA) algorithm. The standard preferential attachment starts with a network with N_0 vertices and m_0 edges. A new vertex is successively added and attached to $m < N_0$ pre-existing vertices. The probability of attaching to a vertex i is proportional to its degree k_i. This algorithm generates the network with power-law degree distribution $p(k) \sim k^{-\gamma}$ with $\gamma = 3$. There are many variations of the PA algorithm in the literature [4–6], all of which have similar complexity and other values of γ. The rigorous network algorithms in [2] generate networks in a wide range of γ values by varying the relevant parameters δ and α.

There are also well-known network models based purely on the notion of relevance. The simplest evolutionary network model based on relevance is the random geometric graph (RGG). In this model, we successively add vertices at random locations in a unit square and link each new vertex to all the nearby vertices within a given radius r. Here the relevance is measured by the geometric distance. Another model based on the notion of relevance is given in [7], in which the

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relevance is determined by a hierarchical structure and tree distance. In these models, a natural way to measure relevance is through an underlying metric space. We will show later in this paper how this metric space affects the global properties of the network. Briefly, due to the triangle inequality in metric spaces, the corresponding relevance relationship satisfies the following important property: the relevance of any two objects or nodes has a lower bound that is a simple function of the pair of relevance values between these objects and a third one. Therefore, network models based on the notion of relevance should have high clustering coefficients. Another way of thinking about the notion of relevance in complex networks is to use geometric embedding [8–10], which is not to provide an evolutionary model but to find the most suitable underlying metric space for the known network. We will not dwell on this here.

In this paper, we propose an evolutionary network model that contains a specific modification to the geometric preferential attachment models in [1, 2]. Like these models, the probability of links between a new node and the older nodes depends on the product of the degrees of older nodes and a geometric factor that depends on their distance under a given metric. Unlike these models, the placement of the new node is not uniformly random, but is rather based on a natural probability measure on the metric space defined by a local partition function that is weighted by the degrees of older nodes and the same metric as in the attachment phase. Next, we will introduce our relevance and importance preferential attachment (RIPA) model given by an evolution process, analyze several network properties and compare this model with other network models and some empirical data.

2. Model

In this section, we describe the algorithm called relevance and importance preferential attachment (RIPA), which generates a class of complex networks. The RIPA, like the classical preferential attachment, starts with a initial connected graph $G_0 = \{V_0, E_0, X_0\}$, with n_0 nodes and m_0 edges. V_0 is the set of nodes, E_0 is the set of edges, and X_0 is the sequence of the locations of the nodes in the metric space Ω. The distance between two points $x, y \in \Omega$ is given by $d(x, y)$. We introduce the relevance $\rho(x, y)$ as a non-increasing function of the distance $d(x, y)$,

$$\rho(x, y) = f\big(d(x, y)\big),$$

satisfying $f\big(0\big) = 1$, $f(\infty) = 0$, $f(r) \in \big(0, 1\big)$ for all $r > 0$. A typical example is $f(r) = e^{-r}$. $f(r)$ can also have a power-law tail, for example,

$f(r) = \min\{1, r^{-\beta}\}$ with $\beta > 0$. In each time step, we add one new node to the graph and have the new node attach to the old nodes m times. The graph at time step n (after the attachment) is $G_n = \{V_n, E_n, X_n\}$. At the n^{th} step, the index of the new node is $j = n + n_0$. The location of the node j, x_j is randomly picked following the probability measure $\mu_n(x)$ on Ω. $\mu_n(x)$ is adaptive to the previous graph G_{n-1}. Then we select m old nodes denoted by $\{W_{n,1}, \ldots, W_{n,m}\} \in V_{n-1}$ and attach the node j to these nodes. The selections of nodes $W_{n,v}$ are independent, so multi-links (i.e., $W_{n,v} = W_{n,u}$ for $1 \le v \ne u \le m$) are possible and allowed, but self-loops are prohibited. For given v, we select the old node i as $W_{n,v}$ by probability

$$\Pi_{ij} = \frac{(k_{i,n-1} + q)\rho_{ij}}{z_{n-1}(x_j)}.$$

Here $k_{i,n}$ denotes the degree of node i in G_n; ρ_{ij} is the shortening of $\rho(x_i, x_j)$. The probability Π_{ij} is proportional to the degree of i in G_{n-1} plus a constant $q \in (-m, \infty)$ and the relevance between i and j. $z_{n-1}(x_j)$ is the normalization constant. Since $\sum_i \Pi_{ij} = 1$, we define the local partition function $z_n(x)$ by

$$z_n(x) = \sum_{i=1}^{n+n_0} (k_{i,n} + q)\rho(x_i, x).$$

The summation here goes over all nodes in G_n. A location $x \in \Omega$ with higher local partition $z_{n-1}(x)$ has more overall relevance to the old nodes in G_{n-1}, therefore may attract more interest of a new node. So we suggest $\mu_n(x)$, the probability measure of the location of node $i = n_0 + n$, is proportional to $z_{n-1}(x)$,

$$\mu_n(x) = \frac{z_{n-1}(x)}{Z_{n-1}},$$

where Z_n is the global partition function of G_n:

$$\begin{aligned} Z_n &= \int_\Omega z_n(x)dx = \int_\Omega \sum_{j=1}^{n_0+n} (k_{j,n} + q)\rho(x_j, x)dx \\ &= \sum_{j=1}^{n_0+n} (k_{j,n} + q)C(x_j). \end{aligned}$$

Here $C(x) = \int_\Omega \rho(x, x')dx'$ is the centrality of x, which measures the total influence of any one degree node at x on the whole space. The

centrality actually gives the "importance" of a location in the metric space Ω instead of the importance of a node in G_n. In the scenario of between-city transportation, centrality measures the physical geographical transportation condition of a location. In the scenario of scientific research, a research topic with high centrality means it is a bridge between many other fields and therefore is important by itself, regardless of how it is recognized by citations. Homogeneous metric spaces have constant centrality $C(x) \equiv C$. Examples are: (1) unit square or cube with periodic boundary condition; (2) n-sphere in $n+1$ dimensional space; and (3) n-dimensional binary vector space with metric induced by the L1 norm.

In a metric space with constant centrality C, we further have

$$Z_n = (K_n + (n_0 + n)q)C = \big[(2m + q)n + (2m_0 + n_0 q)\big]C,$$

where $K_n = \sum_{j=1}^{n_0+n} k_{j,n} = 2(m_0 + mn)$ is the total number of degrees in the network.

We summarize the algorithm of RIPA as follows:

1. Begin with a initial graph G_0.
2. For $n = 1$ to N:
   (a) Add a new node j at the location x with probability $\mu_n(x) = z_{n-1}(x) / Z_{n-1}$.
   (b) Attach j to the old node i with probability $\Pi_{ij} = (k_{i,n-1} + q)\rho_{ij} / z_{n-1}(x_j)$ and repeat m times.

2.1 Growth Rate of Degree

For the node i, the expected increment of degree at time step n is given by

$$\begin{aligned}
&E\big[k_{i,n} - k_{i,n-1} \mid G_{n-1}\big] \\
&\quad = \int_\Omega E\big[k_{i,n} - k_{i,n-1} \mid G_{n-1}, x_j = x\big]\mu_n(x)dx \\
&\quad = \int_\Omega E\left[\sum_{v=1}^{m} \{W_{n,v} = i\} \mid G_{n-1}, x_j = x\right]\mu_n(x)dx \\
&\quad = \int_\Omega m\Pi_{ij}\mu_n(x_j)dx_j \\
&\quad = \int_\Omega m\frac{(k_{i,n-1} + q)\rho_{ij}}{z_{n-1}(x_j)}\frac{z_{n-1}(x_j)}{Z_{n-1}}dx_j \\
&\quad = m\frac{(k_{i,n-1} + q)C(x_i)}{Z_{n-1}}.
\end{aligned}$$

The preceding equation shows that the degree of a node grows at an expected speed proportional to the current degree, which is exactly the relation we have in the classical preferential attachment. Therefore, we expect the RIPA to have the same asymptotic degree distribution.

2.2 Asymptotic Behavior of $z_n(x)$

To analyze the asymptotic behavior of $z_n(x)$, we require that the metric space Ω be bounded, that is, $\sup_{x,y\in\Omega}\{d(x,y)\}$ exists. In bounded metric spaces, we show $\inf z_n(x) \sim O(n)$. But in unbounded metric spaces like $\mathbb{R}^n$, with probability 1, we have $\inf z_n(x) = 0$.

Lemma 1. In bounded metric space Ω with constant centrality C, $z_n(x) \sim O(n)$ uniformly as $n \to \infty$; that is, there exist positive constants C_1, C_2, N independent of x such that when $n \geq N$, for $\forall\, x \in \Omega$, $C_1 \leq z_n(x) \,/\, n \leq C_2$.

Proof. Ω is bounded, so $d_{\max} = \sup_{x,y\in\Omega}\{d(x,y)\}$, and $\rho(x,y) \geq \rho_0 = f(d_{\max}) > 0$. Combined with $\rho(x,y) \leq 1$, we have

$$2(mn + m_0) \geq z_n(x) = \sum_{i=1}^{n+n_0} (k_{i,n} + q)\rho(x_i, x) \geq 2(mn + m_0)\rho_0.$$

Therefore $C_1 = 2m\rho_0$, $C_2 = 2m + \epsilon$, $\epsilon > 0$ is arbitrarily small.

When the centrality is constant C, the expected change of the local partition comes from two parts: the growth of degree of the old nodes and the new node j,

$$\begin{aligned}
&E[z_n(x) - z_{n-1}(x) \mid G_{n-1}] \\
&= \sum_{i=1}^{n_0+n-1} E[k_{i,n} - k_{i,n-1} \mid G_{n-1}]\rho(x_i, x) + E[(k_{j,n} + q)\rho(x_j, x) \mid G_{n-1}] \\
&= \sum_{i=1}^{n_0+n-1} m\frac{(k_{i,n-1} + q)C}{Z_{n-1}}\rho(x_i, x) + (m+q)\int_\Omega \rho(x_j, x)\frac{z_{n-1}(x_j)}{Z_{n-1}}dx_j \\
&= \frac{m}{Z_{n-1}}Cz_{n-1}(x) + \frac{m+q}{Z_{n-1}}\int_\Omega z_{n-1}(x')\rho(x', x)dx' \\
&= \frac{1}{n + \frac{2m_0+n_0q}{2m+q}}z_{n-1}(x) + \frac{(m+q)}{(2m+q)n + (2m_0 + n_0q)}(\bar{z}_{n-1}(x) - z_{n-1}(x)),
\end{aligned}$$

where $\bar{z}_n(x) = 1 \,/\, C\int_\Omega z_n(x')\rho(x', x)dx'$, is the average of $z_n(y)$ weighted by $\rho(x, y)$. The last line of the preceding equation shows the

asymptotic behavior of $E[z_n(x)]$. When $n \to \infty$, the first term indicates a linear growth of $E[z_n(x)]$ with respect to n; the second term is a diffusion term that makes $z_n(x)$ get close to its global average Z_n / S. $S = \int_\Omega dx$ is the volume of the whole space.

We further consider

$$\begin{aligned} & E\left[\frac{z_{n+1}(x)}{n+1} - \frac{z_n(x)}{n} \,\middle|\, G_n \right] \\ &= \frac{1}{n+1} E\left[z_{n+1}(x) - z_n(x) \mid G_n \right] + \left(\frac{1}{n+1} - \frac{1}{n} \right) z_n(x) \\ &= \frac{n(m+q)}{(n+1)\left[(2m+q)(n+1) + (2m_0 + n_0 q) \right]} \left(\overline{z}_n(x) / n - z_n(x) / n \right) + O\left(\frac{1}{n^2} \right). \end{aligned}$$

Taking the expectation, let $u_n(x) = E[z_n(x) / n]$, $\overline{u}_n(x) = E\left[\overline{z}_n(x) / n \right]$; we have

$$u_{n+1}(x) - u_n(x) = \eta_n \left(\overline{u}_n(x) - u_n(x) \right) + O\left(\frac{1}{n^2} \right),$$

where $\eta_n \sim O(1 / n)$.

Neglecting the $O(1 / n^2)$ term, the equation shows that $u_n(x)$ converges to a fixed point $u(x) = u_0$. Using the perturbation method, we obtain $u_n(x) = u_0 + O(1 / n)$, and $z_n(x) = n u_0 + O(1)$. Since $\int_\Omega z_n(x) dx = Z_n = (2m+q) C n$, we conclude

$$z_n(x) = \frac{(2m+q)C}{S} n + O(1). \square$$

3. Degree Distribution

In this section, we investigate the degree distribution of the RIPA model. We follow the approach in [1] and use Lemma 2 proved in [1]:

Lemma 2. For $n \in \mathbb{N}$, let x_n, y_n, η_n, r_n be real numbers such that

$$x_{n+1} - x_n = \eta_n (y_n - x_n) + r_n$$

and

- $\lim_{n \to \infty} y_n = x$
- $\eta_n > 0$ and $\limsup_{n \to \infty} \eta_n < 1$

- $\sum_{n=1}^{\infty} \eta_n = \infty$
- $\lim_{n\to\infty} r_n / \eta_n = 0$;

then $x_n \to x$ as $n \to \infty$.

Then we prove our main theorem about the degree distribution.

Theorem 1. In bounded metric space Ω with constant centrality, as $n \to \infty$, the expected fraction of nodes with degree d:

$$\alpha_{d,n} \to \frac{2 + \frac{q}{m}}{m + q + 2 + \frac{q}{m}} \frac{\Gamma(d+q)\Gamma\left(m + 3 + q + \frac{q}{m}\right)}{\Gamma(m+q)\Gamma\left(d + 3 + q + \frac{q}{m}\right)}.$$

Proof. The probability for the new node $j = n_0 + n + 1$ attaching to node i exactly k times is

$$P(k_{i,n+1} - k_{i,n} = k \mid G_n) = \int_\Omega \binom{m}{k} \left[\frac{(k_{i,n} + q)\rho(x_i, x)}{z_n(x)}\right]^k \left[1 - \frac{(k_{i,n} + q)\rho(x_i, x)}{z_n(x)}\right]^{m-k} \frac{z_n(x)}{Z_n} dx.$$

We denote the number of nodes with degree d at time n (after attachment) by $A_{d,n}$, which has a recursive relation. In the following derivations, we use ρ, z_n as the shortenings of $\rho(x_i, x), z_n(x)$, respectively:

$$E[A_{d,n+1} \mid G_n] = \sum_{k=0}^{m} A_{d-k,n} \binom{m}{k} \int_\Omega \left(\frac{(d-k+q)\rho}{z_n}\right)^k \left(1 - \frac{(d-k+q)\rho}{z_n}\right)^{m-k} \frac{z_n}{Z_n} dx + \mathbf{1}_{\{d=m\}}.$$

$\tilde{A}_{d,n} = A_{d,n} / (n + n_0)$ is the fraction of nodes with degree d at time n. Let $d' = d + q$,

$$\begin{aligned} E[\tilde{A}_{d,n+1} \mid G_n] - \tilde{A}_{d,n} &= \left\{\frac{n+n_0}{n+n_0+1} \int_\Omega \left(1 - \frac{d'\rho}{z_n}\right)^m \frac{z_n}{Z_n} dx - 1\right\} \tilde{A}_{d,n} \\ &+ \left\{\frac{n+n_0}{n+n_0+1} m \int_\Omega \frac{(d'-1)\rho}{z_n} \left(1 - \frac{(d'-1)\rho}{z_n}\right)^{m-1} \frac{z_n}{Z_n} dx\right\} \tilde{A}_{d-1,n} \\ &+ \left\{\frac{n+n_0}{n+n_0+1} \binom{m}{2} \int_\Omega \left(\frac{(d'-2)\rho}{z_n}\right)^2 \left(1 - \frac{(d'-2)\rho}{z_n}\right)^{m-2} \frac{z_n}{Z_n} dx\right\} \tilde{A}_{d-1,n} \\ &+ \cdots + \frac{1}{n+n_0+1} \mathbf{1}_{\{d=m\}}. \end{aligned}$$

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Let $\alpha_{d,n} = E[\tilde{A}_{d,n+1}]$; the expectation of the preceding equation gives

$$\begin{aligned}\alpha_{d,n+1} - \alpha_{d,n} &= \left\{\frac{n+n_0}{n+n_0+1}\int_\Omega\left(1-\frac{d'\rho}{z_n}\right)^m\frac{z_n}{Z_n}dx - 1\right\}\alpha_{d,n}\\ &+\left\{\frac{n+n_0}{n+n_0+1}m\int_\Omega\frac{(d'-1)\rho}{z_n}\left(1-\frac{(d'-1)\rho}{z_n}\right)^{m-1}\frac{z_n}{Z_n}dx\right\}\alpha_{d-1,n}\\ &+\left\{\frac{n+n_0}{n+n_0+1}\binom{m}{2}\int_\Omega\left(\frac{(d'-2)\rho}{z_n}\right)^2\left(1-\frac{(d'-2)\rho}{z_n}\right)^{m-2}\frac{z_n}{Z_n}dx\right\}\alpha_{d-1,n}\\ &+\cdots+\frac{1}{n+n_0+1}\mathbf{1}_{\{d=m\}}.\end{aligned}$$

Since $z_n \sim O(n)$ and $\rho \le 1$, for fixed d, $d\rho/z \sim O(1/n)$ uniformly. For the first term

$$\begin{aligned}&\left\{\frac{n+n_0}{n+n_0+1}\int_\Omega\left(1-\frac{d'\rho}{z_n}\right)^m\frac{z_n}{Z_n}dx - \int_\Omega\frac{z_n}{Z_n}dx\right\}\alpha_{d,n}\\ &= \frac{\alpha_{d,n}}{n+n_0+1}\int_\Omega\left[(n+n_0)\left(1-\frac{d'\rho}{z_n}\right)^m - (n+n_0+1)\right]\frac{z_n}{Z_n}dx\\ &= \frac{-\alpha_{d,n}}{n+n_0+1}\int_\Omega\left[(n+n_0)m\frac{d'\rho}{z_n}+1+O\left(\frac{1}{n}\right)\right]\frac{z_n}{Z_n}dx\\ &= \frac{-\alpha_{d,n}}{n+n_0+1}\left[\frac{(n+n_0)m(d+q)}{(2nm+2m_0+nq+n_0q)}+1\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

The second term:

$$\begin{aligned}&\frac{n+n_0}{n+n_0+1}m\alpha_{d-1,n}\int_\Omega\left[\frac{(d'-1)\rho}{z_n}+O\left(\frac{1}{n^2}\right)\right]\frac{z_n}{Z}dx =\\ &\quad\left((n+n_0)m(d+q-1)\right)/\left((n+n_0+1)(2\mathrm{nm}+2m_0+nq+n_0q)\right)\\ &\qquad\times\alpha_{d-1,n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

The third to the $(m+1)^{\text{th}}$ terms only involve higher-order terms of $d\rho z_n$, hence are no greater than $O(1/n^2)$.

For the $d < m$ case, only the nodes in the initial graph can have degree d. So $\alpha_{d,n} \le n_0/(n_0+n) \sim O(1/n) \to 0$.

For the $d = m$ case, the second term corresponds to the degree $m-1$ and has the order $O(1/n^2)$:

$$\begin{aligned}\alpha_{m,n+1}-\alpha_{m,n} &= \frac{-\alpha_{m,n}}{n+n_0+1}\left[\frac{(n+n_0)m(m+q)}{(2nm+2m_0+nq+n_0q)}+1\right]\\ &\quad+\frac{1}{n+n_0+1}+O\left(\frac{1}{n^2}\right)\\ &= \frac{1}{n+n_0+1}\frac{m+q+2+\frac{q}{m}}{2+\frac{q}{m}}\left(\frac{2+\frac{q}{m}}{m+q+2+\frac{q}{m}}-\alpha_{m,n}\right)\\ &\quad+O\left(\frac{1}{n^2}\right).\end{aligned}$$

According to Lemma 2, when $n \to \infty$, we have

$$\alpha_{m,n} \to \frac{2+\frac{q}{m}}{m+q+2+\frac{q}{m}}.$$

For the $d > m$ case, let $c_n = (2nm+2m_0+nq+n_0q)$; we have

$$\begin{aligned}&\alpha_{d,n+1}-\alpha_{d,n} =\\ &\frac{(n+n_0)m(d+q)+c_n}{(n+n_0+1)c_n}\left[\frac{(n+n_0)m(d+q-1)}{(n+n_0)m(d+q)+c_n}\alpha_{d-1,n}-\alpha_{d,n}\right]+O\left(\frac{1}{n^2}\right).\end{aligned}$$

Invoking Lemma 2 again, we have

$$\alpha_{d,n} \to \frac{d+q-1}{d+q+2+\frac{q}{m}}\alpha_{d-1,n},$$

and

$$\begin{aligned}\alpha_{d,n} &\to \frac{2+\frac{q}{m}}{m+q+2+\frac{q}{m}}\prod_{l=m+1}^{d}\frac{l+q-1}{l+q+2+\frac{q}{m}}\\ &= \frac{2+\frac{q}{m}}{m+q+2+\frac{q}{m}}\frac{\Gamma(d+q)\Gamma\left(m+3+q+\frac{q}{m}\right)}{\Gamma(m+q)\Gamma\left(d+3+q+\frac{q}{m}\right)}.\end{aligned}$$

Particularly, when $q = 0$, we have

$$\alpha_{d,n} \to \frac{2m(m+1)}{d(d+1)(d+2)}. \square$$

4. Edge Length Distribution

In this section, we discuss the length of a randomly picked edge. This is a very important property for all the geometrically embedded

networks. We define the shell volume integral

$$\sigma(l, x) = \int_{\Omega} 1_{\{d(x,x')=l\}} dx'.$$

In spaces where $\sigma(l, x)$ does not depend on x, we use the notation $\sigma(l) = \sigma(l, x)$.

Theorem 2. In the space Ω, $C(x) = C$, $\sigma(l, x) = \sigma(x)$, l_0 is the length of an edge randomly picked from E_n. Then the distribution of l_0 converges to $f(l)\sigma(l) / C$ as $n \to \infty$.

Proof. $\{i, j\}$ is an edge randomly picked from $E_n \backslash E_0$, and by this notation, we always assume $i < j$. One way to pick the edge $\{i, j\}$ with equal probability is to randomly pick two integers s, v satisfying $1 \le s \le n$ and $1 \le v \le m$, so that j is the node added to G_n at time step s, and j attaches to i by the v^{th} attachment; that is, $W_{s,v} = i$. Then $l_{ij}(s, v)$, the length of $\{i, j\}$ with specific choices of i, s, v. Note that $i < j = n_0 + s$, so x_i, k_i is known at G_{s-1}:

$$\begin{aligned}
\mathcal{P}(l_{i,j}(s, v) &= l \mid G_{s-1}, W_{s,v} = i) \\
&= \frac{\mathcal{P}(W_{s,v} = i, l(s, v) = l \mid G_{s-1})}{\mathcal{P}(W_{s,v} = i \mid G_{s-1})} \\
&= \frac{\int_{\Omega} \frac{k_{i,s-1}\rho(x_i,x_j)}{z_{s-1}(x_j)} \frac{z_{s-1}(x_j)}{Z_{s-1}} \mathbf{1}_{\{l_{i,j}(s,v)=l\}} dx_j}{\int_{\Omega} \frac{k_{i,s-1}\rho(x_i,x_j)}{z_{s-1}(x_j)} \frac{z_{s-1}(x_j)}{Z_{s-1}} dx_j} \\
&= \frac{f(l)\sigma(l, x_i)}{C(x_i)}.
\end{aligned}$$

In the space Ω, $\sigma(l, x)$ and $C(x)$ do not depend on x, so we have

$$\mathcal{P}(l_{i,j}(s, v) = l \mid G_{s-1}, j = n_0 + s, i = W_{s,v}) = \frac{f(l)\sigma(l)}{C}.$$

This probability distribution is independent of the choices of i, s, v, so the length of a random edge $\{i, j\} \in E_n \backslash E_0$ with random parameters i, s, v also follows $f(l)\sigma(l) / C$. Assuming l_0 is the length of an edge randomly picked from E_n, since the size of E_0 is fixed as $n \to \infty$, the distribution of l_0 converges to $f(l)\sigma(l) / C$ for large n. □

According to Theorem 2, we observe a phase transition, which is also mentioned in a similar case by [11]. For the large l, the behavior

of $f(l)$ changes from a power law $f(l) \sim l^{-\gamma}$ to exponential decay; the phase transition happens at $\gamma = \dim - 1$, where dim is the dimension of the space. When $\gamma \leq \dim - 1$, the average edge length grows to infinity as the space extends to infinity, while, when $\gamma > \dim - 1$, the average edge length converges to a constant.

5. Clustering Coefficient

In this section, we show that the clustering coefficient of the RIPA model is significantly higher than non-geometric models like the BA network model. First, we prove a slightly different version of Theorem 2, considering the relative location of two neighboring nodes.

Theorem 3. For given node i, j is a node randomly picked from the younger neighbors of i; that is, $j > i$ and $\{i, j\} \in E_n \backslash E_0$. As $n \to \infty$, the probability measure of x_j converges by the L1 norm,

$$\mu_i(x_j = x \mid j \to i) \to \frac{\rho(x, x_i)}{C(x_i)},$$

where $j \to i$ denotes the event that the node j attaches to the node i at least once.

Proof. We first consider this probability measure of x_j under the condition that $j = n_0 + s$:

$$\begin{aligned}
&\mu_i(x_j = x \mid G_{s-1}, j \to i, j = n_0 + s) \\
&\quad = \frac{\mu_i(x_j = x, j \to i \mid G_{s-1}, j = n_0 + s)}{\mu_i(j \to i \mid G_{s-1}, j = n_0 + s)} \\
&\quad = \left(\int_\Omega \left[1 - \left(1 - \frac{k_{i,s-1}\rho(x_i, x')}{z_{s-1}(x')}\right)^m\right] \frac{z_{s-1}(x')}{Z_{s-1}} \delta(x - x') dx'\right) \Big/ \\
&\qquad \left(\int_\Omega \left[1 - \left(1 - \frac{k_{i,s-1}\rho(x_i, x')}{z_{s-1}(x')}\right)^m\right] \frac{z_{s-1}(x')}{Z_{s-1}} dx'\right) \\
&\quad = \frac{\rho(x_i, x)}{C(x_i)} + O\left(\frac{1}{s}\right).
\end{aligned}$$

For arbitrarily small $\epsilon > 0$, there exists a large enough N such that for $n > N$,

$$\left\| \mu_i(x_j = x \mid G_{s-1}, j \to i, j = n_0 + s) - \frac{\rho(x, x_i)}{C(x_i)} \right\|_{L_1} < \epsilon.$$

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Consider the expectation of $\mu_i(x_j = x \mid G_{s-1}, j \to i, j = n_0 + s)$ over all possible $j > N$; we have

$$\left\| \mu_i(x_j = x \mid G_N, j \to i, j > N) - \frac{\rho(x, x_i)}{C(x_i)} \right\|_{L_1} < \epsilon.$$

As $n \to \infty$, the node i will be attached to by younger nodes infinite times. Therefore

$$\left\| \mu_i(x_j = x \mid G_N, j \to i, j > N) - \mu_i(x_j = x \mid j \to i) \right\|_{L_1}$$

is also arbitrarily small. So we prove

$$\mu_i(x_j = x \mid j \to i) \to \frac{\rho(x_i, x)}{C(x_i)}$$

by the L1 norm. □

Theorem 3 shows a distinct feature of the RIPA model. It works even for the metric spaces with nonuniform $C(x)$. Consider the subgraph $H_{i,s_n} = \{V_{i,s_n}, E_{i,s_n}, X_{i,s_n}\}$, which consists of the node i and all its younger neighbors, that is,

$$V_{i,s_n} = \{i\} \cap \{j \mid n_0 + 1 \le j \le n_0 + n, j > i\}.$$

s_n is the number of nodes in H_{i,s_n} and acts as the time variable for the subgraph. According to Theorem 3, for large enough s_n, we can simulate the evolution of the subgraph H_{i,s_n} without knowing any information about the other parts of the network. This property of localization brings a lot of convenience in analysis.

The classical definition of a clustering coefficient is

$$c(G_n) = \frac{3 \times \text{number of triangles}}{\text{number of wedges}}.$$

A wedge is three nodes i, j, k linked as $i - j - k$ with i, k either linked or unlinked. This definition is equivalent to

$$c(G_n) = \mathcal{P}\,(\text{a randomly picked wedge belongs to a triangle}).$$

We denote the set of all wedges in G_n by W_n, and the set of wedges containing at least one edge from E_0 by W_n^0. Obviously $|W_n^0| / |W_n| \to 0$, as $n \to \infty$. So for large n, we only consider the wedges randomly picked from $W_n' = W_n \backslash W_n^0$. Regarding the order of the three nodes, there are three types of wedges in W_n'. Assuming $i < j < k$, the three types of wedges are: (1) $j - i - k$, (2) $i - j - k$, (3) $i - k - j$.

We first analyze the type (1) $j - i - k$ case. Define

$$w_i(x) = \frac{\rho(x_i, x)}{C(x_i)}.$$

Under the condition that j, k both attach to i, the probability measures of x_j, x_k are

$$\mu_j(x) = w_i(x) + O\left(\frac{1}{j}\right)$$

and

$$\mu_k(x) = w_i(x) + O\left(\frac{1}{k}\right),$$

respectively. Note that the dependence of x_k on x_j has already been counted in the error term $O(1/k)$. The probability for a randomly picked type (1) wedge to belong to a triangle is, at time step $n = k - n_0$,

$$\begin{aligned}
&\mathcal{P}(i \to j \mid G_{n-1}, k \to i, j \to i) \\
&\quad = \int\int_{\Omega^2} 1 - \left(1 - \frac{(k_{i,n-1} + q)\rho(x_j, x_k)}{z_{n-1}(x_k)}\right)^m d\mu_j(x_j) d\mu_k(x_k) \\
&\quad = (k_{i,n-1} + q)m \int\int_{\Omega^2} \frac{\rho(x_j, x_k)}{z_{n-1}(x_k)} \\
&\qquad + O\left(\frac{1}{n^2}\right) d\left(w_i(x_j) + O\left(\frac{1}{j}\right)\right) d\left(w_i(x_k) + O\left(\frac{1}{k}\right)\right) \\
&\quad = (k_{i,n-1} + q)m\left(1 + O\left(\frac{1}{j}\right)\right) \int\int_{\Omega^2} \frac{\rho(x_j, x_k)}{\frac{Z_{n-1}}{S} + O(1)} dw_i(x_j) dw_i(x_k) \\
&\quad = \frac{(k_{i,n-1} + q)mS}{Z_{n-1}}\left(1 + O\left(\frac{1}{j}\right)\right) \int\int_{\Omega^2} \rho(x_j, x_k) dw_i(x_j) dw_i(x_k).
\end{aligned}$$

For the classical preferential attachment, the probability for the same event is

$$\mathcal{P}'(i \to j \mid G_{n-1}, k \to i, j \to i) = \frac{(k_{i,n-1} + q)m}{K_{n-1} + (n + n_0 - 1)q}\left(1 + O\left(\frac{1}{n}\right)\right).$$

With $C(x) \equiv C$, we calculate the ratio of these two probabilities:

$$\frac{\mathcal{P}(i \to j \mid G_{n-1}, k \to i, j \to i)}{\mathcal{P}'(i \to j \mid G_{n-1}, k \to i, j \to i)} \sim \frac{\rho_1}{\rho_0},$$

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where $\rho_0 = C / S$ and

$$\rho_1 = \int\int_{\Omega^2} \rho(x_j, x_k) dw_i(x_j) dw_i(x_k).$$

Next, we give a heuristic argument to show $\rho_1 \gg \rho_0$ in most cases. Consider l_0 is the distance between two independent random locations with uniform distribution, and l_1 is the distance between two independent random locations with distribution

$$w_i(x) = \frac{\rho(x_i, x)}{C(x_i)}.$$

Because $\rho_0 = E[f(l_0)]$ and $\rho_1 = E[f(l_1)]$, we estimate ρ_0 and ρ_1 by $\rho_0 \sim f(E[l_0])$ and $\rho_1 \sim f(E[l_1])$. $E[l_0]$ is the typical length scale of the whole metric space. $E[l_1]$ is the typical length scale of the neighborhood of a node containing most of its neighbors. For most of the location-based real-world networks, there is a local-global scale separation; that is, $E[l_1] \ll E[l_0]$, which leads to $\rho_1 \gg \rho_0$. The opposite case, $E[l_1] \sim E[l_0]$, actually implies that the effect of location is not significant on the network topology.

The type (2) $i - j - k$ case is similar to the type (1) case, given that the space Ω is isotropic. In isotropic spaces, the relative locations $x_i - x_j$ and $x_j - x_i$ have the same probability distribution. Then in the probability condition, we can replace the event $i \to j$ by $j \to i$.

Finally, we analyze the type (3) $i - k - j$ case. When the node k is added to the graph and attaches to the old nodes m times, it always brings exactly $m(m-1)/2$ number of type (3) wedges. At the same time, with probability 1, it brings at least m^2 number of type (1) or (2) wedges. So the fraction of type (3) wedges in W'_n is at most $1/3$ for large n. Therefore we conclude, for large n,

$$c(G_n) \geq \frac{2\rho_1}{3\rho_0} c(G_{PA}(n))$$

where $G_{PA}(n)$ is the network generated by the classical preferential attachment by time step n started with the same initial graph G_0.

6. Average Path Length

In the area of complex networks, we say a network is a "small world" if the average path length of two arbitrary nodes in the network is no more than the order $O(\ln(N))$ as the network size N

grows. There are two different large N limits of a geometrically embedded network model. One is the non-extensive limit, for which the metric space stays the same and the density of nodes increases to infinity. The other is the extensive limit, for which the density of nodes stays the same and the metric space extends to infinity. In the latter case, an equivalent way is to keep the metric space the same and rescale the metric. For instance, on the unit square, the metric $d(x, y)$ should be rescaled as $d_N(x, y) = \sqrt{N}\, d(x, y)$, so that the average density of nodes stays constant as N grows.

According to Figure 1, the RIPA under a non-extensive limit is always a small world. The average path length even lightly decays as N grows. This observation can be interpreted as: the transportation in a fixed area becomes more convenient when you have more choices of transition points. We also observe that the RIPA under an extensive limit is a small world when the relevance function f has the power-law decay $(f(d) = d^{-2})$, but is not when f has a exponential decay $(f(d) = e^{-\lambda d})$. From the physics aspect, the two relevance functions are analogs of long-range and short-range correlations. The RIPA network is a small world when the relevance function represents a long-range correlation.

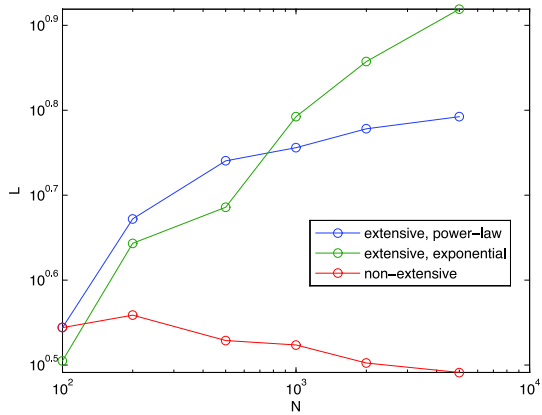

Figure 1. Average path length L in the RIPA network as network size N grows. Red plots are for the RIPA under the non-extensive large N limit. Blue and green plots are for the RIPA under the extensive large N limit. The blue plot is for the relevance function with power-law decay; the green one is for the relevance function with exponential decay.

The following theorem gives a criterion for the RIPA network in two-dimensional space that is not a small world.

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Theorem 4. The network is not a small-world network if the

$$\lim_{a\to\infty} a^2 \int_{L=a}^{\infty} Lf(L)dL = 0. \tag{1}$$

Proof. First, we show that the probability distribution $p(L)$ of L, the length of the links, is proportional to $Lf(L)$. For a fixed vertex i at location x_i at an arbitrary time step, consider the length of the next link attached to it. Ignoring the boundary effect of the two-dimensional space (for unit square it means $L < 1/2$), the probability that the new vertex j appears at x_j, which is apart from x_i with the distance L and attaches to the vertex i, is

$$\oint_{R(x_i,L)} \frac{z(x_j)}{Z} \frac{k_i \rho(x_i, x_j)}{z(x_j)} dx_j = \oint_{R(x_i,L)} \frac{k_i f(L)}{Z} dx_j \sim Lf(L)$$

where $R(x_i, L)$ is the circle centered at x_i with radius L. Since for all of the pre-existing vertices, the distribution of the length of the next new link is the same, so is the overall length distribution of the next new link at an arbitrary time step. So except for the m_0 initial links, which can be neglected in the large N limit, the length distribution $p(L)$ is proportional to $Lf(L)$.

Then we divide the two-dimensional space into blocks with edge length a. In the extensive large N limit, the density of vertices ρ_0 remains constant, so the expected number of links that are attached to the given block and longer than a is

$$\rho_0 a^2 \int_{L=a}^{\infty} Lf(L)dL.$$

If equation (1) holds, for big enough a, the probability of finding a link longer than a in a given block can be controlled by arbitrarily small $\epsilon > 0$; that is, with probability $1 - \epsilon$, one can only move to its neighboring blocks by one step along the path. Therefore, the shortest path length between two vertices with distance D has a lower bound of $D / a(1 - \epsilon)^{D/a}$, which obviously is not a small world. A similar criterion is easy to establish for R^n space. □

7. Equivalent Model: The Invitation Process

In this section, we propose an equivalent model of RIPA called the relevance and importance invitation process (RIIP). The RIIP model is much faster than RIPA as a computer algorithm and is useful in analyzing some important network properties, especially the degree-degree correlation. The algorithm of RIIP is described as follows:

1. Begin with an initial graph G_0.
2. For $n = 1$ to N:
   (a) Pick a node i as the generator with probability $(k_{i,n-1} + q)C(x_i) / Z_{n-1}$.
   (b) Locate the new node j at location x with probability $\rho(x, x_i) / C(x_i)$, and attach j to i.
   (c) Attach j to preexisting nodes for $m - 1$ independent times with probability $\Pi_{ij} = (k_{i,n-1} + q)\rho_{ij} / z_{n-1}(x_j)$.

The notations like Z_n, $z_n(x)$ are defined the same as in RIPA. The crucial difference between RIIP and RIPA lies in the arrival of new nodes. For RIPA, the arrival of new nodes follows a global probability distribution $\mu_n(x)$ that is affected by all the existing nodes and thus is very complicated. For RIIP, the arrival of new nodes is more like what happened in some private clubs: the membership of a new guest requires the invitation of an existing member and there is a default social link between the new member and his/her inviter. The RIIP is more parallelizable because each existing node invites new nodes to join the network independently and the location of the new node is only affected by its inviter. When implementing RIIP on the computer, instead of $\mu_n(x)$, we only need to evaluate $\rho(x, x_i)$, whose computational complexity does not depend on n.

As we will show, by carefully choosing the rate of the invitation for each node, we build up the RIIP, which is essentially a different stochastic process from RIPA but generates the same random network ensemble.

To prove the equivalence, we consider the process of network G_n. At time step $n + 1$, G_n is known. G' is a specific realization of G_{n+1}. $\mathcal{P}_{\mathrm{RIPA}}(G_{n+1} = G' \mid G_n)$ is the probability for $G_{n+1} = G'$ under the filtration G_n using the RIPA algorithm. $\mathcal{P}_{\mathrm{RIIP}}$ is defined in the same way. We just need to prove

$$\mathcal{P}_{\mathrm{RIPA}}(G_{n+1} = G' \mid G_n) = \mathcal{P}_{\mathrm{RIIP}}(G_{n+1} = G' \mid G_n).$$

For simplicity, we first consider the $m = 1$ case. Without loss of generality, let G' be the network state that the last new node j locates at x_0 and attaches to the node i. Therefore

$$\begin{aligned}\mathcal{P}_{\mathrm{RIPA}}(G_{n+1} = G' \mid G_n) &= \mu(x_0)\Pi_{ij} = \frac{z_n(x_0)}{Z_n}\frac{(k_{i,n} + q)\rho(x_i, x_0)}{z_n(x_0)} \\ &= \frac{(k_{i,n} + q)\rho(x_i, x_0)}{Z_n}.\end{aligned}$$

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$$\mathcal{P}_{\mathrm{RIIP}}\big(G_{n+1} = G' \mid G_n\big) = \frac{\big(k_{i,n}+q\big)C(x_i)}{Z_n}\frac{\rho(x_i, x_0)}{C(x_i)} = \frac{\big(k_{i,n}+q\big)\rho(x_i, x_0)}{Z_n}.$$

More generally, when $m \geq 1$, G' is the network state that the last new node j locates at x_0 and attaches to the nodes $i_1, i_2, \ldots, i_m$:

$$\begin{aligned}\mathcal{P}_{\mathrm{RIPA}}\big(G_{n+1} = G' \mid G_n\big) &= \mu_{n+1}(x_0)m!\Pi_{i_1 j}\ldots\Pi_{i_m j}\\ &= \frac{m!}{Z_n z_n^{m-1}(x_0)}\prod_{i=i_1\ldots i_m}\big(k_{i,n}+q\big)\rho(x_i, x_0).\end{aligned}$$

To calculate $\mathcal{P}_{\mathrm{RIIP}}\big(G_{n+1} = G' \mid G_n\big)$, we first consider the case that i_1 is the generator. The probability for this case is

$$\begin{aligned}\mathcal{P}_{i_1} &= \frac{\big(k_{i_1,n}+q\big)C\big(x_{i_1}\big)}{Z_n}\frac{\rho\big(x_0, x_{i_1}\big)}{C\big(x_{i_1}\big)}\big(m-1\big)!\Pi_{i_2 j}\ldots\Pi_{i_m j}\\ &= \frac{\big(m-1\big)!}{Z_n z_n^{m-1}(x_0)}\prod_{i=i_1\ldots i_m}\big(k_{i,n}+q\big)\rho(x_i, x_0).\end{aligned}$$

Summing up the probabilities of all such cases, we have

$$\mathcal{P}_{\mathrm{RIIP}}\big(G_{n+1} = G' \mid G_n\big) = m\mathcal{P}_{i_1} = \mathcal{P}_{\mathrm{RIPA}}\big(G_{n+1} = G' \mid G_n\big).$$

7.1 Degree-Degree Correlation

When $m = 1$, it is easy to show that all the degrees $k_{i,n}$ are independent by the construction of the RIIP model.

When $m \geq 2$, calculate the probability that the new node k attaches to both nodes i and j at the time step $n+1$:

$$\begin{aligned}P_1 &= \mathcal{P}\big(k \to i, k \to j \mid G_n\big)\\ &= \int\frac{2}{Z_n z_n(x_0)}\big(k_{i,n}+q\big)\rho(x_i, x_0)\big(k_{j,n}+q\big)\rho\big(x_j, x_0\big)dx_0\\ &= \frac{2\big(k_{i,n}+q\big)\big(k_{j,n}+q\big)}{Z_n}\int\frac{\rho(x_i, x_0)\rho\big(x_j, x_0\big)}{z_n(x_0)}dx_0.\end{aligned}$$

As a baseline, we also calculate the probability for the same event under the assumption that $k_{i,n}$, $k_{j,n}$ are independent:

$$\begin{aligned}P_2 &= \mathcal{P}\big(k \to i \mid G_n\big)\mathcal{P}\big(k \to j \mid G_n\big)\\ &= 2\int\frac{\big(k_{i,n}+q\big)\rho(x_i, x_0)}{Z_n}dx_0\int\frac{\big(k_{j,n}+q\big)\rho\big(x_j, x_0\big)}{Z_n}dx_0\\ &= \frac{2\big(k_{i,n}+q\big)\big(k_{j,n}+q\big)}{Z_n^2}C(x_i)C\big(x_j\big).\end{aligned}$$

We define

$$\Delta = P_1 - P_2 = \mathcal{P}(k \to i \mid G_n)\left[\mathcal{P}(k \to j \mid G_n, k \to i) - \mathcal{P}(k \to j \mid G_n)\right].$$

Δ implies the correlation between k_i and k_j. When $\Delta > 0$, k_i and k_j are correlated; when $\Delta < 0$, k_i and k_j are anti-correlated:

$$\Delta = \frac{2(k_{i,n} + q)(k_{j,n} + q)}{Z_n}\left[\int \frac{\rho(x_i, x_0)\rho(x_j, x_0)}{z(x_0)} dx_0 - \frac{C(x_i)C(x_j)}{Z_n}\right].$$

Assume n is large enough and $C(x) \equiv C$,

$$z_n(x) = \frac{Z_n}{S} + O(1);$$

we obtain

$$\Delta = B\left[\langle \rho(x_i, x)\rho(x_i, x)\rangle_\Omega - \langle \rho(x_i, x)\rangle_\Omega \langle \rho(x_j, x)\rangle_\Omega + O(1)\right],$$

where $B > 0$, $\langle \cdot \rangle_\Omega$ is the average over the space Ω. In this case, we conclude the degree-degree correlation between i, j goes along with the correlation between $\rho(x_i, x)$, $\rho(x_j, x)$, which mainly depends on the distance between the two nodes. For two close-enough nodes i and j, $\rho(x_i, x)$ and $\rho(x_j, x)$ are positively correlated. For two nodes far apart enough such that relevant to one means irrelevant to the other, $\rho(x_i, x_0)$ and $\rho(x_j, x_0)$ are negatively correlated.

8. Between-City Transportation

In this section, we focus on RIPA on a two-dimensional surface with respect to the case of between-city transportation. First, we consider networks generated by RIPA on the unit square D with periodic boundary conditions. The relevance $\rho(x, y) = f(d(x, y))$ is given by $f(x) = \exp(-\lambda x)$. In this case, the total partition function is:

$$Z_n = \int_{x \in D} \sum_{j=1}^{n_0+n} (k_{j,n} + q) e^{-\lambda d(x_j, x)} dx.$$

Figure 2 represents a special realization of the network. Each circle in the figure represents a city, the center of the circle indicates the locations of the city, and the radius indicates the degree; the color (brightness) in the background indicates the logarithm of the local partition function $z(x)$. In Figure 2, we observe the phenomenon that cities tend to gather but big cities tend to separate. For example, around the most important city (the capital), we can find a bigger city in the area further from the capital. This is because a huge city has

two effects: (1) the local partition in its neighbor area is bigger, therefore it attracts more new cities; and (2) it will attract more links from new cities, therefore inhibit the nearby cities from growing. The second effect is the more significant when we choose smaller m.

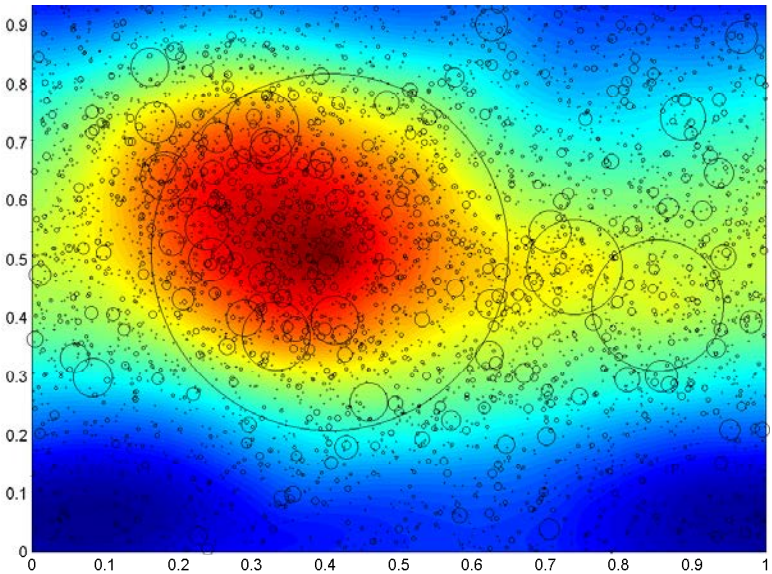

Figure 2. Network generated on a unit square with periodic boundary condition. $m = 1$, $N = 5000$, $q = 0$, $\lambda = 10$. The circles are centered at the locations of the cities and the radii represent their degrees. The background color indicates the logarithm of local partition.

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