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Automatic Computing Methods for Special Functions. Part III. The Sine, Cosine, Exponential Integrals, and Related Functions

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Accurate, efficient, automatic methods for computing the sine, cosine, exponential integrals and hyperbolic sine and cosine integrals are detailed and implemented in an American National Standard FORTRAN program. The functions are also tabulated to 35 significant figures for arguments 0, $10^J\,(10^J)\,10^{J+1}$ with $J = -2(1)2$.

Key words: Continued fraction; cosine integral; exponential integral; FORTRAN program; hyperbolic sine and cosine integrals; key values; recurrence relations.

1. Introduction

Since the sine, cosine, exponential integrals and hyperbolic sine, and cosine integrals are frequently encountered together in physical problems and their expansions have terms in common, we have incorporated these functions into Part III. (For Parts I and II, see.[1]).

While accuracy over the entire domain of definition remains our main concern, we have tended toward methods that also ensure efficiency, portability, and ease of programming and modification. The number of terms in series, the number of convergents in an iterative process, the starting arguments for different methods, are all determined by the program as a function of word length, arguments, accuracy desired, etc. More realistic results are returned when error conditions are encountered. The proper analytic behavior of the function will always be retained to further ensure correct limiting values, in particular of related functions and for purposes of differentiation and integration.

In Parts I and II in addition to the implementing ANS FORTRAN program, we had included a driver (test) program and its results. Since either of these driver programs can be readily modified to compute other functions, we have omitted the driver program and in place of its results have included a table of correct results to 35 significant figures covering essentially the functional range of present computers.

2. Mathematical Properties

Relevant formulas are collected here for completeness and ease of reference. In keeping with the convention of the Handbook [1],[2] x here is a real variable.

[1] Automatic Computing Methods for Special Functions. Part I. Error, Probability, and Related Functions, J. Res. Nat. Bur. Stand. (U.S.), **74B**, (Math. Sci), No. 3, 211–224 (July–Sept. 1970). Automatic Computing Methods for Special Functions. Part II. The Exponential Integral $E_n(x)$. J. Res. Nat. Bur. Stand. (U.S.), **78B**, (Math. Sci), No. 4, 199–216 (Oct.–Dec. 1974).

[2] Figures in brackets indicate the literature references on page

A. Definitions

$$Si(x) = \int_0^x \frac{\sin t}{t}\,dt$$

$$Ci(x) = \gamma + \ln x + \int_0^x \frac{\cos t - 1}{t}\,dt$$

$$Ei(x) = -\,-\!\!\!\!\!\!\int_{-x}^{\infty} \frac{e^{-t}}{t}\,dt = -\!\!\!\!\!\!\int_{-\infty}^{x} \frac{e^{t}}{t}\,dt \qquad (x > 0)$$

(For $\int_x^{\infty} \frac{e^{-t}}{t}\,dt = E_1(x)$, often denoted by $-Ei(-x)$, see Part II.)

$$Shi(x) = \int_0^x \frac{\sinh t}{t}\,dt \qquad = \frac{Ei(x) + E_1(x)}{2}$$

$$Chi(x) = \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}\,dt = \frac{Ei(x) - E_1(x)}{2}$$

γ (Euler's constant) = 0.57721 56649 . . ..

B. Series Expansions

$$Si(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}$$

$$Ci(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{(2k)(2k)!}$$

$$Ei(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^k}{(k)(k)!} \qquad (x > 0)$$

$$Shi(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)(2k+1)!}$$

$$Chi(x) = \gamma + \ln x + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)(2k)!}$$

C. Continued Fraction

$$-Ci(x) + i[Si(x) - \pi/2] = e^{-ix}\left[\frac{1}{ix+}\,\frac{1}{1+}\,\frac{1}{ix+}\,\frac{2}{1+}\,\frac{2}{ix+\ldots}\right] \qquad (0 < x)$$

$$= E_1(ix)$$

D. Asymptotic Expansions

$$Si(x) = \pi/2 - f(x)\cos x - g(x)\sin x$$

$$Ci(x) = f(x)\sin x - g(x)\cos x$$

where $f(x) \sim \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{x^{2k}}$

and $g(x) \sim \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)!}{x^{2k+1}}$

$$Ei(x) \sim \frac{e^x}{x}\sum_{k=0}^{\infty}\frac{k!}{x^k} \qquad (x > 0)$$

$$Shi(x) = \frac{1}{x}[p(x)\cosh x + q(x)\sinh x]$$

$$Chi(x) = \frac{1}{x}[p(x)\sinh x + q(x)\cosh x]$$

where $p(x) \sim \sum_{k=0}^{\infty}\frac{(2k)!}{x^{2k}}$

and $q(x) \sim \sum_{k=0}^{\infty}\frac{(2k+1)!}{x^{2k+1}}$

E. Special Values

$Si(0) = 0$ $\quad Ci(0) = -\infty$ $\quad Ei(0) = -\infty$

$Shi(0) = 0$ $\quad Chi(0) = -\infty$

F. Symmetry Relations

$(x > 0)$

$Si(-x) = -Si(x)$ $\quad Ci(-x) = Ci(x) - i\pi$

$Shi(-x) = -Shi(x)$ $\quad Chi(-x) = Chi(x) - i\pi$

G. Interrelations

$$Si(x) = \frac{1}{2i}[E_1(ix) - E_1(-ix)] + \pi/2$$

$$Ci(x) = -\frac{1}{2}[E_1(ix) + E_1(-ix)]$$

$$Ei(x) = -\frac{1}{2}[E_1(-x + i0) + E_1(-x - i0)] \qquad (x > 0)$$

H. Value at Infinity

$$\lim_{x \to \infty} Si(x) = \pi/2$$

I. Related Function
Logarithmic Integral

$$li(x) = Ei(\ln(x)) \qquad (x > 1)$$

3. Method

Evaluation of the integrals by means of quadrature formulas suited to the particular type of integrand tends to be inefficient and inaccurate. For $Si(x)$ and $Ci(x)$, the use of the asymptotic expansion is not valid for moderate values of x, while the use of the continued fraction is inefficient and also inaccurate for small values of x. An examination of the series expansion for the functions indicates several difficulties. Summation of the alternating series for $Si(x)$ and $Ci(x)$ will lead to greater round-off errors as x increases. The partial sum at a particular value of k may be zero. Additionally there may be cancellations in adding the logarithmic term and/or Euler's constant for $Ci(x)$, $Ei(x)$ and $Chi(x)$. The more rapidly accumulating round-off errors, in particular when summations are limited to a single register, eliminate the prolonged use of the series expansion. Since the maximum of $Ci(x)$ occurs at $\pi/2$ and $Si(x) = \pi/2$ at $x \simeq 1.92$, testing indicates x = 2(PSLSC) as a reasonable starting point for the use of the continued fraction. The starting point for the valid use of the asymptotic expansion for $Si(x)$ and $Ci(x)$ does not coincide with the starting point for $Ei(x)$ ($Shi(x)$ and $Chi(x)$). Testing also indicates that fewer terms are needed in the continued fraction than in the asymptotic expansion.

The following table gives an indication of the number of terms needed to obtain maximum machine accuracy for particular values of x with the various methods of computation. Throughout the paper, NBM is the maximum number of binary digits in the mantissa of a floating point number, and TOLER = $2^{-\text{NBM}}$.

Method	Number of Terms					
	$NBM = 27$			$NBM = 60$		
$x = 2$	$Si(x)$	$Ci(x)$	$Ei(x)$	$Si(x)$	$Ci(x)$	$Ei(x)$
Power Series	7	7	13	12	12	23
Continued Fraction (Even Form)	24	24	—	106	106	—
Numerical Integration (Trapezoidal or Simpson's Rule)	64	128	64	—	—	—
$x = 24$						
Power Series	—	—	55	—	—	80
Asymptotic Expansion	[a]7,11	7,11	13	—	—	—
Continued Fraction	5	5	—	14	14	—
Numerical Integration	512	512	1024	—	—	—
$x = 48$						
Power Series	—	—	—	—	—	119
Asymptotic Expansion	4,5	4,5	8	16,23	16,23	31
Continued Fraction	4	4	—	9	9	—
$x = 88$						
Asymptotic Expansion	3,4	3,4	6	9,10	9,10	17
Continued Fraction	5	5	—	8	8	—

[a] We indicate the number of odd and even terms of the respective series.

The most accurate, efficient, automatic methods for $Si(x)$ and $Ci(x)$ then are the power series and the continued fraction; for $Ei(x)$, $Shi(x)$ and $Chi(x)$ the power series and the asymptotic expansion. The lower limit (AELL) for the use of the asymptotic expansion may be shown to approximate |ln TOLER|=NBM(ln 2), where TOLER is the requested upper limit for the relative error. With this choice of the lower limit, one can also show that $Shi(x) \simeq Chi(x) \simeq \frac{1}{2} Ei(x)$. It is necessary then to consider only the asymptotic expansion for $Ei(x)$.

The series computations have been so arranged that the maximum number of functions may be obtained in a minimum of time. The even and odd terms of the series are summed independently both with and without the factor $(-1)^k$. Since $Si(x) - \pi/2$ and $Ci(x)$ are the imaginary and real parts respectively of the continued fraction expansion for $E_1(ix)$, there would be a saving in computing time with options on the functions to be computed. Invalid results are initially supplied for all functions. With the parameter $IC = 1$, $Si(x)$ and $Ci(x)$ only are computed; with $IC = 2$, $Ei(x)$ and $e^{-x}Ei(x)$ only; with $IC = 3$, $Ei(x)$, $e^{-x}Ei(x)$, $Shi(x)$ and $Chi(x)$ only and with $IC = 4$, all functions are computed.

The implementing program checks the input parameters. If IC is outside the range 1–4, the working indicator IND is automatically set equal to 4. Since $Ei(x)$ is defined for positive x only, if $IC = 2$ and $x < 0$, there is an error return and the indicator IERR is set equal to 1. If $x < 0$, $IC = 3$, $Shi(x)$ and $Chi(x)$ are computed; for $IC = 4$, $Si(x)$ and Ci(x) are also computed, invalid results are returned for Ei(x) and $e^{-x}Ei(x)$ and the indicator IERR is set equal to 1. For $x > 0$, the indicator IERR is set equal to zero and valid results are returned only for the functions requested by the parameter IC (or IND).

The computations are performed for positive x (=T) only. Before the return, use is then made of the symmetry relations. Various cases are treated independently. Among these are $x = 0$ and x equal to or greater than the supplied upper limit (ULSC) for the sine and cosine routine. The appropriate values of the functions are supplied. To avoid unnecessary computations of the exponential function and possible overflows and underflows in the final results, if T is equal to or less than the upper limit for the relative error, the exponential of half the argument is set equal to 1. If T is equal to or greater than the computed limiting argument (XMAXHF) for $Shi(x)$ and $Chi(x)$, the maximum machine value (RINF) is supplied for the exponential of half the argument as well as for $Shi(x)$ and $Chi(x)$. The computed limiting argument (XMAXEI) for $Ei(x)$ can be shown to be approximately ln RINF + ln[ln RINF + ln(ln RINF)] −1/ln RINF. The value of XMAXHF is approximately XMAXEI + ln 2. The computation of $e^{x/2}$ provides a slight improvement in accuracy and an extension of the range of x. Throughout the program, overflows are avoided as well as underflows affecting accuracy. Other underflows are assumed to be set equal to zero.

For $|x| \leq$ PSLSC (=2), all functions are computed by means of the power series. For $|x| >$ PSLSC, $Si(x)$ and $Ci(x)$ are computed by means of the continued fraction. Only if $IC = 4$ is the working indicator IND set equal to 3. The functions $Ei(x)$, $e^{-x}Ei(x)$, $Shi(x)$ and $Chi(x)$ are then computed by means of the power series or the asymptotic expansion depending on whether $|x| \leq$ AELL or $>$ AELL respectively. With NBM = 27, AELL $\approx$ 18.7 and with NBM = 60, AELL $\approx$ 41.6. To avoid underflow, $|x|$ is tested against a lower limit argument PSLL(=$2\sqrt{\text{AMIN}}$). To simplify computation, AMIN, a minimum machine value is computed as the reciprocal of the maximum machine value (RINF). If $|x| \leq$ PSLL, only the first term of the series of odd terms is used.

The following series definitions are in use

$Si(x)$ = SI = SUMS = $\sum$ SGN (RK)*$TM(RK)$	$IP = -1$ (RK = 1,3, . . .)
$Ci(x)$ = CI = SUMC + XLOG + EULER	
where SUMC = $\sum$ SGN (RK)*$TM(RK)$	$IP = 1$ (RK = 2,4, . . .)
$Ei(x)$ = EI = SUMET + SUMOT + XLOG + EULER	
$Shi(x)$ = SHI = SUMOT = $\sum TM(RK)$	$IP = -1$ (RK = 1,3, . . .)
$Chi(x)$ = CHI = SUMET[= $\sum TM(RK)$] + XLOG + EULER	$IP = 1$ (RK = 2,4,. . .)

with SGN(1) = 1, SGN(RK + 1) = −SGN(RK) for RK = 1,3,. . ., and SGN(RK + 1) = SGN(RK) for RK = 2,4, ... The term $TM(RK) = [T^k/k!]/k$ = PTM(RK)/RK where PTM(1) = T and PTM(RK + 1) = PTM(RK)[T/(RK + 1)], $RK \geq 1$.

The series of even and odd terms are always computed together. If the relative error RE computed as $TM/|SUM|$ is less than the prescribed tolerance both series are considered to have converged. If IND = 1 or 4, SUM is replaced by SUMS or SUMC; otherwise by SUMET or SUMOT. To avoid underflow, in generating the terms for $|x| \leq 2$, if PTM $\leq$ AMIN $(RK)^2/T$, the series are likewise considered to have converged. If the sum of terms is zero, the relative RE is automatically set equal to the maximum machine value. This condition is not encountered if the power series for $Si(x)$ and $Ci(x)$ is restricted to the region $|x| \leq 2$. It has been retained to permit the program's use for experimental purposes.

To enable the continued fraction computations to be performed in double precision, since complex quantities are involved, the real notation only is used. Testing has also confirmed the improved accuracy and efficiency of this course. The continued fraction for $Si(x)$ and $Ci(x)$ in its "even" form

$$E_1(ix) = -Ci(x) + i[Si(x) - \pi/2] = e^{-ix}\left[\frac{1}{1+ix} - \frac{1}{3+ix} - \frac{4}{5+ix} - \ldots\right]$$
$$e^{-ix}[F]$$

is evaluated in the forward direction. The first convergent $F_1/G_1 = A_1/B_1$ where $A_1 = 1$, $A_M = -(M - 1)^2$, $B_M = 2M - 1 + ix$. If we define

$$F_{-1} = 1,\ F_0 = 0,\ G_{-1} = 0 \text{ and } G_0 = 1$$

then successive convergents F_M/G_M for $M = 1,2, \ldots$ may be obtained by the following recurrence relation

$$F_M = B_M F_{M-1} + A_M F_{M-2}$$
$$G_M = B_M G_{M-1} + A_M G_{M-2}$$

The continued fraction is considered to have converged either if in effect the relative error is equal to or less than the prescribed tolerance or the relative error increases.

Since the successive convergents are complex, $(RE)^2$ is compared with $(TOLER)^2$ where $(RE)^2 = \left[\text{mod}\left(1 - \frac{F_{M-1}/G_{M-1}}{F_M/G_M}\right)\right]^2$. Throughout the computation, to avoid overflow, there is scaling by the absolute maximum (=TMAX) of the real and imaginary parts of the numerator and denominator of the successive convergents. In addition, there is scaling by |TMAX| if the product of the real part of $(B_M - A_M)$ and |TMAX| is equal to or greater than 1/4 the maximum machine value.

The successive terms of the asymptotic expansion are likewise obtained by recurrence with $T_0 = 1$ and $T_K = [K/T]T_{K-1}$ for $K \geq 1$. Since the sum of terms for $Ei(x)$ is always greater than one, the term itself is a good approximation to the relative error. The summation is terminated when a term is less than the prescribed tolerance or the term is equal to or greater than the preceding term. In the latter case, the preceding term is subtracted from the summation to minimize the error.

4. Range

The range for $Si(x)$ and $Ci(x)$ (as well as the accuracy) is limited to the range (and accuracy) of the sine and cosine routine ($|x| <$ ULSC). For the UNIVAC 1108, namely, $x < 2^{21}$ in single precision and $x<2^{56}$ in double precision. For the function $Ei(x)$, the range of x is essentially the range of the exponential routine. The function $Ei(x)$ is set equal to the machine maximum (RINF) for x beyond XMAXEI, approximately 92.5 in single precision and 715.6 in double precision. For the function $e^{-x}Ei(x)$ beyond x = ULSC only the first two terms of the asymptotic expansion are used. The functions $Shi(x)$ and $Chi(x)$ are set equal to the maximum machine value for x beyond XMAXHF, approximately 93.2 in single precision and 716.3 in double precision.

5. Accuracy and Precision

Using the UNIVAC 1108 to compute the functions, the maximum relative error, except for regions in the immediate neighborhood of zeros, is 4.5 (−7) for single precision computations and 7.5 (−17) for double precision computations. Various auxiliary functions are available to greater accuracy at intermediate points in the subroutine. For example, since $Si\,(x) \underset{x\to\infty}{\to} \pi/2$, $Si(x) - \pi/2$ should be taken as the imaginary part of the continued fraction. The functions $Ci(x)$, $Ei(x)$ and $Chi(x) - \gamma - \ln x$ are available from the sum of the appropriate series.

The precision may be set lower than the maximum by varying the value of NBM or deleting NBM and setting a precomputed value of TOLER. The above relative errors give an indication of the allowance for round-off errors.

6. Timing—UNIVAC 1108 Time/Sharing Executive System

(The time estimates given below are highly dependent on the operating system environment and consequently should not be relied on for critical timing measurements.)

Single Precision $NBM = 27$		Double Precision $NBM = 60$		
For $Si(x)$ and $Ci(x)$				
Region	Time (seconds)	Region	Time (seconds)	Method
0(0.01)2 (201 values)	0.40	0(.01)2 (201 values)	0.98	Power Series
2(.5)100 (197 values)	.56	2(.5)100 (197 values)	2.06	Continued Fraction
Maximum Time/Evaluation	.0023		0.0059	Power Series
($x = 2$)	.0093	($x = 2$)	.070	Continued Fraction
For $Ei(x)$				
0(.1)18 (181 values)	0.54	0(.2)41 (206 values)	2.05	Power Series
18(.1)41 (231 values)	.28			Asymptotic Expansion
41(.25)100 (237 values)	.25	41(.25)100 (237 values)	0.70	Asymptotic Expansion
Maximum Time/Evaluation	.0044		.016	Power Series
($x = 18$)		($x = 41$)		
	.0015		.0040	Asymptotic Expansion

7. Testing

The double precision results obtained were compared against available published values. Check values were obtained, where appropriate, by overlapping the power series with either the asymptotic expansion or the continued fraction. Various forms of the continued fraction were also employed as well as numerical integration. Two multi-precision packages[3] were also utilized with varied precision. The single and double precision results agreed with the multi-precision results within the reported accuracy.

8. Many-Place Tables

In the appendix, we have included three tables; one for $Si(x)$ and $Ci(x)$, one for $Ei(x)$ and $e^{-x}Ei(x)$ and one for $Shi(x)$ and $Chi(x)$. The functions are tabulated to 35 significant figures for $x = 0, 10^{J}\,(10^{J})\,10^{J+1}$ with J = −2(1)2.

[3] (Private Communication) Peavy, Bradley A, A Multi-Precision Arithmetic Package for Use with the UNIVAC 1108.
Wyatt, W. T. Jr., Lozier, D. W. and Orser, D. J., A Portable Extended-Precision Arithmetic Package and Library with FORTRAN Precompiler, ACM TOMS, Sept. 1975.

9. Special Values

Zeros

$Si(x_s) = \pi/2$

$x_0 = 1.92644\ 7660$

$x_1 = 4.89383\ 5953$

$x_2 = 7.97268\ 2624$

$Ci(x_s) = 0$

$x_0 = 0.61650\ 5486$

$x_1 = 3.38418\ 0423$

$x_2 = 6.42704\ 7744$

$Ei(x) = 0$

$x = 0.37250\ 74107\ 81366\ 63446\ 19918\ 66580\ 11913$

$Chi(x) = 0$

$x = 0.52382\ 25713\ 89864$

Maxima

$Si(\pi) = 1.85193\ 70519\ 82466$

$Ci(\pi/2) = 0.47200\ 06514\ 39569$

Minima

$Si(2\pi) = 1.41815\ 15761\ 32628$

$Ci(3\pi/2) = -0.19840\ 75606\ 92358$

Related Constants

$\sum_{N=0}^{\infty} \frac{(-1)^N}{(2N+1)(2N+1)!} = Si(1) = 0.94608\ 30703\ 67183\ 01494\ 13533\ 13823\ 17965$

$\sum_{N=1}^{\infty} \frac{(-1)^N}{(2N)(2N)!} = Ci(1) - \gamma = -0.23981\ 17420\ 00564\ 72594\ 38658\ 86193\ 25166$

$Ci(1) = .33740\ 39229\ 00968\ 13466\ 26462\ 03889\ 15076$

$\sum_{N=1}^{\infty} \frac{1}{(N)(N)!} = Ei(1) - \gamma = 1.31790\ 21514\ 54403\ 89486\ 00088\ 44249\ 23183$

$Ei(1) = 1.89511\ 78163\ 55936\ 75546\ 65209\ 34331\ 63426$

$\sum_{N=0}^{\infty} \frac{1}{(2N+1)(2N+1)!} = Shi(1) = 1.05725\ 08753\ 75728\ 51457\ 18423\ 54895\ 87795$

$\sum_{N=1}^{\infty} \frac{1}{(2N)(2N)!} = Chi(1) - \gamma = 0.26065\ 12760\ 78675\ 38028\ 81664\ 89353\ 35387$

$Chi(1) = .83786\ 69409\ 80208\ 24089\ 46785\ 79435\ 75630$

γ(Euler's constant) $= 0.57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243$

$\pi/2 = 1.57079\ 63267\ 94896\ 61923\ 13216\ 91639\ 75144$

$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288$

$\log_e 2 = 0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656$

Typical Tolerances and Their Natural Logarithms

2^{-24} = 0.59604 64477 53906 25 (−7)
2^{-27} = .74505 80596 92382 8125 (−8)
2^{-36} = .14551 91522 83668 51806 64062 5 (−10)
2^{-48} = 35527 13678 80050 09293 55621 33789 0625 (−14)
2^{-56} = .13877 78780 78144 56755 29539 58511 35253 90625 (−16)
2^{-60} = .86736 17379 88403 54720 59622 40695 95336 91406 25 (−18)
2^{-108} = .30814 87911 01957 73648 89564 70813 58837 09660 96263 71446 21112 38390 20729 06494 14062 5 (−32)

$\log_e(2^{-24})$ = −16.63553 23334 38687 42601 35709 14996 23763
$\log_e(2^{-27})$ = −18.71497 38751 18523 35426 52672 79370 76733
$\log_e(2^{-36})$ = −24.95329 85001 58031 13902 03563 72494 35645
$\log_e(2^{-48})$ = −33.27106 46668 77374 85202 71418 29992 47526
$\log_e(2^{-56})$ = −38.81624 21113 56937 32736 49988 01657 88781
$\log_e(2^{-60})$ = −41.58883 08335 96718 56503 39272 87490 59408
$\log_e(2^{-108})$ = −74.85989 55004 74093 41706 10691 17483 06935

Maximum and Minimum Machine Values and Their Natural Logarithms
NBC = Number of binary digits in the (biased) characteristic of a floating point number

$$2^{-(2^{NBC-1}+1)} \leq x < 2^{2^{NBC-1}-1}$$

NBC = 8

2^{127} = 0.17014 11834 60469 23173 16873 03715 88410 (39)
2^{-129} = .14693 67938 52785 93849 60920 67152 78070 (−38)
$\log_e(2^{127})$ = 88.02969 19311 13054 29598 84794 25188 42414
$\log_e(2^{-129})$ = −89.41598 62922 32944 91482 29436 68104 77728

NBC = 11

2^{1023} = 0.89884 65674 31157 95386 46525 95394 51236 (308)
2^{-1025} = .27813 42323 13400 17288 62790 89666 55050 (−308)
$\log_e(2^{1023})$ = 709.08956 57128 24051 53382 84602 51714 62914
$\log_e(2^{-1025})$ = −710.47586 00739 43942 15266 29244 94630 98227

10. References

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Nat. Bur. Stand. (U. S.), Appl. Math. Ser. 55, (1964).

[2] Blanch, G., Numerical evaluation of continued fractions, SIAM Review **6,** 4, 383–421(1964).

[3] British Association for the Advancement of Science, Mathematical Tables, Vol. 1, 3d ed. (Cambridge University Press, Cambridge, England, 1951).

[4] Glaisher, J. W. L., Tables of the numerical values of the sine-integral, cosine-integral and exponential-integral, Phil. Trans. Roy. Soc. London **160,** 367–388(1870).

[5] Harris, F. E., Tables of the exponential integral $Ei(x)$, MTAC **11,** 9–16(1957).

[6] Lee, Kin L., High-precision Chebyshev series approximation to the exponential integral, NASA TN D–5953, 1970.

[7] Miller, James and Hurst, R. P., Simplified calculation of the exponential integral, MTAC **12,** 187–193(1958).

[8] Murnaghan, Francis D., and Wrench, John W. Jr., The Converging Factor for the Exponential Integral, Report 1535, David Taylor Model Basin, 1963.

[9] National Bureau of Standards, Table of sine and cosine integrals for arguments from 10 to 100, Appl. Math. Ser. **32,** (1954).

[10] National Bureau of Standards, Tables of Sine, Cosine and Exponential integrals, Vol. I & II, MT5 and MT6, (1940).

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C                                   APPENDIX
C
C                              IMPLEMENTING PROGRAM
C LANGUAGE. AMERICAN NATIONAL STANDARD FORTRAN
C DEFINITIONS. X, A REAL VARIABLE
C     SI(X) =INTEGRAL(SIN T/T)DT FROM 0 TO X
C     SI(-X)=-SI(X)
C     CI(X) =GAMMA+LN X+INTEGRAL((COS T-1)/T)DT FROM 0 TO X
C     CI(-X)=CI(X)-I PI
C     EI(X) =-P.V.INTEGRAL(EXP(-T)/T)DT FROM -X TO INFINITY
C     EXNEI(X)=EXP(-X)*EI(X)                           (X .GT. 0)
C         INTEGRAL(EXP(-T)/T) DT FROM X TO INFINITY, OFTEN
C         DENOTED BY -EI(-X)=E1(X). (SEE AUTOMATIC COMPUTING
C         METHODS FOR SPECIAL FUNCTIONS, PART II. THE EXPO-
C         NENTIAL INTEGRAL EN(X), J. OF RESEARCH NBS, 78B,
C         OCTOBER-DECEMBER 1974, PP. 199-216.)
C     SHI(X) =INTEGRAL(SINH T/T)DT FROM 0 TO X
C     SHI(-X)=-SHI(X)
C     CHI(X)=GAMMA+LN X+INTEGRAL((COSH T-1)/T)DT FROM 0 TO X
C     CHI(-X)=CHI(X)-I PI
C                 GAMMA(EULER'S CONSTANT)=.5772156649...
C   SPECIAL CASES
C     X=0
C       SI(0)=SHI(0)=0
C       CI(0)=EI(0)=EXNEI(0)=CHI(0)=-INFINITY
C                                  =-MAX. MACH. VALUE (RINF)
C     LIMITING VALUES - X APPROACHES INFINITY
C       SI(X)=PI/2
C       CI(X)=0
C       EI(X)=SHI(X)=CHI(X)=INFINITY (RINF)
C       EXNEI(X)=0
C USAGE. CALL SICIEI (IC,X,SI,CI,CII,EI,EXNEI,SHI,CHI,CHII,
C                                                         IERR)
C     FORMAL PARAMETERS
C         IC       INTEGER TYPE                           INPUT
C                      IC  FUNCTIONS TO BE COMPUTED
C                       1    SI,CI
C                       2    EI,EXNEI
C                       3    EI,EXNEI,SHI,CHI
C                       4    SI,CI,EI,EXNEI,SHI,CHI
C         X        REAL OR DOUBLE PRECISION TYPE          INPUT
C         SI=SI(X)               (SAME TYPE AS X)         OUTPUT
C         CI+I CII=CI(X)                ''                OUTPUT
C         EI=EI(X)                      ''                OUTPUT
C         EXNEI=EXP(-X)*EI(X)           ''                OUTPUT
C         SHI=SHI(X)                    ''                OUTPUT
C         CHI+I CHII=CHI(X)             ''                OUTPUT
C         IERR     INTEGER TYPE                           OUTPUT
C                      IERR=0    X .GE. 0, NORMAL RETURN
C                      IERR=1    X .LT. 0, ERROR RETURN IF
C                                               IC=2
C MODIFICATIONS.
C     THE CODE IS SET UP FOR DOUBLE PRECISION COMPUTATION
C     WITH DOUBLE PRECISION TYPE STATEMENTS
C          DOUBLE PRECISION FUNCTION REFERENCES AND,PARTICU-
C     LARLY,FOR THE UNIVAC 1108 WITH (SEE DEFINITIONS BELOW)
C          RINF APPROX. 2**1023,ULSC=2**56,NBM=60 AND OTHER
C     CONSTANTS IN DOUBLE PRECISION FORMAT TO 19 SIGNIFICANT
```

```
C     FIGURES. ALL ABOVE ITEMS MUST BE CHANGED FOR SINGLE
C     PRECISION COMPUTATIONS WITH DATA ADJUSTMENTS FOR OTHER
C     COMPUTERS.
C   AUXILIARY FUNCTIONS
C       VARIOUS FUNCTIONS ARE AVAILABLE TO GREATER ACCURACY
C       AT INTERMEDIATE POINTS IN THE SUBROUTINE,NAMELY,
C           SI-(PI/2)=IMAG. PART OF THE CONTINUED FRACTION
C           CI(EI AND CHI)-GAMMA-LN X=SUM OF SERIES
C   CAUTION - THE SUBROUTINE CANNOT READILY BE ADAPTED TO
C             COMPUTE THE FUNCTIONS FOR COMPLEX ARGUMENTS.
C METHOD.   T=ABS(X)
C     POWER SERIES   T .LE. PSLSC(=2) FOR SI,CI
C                    T .LE. AELL(=-LN(TOLER)) FOR EI,SHI,CHI
C         SI=SUMS(SGN(RK)*TM(RK))  IP=-1  RK=1,3,....,RKO
C         CI=SUMC(SGN(RK)*TM(RK))  IP=+1  RK=2,4,....,RKE
C              +EULER+XLOG
C         SHI=SUMOT(TM(RK))        IP=-1  RK=1,3,....,RKO
C         CHI=SUMET(TM(RK))        IP=+1  RK=2,4,....,RKE
C              +EULER+XLOG
C         EI=SUMOT+SUMET+EULER+XLOG                (X .GT. 0)
C               SGN(1)=1
C               SGN(RK+1)=-SGN(RK)          RK=1,3,...
C               SGN(RK+1)=+SGN(RK)          RK=2,4,...
C               TM(RK)=((T**RK)/(1*2...RK))/RK
C                    =PTM(RK)/RK
C                  PTM(1)=T
C                  PTM(RK+1)=PTM(RK)*(T/(RK+1))   RK .GE. 1
C               IF TM(RK)/SUM .LT. TOLER
C                 RKE=RK WHERE SUM=ABS(SUMC)       IC=1 OR 4
C                              SUM=SUMET           IC=2 OR 3
C                                            IC=4,X .GT. PSLSC
C                 RKO=RK WHERE SUM=ABS(SUMS)       IC=1 OR 4
C                              SUM=SUMOT           IC=2 OR 3
C                                            IC=4,X .GT. PSLSC
C         EXNEI= EI/EXP(T/2)/EXP(T/2)
C              =(EI/EXPHT)/EXPHT
C     CONTINUED FRACTION    T .GT. PSLSC
C         -CI+I(SI-PI/2)=E1(IT)
C                     =EXP(-IT)*(1 I/I (1+IT)-
C                            1**2 I/I (3+IT)-
C                            2**2 I/I (5+IT)-...)
C                     =EXP(-IT)*II(AM(RM) I/I BM(RM))
C                                          RM=1,2,....,RMF
C                         AM(1)=1
C                         AM(RM)=-(RM-1)**2     RM .GT. 1
C                         BM(RM)=2*RM-1+IT=BMR+I BMI
C                     =EXP(-IT)*(FM/GM)
C                     =EXP(-IT)*(FMR+I FMI)/(GMR+I GMI)
C                     =EXP(-IT)*F(RM)
C                     =(COST-I SINT)*(FR+I FI)
C         -CI+I(SI-PI/2)=(FR*COST+FI*SINT)+
C                                        I(FI*COST-FR*SINT)
C                       IF RESQ(RM) .LE. TOLSQ(=TOLER**2)
C                          OR RESQ(RM) .GE. RESQ(RM-1)
C                            (RESQ .GE. RESQP)
C                       RMF=RM  WHERE
C                          RESQ=(MOD(1-F(RM-1)/F(RM)))**2
C     ASYMPTOTIC EXPANSION    T .GT. AELL
```

```
C          EI=(EXNEI*EXPHT)*EXPHT
C          EXNEI=(1+SUME(TM(RK)))/T               RK=1,2,...,RKF
C          SHI=CHI=EI/2
C              TM(RK)=(1*2...RK)/(T**RK)
C              TM(0)=1
C              TM(RK)=(RK/T)*TM(RK-1)                   RK .GE. 1
C                IF TM(RK) .LT. TOLER (CONVERGENCE) RKF=RK OR
C                   TM(RK) .GE. TM(RK-1)(DIVERGENCE) RKF=RK-1
C RANGE.
C      FOR SI(X),CI(X), ABS(X) .LT. ULSC(UPPER LIMIT FOR
C                                          SIN,COS ROUTINE)
C          X=APPROXIMATELY 2**21, NBM=27
C                          2**56, NBM=60
C      FOR EXP(-X)*EI(X), X .LE. RINF
C      FOR EI(X), X .LT. XMAXEI (APPROXIMATELY 92.5,  NBC=8,
C                                             715.6,  NBC=11)
C                  NBC=NUMBER OF BINARY DIGITS IN THE BIASED
C                  CHARACTERISTIC OF A FLOATING POINT NUMBER
C      FOR SHI(X),CHI(X), ABS(X) .LT. XMAXHF
C          X=APPROXIMATELY 93.2,    NBC=8
C                         716.3,    NBC=11
C ACCURACY. THE MAXIMUM RELATIVE ERROR, EXCEPT FOR REGIONS
C           IN THE IMMEDIATE NEIGHBORHOOD OF ZEROS,ON THE
C           UNIVAC 1108 IS 4.5(-7) FOR SINGLE PRECISION COM-
C           PUTATION AND 7.5(-17) FOR DOUBLE PRECISION COM-
C           PUTATION.
C PRECISION. VARIABLE - BY SETTING THE DESIRED VALUE OF NBM
C                         OR A PREDETERMINED VALUE OF TOLER
C MAXIMUM      UNIVAC 1108 TIME/SHARING EXECUTIVE SYSTEM
C TIMING.       NBM=27   NBM=60
C (SECONDS)      .0093    .070
C STORAGE. 954 WORDS REQUIRED BY THE UNIVAC 1108 COMPILER
C
C
       SUBROUTINE SICIEI(IC,X,SI,CI,CII,EI,EXNEI,SHI,CHI,
      1                                           CHII,IERR)
C            MACHINE DEPENDENT STATEMENTS
C                TYPE STATEMENTS
       DOUBLE PRECISION X,SI,CI,CII,EI,EXNEI,SHI,CHI,CHII
       DOUBLE PRECISION A,AELL,AM,AMIN,ASUMSC,
      1         BMI,BMR,COST,EXPL,EXPHT,
      2         FI,FIP,FMI,FMM1I,FMM1R,FMM2I,FMM2R,FMR,FR,FRP,
      3         GMI,GMM1I,GMM1R,GMM2I,GMM2R,GMR,
      4         PSLL,PSLSC,PTM,RE,RESQ,RESQP,RK,RM,
      5         SCC,SFMI,SFMR,SGMI,SGMR,SGN,
      6         SINT,SUMC,SUME,SUMEO,SUMET,SUMOT,SUMS,SUMSC,
      7         T,TEMP,TEMPA,TEMPB,TM,TMAX,TMM1,TOLER,TOLSQ,
      8         XLOG,XMAXEI,XMAXHF
       DOUBLE PRECISION RINF,ULSC,EULER,HALFPI,PI,ALOG2,
      1                 ZERO,ONE,TWO,FOUR
       DIMENSION A(4)
       EQUIVALENCE (FMR,A(1)), (FMI,A(2)), (GMR,A(3)),
      1             (GMI,A(4))
C                CONSTANTS
       DATA EULER/.57721566490153286060D0/
       DATA HALFPI/1.5707963267948966190D0/
       DATA PI/3.1415926535897932380D0/
       DATA ALOG2/.69314718055994530940D0/
```

```
      DATA ZERO,ONE,TWO,FOUR /
     1     0.0D0,1.D0,2.D0,4.D0/
C                    RINF=MAXIMUM MACHINE VALUE
C                    ULSC=MAXIMUM ARGUMENT FOR SIN,COS ROUTINE
C                           APPROX. 2**(NBM-6) OR 10**(S-2)
C                                      (S=SIGNIFICANT FIGURES)
C                    NBM=ACCURACY DESIRED OR THE
C                        MAXIMUM NUMBER OF BINARY DIGITS IN THE
C                          MANTISSA OF A FLOATING POINT NUMBER
C                    TOLER=UPPER LIMIT FOR RELATIVE ERRORS
C                         =2**(-NBM)=APPROX. 10**(-S)
C TOLER PRECOMPUTED MAY BE INSERTED IN A DATA STATEMENT AND
C THE NBM DATA STATEMENT ELIMINATED
      DATA RINF/.8988465674311579538D308 /
      DATA ULSC/.72057594037927936D17/
      DATA NBM / 60 /
      TOLER=TWO**(-NBM)
C NOTE - ARGUMENT CHECKS PRECEDING FUNCTION REFERENCES
C            NECESSITATE ADDITIONAL MACHINE DEPENDENT STATEMENTS
C            IN THE STATEMENT NUMBER RANGE 140-150
C              INITIALIZATION OF OUTPUT FUNCTIONS
      SI=RINF
      CI=RINF
      CII=RINF
      EI=ZERO
      EXNEI=RINF
      SHI=ZERO
      CHI=ZERO
      CHII=RINF
C              VALIDITY CHECK ON INPUT PARAMETERS
C                   INDICATOR CHECK
C                    SET IND=IC
C                      CHANGE IND=4 IF IC .LT. 1 OR .GT. 4
      IND=IC
      IF (IND .LT. 1) GO TO 10
      IF (IND .GT. 4) GO TO 10
      GO TO 20
   10 IND=4
C                   ARGUMENT CHECK
C                    X .GE. 0     IERR=0
C                    X .LT. 0     IERR=1
C                                 (ERROR RETURN IF IC=2)
   20 IERR=0
      T=X
   30 IF (T) 40,50,90
   40 T=-T
      IF (IND .EQ. 1) GO TO 30
      IERR=1
      IF (IND .NE. 2) GO TO 30
      IF (X .LT. ZERO) RETURN
C              SPECIAL CASES
C                   X=0
   50 IF (IND-2) 80,70,60
   60 SHI=ZERO
      CHI=-RINF
      CHII=ZERO
   70 EI=-RINF
      EXNEI=-RINF
```

```
      IF (IND .NE. 4) RETURN
   80 SI=ZERO
      CI=-RINF
      CII=ZERO
      RETURN
   90 IF (T .LT. ULSC) GO TO 140
C                ABS(X) .GE. ULSC
      IF (IND-2) 130,110,100
  100 SHI=RINF
      CHI=RINF
      CHII=ZERO
      IF (IERR .EQ. 1) GO TO 120
  110 EI=RINF
      EXNEI=(ONE+(ONE/T))/T
  120 IF (IND .NE. 4) GO TO 1000
  130 SI=HALFPI
      CI=ZERO
      CII=ZERO
      GO TO 1000
C          EVALUATIONS FOR ABS(X)(=T) .GT. 0 AND .LT. ULSC
C               ADDITIONAL MACHINE DEPENDENT STATEMENTS
C                    FUNCTION REFERENCES
C                    CONTROL VARIABLES
  140 XLOG=DLOG(T)
      SINT=DSIN(T)
      COST=DCOS(T)
      EXPL =DLOG(RINF)
      XMAXEI=EXPL+DLOG(EXPL+DLOG(EXPL)) -ONE/EXPL
      XMAXHF=XMAXEI+ALOG2
      AELL=-DLOG(TOLER)
      AMIN=ONE/RINF
      PSLL=TWO*DSQRT(AMIN)
      PSLSC=TWO
C               EXPONENTIAL FUNCTION DETERMINATION
      IF (T .LE. TOLER) GO TO 150
      IF (T .GE. XMAXHF) GO TO 160
      EXPHT=DEXP(T/TWO)
      GO TO 170
  150 EXPHT=ONE
      GO TO 170
  160 EXPHT=RINF
C               METHOD SELECTION
  170 IF (T .LE. PSLSC) GO TO 200
      IF (IND .EQ. 1) GO TO 500
      IF (IND .EQ. 4) GO TO 500
  180 IF (T .GT. AELL) GO TO 800
      GO TO 230
C                    INDICATOR TO COMPUTE EI,SHI,CHI
  190 IF (IND .EQ. 1) GO TO 1000
      IND=3
      GO TO 180
C                    METHOD --- POWER SERIES
C                      SI(X),CI(X),              T .LE. PSLSC
C                      EI(X),SHI(X),CHI(X),      T .LE. AELL
C                         LIMITING VALUES, T NEAR ZERO
  200 IF (T .GT. PSLL) GO TO 210
      SUMC=ZERO
      SUMET=ZERO
```

```
            SUMS=T
            SUMOT=T
            GO TO 360
      C                                INITIALIZATION FOR SI,CI
       210  IF (IND .NE. 1) GO TO 230
       220  SUMS=ZERO
            SUMC=ZERO
            SUMSC=ZERO
            SGN=ONE
            GO TO 240
      C                                INITIALIZATION FOR SHI,CHI(AND EI)
       230  SUMOT=ZERO
            SUMET=ZERO
            SUMEO=ZERO
            IF (IND .EQ. 4) GO TO 220
      C                                       IP -  INDICATOR FOR ODD OR
      C                                               EVEN TERMS
       240  IP=-1
            RK=ONE
            PTM=T
      C                                COMPUTATION OF (T**K)/(1*2...K)/K
       250  TM=PTM/RK
      C                                SUMMATION FOR SI(CI)
               IF (IND .NE. 1) GO TO 310
       260     SUMSC=SGN*TM+SUMSC
      C                                RELATIVE ERROR FOR SI(CI)
      C PARTIAL SUM OF ALTERNATING ODD(EVEN) TERMS MAY EQUAL ZERO
               ASUMSC=SUMSC
       270     IF (ASUMSC) 280,300,290
       280     ASUMSC=-ASUMSC
               GO TO 270
       290     RE=TM/ASUMSC
               GO TO 320
       300     RE=RINF
               GO TO 320
      C                                SUMMATION FOR SHI(CHI)(AND EI)
       310     SUMEO=TM+SUMEO
               IF (IND .EQ. 4) GO TO 260
      C                                RELATIVE ERROR FOR SHI(CHI)
               RE=TM/SUMEO
      C                                SIGN CHANGE AND SELECTION
      C                                OF SUMS OF ODD(EVEN) TERMS
       320     IF (IP .EQ. 1) GO TO 330
               SGN=-SGN
               SUMS=SUMSC
               SUMSC=SUMC
               SUMOT=SUMEO
               SUMEO=SUMET
               GO TO 340
       330     SUMC=SUMSC
               SUMSC=SUMS
               SUMET=SUMEO
               SUMEO=SUMOT
      C                                RELATIVE ERROR CHECK
       340     IF (RE .LT. TOLER) GO TO 360
      C                                ADDITIONAL TERMS
               RK=RK+ONE
      C                                       UNDERFLOW TEST
```

```
      C UNDERFLOWS AFFECTING ACCURACY ARE AVOIDED. ALL OTHER
      C UNDERFLOWS ARE ASSUMED TO BE SET EQUAL TO ZERO
              IF (T .GT. PSLSC) GO TO 350
              IF (PTM .LE. (AMIN*RK*RK)/T ) GO TO 360
        350   PTM=(T/RK)*PTM
              IP=-IP
              GO TO 250
      C                                SI,CI EVALUATION
        360 IF (IND .NE. 1) GO TO 380
        370 SI=SUMS
            CI=(SUMC+XLOG)+EULER
            CII=ZERO
            GO TO 1000
      C                                EI EVALUATION
        380 IF (X .LE. ZERO) GO TO 390
            EI=(SUMET+SUMOT+XLOG)+EULER
            EXNEI=(EI/EXPHT)/EXPHT
            IF (IND .EQ. 2) RETURN
      C                                SHI,CHI EVALUATION
        390 SHI=SUMOT
            CHI=(EULER+SUMET)+XLOG
            CHII=ZERO
            IF (IND .NE. 4) GO TO 1000
            GO TO 370
      C                           METHOD --- CONTINUED FRACTION
      C                             SI(X),CI(X),        T .GT. PSLSC
      C                             -CI(T) + I (SI(T)-HALFPI)=E1(IT)
      C                                INITIALIZATION
        500 SCC=RINF/FOUR
            TOLSQ=TOLER*TOLER
            RM=ONE
            AM=ONE
            BMR=ONE
            BMI=T
            FMM2R=ONE
            FMM2I=ZERO
            GMM2R=ZERO
            GMM2I=ZERO
            FMM1R=ZERO
            FMM1I=ZERO
            GMM1R=ONE
            GMM1I=ZERO
            RESQP=RINF
            FRP=ZERO
            FIP=ZERO
      C                                RECURRENCE RELATION
      C                                  FM=BM*FMM1 + AM*FMM2
      C                                  GM=BM*GMM1 + AM*GMM2
        510 FMR=BMR*FMM1R-BMI*FMM1I+AM*FMM2R
              FMI=BMI*FMM1R+BMR*FMM1I+AM*FMM2I
              GMR=BMR*GMM1R-BMI*GMM1I+AM*GMM2R
              GMI=BMI*GMM1R+BMR*GMM1I+AM*GMM2I
      C                                CONVERGENT F=FM/GM
      C                                  TESTS TO AVOID INCORRECT RESULTS
      C                                      DUE TO OVERFLOWS(UNDERFLOWS)
      C                                    FINDING MAXIMUM(=TMAX) OF
      C                                      ABSOLUTE OF FMR,GMR,FMI,GMI
      C                                      FOR SCALING PURPOSES
```

```
         TMAX=ZERO
         I=1
  520    TEMP=A(I)
  530      IF (TEMP) 540,560,550
  540      TEMP=-TEMP
           GO TO 530
  550      IF (TEMP .LE. TMAX) GO TO 560
           TMAX=TEMP
  560      IF (I .GE. 4) GO TO 570
           I=I+1
           GO TO 520
  570    SFMR=FMR/TMAX
         SFMI=FMI/TMAX
         SGMR=GMR/TMAX
         SGMI=GMI/TMAX
         TEMP=SGMR*SGMR + SGMI*SGMI
         FR=(SFMR*SGMR+SFMI*SGMI)/TEMP
         FI=(SFMI*SGMR-SFMR*SGMI)/TEMP
C                                  RELATIVE ERROR CHECK
         TEMP=FR*FR+FI*FI
         TEMPA=(FRP*FR+FIP*FI)/TEMP
         TEMPB=(FIP*FR-FRP*FI)/TEMP
         TEMP=ONE-TEMPA
         RESQ =TEMP*TEMP+TEMPB*TEMPB
         IF (RESQ  .LE. TOLSQ) GO TO 590
         IF (RESQ .GE. RESQP) GO TO 580
C                                  ADDITIONAL CONVERGENTS
         AM=-RM*RM
         RM=RM+ONE
         BMR=BMR+TWO
         FMM2R=FMM1R
         FMM2I=FMM1I
         GMM2R=GMM1R
         GMM2I=GMM1I
         FMM1R=FMR
         FMM1I=FMI
         GMM1R=GMR
         GMM1I=GMI
         FRP=FR
         FIP=FI
         RESQP=RESQ
C                                  SCALING
C SCALING SHOULD NOT BE DELETED AS THE VALUES OF FMR,FMI AND
C GMR,GMI MAY OVERFLOW FOR SMALL VALUES OF T
         IF (TMAX .LT. SCC/(BMR-AM ) ) GO TO 510
         FMM2R=FMM2R/TMAX
         FMM2I=FMM2I/TMAX
         GMM2R=GMM2R/TMAX
         GMM2I=GMM2I/TMAX
         FMM1R=FMM1R/TMAX
         FMM1I=FMM1I/TMAX
         GMM1R=GMM1R/TMAX
         GMM1I=GMM1I/TMAX
         GO TO 510
C                                  DIVERGENCE OF RELATIVE ERROR
C                                    ACCEPT PRIOR CONVERGENT
  580  FR=FRP
       FI=FIP
```

```
      C                               SI,CI EVALUATION
       590  SI=FI*COST-FR*SINT+HALFPI
            CI=-(FR*COST+FI*SINT)
            CII=ZERO
            GO TO 190
      C                          METHOD --- ASYMPTOTIC EXPANSION
      C                            EI(X),EXNEI(X)         X .GT. AELL
      C                            SHI(T)=CHI(T)=EI(T)/2  T .GT. AELL
      C                               INITIALIZATION
       800  IF (IND .NE. 2) GO TO 880
       810  SUME=ZERO
            RK=ZERO
            TM=ONE
      C                               ADDITIONAL TERMS
       820  TMM1=TM
              RK=RK+ONE
              TM=(RK/T)*TM
      C                               TOLERANCE CHECK
              IF (TM .LT. TOLER) GO TO 840
              IF (TM .GE. TMM1) GO TO 830
              SUME=SUME+TM
              GO TO 820
      C                               DIVERGENT PATH
       830  SUME=SUME-TMM1
      C                               EXNEI EVALUATION
       840  IF (X .LT. ZERO) GO TO 870
            EXNEI=(ONE+SUME)/T
      C                               EI EVALUATION - X .LT. XMAXEI
            IF (T .GE. XMAXEI) GO TO 850
            EI=(EXNEI*EXPHT)*EXPHT
            GO TO 860
      C                               EI - LIMITING VALUE, X .GE. XMAXEI
       850  EI=RINF
      C                               SHI,CHI EVALUATION - T .LT. XMAXHF
       860  IF (IND .EQ. 2) RETURN
       870  IF (T .GE. XMAXHF) GO TO 1000
            SHI=(((( ONE+SUME)/T)/TWO)*EXPHT)*EXPHT
            CHI=SHI
            CHII=ZERO
            GO TO 1000
      C                               SHI,CHI - LIMITING VALUE
      C                                                    T .GE. XMAXHF
       880  IF ( T .LT. XMAXHF) GO TO 810
            SHI=RINF
            CHI=RINF
            CHII=ZERO
            IF ( X .GT. ZERO) GO TO 810
            GO TO 1010
      C          ADJUSTMENTS FOR X .LT. 0
      1000  IF (X .GT. ZERO) RETURN
      1010  IF (IC .EQ. 3) GO TO 1020
            SI=-SI
            CII=-PI
            IF (IC .EQ. 1) RETURN
      1020  SHI=-SHI
            CHII=-PI
            RETURN
            END
```

TABLE 1

X	SI(X)				CI(X)			
.0	.0				− ∞			
.1−001	.9999944444	6111108276	6470528517	85973−002	−.4027979520	9823920722	4397561453	54440+001
.2−001	.1999955556	0888852607	8665206276	10625−001	−.3334907338	8599613460	8102204879	74545+001
.3−001	.2999850004	0499380108	0675616964	14691−001	−.2929567223	9811175640	0020167602	13443+001
.4−001	.3999644461	5106467200	4469605623	17391−001	−.2642060133	3009493497	5931669690	80028+001
.5−001	.4999305607	6366745212	5334525894	59473−001	−.2419141543	5519082432	1138357349	19704+001
.6−001	.5998800129	5920656146	8561513856	83629−001	−.2237094916	8693029859	1937022506	78885+001
.7−001	.6998094724	5377692911	0744599342	97997−001	−.2083269121	9543103133	4741544991	04089+001
.8−001	.7997156101	6294499144	3834290267	56205−001	−.1950112552	8007321930	1863050699	50094+001
.9−001	.8995950984	0144401781	3419097919	40853−001	−.1832754260	4358445302	6218391310	18002+001
.1+000	.9994446110	8276950160	5921185541	90930−001	−.1727868386	6572966389	9772515290	65736+001
.2+000	.1995560885	2623382140	0456944764	16595+000	−.1042205595	6727819753	6291690667	58895+001
.3+000	.2985040438	0704316138	6446229574	64345+000	−.6491729329	7116174495	6181105135	48965+000
.4+000	.3964614647	5137288302	0334263135	17445+000	−.3788093464	2524433208	4661757229	44184+000
.5+000	.4931074180	4306668916	1626707572	76465+000	−.1777840788	0661290133	5810271070	56908+000
.6+000	.5881288096	0808006689	9647904006	83682+000	−.2227070695	9279762526	6027439020	62561−001
.7+000	.6812222391	1661131088	9506811453	94252+000	.1005147070	0889783268	9135892810	28616+000
.8+000	.7720957854	8199656025	3889712479	89549+000	.1982786159	5246717701	5833582233	84097+000
.9+000	.8604707107	4529293277	4085411696	29011+000	.2760678304	6777286015	0434115538	80334+000
.1+001	.9460830703	6718301494	1353313823	17966+000	.3374039229	0096813466	2646203889	15077+000
.2+001	.1605412976	8026948485	7672014819	85889+001	.4229808287	7486499569	8565153198	25589+000
.3+001	.1848652527	9994682563	9773025111	19732+001	.1196297860	0800032762	6472281176	67785+000
.4+001	.1758203138	9490530581	0555930335	85016+001	−.1409816978	8693041163	9144898694	03593+000
.5+001	.1549931244	9446741372	7440840073	06390+001	−.1900297496	5664387861	8458900116	30081+000
.6+001	.1424687551	2805065357	6903102791	71420+001	−.6805724389	3247126204	1683048406	17428−001
.7+001	.1454596614	2480935906	1476849383	61604+001	.7669527848	2184518382	9157630314	36847−001
.8+001	.1574186821	7069420520	8297145120	66585+001	.1224338825	3200955729	2295958268	73503+000
.9+001	.1665040075	8296024951	0665342789	71085+001	.5534753133	3133607085	6416484497	93252−001
.1+002	.1658347594	2188740493	3097187938	96725+001	−.4545643300	4455372634	5328299526	27853−001
.2+002	.1548241701	0434398401	6364334212	95137+001	.4441982084	5353316539	7687169925	70578−001
.3+002	.1566756540	0303511109	8373130900	67982+001	−.3303241728	2071143779	2264409630	03714−001
.4+002	.1586985119	3547845067	7566596201	46420+001	.1902000789	6208766961	9812034202	67962−001
.5+002	.1551617072	4859358947	2798559485	93775+001	−.5628386324	1163054401	8589549846	40996−002
.6+002	.1586745616	2599474123	2644013231	99104+001	−.4813243377	4432152888	7234385245	11961−002
.7+002	.1561594849	1780061055	2298220467	79853+001	.1092198847	3464977037	8113544132	70120−001
.8+002	.1572330886	9124873153	5125172966	74798+001	−.1240250115	5070958192	0458829452	17856−001
.9+002	.1575663406	6574562607	3805334080	46545+001	.9986124071	6431336804	5713422489	45291−002
.1+003	.1562225466	8890562933	5234513880	45027+001	−.5148825142	6104921444	4355390534	44979−002
.2+003	.1568382339	3394698333	5878557542	35465+001	−.4378446093	0278256791	6569771749	32552−002
.3+003	.1570881088	2137495192	5231225344	08620+001	−.3332199918	5921117799	7045482583	87851−002
.4+003	.1572114869	2738117518	0132144796	40848+001	−.2123988830	8463489343	4793666984	71878−002
.5+003	.1572565882	2431687035	3434162096	10243+001	−.9320008144	0429025451	7261962053	86680−003
.6+003	.1572461233	9493979398	3169426317	07478+001	.7641202377	3809449584	1487520193	67238−004
.7+003	.1571993932	2374915706	3702809228	14464+001	.7788100127	3975633981	0132017166	41226−003
.8+003	.1571355087	6214727479	0846382718	80577+001	.1118158760	9180513118	2640905181	04941−002
.9+003	.1570721487	6829785964	0335292159	62388+001	.1108585782	0500159240	9398971297	71935−002
.1+004	.1570233121	9687712181	4796277803	63344+001	.8263155110	9068228200	1773882343	20723−003

$\pi/2 = 1.57079 \quad 63267 \quad 94896 \quad 61923 \quad 13216 \quad 91639 \quad 75144$

Table 2

X	EI(X)				EXP(−X)*EI(X)			
.0	− ∞				− ∞			
.1−001	−.4017929465	4266693867	7534341058	83082+001	−.3977950399	2615576971	3408360703	19476+001
.2−001	−.3314706894	4101539023	9227816770	05873+001	−.3249071300	3015878880	1888549128	04425+001
.3−001	−.2899115723	9402794440	9618312164	59357+001	−.2813433905	5380940257	4977395174	78007+001
.4−001	−.2601256577	5728261871	6902308916	05352+001	−.2499259848	2574874044	8538470778	63194+001
.5−001	−.2367884598	5793745242	5950757147	49992+001	−.2252401503	9907626374	1216434311	86807+001
.6−001	−.2175282915	5516236358	6147208749	96411+001	−.2048604300	3782595071	0809271777	33217+001
.7−001	−.2010800063	5428875184	2790326756	89313+001	−.1874857552	3138763197	4860337705	30829+001
.8−001	−.1866884102	7729969797	4868911203	84179+001	−.1723351232	0791016360	9393503175	42803+001
.9−001	−.1738663750	3469511928	7135014650	31175+001	−.1589019022	1427078812	0803355460	85013+001
.1+000	−.1622812813	9692766749	6568299922	74753+001	−.1468381756	5476302940	1938294633	01893+001
.2+000	−.8217605879	0240031565	3310869899	97772+000	−.6728006649	8313731896	8600992150	20087+000
.3+000	−.3026685392	6582588446	8136901536	93942+000	−.2242223687	1524378927	5487443901	40039+000
.4+000	.1047652186	1932479322	8929763883	46026+000	.7022622616	7839611020	0870681109	89467−001
.5+000	.4542199048	6317357992	0523812662	80237+000	.2754982985	5127026213	2219977961	61145+000
.6+000	.7698812899	3735943709	3137330257	85125+000	.4225198103	2870176143	1329955834	52537+000
.7+000	.1064907194	6242905405	7162359274	67593+001	.5288172627	5216097535	3053374545	58451+000
.8+000	.1347396548	2123259381	1899638330	35829+001	.6054242952	6336443347	4212274854	28025+000
.9+000	.1622811713	6968674413	4419142440	54222+001	.6597860062	6079393891	3476848207	67092+000
.1+001	.1895117816	3559367554	6652093433	16343+001	.6971748832	3506606876	5478681919	55160+000
.2+001	.4954234356	0018901633	7950513022	70353+001	.6704827097	9007328104	3223808083	90335+000
.3+001	.9933832570	6254165580	0833601921	67653+001	.4945764013	4864123502	8769696756	17092+000
.4+001	.1963087447	0056220022	6457202797	23839+002	.3595520078	6362069617	7421675826	60745+000
.5+001	.4018527535	5803177455	0914217937	95867+002	.2707662554	9105719558	0378348094	65324+000
.6+001	.8598976214	2439204803	5834003079	90690+002	.2131473100	8159360315	3856832629	71120+000
.7+001	.1915047433	3550139595	3063148272	45695+003	.1746297217	6579015129	0938352318	25166+000
.8+001	.4403798995	3483826899	7424596659	39339+003	.1477309983	7340099664	7174203305	76383+000
.9+001	.1037878290	7170895876	5757322679	36222+004	.1280843565	2321386813	5996461057	99933+000
.1+002	.2492228976	2418777591	3844014399	85248+004	.1131470204	7341077803	4051681354	47701+000
.2+002	.2561565266	4056588820	4811208040	98072+008	.5279779527	9648132254	5779817983	87276−001
.3+002	.3689732094	0727419706	4006328910	84575+012	.3452712179	2361846131	5818701748	98032−001
.4+002	.6039718263	6112415783	5923141851	06913+016	.2565886278	5975145205	8133890489	43081−001
.5+002	.1058563689	7131690963	0615414332	29987+021	.2041704555	5943987333	5659940616	10444−001
.6+002	.1936182213	9292765388	2072596687	52373+025	.1695420039	4813286610	0796814766	65768−001
.7+002	.3646352759	5797356367	1345154856	77168+029	.1449589211	5101025543	4076487427	26111−001
.8+002	.7014600004	9047999696	2996948047	97919+033	.1266031055	4032884299	9914561805	60367−001
.9+002	.1371416869	5072519995	0222595544	66918+038	.1123740714	1495976244	6676886934	83357−001
.1+003	.2715552744	8538798219	1401464231	08254+042	.1010206252	7748357112	3003599184	99944−001
.2+003	.3631235233	1593568523	9671004384	64250+085	.5025253826	9333012313	0841542901	12999−002
.3+003	.6496482508	0886657890	2569189493	42473+128	.3344519269	3037826333	1099042135	67540−002
.4+003	.1308647281	7074277342	4939487839	11718+172	.2506281486	7484941759	1294495095	34843−002
.5+003	.2812821397	8862943374	7493151789	64387+215	.2004016096	7757734712	6302843056	26131−002
.6+003	.6298882891	3879314245	2569543511	94621+258	.1669453750	3112400160	6469063742	31301−002
.7+003	.1450978736	0525608526	2088252210	93011+302	.1430618100	9351634011	0181288435	03898−002
.8+003	.3412238865	4483770461	9667674034	14661+345	.1251566420	9721409147	4398485683	85823−002
.9+003	.8152195006	2752522682	3560124386	87339+388	.1112348431	6823895903	3663223603	11253−002
.1+004	.1972045137	1412383028	0964504841	20236+432	.1001002006	0241207250	8068654920	21171−002

Table 3

X	SHI(X)				CHI(X)			
.0	.0				− ∞			
.1−001	.1000005555	5722222505	6692404330	26380−001	−.4027929520	9823916092	8101265102	13346+001
.2−001	.2000044444	9777814059	1136870933	39861−001	−.3334707338	8599317164	5139185479	39271+001
.3−001	.3000150004	0500619903	9860097325	50634−001	−.2929117223	9807800640	0016913137	84864+001
.4−001	.4000355572	6226866293	4193925798	75059−001	−.2641260133	2990530534	6244248174	04103+001
.5−001	.5000694496	5299922650	1974168087	39980−001	−.2417891543	5446744469	0970498828	37392+001
.6−001	.6001200129	6079350024	5724232696	91766−001	−.2235294916	8477029858	8604451076	93328+001
.7−001	.7001905835	6955665134	1973404093	50990−001	−.2080819121	8998431835	6210060797	82823+001
.8−001	.8002844990	6372249715	1895071106	72843−001	−.1946912552	6793692294	6387861914	90908+001
.9−001	.9004050984	2855835469	1051467784	84646−001	−.1828704260	1898070283	4045529328	16021+001
.1+000	.1000555722	2505699555	7615329532	17784+000	−.1722868386	1943336705	2329832875	96531+001
.2+000	.2004449781	4074638634	0730853837	22252+000	−.1022205566	0431467019	9404172373	72002+001
.3+000	.3015040562	0501041398	1095310302	38247+000	−.6041725954	7083629844	9232211839	32189+000
.4+000	.4035726687	4249363590	5979378947	55253+000	−.2988074501	2316884267	7049615064	09227+000
.5+000	.5069967498	1966719583	3659875988	94380+000	−.5277684495	6493615913	1360633261	41435−001
.6+000	.6121303965	6338077262	4562784597	54146+000	.1577508933	7397866446	8574545660	30978+000
.7+000	.7193380189	2889984241	9121259127	50575+000	.3455691756	9539069815	2502333619	25356+000
.8+000	.8289965633	7893448638	6910469189	60092+000	.5183999848	3339145173	2085914113	98201+000
.9+000	.9414978265	1143354092	2701645733	42970+000	.6813138871	8543390042	1489778671	99251+000
.1+001	.1057250875	3757285145	7184235489	58780+001	.8378669409	8020824089	4678579435	75631+000
.2+001	.2501567433	3549756414	7337248272	75424+001	.2452666922	6469145219	0613264749	94929+001
.3+001	.4973440475	8598067977	1041838252	27051+001	.4960392094	7656097602	9791763669	40601+001
.4+001	.9817326911	2330344645	6229756992	81526+001	.9813547558	8231855580	8342270979	56862+001
.5+001	.2009321182	5697226390	4443761778	82843+002	.2009206353	0105951064	6470456159	13024+002
.6+001	.4299506111	2445683731	1213478510	53231+002	.4299470102	9993521072	4620524569	37459+002
.7+001	.9575242940	8616503145	6397896419	56496+002	.9575231392	6884892807	4233586305	00452+002
.8+001	2201899686	0023055646	1163184608	69467+003	.2201899309	3460771253	6261412050	69872+003
.9+001	.5189391515	8222188283	1922673971	09373+003	.5189391391	3486770482	5650552822	52849+003
.1+002	.1246114490	1994233444	1188221070	06923+004	.1246114486	0424544147	2655793329	78325+004
.2+002	.1280782633	2028294459	4181868552	98444+008	.1280782633	2028294361	0629339487	99627+008
.3+002	.1844866047	0363709853	2003165966	19888+012	.1844866047	0363709853	2003162944	64687+012
.4+002	.3019859131	8056207891	7961570925	53457+016	.3019859131	8056207891	7961570925	53456+016
.5+002	.5292818448	5658454815	3077071661	49936+020	.5292818448	5658454815	3077071661	49936+020
.6+002	.9680911069	6463826941	0362983437	61866+024	.9680911069	6463826941	0362983437	61866+024
.7+002	.1823176379	7898678183	5672577428	38584+029	.1823176379	7898678183	5672577428	38584+029
.8+002	.3507300002	4523999848	1498474023	98960+033	.3507300002	4523999848	1498474023	98960+033
.9+002	.6857084347	5362599975	1112977723	34589+037	.6857084347	5362599975	1112977723	34589+037
.1+003	.1357776372	4269399109	5700732115	54127+042	.1357776372	4269399109	5700732115	54127+042
.2+003	.1815617616	5796784261	9835502192	32125+085	.1815617616	5796784261	9835502192	32125+085
.3+003	.3248241254	0443328945	1284594746	71236+128	.3248241254	0443328945	1284594746	71236+128
.4+003	.6543236408	5371386712	4697439195	58592+171	.6543236408	5371386712	4697439195	58592+171
.5+003	.1406410698	9431471687	3746575894	82193+215	.1406410698	9431471687	3746575894	82193+215
.6+003	.3149441445	6939657122	6284771755	97311+258	.3149441445	6939657122	6284771755	97311+258
.7+003	.7254893680	2628042631	0441261054	65053+301	.7254893680	2628042631	0441261054	65053+301
.8+003	.1706119432	7241885230	9833837017	07331+345	.1706119432	7241885230	9833837017	07331+345
.9+003	.4076097503	1376261341	1780062193	43669+388	.4076097503	1376261341	1780062193	43669+388
.1+004	.9860225685	7061915140	4822524206	01178+431	.9860225685	7061915140	4822524206	01178+431