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Abstract

Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicator functions defined on a probability space $(\Omega, \mathcal{A}, P)$. We say that index k is a success time if $I_k = 1$. The sequence $I_1, I_2, \ldots, I_n$ is observed sequentially.

The objective of this article is to predict the l-th last success, if any, with maximum probability at the time of its occurence. We find the optimal rule and discuss briefly an algorithm to compute it in an efficient way. This generalizes the result of Bruss (1998) for $l = 1$, and is equivalent to the problem of (multiple) stopping with l stops on the last l successes. We extend then the model to a larger class allowing for an unknown number N of indicator functions, and present, in particular, a convenient method for an approximate solution if the success probabilities are small. We also discuss some applications of the results.

Keywords: "Sum the odds" algorithm, optimal stopping, multiple stopping, stopping islands, generating functions, modified secretary problems, unimodality.

AMS subject classification: 60G40

1 Introduction.

Let $E_1, E_2, \ldots, E_n$ be a sequence of n independent well-defined experiments. The outcome of each E_k is classified either as a success or a failure. Further, let $I_k = 1$ if E_k results in a success, and $I_k = 0$ otherwise, i.e. I_k is the success indicator of the k-th experiment.

We study the following problem. A decision maker observes sequentially I_1, $I_2, \ldots, I_n$ with the objective to predict, with maximum probability, correctly the occurences of the last l $(1 \leq l \leq n)$ successes at their respective occurence times.

This is clearly equivalent to predict, with maximum probability, the occurence of the l-th last success and to enumerate simply all following successes. If there are, from the first announcement onwards, exactly l successes up to step n, then the decision maker's game is defined to be a success, otherwise a failure.

This problem has been studied in detail for the case $l = 1$ by Bruss (1998). This case stands out for two reasons. Firstly, viewing applications, it is, in many ways, the most natural problem. If we define for instance I_k as the indicator of the event that E_k results in a record observation of a certain variable observed in all experiment, then stopping on (or predicting the) last success means stopping on the largest observation without recall of preceding observations. This is the kernel of many problems like investment problems, secretary problems and others. Secondly, the case $l = 1$ allows for a quick recursive solution by the so-called "Sum the odds" algorithm, which is so simple that it often does not even require paper and pencil.

In this paper, we first prove that the optimal prediction rule is always of the following form: for each l, $1 \le l \le n$, there exists a non-random index s_l from which onwards one should announce the first success as being the l-th last one. After such an announcement, it is understood that the next $l-1$ successes are automatically announced as being the $l-1$ last ones (we say that we "stop" from s_l onwards). We then describe the corresponding solution algorithm to find s_l. Although it becomes more and more computationally involved as l increases, many asymptotic cases (as e.g. $E(I_k) \to 0$ for all k as $n \to \infty$) are equally nice for $l > 1$ as for $l = 1$.

The last section discusses possible applications of our results.

2 Mathematical Formulation.

Let $I_1, I_2, \ldots, I_n$ be a sequence of independent indicator functions defined on a probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{F}_k = \sigma(I_1, I_2, \ldots, I_k)$ denote the σ-field generated by $I_1, I_2, \ldots, I_k$. Further let $\mathcal{T}$ be the set of stopping rules τ such that $\{\tau = k\} \in \mathcal{F}_k$. Given $p_j = E(I_j)$, $j = 1, \ldots, n$, the problem is to show that there exists $\tau_l^\star \in \mathcal{T}$ maximizing the probability of the event $I_{\tau_l} = 1$ and $I_{\tau_l+1} + I_{\tau_l+2} + \ldots + I_n = l-1$, and to find its value. Thus we want to find

$$\tau_l^\star = \arg \sup_{\tau \in \mathcal{T}} P\Big(\sum_{k=\tau}^{n} I_k = l \Big)$$

and the corresponding value which we denote by $V(l, n)$.

3 Results.

Recall that $p_k = E(I_k)$, and put $q_k := 1 - p_k$ and $r_k := p_k/q_k$. Further, let

$$R_{0,k} := 1$$

and, for $l \geq 1$,

$$R_{l,k} := \sum_{\substack{i_1,\ldots,i_l=k \\ all \neq}}^{n} r_{i_1} \ldots r_{i_l}, \tag{1}$$

where "all $\neq$" means that the sum is taken over all different ordered sets $\{i_1, i_2, \ldots, i_l\}$. We also define

$$\tilde{R}_{l,k} = \sum_{\substack{i_1,\ldots,i_l=k \\ i_1<\ldots<i_l}}^{n} r_{i_1} \ldots r_{i_l} \quad \Big(= R_{l,k}/(l!)\Big) \tag{2}$$

and

$$Q_k := \prod_{j=k+1}^{n} q_j. \tag{3}$$

Form of the optimal strategy.

We now state the first part of the main result. It will show that the optimal strategy is of the simple form as announced in the Introduction.

Theorem 1 *An optimal rule for stopping on the l-th last success exists and is to stop on the first index (if any) k with $I_k = 1$ and $k \geq s_l$ for some fixed $1 \leq s_l \leq n - l + 1$.*

Proof. Let $g_j(t)$ and $G_k(t)$ denote respectively the probability generating functions of I_j and $S_k := I_{k+1} + I_{k+2} + \ldots + I_n$. Independence of the I_j's yields

$$g_j(t) = q_j + p_j\, t\,; \quad G_k(t) = \prod_{j=k+1}^{n} (q_j + p_j\, t) = Q_k \prod_{j=k+1}^{n} (1 + r_j\, t),$$

where, as before, $Q_k := \prod_{j=k+1}^{n} q_j$.

Let $G_k^{(l)}(t_0)$ denote the derivative of order l of $G_k(t)$ evaluated in $t = t_0$. For $l = 1$ we obtain

$$G_k'(s) = Q_k \Big(\sum_{j=k+1}^{n} r_j \prod_{\substack{i=k+1 \\ i \neq j}}^{n} (1 + r_i s) \Big), \quad k = 0, 1, \ldots, n-1, \tag{4}$$

and then by induction on l

$$G_k^{(l)}(s) = Q_k \Big(\sum_{\substack{j_1, \ldots, j_l = k+1 \\ all \neq}}^{n} r_{j_1} \ldots r_{j_l} \prod_{\substack{i=k+1 \\ i \neq j_1, \ldots, j_l}}^{n} (1 + r_i s) \Big), \quad k = 0, 1, \ldots, n-l. \tag{5}$$

From (4) and (5), we get

$$P(S_k = 1) = G_k'(0) = Q_k \, R_{1,k+1}, \quad k = 0, 1, \ldots, n-1,$$

and

$$l! \; P(S_k = l) = G_k^{(l)}(0) = Q_k \, R_{l,k+1}, \quad k = 0, 1, \ldots, n-l,$$

or

$$P(S_k = l) = Q_k \, \tilde{R}_{l,k+1}, \quad k = 0, 1, \ldots, n-l.$$

The crucial step is to prove that $P(S_k = l)$ is unimodal in k. For the rest, the proof of the corresponding result in Bruss (1998) for $l = 1$ extends immediately for general l. We summarize these arguments. If $P(S_k = l)$ is unimodal in k then the optimal rule must be simple (without stopping island) and of the form: announce the first success after some, possibly random, waiting time W as being the l-th last. However, since S_W is, by construction, independent of $I_1, \ldots, I_W$, the optimal waiting time W must coincide with the deterministic optimal waiting time $s_l - 1$.

To prove unimodality, it will be helpful to look again first at the case $l = 1$. Let $k \leq n - 1$. In order to understand the evolution of $G_k'(0)$ for k's decreasing from $n - 1$ to 0, let us study the transition between $G_k'(0)$ and $G_{k-1}'(0)$:

$$\begin{aligned} G_{k-1}'(0) \;=\; Q_{k-1} \, R_{1,k} \;&=\; Q_k \, q_k \, (R_{1,k+1} + r_k) \\ &=\; Q_k \Big[(R_{1,k+1} - 1) \, q_k + 1 \Big] \end{aligned}$$

Hence, with $q_k < 1$ (if $q_k = 1$, we simply have $G'_{k-1}(0) = G'_k(0)$), we have $G'_{k-1}(0) > G'_k(0) = Q_k\, R_{1,k+1}$ if and only if $(R_{1,k+1} - 1)\, q_k + 1 > R_{1,k+1}$, i.e. if and only if $R_{1,k+1} < 1$.

Similarly we obtain, assuming that $q_k < 1$,

$$\begin{array}{lcl} R_{1,k+1} = 1 & \Leftrightarrow & G'_{k-1}(0) = G'_k(0) \\ R_{1,k+1} > 1 & \Leftrightarrow & G'_{k-1}(0) < G'_k(0). \end{array}$$

Viewing these relations, $P(S_k = 1)$ is clearly unimodal. Working in the same way for $l \geq 2$, we obtain

$$\begin{aligned} G^{(l)}_{k-1}(0) &= Q_k\, q_k\, (R_{l,k+1} + l\, r_k\, R_{l-1,k+1}) \\ &= Q_k \Big[(R_{l,k+1} - l\, R_{l-1,k+1})\, q_k + l\, R_{l-1,k+1} \Big]. \end{aligned}$$

Again, if $q_k < 1$ (if $q_k = 1$, we still have $G^{(l)}_{k-1}(0) = G^{(l)}_k(0)$), we have

$$\begin{array}{lcl} A_{l,k} := R_{l,k+1} - l\, R_{l-1,k+1} < 0 & \Leftrightarrow & G^{(l)}_{k-1}(0) > G^{(l)}_k(0) \\ A_{l,k} := R_{l,k+1} - l\, R_{l-1,k+1} = 0 & \Leftrightarrow & G^{(l)}_{k-1}(0) = G^{(l)}_k(0) \\ A_{l,k} := R_{l,k+1} - l\, R_{l-1,k+1} > 0 & \Leftrightarrow & G^{(l)}_{k-1}(0) < G^{(l)}_k(0). \end{array} \tag{6}$$

Remark. If we define $A_{1,k} := R_{1,k+1} - R_{0,k+1}$, i.e. $A_{1,k} = R_{1,k+1} - 1$, we also obtain, if $q_k < 1$,

$$\begin{array}{lcl} A_{1,k} < 0 & \Leftrightarrow & G'_{k-1}(0) > G'_k(0) \\ A_{1,k} = 0 & \Leftrightarrow & G'_{k-1}(0) = G'_k(0) \\ A_{1,k} > 0 & \Leftrightarrow & G'_{k-1}(0) < G'_k(0). \end{array}$$

so that the properties (6) are true for all $l \geq 1$.

To prove unimodality of $G^{(l)}_k(0)$, it suffices, from (6), to show that, if $A_{l,K} > 0$, $K \leq n - l$, then we have $A_{l,k} > 0$ for all $k \leq K$.

But, since

$$A_{l,k-1} = A_{l,k} + l\, r_k\, A_{l-1,k}, \tag{7}$$

it is sufficient to prove that

$$A_{l,k} > 0 \Rightarrow A_{l-1,k} \geq 0 \quad \forall\, 2 \leq k \leq n-l.$$

We will in fact prove something slightly stronger, namely

$$A_{l,k} > 0 \Rightarrow A_{l-1,k} > 0 \quad \forall\, 0 \leq k \leq n-l. \tag{8}$$

We will denote by $(I_{l,k})$ the implication (8). We will now prove unimodality by proving all implications $(I_{l,k})$ by double induction on k and l.

Let us first prove the implications $(I_{2,k})$ by an induction on $k \leq n-2$.

i) For $k = n-2$, $A_{1,k} = r_{n-1} + r_n - 1 \leq 0$ implies $r_{n-1} r_n \leq r_{n-1}$ and so we have $2\, r_{n-1} r_n - 2\,(r_{n-1} + r_n) \leq 0$, i.e. $A_{2,k} \leq 0$.

ii) Assume statement $(I_{2,k})$ is true. If $A_{1,k-1} \leq 0$, we have $A_{1,k} \leq 0$. So, using the induction hypothesis, $A_{2,k} \leq 0$ and $2\, r_k\, A_{1,k} \leq 0$. By summing up, we therefore obtain $A_{2,k-1} \leq 0$.

It remains to be shown that the statement $(I_{l,k})$ holds for $l > 2$ $(k \leq n-l)$. Using an induction on l, assume that the implications $(I_{l-1,k})$ are true for $0 \leq k \leq n-l+1$. We will first prove implications $(I_{l,n-l})$ for $3 \leq l \leq n$. Then, we will show that if the implication $(I_{l,\tilde{k}})$ and the implications $(I_{l-1,k})$ are true for all k, then $(I_{l,\tilde{k}-1})$ is true.

i) Note that

$$\begin{aligned}
R_{l,n-l} &= l!\,(r_{n-l+1} \ldots r_n) \\
R_{l-1,n-l} &= (l-1)! \sum_{i=n-l+1}^{n} r_{n-l+1} \ldots \hat{r}_i \ldots r_n \\
&= (l-1)!\,(r_{n-l+1} \ldots r_n) \left(\frac{1}{r_{n-l+1}} + \ldots + \frac{1}{r_n}\right) \\
R_{l-2,n-l} &= (l-2)! \sum_{i,j=n-l+1,\, i \neq j}^{n} r_{n-l+1} \ldots \hat{r}_i \ldots \hat{r}_j \ldots r_n,
\end{aligned}$$

where $\hat{r}_i$ means that the factor r_i is replaced by 1. Using (6) and (8), the implication $(I_{l,n-l})$ turns into

$$(r_{n-l+1}\ldots r_n) > (r_{n-l+1}\ldots r_n)\left(\frac{1}{r_{n-l+1}}+\ldots+\frac{1}{r_n}\right)$$

which implies

$$(r_{n-l+1}\ldots r_n)\left(\frac{1}{r_{n-l+1}}+\ldots+\frac{1}{r_n}\right) > \sum_{i,j=n-l+1,i\neq j}^{n} r_{n-l+1}\ldots\hat{r}_i\ldots\hat{r}_j\ldots r_n.$$

Note that the hypothesis of statement $(I_{l,n-l})$ ensures $r_j > 0$ for all $j \geq n-l+1$. To prove this implication, it suffices to check that

$$\left(\frac{1}{r_{n-l+1}}+\ldots+\frac{1}{r_n}\right)^{-1}\sum_{i,j=n-l+1,i\neq j}^{n} r_{n-l+1}\ldots\hat{r}_i\ldots\hat{r}_j\ldots r_n$$
$$\leq (r_{n-l+1}\ldots r_n)\left(\frac{1}{r_{n-l+1}}+\ldots+\frac{1}{r_n}\right).$$

This inequality is true since it is equivalent to

$$\sum_{i,j=n-l+1,i\neq j}^{n} r_{n-l+1}\ldots\hat{r}_i\ldots\hat{r}_j\ldots r_n \leq (r_{n-l+1}\ldots r_n)\,\frac{\left(\sum_{i=n-l+1}^{n} r_{n-l+1}\ldots\hat{r}_i\ldots r_n\right)^2}{\left(r_{n-l+1}\ldots r_n\right)^2},$$

or again,

$$\sum_{i,j=n-l+1,i\neq j}^{n} r_{n-l+1}^2\ldots r_i\ldots r_j\ldots r_n^2 \leq \left(\sum_{i=n-l+1}^{n} r_{n-l+1}\ldots\hat{r}_i\ldots r_n\right)^2.$$

This proves implication $(I_{l,n-l})$.

ii) Assume now that the implications $(I_{l,\tilde{k}})$ and $(I_{l-1,k})$ hold for all $0 \leq k \leq n-l+1$. We will show that the implication $(I_{l,\tilde{k}-1})$ is then also true.

From the $(I_{l-1,k})$'s and (7), we deduce that

$$A_{l-1,\tilde{k}-1} \leq 0 \Rightarrow A_{l-1,\tilde{k}} \leq 0.$$

Furthermore, $A_{l-1,\tilde{k}} \leq 0 \Rightarrow A_{l,\tilde{k}} \leq 0$, since this is the contraposition of implication $(I_{l,\tilde{k}})$. Using (7), we therefore obtain

$$A_{l-1,\tilde{k}-1} \leq 0 \Rightarrow A_{l,\tilde{k}-1} \leq 0 \ ,$$

which is the contraposition of implication $(I_{l,\tilde{k}-1})$.

This proves unimodality and completes the proof.

□

A formula for s_l.

We now state the second part of the main result which is the value of s_l. Note that we have made no distinction so far between the case $p_j > 0$ for all j, respectively $p_j = 0$ for some j. Indeed this was not necessary for the proof of unimodality in Theorem 1. However, to give a general formula for s_l and the optimal value, we now define

$$\pi_k := \#\{\, j \geq k \,|\, r_j > 0 \,\},$$

and prove the following result.

Theorem 2 *1. Let $p_j < 1$ for all $j = 1, 2, \ldots, n$. Then, with $R_{l,k}$ defined in (1),*

$$s_l = \sup \Big\{ 1, \, \sup \{\, 1 \leq k \leq n-l+1 \ : \ R_{l,k} \geq l\, R_{l-1,k} \text{ and } \pi_k \geq l \,\} \Big\},$$

and the optimal reward is given by

$$V(l,n) = Q_{s_l}\, \tilde{R}_{l,s_l}.$$

2. $k_l^\star := s_l - 1$ is the largest index maximizing $P(S_k = l)$, provided that the second sup *in the definition of s_l is not taken on an empty set.*

Note also that it is not obvious that $k_l^\star$ should maximize $P(S_{k_l^\star} = l) = G_{k_l^\star}^{(l)}(0)/(l!)$. But $k_l^\star$ satisfies $A_{l,k_l^\star} \geq 0$. If $A_{l,k_l^\star} > 0$, the proof of unimodality showed that $A_{l,j} > 0$ for all $j < k_l^\star$, so that, from properties (6), $k_l^\star$ maximizes clearly $P(S_{k_l^\star} = l)$ in that case. But what could happen if $A_{l,k_l^\star} = 0$? Could we then have $A_{l,j} < 0$ for some $j < k_l^\star$? As shown in Lemma 1 below, the answer is no, provided that $\pi_{k_l^\star+1} \geq l$. That's why we required $\pi_{s_l} \geq l$ in Theorem 2.

Lemma 1 *Let $l \geq 2$ and $k \leq n - l$. Assume that $A_{l,k} = 0$ and that $A_{l,j} < 0$ for some $j < k$. Then, we have $\pi_{k+1} < l - 1$.*

Lemma 1 will be proved in the Appendix. Using Lemma 1 we now prove Theorem 2.

Proof. Recall $k_l^\star$ defined in Theorem 2. We denote by K_l the largest index k such that $P(S_k = l)$ is maximal.

$k_l^\star \leq K_l$: If $A_{l,k_l^\star} > 0$, we have $A_{l,j} > 0$ for all $j \leq k_l^\star$. Thus, from properties (6), $G_j^{(l)}(0) \leq G_{k_l^\star}^{(l)}(0)$ for all $j \leq k_l^\star$. If $A_{l,k_l^\star} = 0$, we have $A_{l,j} \geq 0$ for all $j \leq k_l^\star$ (it is trivial for $l = 1$, and it follows from Lemma 1 for $l \geq 2$ since $\pi_{k_l^\star+1} \geq l$). Therefore, we have again $G_j^{(l)}(0) \leq G_{k_l^\star}^{(l)}(0)$ for all $j \leq k_l^\star$.

$K_l \leq k_l^\star$: If $\pi_{k_l^\star+2} \geq l$, we have necessarily $A_{l,k_l^\star+1} < 0$. It follows that $r_{k_l^\star+1} > 0$ (because we must have $A_{l,k_l^\star} \geq 0$). From properties (6), we thus have $G_{k_l^\star}^{(l)}(0) > G_{k_l^\star+1}^{(l)}(0)$. Hence $K_l \leq k_l^\star$. If $\pi_{k_l^\star+2} < l$, $\pi_{k_l^\star+1} = l$ and $\pi_{k_l^\star+2} = l - 1$. We conclude respectively that $G_{k_l^\star}^{(l)}(0) > 0$ and $G_{k_l^\star+1}^{(l)}(0) = 0$. Hence $K_l \leq k_l^\star$. $\square$

4 The algorithm.

In this short section, we will make some remarks about efficient algorithms to compute s_l . First of all, recalling that s_l is the largest k which satisfies $R_{l,k} \geq l\, R_{l-1,k}$, one could think that the computation of the sums $R_{l,k}$, $k \geq s_l$, could be avoided by using the "Ersatz" sums $R_k^{(l)} = \sum_{i=k}^n (r_i)^l$, which are easier to compute. In terms of these sums, the stopping conditions for $l = 2$ and 3, for instance, can be rewritten respectively

$$(R_k^{(1)})^2 - R_k^{(2)} \geq 2\, R_k^{(1)},$$

$$(R_k^{(1)})^3 - 3\, R_k^{(2)} R_k^{(1)} + 2\, R_k^{(3)} \geq 3 \left((R_k^{(1)})^2 - R_k^{(2)} \right).$$

The relation between the sums $R_{l,k}$ and the power sums $R_k^{(l)}$ is due to Newton (see David and Kendall (1966)). It can be shown that, as we would write today in form of a determinant,

$$R_{l,k} = \begin{vmatrix} R_k^{(1)} & 1 & 0 & 0 & \dots & 0 \\ R_k^{(2)} & R_k^{(1)} & 2 & 0 & \dots & 0 \\ R_k^{(3)} & R_k^{(2)} & R_k^{(1)} & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_k^{(l)} & R_k^{(l-1)} & R_k^{(l-2)} & R_k^{(l-3)} & \dots & R_k^{(1)} \end{vmatrix}.$$

Although the power sums are more convenient, the associated algorithm will not be more efficient since we would have to incorporate the computation of these new stopping conditions. Note that alternatively, we can compute the sums $R_{l,k}$ in an efficient way, using the relation $R_{l,k} = R_{l,k+1} + l\, r_k\, R_{l-1,k+1}$. We could still slightly improve the algorithm by working with the sums *without repetitions* $\tilde{R}_{l,k}$. Indeed, the stopping conditions turn then into $\tilde{R}_{l,k} \geq \tilde{R}_{l-1,k}$, while the recursive relation to compute the sums becomes $\tilde{R}_{l,k} = \tilde{R}_{l,k+1} + r_k\, \tilde{R}_{l-1,k+1}$.

5 Asymptotics and trials on Poisson arrivals.

Since

$$R_{1,k}\, R_{l-1,k} = R_{l,k} + (l-1) \sum_{\substack{i_1,\dots,i_l=k \\ all \neq}}^{n} (r_{i_1})^2\, r_{i_2} \dots r_{i_{l-1}}, \tag{9}$$

the stopping condition defining s_l can be rewritten in the form

$$R_{1,k} - (l-1)\, \frac{\sum (r_{i_1})^2\, r_{i_2} \dots r_{i_{l-1}}}{R_{l-1,k}} \geq l. \tag{10}$$

Clearly, this means that $R_{1,k} - l\, O(\|r_j\|_\infty) \geq l$ for $\|r_j\|_\infty \to 0$, where the $O(\|r_j\|_\infty)$ is positive and does not depend on l. So if we define

$$s_l^a := \sup \Big\{ 1, \sup \{ 1 \leq k \leq n-l+1 \ : \ R_{1,k} \geq l \} \Big\}, \tag{11}$$

the error $s_l^a - s_l$ tends to 0 when $\|r_j\|_\infty$ tends to 0. Furthermore, notice from (10), that we always have $s_l^a \geq s_l$, and that, for fixed values p_k, $s_l^a - s_l$ increases with l.

For the following problem, the situation for $l > 1$ is equally nice as for $l = 1$. Let us consider $(N(t))$ an inhomogeneous Poisson process with intensity rate $\lambda(t)$. Let $h(t)$ be the success parameter function for an experiment occuring at time t, where a success at time t is independent of preceding outcomes. Assume that $\phi(t) = \lambda(t)\,h(t)$ has at most finitely many discontinuities on a given horizon $[0, T]$. The objective is to stop the process with maximum probability on the l-th last success.

We consider an arbitrary partition $[0, t_1] \cup]t_1, t_2] \cup \ldots \cup]t_{m-1}, t_m]$ of $[0, T]$ ($t_m = T$). If p_k denotes the probability of at least one success in $]t_{k-1}, t_k]$, then it follows from our assumption that $p_k = \lambda(t_k)\,h(t_k)\,(t_k - t_{k-1}) + o(t_k - t_{k-1})$, with at most a fixed number of exceptions for k. When the caliber of the partition tends to 0 we obtain a well-defined limiting success intensity ϕ at time t, which is $\phi(t) = \lambda(t)\,h(t)$. Note that the limiting "odds-intensity" for successes is here just the limiting success intensity $\phi(t)$ at time t. Further, since the O-term in (10) vanishes in this case, it suffices to define, similarly as for $l = 1$,

$$\tau_l := \sup \left\{ 0, \sup \left\{ 0 \le t \le T \;:\; \int_t^T \phi(u)\,du = l \right\} \right\} \tag{12}$$

to obtain the optimal rule which is to announce the first success (if any) after time τ_l as being the l-th last success.

6 Some applications.

1. Let us look at the secretary problem, but now with the objective of selecting the l top records. Let $I_k = 1$ if the k-th observation is a record ($I_k = 0$ otherwise). It is well known that $E(I_k) = 1/k$ and that the I_k are independent random variables (see Rënyi (1962)). So $r_k = 1/(k-1)$, and s_l^a is simply given by $R_{1,s_l^a} = 1/(n-1) + 1/(n-2) + \ldots + 1/(s_l^a - 1)$, stopped at l. Here, "stopped at l" is shorthand for stopped as soon as this sum reaches or exceeds l. So $s_l^a/n \to (1/e)^l$ as $n \to \infty$, since $\sum_k 1/k$ diverges and thus $1/s_l^a \to 0$. Since $\sup_{j \ge s_l^a} r_j \to 0$, we have $s_l^a - s_l \to 0$ as $n \to \infty$, as we have seen after equation (11). So the well-known limiting result $s_l/n \to (1/e)^l$ as $n \to \infty$ appears as a direct consequence of Theorem 2. We can also see from Theorem 2 that the asymptotic optimal reward for the top-l-records-objective is given by

$$V(l, n) \to \frac{l^l}{(l!)\,e^l} \qquad \text{as } n \to \infty. \tag{13}$$

Indeed, if $\tilde{R}_{l,k}$ denotes the sum defined in (2), stopping from index s_l^a onwards yields the reward $(s_l^a - 1)/n)\, \tilde{R}_{l,s_l^a} = ((s_l^a - 1)/n)\, R_{l,s_l^a}/(l!)$, where $(s_l^a - 1)/n \to (1/e)^l$. So it suffices to show that $R_{l,s_l^a} \to l^l$ as $n \to \infty$. Using the relation (9), we see that R_{l,s_l^a}, $R_{1,s_l^a}R_{l-1,s_l^a}$, $(R_{1,s_l^a})^2 R_{l-2,s_l^a}, \ldots$, $(R_{1,s_l^a})^l$ have the same limit as $n \to \infty$. Since R_{1,s_l^a} is stopped at l, we get the result.

2. For the case $l = 1$, several applications were given in Bruss (1998), including a dice problem, the secretary problem and the group interview problem. Clearly, this case is in many ways the most natural one because optimal stopping problems often allow for just one stop.

In the secretary problem with two choices, say, suppose we succeed in stopping on the last two records (for a detailed study of records see Arnold et al. (1998)). We then succeed in getting the best of all, because the latter is always the last record, and we also get the second last record, i.e. together, we get the two top records. Thus, stopping on the last two records is more demanding than just stopping on two occasions to get the best. Note that if we want to obtain the best with two stops, having used one choice already, we have to change to the optimal l=1-rule (see Gilbert and Mosteller (1966) and Sakaguchi (1978)). However, we do not have this option if we want to obtain the last two records, and so the success probability for the latter is essentially smaller. As $n \to \infty$ for instance, we obtain from (13)

$$P(S_{s_2-1} = 2) \longrightarrow 2\, e^{-2} \approx 0.27067, \quad \text{as } n \to \infty,$$

compared with $P(\text{win if either selection is best}) \longrightarrow e^{-1} + e^{-3/2} \approx 0.59101$, as $n \to \infty$ (see also Tamaki (1979 a) and (1979 b)).

Ano (1989) studied optimal selection problems with three choices; for a recursion solving the (asymptotic) problem for l choices, see Bruss (1988); for a strongly related problem of choice with assignments, see Rose (1982). Multiple stopping rules for secretary problem are studied by Praeter (1994). More general multiple stopping rules are investigated by Stadje (1985).

Appendix : Proof of Lemma 1.

We will use an induction on l.

a) $l = 2$**:** Let $\tilde{j} = \sup\{\, j < k \,|\, A_{2,j} < 0 \,\}$. So we have $A_{2,\tilde{j}} < 0$ and $A_{2,j} = 0$ for all $j = \tilde{j}+1, \ldots, k$. Equality (7) implies that $A_{1,\tilde{j}+1} < 0$, thus $A_{1,j} < 0$ for all $j = \tilde{j}+1, \ldots, n-1$. By using $(I_{2,j})$, we therefore obtain $A_{2,j} \leq 0$ for all $j = \tilde{j}+1, \ldots, n-2$. From (7), we cannot have $A_{2,k} = 0$, $A_{2,k+1} \leq 0$, $A_{1,k+1} < 0$ and $r_{k+1} > 0$ (if $k \leq n-3$). We may therefore assume that $r_{k+1} = 0$, and we get that $A_{2,k+1} = 0$. Repeating this argument, we obtain

$$\begin{aligned} r_j &= 0 \quad \forall j = k+1, \ldots, n-2 \\ A_{2,j} &= 0 \quad \forall j = k+1, \ldots, n-2. \end{aligned} \tag{14}$$

Hence $\pi_{k+1} \leq 2$.

Assume now that $\pi_{k+1} = 2$, i.e. $r_{n-1} > 0$ and $r_n > 0$. Then from $A_{1,n-1} < 0$ ($r_n < 1$) we get $r_{n-1}\, r_n < r_{n-1}$, and so $A_{2,n-2} < 0$. This contradicts (14). Assume next that $\pi_{k+1} = 1$, i.e. $r_{n-1}\, r_n = 0$ and $r_{n-1} + r_n > 0$. Then $r_{n-1}\, r_n < r_{n-1} + r_n$, i.e. $A_{2,n-2} < 0$, which again contradicts (14).

Thus $\pi_{k+1} = 0$, which proves Lemma 1 for $l = 2$.

b) $l \geq 3$**:** Let $\tilde{j} = \sup\{\, j < k \,|\, A_{l,j} < 0 \,\}$. So we have $A_{l,\tilde{j}} < 0$ and $A_{l,j} = 0$ for all $j = \tilde{j}+1, \ldots, k$. Equality (7) implies that $A_{l-1,\tilde{j}+1} < 0$, thus $A_{l-1,j} \leq 0$ for all $j = \tilde{j}+1, \ldots, n-l+1$. By using $(I_{l,j})$, we therefore obtain $A_{l,j} \leq 0$ for all $j = \tilde{j}+1, \ldots, n-l$.

If $A_{l-1,k+1} = 0$, the induction argument yields $\pi_{k+2} < l-2$. Hence $\pi_{k+1} < l-1$. We may therefore assume that $A_{l-1,k+1} < 0$. But from (7), we cannot have $A_{l,k} = 0$, $A_{l,k+1} \leq 0$, $A_{l-1,k+1} < 0$ and $r_{k+1} > 0$ (if $k \leq n-l-1$). Hence $r_{k+1} = 0$, and $A_{l,k+1} = 0$.

Again, if $A_{l-1,k+2} = 0$, the induction argument yields $\pi_{k+3} < l-2$. Then $\pi_{k+1} < l-1$ since $r_{k+1} = 0$. We may therefore assume that $A_{l-1,k+2} < 0$. Then we must have $r_{k+2} = 0$, and $A_{l,k+2} = 0$ (if $k \leq n-l-2$).

By repeating this argument, we either obtain the proof of Lemma 1, or else

$$r_j = 0, A_{l-1,j} < 0, A_{l,j} = 0 \tag{15}$$

for all $j = k+1, \ldots, n-l$. Hence $\pi_{k+1} \leq l$.

Assume now that $\pi_{k+1} = l$, i.e. $r_j > 0$ for all $j \geq n-l+1$. Then $A_{l-1,n-l+1} \leq 0$ yields after straightforward simplifications

$$(r_{n-l+1} \dots r_n) < \sum_{i=n-l+1}^{n} r_{n-l+1} \dots \hat{r}_i \dots r_n,$$

i.e. $A_{l,n-l} < 0$. This contradicts (15).

Assume now that $\pi_{k+1} = l-1$, i.e. there exists a unique $j \geq n-l+1$ such that $r_j = 0$. Then we have

$$0 = (r_{n-l+1} \dots r_n) < \sum_{i=n-l+1}^{n} r_{n-l+1} \dots \hat{r}_i \dots r_n,$$

i.e. $A_{l,n-l} < 0$, which contradicts (15).

Therefore $\pi_{k+1} < l-1$, and Lemma 1 is proved.

$\square$

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