Automorphic forms on $O_{s+2,2}(\mathbf{R})^+$ and generalized Kac-Moody algebras.

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We discuss how modular forms and automorphic forms can be written as infinite products, and how some of these infinite products appear in the theory of generalized Kac-Moody algebras. This paper is based on my talk at the ICM, and is an exposition of [B5].

1. Product formulas for modular forms.

We will start off by listing some apparently random and unrelated facts about modular forms, which will begin to make sense in a page or two. A modular form of level 1 and weight k is a holomorphic function f on the upper half plane $\{\tau \in \mathbf{C} | \Im(\tau) > 0\}$ such that $f((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f(\tau)$ for $\binom{ab}{cd} \in SL_2(\mathbf{Z})$ that is "holomorphic at the cusps". Recall that the ring of modular forms of level 1 is generated by $E_4(\tau) = 1+240\sum_{n>0}\sigma_3(n)q^n$ of weight 4 and $E_6(\tau) = 1-504\sum_{n>0}\sigma_5(n)q^n$ of weight 6, where $q = e^{2\pi i\tau}$ and $\sigma_k(n) = \sum_{d|n} d^k$. There is a well known product formula for $\Delta(\tau) = (E_4(\tau)^3 - E_6(\tau)^2)/1728$

$$\Delta(\tau) = q\prod_{n>0}(1-q^n)^{24}$$

due to Jacobi. This suggests that we could try to write other modular forms, for example E_4 or E_6, as infinite products. At first sight this does not seem to be very promising. We can formally expand any power series as an infinite product of the form $q^h\prod_{n>0}(1-q^n)^{a(n)}$, and if we do this for E_4 we find that

$$\begin{aligned} E_4(\tau) &= 1+240q+2160q^2+6720q^3+\cdots \\ &= (1-q)^{-240}(1-q^2)^{26760}(1-q^3)^{-4096240}\cdots \end{aligned} \tag{1.1}$$

but this infinite product does not even converge everywhere because the coefficients are exponentially increasing. In fact such an infinite product can only converge everywhere if the function it represents has no zeros in the upper half plane, and the only level 1 modular forms with this property are the powers of Δ. On the other hand there is a vague principle that a function should have a nice product expansion if and only if its zeros and poles are arranged nicely. Well known examples of this are Euler's product formulas for the gamma function and zeta function, and Jacobi's product formulas for theta functions. (Of course the region of convergence of the infinite product will usually not be the whole region where the function is defined, because it cannot contain any zeros or poles.) The zeros of any modular form are arranged in a reasonably regular way which suggests that some modular forms with zeros might still have nice infinite product expansions.

For a reason that will appear soon we will now look at modular forms of level 4 and weight 1/2 which are holomorphic on the upper half plane but are allowed to have poles at cusps. Kohnen's work [Ko] on the Shimura correspondence suggests that we should look at the subspace A of such forms $f = \sum_{n\in\mathbf{Z}} c(n)q^n$ whose Fourier coefficients $c(n)$ are all integers and vanish unless $n \equiv 0$ or 1 mod 4. It is easy to find the structure of A: it is a 2-dimensional free module over the ring of polynomials $\mathbf{Z}[j(4\tau)]$,where $j(\tau)$ is the elliptic modular function $j(\tau) = E_4(\tau)^3/\Delta(\tau) = q^{-1}+744+196884q+\cdots$. Equivalently, any sequence of numbers $c(n)$ for $n \le 0$ such that $c(n) = 0$ unless $n \equiv 0$ or 1 mod 4 is the set of coefficients of q^n for $n \le 0$ of a unique function in A. The space A has a basis consisting of the following two elements:

$$\begin{aligned} \theta(\tau) &= \sum_{n\in\mathbf{Z}} q^{n^2} = 1+2q+2q^4+\cdots \\ \psi(\tau) &= F(\tau)\theta(\tau)(\theta(\tau)^4-2F(\tau))(\theta(\tau)^4-16F(\tau))E_6(4\tau)/\Delta(4\tau)+60\theta(\tau) \\ &= q^{-3}+4-240q+26760q^4-85995q^5+1707264q^8-4096240q^9+\cdots \end{aligned} \tag{1.2}$$

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where

$$F(\tau) = \sum_{n>0, n \text{ odd}} \sigma_1(n) q^n = q + 4q^3 + 6q^5 \cdots .$$

The reader will now understand the reason for all these odd definitions by comparing the coefficients of ψ in (1.2) with the exponents in (1.1). This is a special case of the following theorem.

Theorem 1.3. *([B5]) Suppose that B is the space of meromorphic modular forms Φ of integral weight and level 1 for some character of $SL_2(\mathbf{Z})$ such that Φ has integral coefficients, leading coefficient 1, and all the zeros and poles of Φ are at cusps or imaginary quadratic irrationals. Then the map taking $f(\tau) = \sum_{n\in\mathbf{Z}} c(n)q^n \in A$ to*

$$\Phi(\tau) = q^h \prod_{n>0} (1-q^n)^{c(n^2)}$$

is an isomorphism from A to B. (Here h is a certain rational number in $\frac{1}{12}\mathbf{Z}$ depending linearly on f.) The weight of Φ is $c(0)$, and the multiplicity of the zero of Φ at an imaginary quadratic number of discriminant $D<0$ is $\sum_{n>0} c(n^2 D)$.

Example 1.4 If $f(\tau) = 12\theta(\tau) = 12 + 24q + 24q^4 + \cdots$ then $c(n^2) = 24$ for $n > 0$, so $\Phi(\tau) = q\prod_{n>0}(1-q^n)^{c(n^2)} = q\prod_{n>0}(1-q^n)^{24} = \Delta(\tau)$ which is the usual product formula for the Δ function. The fact that $f(\tau)$ is holomorphic corresponds to the fact that $\Delta(\tau)$ has no zeros, and the constant term 12 of $f(\tau)$ is the weight of Δ.

Example 1.5 Most of the common modular forms or functions, for example the Eisenstein series E_4, E_6, E_8, E_{10}, and E_{14}, the delta function $\Delta(\tau)$, and the elliptic modular function $j(\tau)$, all belong to the space B and can therefore be written explicitly as infinite products. It is easy to work out the function f corresponding to and Φ by using the remarks about weights and multiplicities of zeros at the end of theorem 1.3; for example, if $\Phi(\tau) = j(\tau) - 1728$, then Φ has a zero of order 2 at every imaginary quadratic irrational of discriminant -4 and has weight 0, so the corresponding function $f \in A$ must be of the form $2q^{-4} + 0q^0 + O(q)$ and must therefore be $2\theta(\tau)(j(4\tau) - 738) - 4\psi(\tau)$.

Theorem 1.3 looks superficially similar to the Shimura correspondence; both correspondences use infinite products to take certain modular forms of half integral weight to modular forms of integral weight. However there are several major differences: the Shimura correspondence uses Euler product expansions, only works for holomorphic modular forms, and is an additive rather than a multiplicative correspondence.

2. Product formulas for $j(\sigma) - j(\tau)$. *In this section we describe 3 different product formulas for $j(\sigma) - j(\tau)$ (where $j(\tau) = \sum_n c(n)q^n = q^{-1} + 744 + \cdots$ is the elliptic modular function).*

The simplest one is valid for any σ, τ with large imaginary part (>1 will do), and is

$$j(\sigma) - j(\tau) = p^{-1} \prod_{m>0, n\in\mathbf{Z}} (1 - p^m q^n)^{c(mn)} \tag{2.1}$$

where $p = e^{2\pi i\sigma}$. This is the denominator formula for the monster Lie algebra; see 5.2.

The next product formula was found by Gross and Zagier [GZ]. We let d_1 and d_2 be negative integers which are 0 or 1 mod 4, and for simplicity we suppose that they are both less than -4. Then

$$\prod_{[\tau_1],[\tau_2]} (j(\tau_1) - j(\tau_2)) = \pm \prod_{x\in Z, n, n'>0, x^2+4nn'=d_1 d_2} n^{\epsilon(n')}$$

where the first product is over representatives of equivalence classes of imaginary quadratic irrationals of discriminants d_1, d_2, and $\epsilon(n') = \pm 1$ is defined in [GZ]. An example of this they give is

$$j\left(\frac{1+i\sqrt{67}}{2}\right) - j\left(\frac{1+i\sqrt{163}}{2}\right) = -2^{15}3^3 5^3 11^3 + 2^{18}3^3 5^3 23^3 29^3$$
$$= 2^{15}3^7 5^3 7^2 13 \times 139 \times 331.$$

In the first product formula σ and τ were both arbitrary complex numbers with large imaginary part, and in the second they were both fixed to run over imaginary quadratic irrationals. The third product formula is a sort of cross between these, because we allow τ to be any complex number with large imaginary part, and make σ run over a set of representatives of imaginary quadratic irrationals of some fixed discriminant d. For simplicity we will assume $d < -4$ and d squarefree. In this case we find that

$$\prod_{[\sigma]}(j(\tau)-j(\sigma)) = q^{-h}\prod_{n\geq 1}(1-q^n)^{c(n^2)},$$

where the numbers $c(n)$ are the coefficients of the unique power series in the space A of 1.3 of the form $q^{-d}+O(q)$, and h is the class number of the imaginary quadratic field of $\sqrt{d}$. This follows from theorem 1.1 because it is easy to see that the product on the left lies in the space B. Conversely the analogue this product formula for all values of d together with the Jacobi product formula for the eta function implies theorem 1.1.

Strangely enough, there seems to be no obvious direct connection between these 3 product formulas. In particular assuming any two of them does not seem to be any help for proving the third one. (Proposition 5.1 of [GZ] almost gives a fourth product formula: it expresses $\log|j(\sigma)-j(\tau)|$ as a limit of an infinite sum.)

3. Automorphic forms for $O_{s+2,2}(\mathbf{R})$.

Theorem 1.3 is essentially a specialization of a product formula for automorphic forms on higher dimensional orthogonal groups $O_{s+2,2}(\mathbf{R})^+$. Before giving this generalization we recall the definitions of automorphic forms on orthogonal groups and of rational quadratic divisors.

We will show how to construct the analogue of the upper half plane for these groups. Suppose that L is a Lorentzian lattice of dimension $s+2$, in other words, a nonsingular lattice of dimension $s+2$ and signature s. The negative norm vectors in $L\otimes\mathbf{R}$ form two open cones; we choose one of these cones and call it C. We define H to be the subset of vectors $\tau\in L\otimes\mathbf{C}$ such that $\Im(\tau)\in C$, so that if L is one dimensional then H is isomorphic to the upper half plane. There is an obvious discrete group acting on H generated by the translations $\tau\to\tau+\lambda$ for $\lambda\in L$ and the automorphisms $O_L(\mathbf{Z})^+$ of L that map C into itself. When H is the upper half plane this group is just the group of translations $\tau\to\tau+n$ for $n\in\mathbf{Z}$. In this case we can enlarge the group to $SL_2(\mathbf{Z})$ acting on the upper half plane (by $\binom{ab}{cd}(\tau)=(a\tau+b)/(c\tau+d)$) by adding an extra automorphism $\tau\to -1/\tau$. The analogue of this for unimodular Lorentzian lattices L is the automorphism $\tau\to 2\tau/(\tau,\tau)$.

An automorphic form of weight k on the upper half plane H is a function satisfying the two functional equations

$$f(\tau+n) = f(\tau)\quad (n\in\mathbf{Z})$$
$$f(-1/\tau) = \tau^k f(\tau)$$

(and some conditions about being holomorphic). By analogy with this, if L is an even unimodular Lorentzian lattice, we define an automorphic form on H to be a holomorphic function f on H satisfying the functional equations

$$f(\tau+\lambda) = f(\tau)\quad (\lambda\in L)$$
$$f(w(\tau)) = \pm f(\tau)\quad (w\in O_L(\mathbf{Z})^+)$$
$$f(2\tau/(\tau,\tau)) = \pm((\tau,\tau)/2)^k f(\tau).$$

The group generated by all these transformations is isomorphic to a subgroup of index 2 of the automorphism group of the lattice $M = L\oplus II_{1,1}$, where $II_{1,1}$ is the 2-dimensional even Lorentzian lattice (with inner product matrix $\binom{01}{10}$).

Suppose that $b\in L$ and $a,c\in\mathbf{Z}$, with $(b,b)>2ac$. The set of vectors $y\in L\otimes\mathbf{C}$ with $a(y,y)/2+(b,y)+c=0$ is called a rational quadratic divisor. A rational quadratic divisor in the upper half plane is the same as an imaginary quadratic irrational.

We choose some vector in $-C$ which has nonzero inner product with all vectors of L, and we write $r>0$ to mean that $r\in L$ has positive inner product with this vector.

We have seen in theorem 1.3 that a modular form with integer coefficients tends to have a nice infinite product expansion if all its zeros are imaginary quadratic irrationals. The next theorem shows that a similar phenomenon occurs for automorphic forms on $O_{s+2,2}(\mathbf{R})^+$, provided we replace imaginary quadratic irrationals by rational quadratic divisors.

Theorem 3.1. *([B5]) Suppose that $f(\tau) = \sum_n c(n)q^n$ is a meromorphic modular form with all poles at cusps. Suppose also that f is of weight $-s/2$ for $SL_2(\mathbf{Z})$ and has integer coefficients, with $24|c(0)$ if $s = 0$. There is a unique vector $\rho \in L$ such that*

$$\Phi(v) = e^{-2\pi i(\rho,v)} \prod_{r>0} (1 - e^{-2\pi i(r,v)})^{c(-(r,r)/2)}$$

is a meromorphic automorphic form of weight $c(0)/2$ for $O_M(\mathbf{Z})^+$. (Or more precisely, can be analytically continued to a meromorphic automorphic form, because the infinite product does not converge everywhere.) All the zeros and poles of Φ lie on rational quadratic divisors, and the multiplicity of the zero of Φ at the rational quadratic divisor of the triple (b, a, c) (with no common factors) is

$$\sum_{n>0} c(n^2(ac - (b, b)/2)).$$

We see that just as in theorem 1.3, the coefficients of negative powers of q in f determine the zeros of Φ, and the constant term of f determines the weight of Φ.

Notice that the zeros of rational quadratic divisors with $a = c = 0$ can be seen as zeros of factors of the infinite product, but the other zeros cannot be seen so easily; they are not zeros of any of the factors of the infinite product and therefore lie outside the region where the infinite product converges.

4. Generalized Kac-Moody algebras.

Some of the infinite products giving automorphic forms appear in the theory of generalized Kac-Moody algebras, so in this section we briefly recall some facts about these.

Generalized Kac-Moody algebras are best thought of as infinite dimensional analogues of finite dimensional reductive Lie algebras. They can almost be defined as Lie algebras G having the following structure [B4]

1. G should have a nonsingular invariant bilinear form $(\,,\,)$.
2. G should have a self centralizing subalgebra H, called the Cartan subalgebra, such that G is the sum of eigenspaces of H.
3. The roots of G (i.e., the eigenvalues of H acting on G) should have properties similar to those of the roots of a finite dimensional reductive Lie algebra. In particular it should be possible to choose a set of "positive" roots $\alpha > 0$ with good properties, a set of "simple roots", and there should be a "Weyl group" W generated by reflections of real (norm ≥ 0) simple roots. Also G has a "symmetrized Cartan matrix", whose entries are the inner products of the simple roots.

An earlier characterization [B1] identified generalized Kac-Moody algebras as Lie algebras with an "almost positive definite contravariant bilinear form", but the one summarized above is easier to use in practice because it avoids the rather difficult problem of proving positive definiteness.

There is a generalization of the Weyl character formula for the characters of some irreducible highest weight representations of generalized Kac-Moody algebras, and in particular there is a generalization of the denominator formula (coming from the character formula for the trivial representation) which is

$$\sum_{w\in W} \det(w)w(e^\rho S) = e^\rho \prod_{\alpha>0} (1 - e^\alpha)^{\mathrm{mult}(\alpha)} \tag{4.1}$$

where $\mathrm{mult}(\alpha)$ is the multiplicity of the root α, i.e., the dimension of the corresponding eigenspace. The vector ρ is a "Weyl vector", and S is a correction term depending on the imaginary (norm ≤ 0) simple roots. For finite dimensional reductive Lie algebras, and more generally for Kac-Moody algebras, there are no imaginary simple roots so $S = 1$ and we recover the usual Weyl-Kac denominator formula.

The best known examples of generalized Kac-Moody algebras are the finite dimensional reductive Lie algebras, the affine Lie algebras, and the Heisenberg Lie algebra (which should be thought of as a sort of degenerate affine Lie algebra). Beyond these there are an enormous number of nonaffine generalized Kac-Moody algebras, which can be constructed by writing down a random symmetrized Cartan matrix, and then writing down some generators and relations corresponding to it. Most of these Lie algebras seem to be of little interest, and it does not usually seem possible to find a clean description of both the root multiplicities and the simple roots. (It is not difficult to find large alternating sums for these numbers by using the denominator formula, but these sums seem too complicated to be of much use; for example, they do not lead to good bounds for the root multiplicities.) There are a handful of good nonaffine generalized Kac-Moody algebras, for which we can describe both the simple roots and the root multiplicities explicitly. (See the next section for some examples.) These all turn out to have the property that the product in the denominator formula is an automorphic form for an orthogonal group $O_{s+2,2}(\mathbf{R})^+$, where $s+2$ is the dimension of the Cartan subalgebra. This suggests that this property of the denominator function being an automorphic form can be used to separate out the "interesting" generalized Kac-Moody algebras from the rest. (Something similar happens for the affine Kac-Moody algebras: in this case the denominator function is a Jacobi form [EZ].)

5. Examples.

We finish by giving some applications and special cases of theorem 3.1.

Example 5.1 If we have an automorphic form for the group $O_{s+2,2}(\mathbf{R})^+$ with an infinite product expansion we can restrict it to smaller subspaces to obtain automorphic forms for smaller groups $O_{s-n+2,2}(\mathbf{R})^+$ with infinite product expansions. For example if we restrict Φ to the multiples τv of a fixed norm $-2N$ vector $v \in L$ (for τ in the upper half plane) we get a modular form of level N. In particular by specializing the forms in theorem 3.1 we obtain many ordinary modular forms for $SL_2(\mathbf{R})$ (which is locally isomorphic to $O_{1,2}(\mathbf{R})$) with infinite product expansions, and this can be used to prove theorem 1.3. Similarly we can get examples of Hilbert modular forms and genus 2 Siegel modular forms with infinite product expansions by using the fact that the groups $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$ and $Sp_4(\mathbf{R})$ are locally isomorphic to $O_{2,2}(\mathbf{R})$ and $O_{3,2}(\mathbf{R})$.

Example 5.2 The simplest nontrivial case of theorem 3.1 is when L is the lattice $II_{1,1}$ and $f(\tau)$ is the elliptic modular function $j(\tau) - 744 = \sum_n c(n)q^n$. In this case theorem 3.1 says that the infinite product

$$p^{-1} \prod_{m>0, n\in\mathbf{Z}} (1 - p^m q^n)^{c(mn)}$$

is an automorphic function on $H \times H$ (where H is the upper half plane). This product is just the right hand side of 2.1, and using the fact that it is an automorphic function with known zeros it is easy to identify it as $j(\sigma) - j(\tau)$. This identity 2.1 is the denominator formula 4.1 for the monster Lie algebra, a generalized Kac-Moody algebra acted on by the monster simple group which is the Lie algebra of physical states of a chiral string on an orbifold of a 26-dimensional torus [B3].

Example 5.3 Suppose we take L to be the 26-dimensional even unimodular lattice $II_{25,1}$, and take $f(\tau)$ to be $1/\Delta(\tau) = \sum_n p_{24}(n+1)q^n = q^{-1} + 24 + 324q^2 + \cdots$. Then by theorem 3.1 we know that

$$\Phi(v) = e^{-2\pi i(\rho,v)} \prod_{r>0} (1 - e^{-2\pi i(r,v)})^{p_{24}(1-(r,r)/2)} \tag{5.4}$$

is a holomorphic automorphic form of weight $24/2 = 12$. We can identify it explicitly using some facts about singular automorphic forms on $O_{s+2,2}(\mathbf{R})^+$. Any holomorphic automorphic form on $O_{s+2,2}(\mathbf{R})^+$ can be expanded as a power series $\Phi(v) = \sum_{r\in\bar{C}} c(r)e^{-2\pi i(r,v)}$ where the coefficients $c(r)$ are zero unless r lies in the closure $\bar{C}$ of the cone C. If the coefficients $c(r)$ are zero unless r lies on the boundary of C then we say that Φ is singular. It turns out that Φ is singular if and only if its weight is a "singular weight", and for $O_{s+2,2}(\mathbf{R})^+$ the singular weights are 0 and $s/2$. (Moreover any automorphic form of weight less than $s/2$ must be constant of weight 0.) In particular the form $\Phi(v)$ above has singular weight $12 = 24/2$ so its coefficients $c(r)$ vanish unless $(r,r) = 0$. But for any automorphic form it is easy to find the multiplicities of the coefficients $c(r)$ with $(r,r) = 0$, and if we do this for Φ we find that

$$\Phi(v) = \sum_{w\in W} \det(w)\Delta((v, w(\rho))) \tag{5.5}$$

where ρ is a norm 0 vector and W is the reflection group of the lattice $II_{25,1}$. If we compare 5.4 with 5.5 we obtain the denominator formula for another Lie algebra called the fake monster Lie algebra [B2], which is the Lie algebra of physical states of a chiral string on the torus $\mathbf{R}^{25,1}/II_{25,1}$ [B3].

Incidentally we also get a short proof of the existence of the Leech lattice (a 24-dimensional even unimodular lattice with no roots), because it is not hard to show that if ρ has norm 0 then the lattice $\rho^\perp/\rho$ is extremal (i.e., has no vectors of norm $\leq s/12$), and the fact that $c(\rho) = 1$ is nonzero implies that ρ must have norm 0 because Φ is singular. It is also possible to prove the uniqueness of the Leech lattice and the fact that it has covering radius $\sqrt{2}$ using similar arguments. Unfortunately this argument does not seem to produce examples of extremal lattices in higher dimensions because the forms Φ no longer have singular weight.

The two examples above are particularly simple because the coefficients $c(r)$ vanish for $(r,r) \neq 0$, so it is easy to identify them all. Most of the automorphic forms constructed in theorem 3.1 do not have this property and seem to be harder to describe explicitly. Moreover most of them do not seem to be related to generalized Kac-Moody algebras, because all the positive norm roots of a generalized Kac-Moody algebra with root lattice $II_{s+1,1}$ must have norm 2, which means that the function $f(\tau)$ cannot have any terms in q^n for $n \leq -2$.

We conclude with an example of a generalized Kac-Moody algebra related to one of the modular forms in theorem 1.3. Example 5.6. The product formula

$$\begin{aligned} E_6(\tau) &= 1 - 504\sum_{n>0}\sigma_5(n)q^n \\ &= 1 - 504q - 16632q^2 - 122976q^3 - \cdots \\ &= \prod_{n>0}(1-q^n)^{c(n^2)} \\ &= (1-q)^{504}(1-q^2)^{143388}(1-q^3)^{51180024}\cdots \end{aligned}$$

where

$$\begin{aligned} &\sum_n c(n^2)q^n \\ =&\theta(\tau)(j(4\tau) - 1470) - 2\phi(\tau) \\ =&q^{-4} + 6 + 504q + 143388q^4 + 565760q^5 + 18473000q^8 + 51180024q^9 + O(q^{12}) \end{aligned}$$

is the denominator formula for a generalized Kac-Moody algebra of rank 1 whose simple roots are all multiples of some root α of norm -2, the simple roots are $n\alpha$ ($\alpha > 0$) with multiplicity $504\sigma_5(n)$, and the multiplicity of the root $n\alpha$ is $c(n^2)$. The positive subalgebra of this generalized Kac-Moody algebra is a free Lie algebra, so we can also state this result by saying that the free graded Lie algebra with $504\sigma_5(n)$ generators of each positive degree n has a degree n piece of dimension $c(n^2)$. There are similar examples corresponding to the infinite products for the Eisenstein series E_{10} and E_{14}. The identity for E_{14} is easy to prove directly because it follows from 2.1 by dividing both sides by $p-q$, setting $p = q$, and using the fact that $j'(\tau) = E_{14}(\tau)/\Delta(\tau)$.

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