Japanese Mathematical Olympiad 1993

Final Round – February 11

1. Suppose that two different words A and B have the same length $n > 1$ and that they differ in the first letter only. Prove that A or B is not periodic.

2. Let $d(n)$ denote the largest odd divisor of $n \in \mathbb{N}$. Define
$$D(n) = d(1) + d(2) + \cdots + d(n),$$
$$T(n) = 1 + 2 + \cdots + n.$$
Show that there exist infinitely many numbers n such that $3D(n) = 2T(n)$.

3. In a contest, x students took part and y problems were posed. Each student solved $y/2$ problems and every problem was solved by the same number of students. For any two students, only three problems were solved by both of them. Determine all possible pairs (x,y), and for each such (x,y) give an example of the matrix (a_{ij}) defined by $a_{ij} = 1$ if $i-$th student solved the $j-$th problem and $a_{ij} = 0$ otherwise.

4. Five diameters of a sphere are given, no three of which are in a plane. Among the 32 possible choices of an endpoint from each segment, find the number of choices for which the 5 points are in a hemisphere.

5. Prove that there is a constant $C > 0$ such that the inequality
$$\max_{0 \le x \le 2} \prod_{j=1}^{n} |x - a_j| \le C^n \max_{0 \le x \le 1} \prod_{j=1}^{n} |x - a_j|$$
holds for any $n \in \mathbb{N}$ and any real numbers $a_1, a_2, \ldots, a_n$.

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