1

Trees ($v \leq 13$)

No subclass of simple graphs has been studied more carefully than the trees — the connected *acyclic* simple graphs — mainly because trees have been used for modeling problems ranging freely from the real to the numerical, and because computer design and use rely so heavily on abstract branching structures.

Every connected graph contains at least one tree that spans all the vertices of the graph, just as a skeleton spans a body. This is called a spanning tree. It is not obvious that fleshing out all trees with additional edges can generate all connected graphs. But it is easy to see that the deletion of edges from a connected graph can eliminate its cycles without disconnecting it. To prune a graph down to a tree, choose a cycle and erase one of its edges. This cannot disconnect the graph, since a pair of vertices **A** and **B** cannot have depended on that missing edge to be connected by a path. There is still an **A**-to-**B** path the other way around that cycle. Continue the process until no cycles remain. The result will not be a *forest* of disconnected trees but a single connected tree.

There are many interesting and useful facts about trees. Here are six of the best.

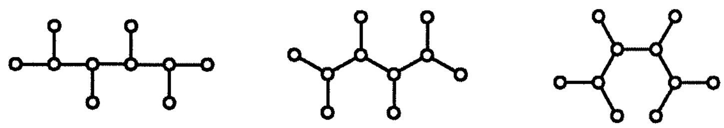
Theorem In any tree:

(1) $v = e + 1$ (the number of vertices is one more than the number of edges).
(2) any two vertices are connected by a unique path.
(3) the removal of any edge disconnects the tree, and the removal of any vertex of degree at least 2 disconnects the tree.
(4) the addition of any one edge creates one cycle.
(5) there are one or two centers and one or two centroids (the subjects of Chapter 2).
A CENTER of a tree is a vertex that minimizes the maximum distance to the other vertices; a CENTROID or *barycenter* of a tree is a vertex that minimizes the maximum weight of the branches that emanate from any vertex.

Theorem The number of labeled trees on v vertices is v^{v-2}.

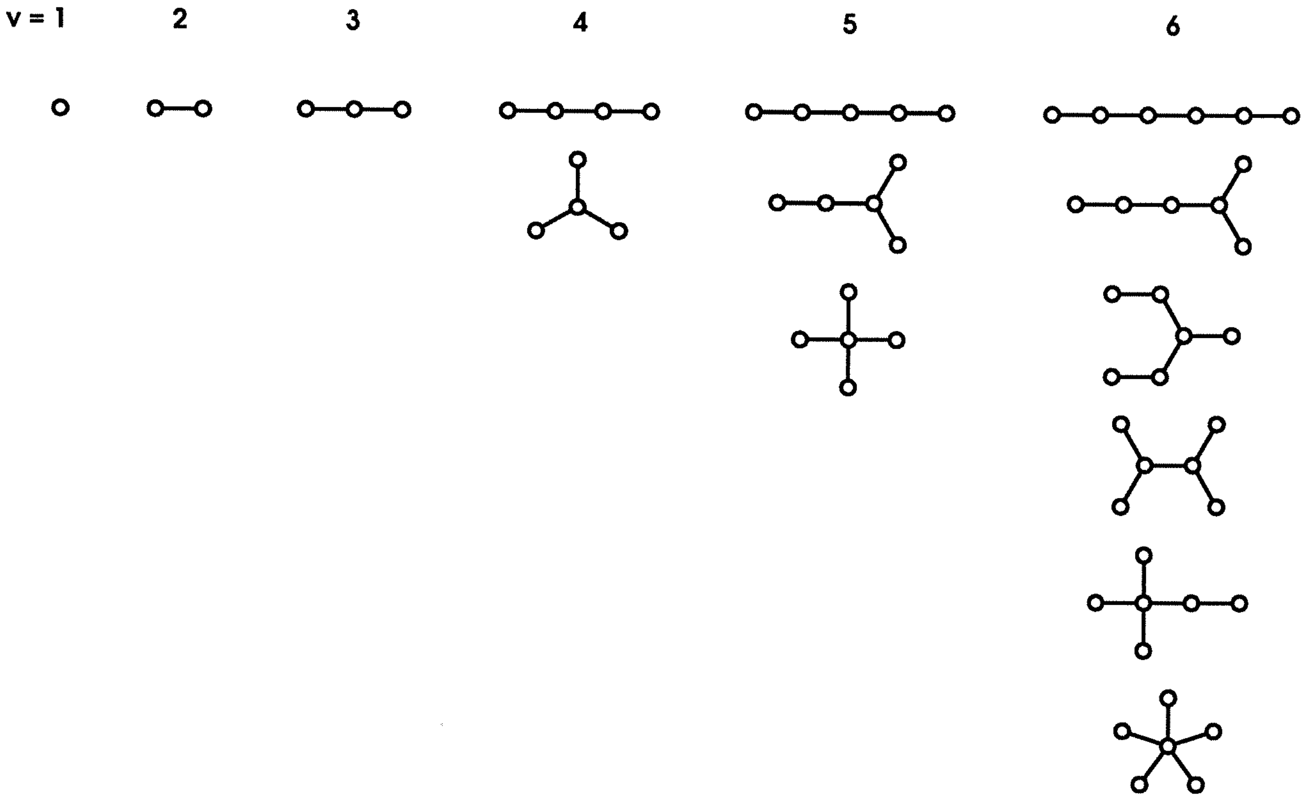
Since the removal of any edge disconnects a tree, trees are minimally connected. Then a spanning tree of a graph is a smallest (sparsest) connected subgraph that uses all of its vertices. For example, suppose a town with dirt roads paves some of its roads so that one can always go from A to B on paved roads, but no unnecessary paving is done. Then the paved roads form a spanning tree of the town. Since the addition of one edge to a tree forms one cycle, any more paving in this town would be redundant and unnecessary. As another example, a minimal subnetwork in a communication network is a spanning tree. With no cycles it has no redundant connections, so it is simplest and cheapest. But the cost of eliminating redundancy is vulnerability to disconnection, since the removal of any edge will disconnect the tree. A tree is where zero-redundancy meets total vulnerability.

The picture is not the graph. For a thorough explanation of this see Volume 1, Introduction. Lengths of edges, angles between edges, and straightness or curvature of edges are all irrelevant. The three pictures below are isomorphic — they depict the same tree. Occasionally, a cartouche will encapsulate isomorphs.

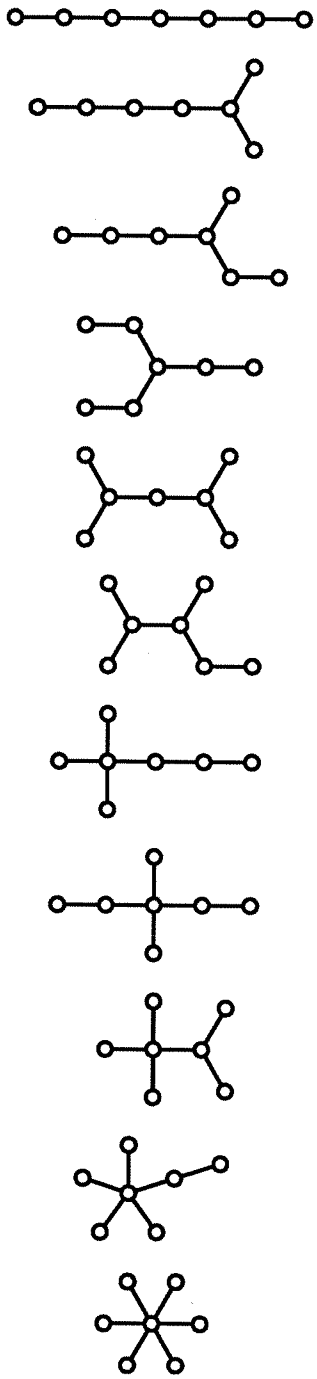


The following trees are ordered according to the number of vertices of degree at least three. The drawings have been left unmarked so that readers can assign diverse notations to them, and the degree sequences are listed at the end. In order to find the trees of a given sequence, start with the maximum degree, then descend the sequence.

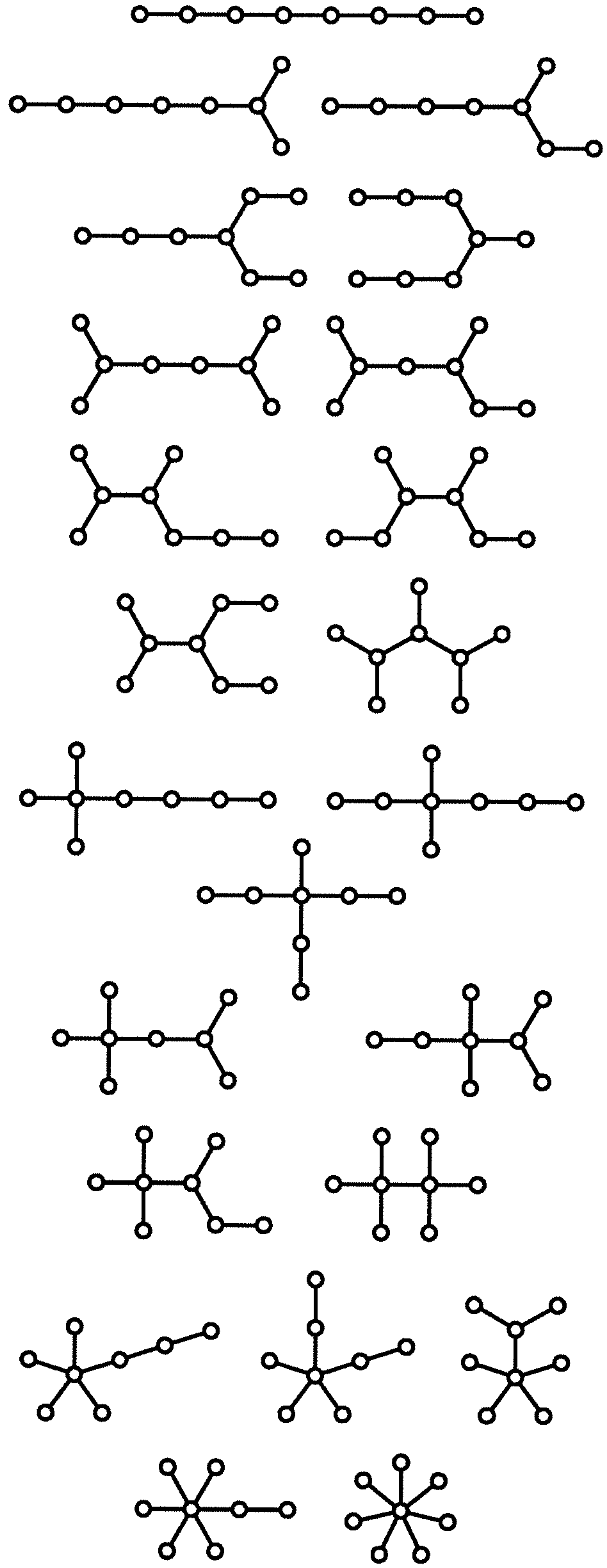
Chapter 2 uses the same set of drawings, but shows centers (filled with black) and centroids (circled).


v = 1
2
3
4
5
6

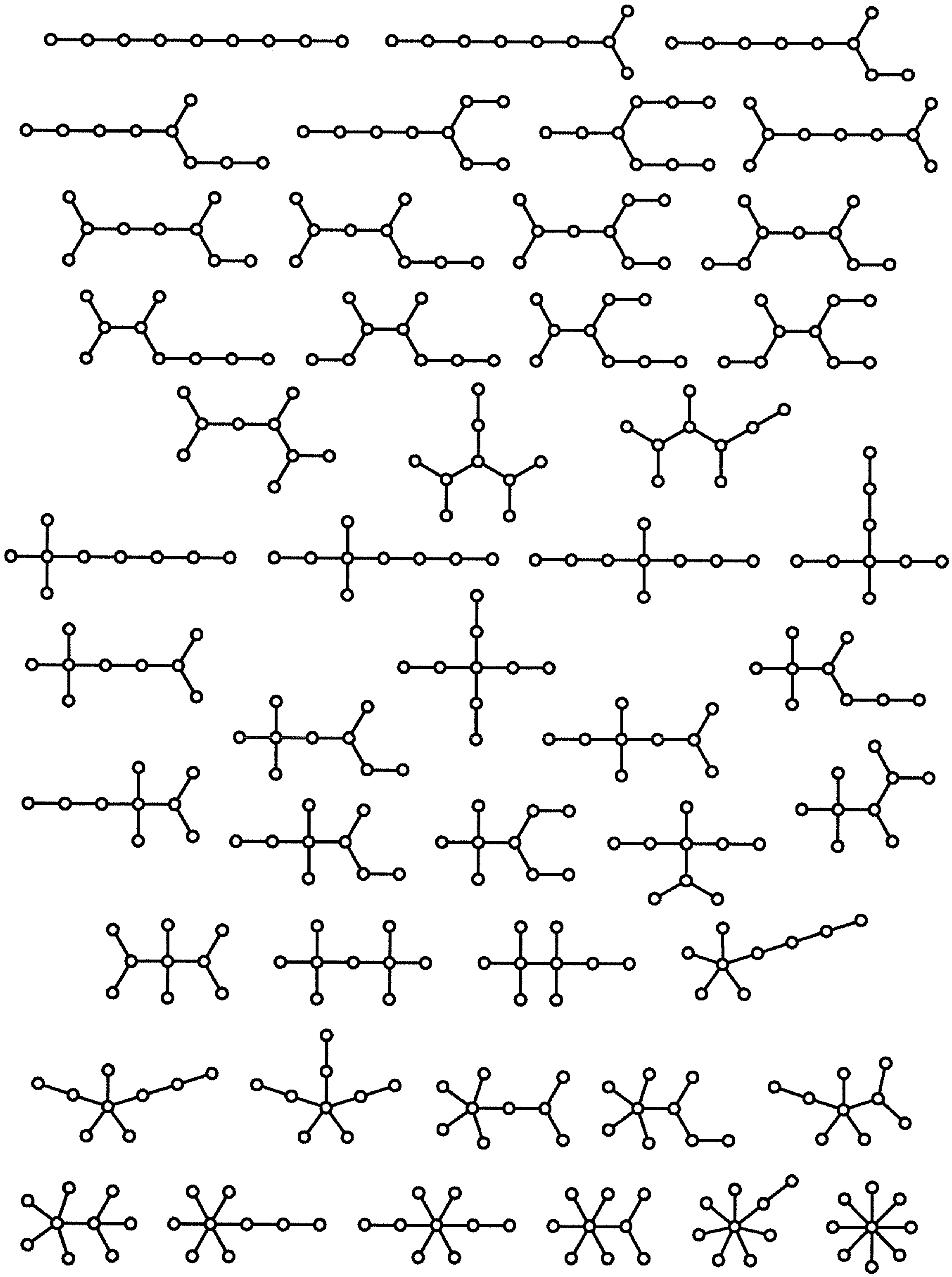
v = 7



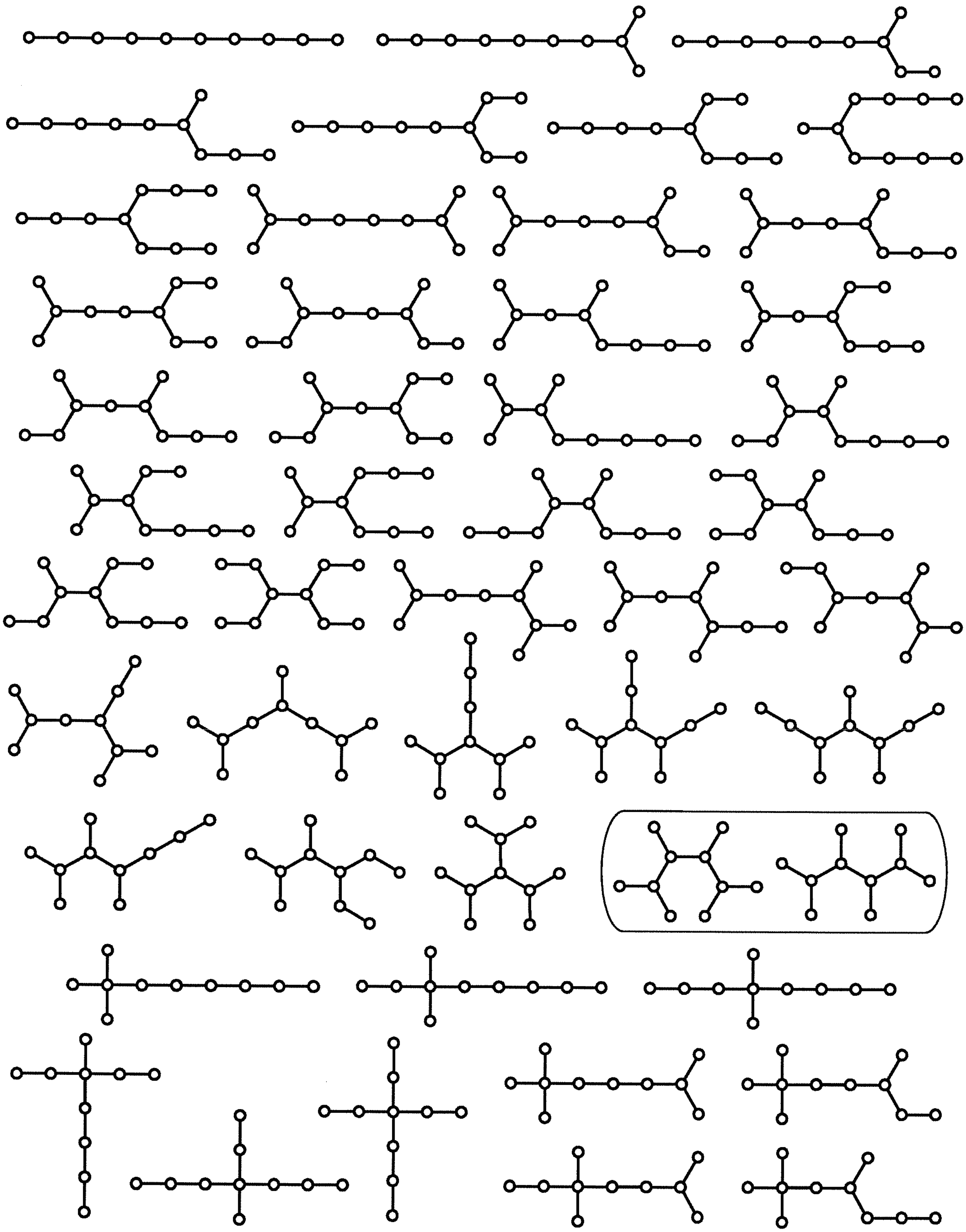
v = 8



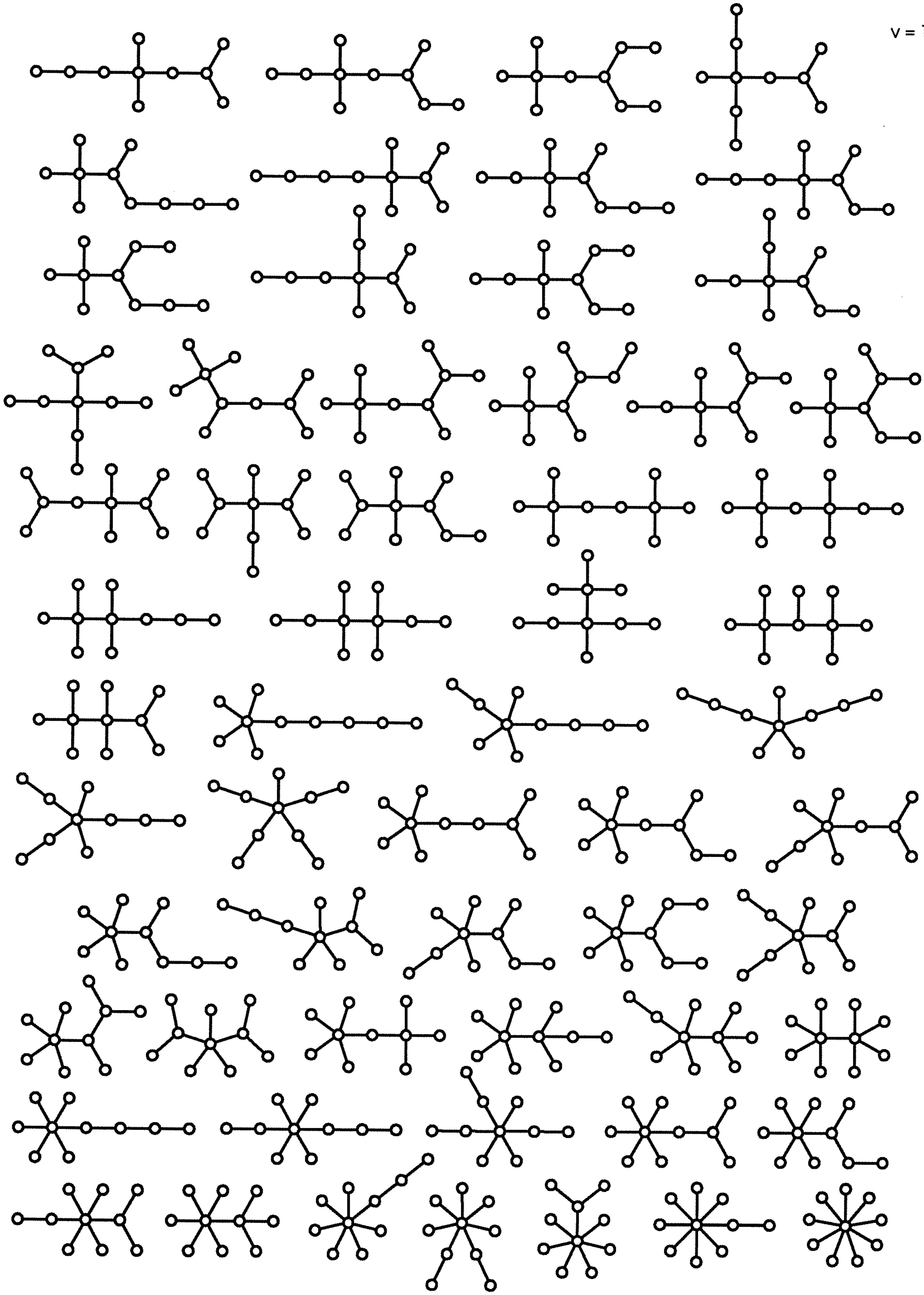
v = 9



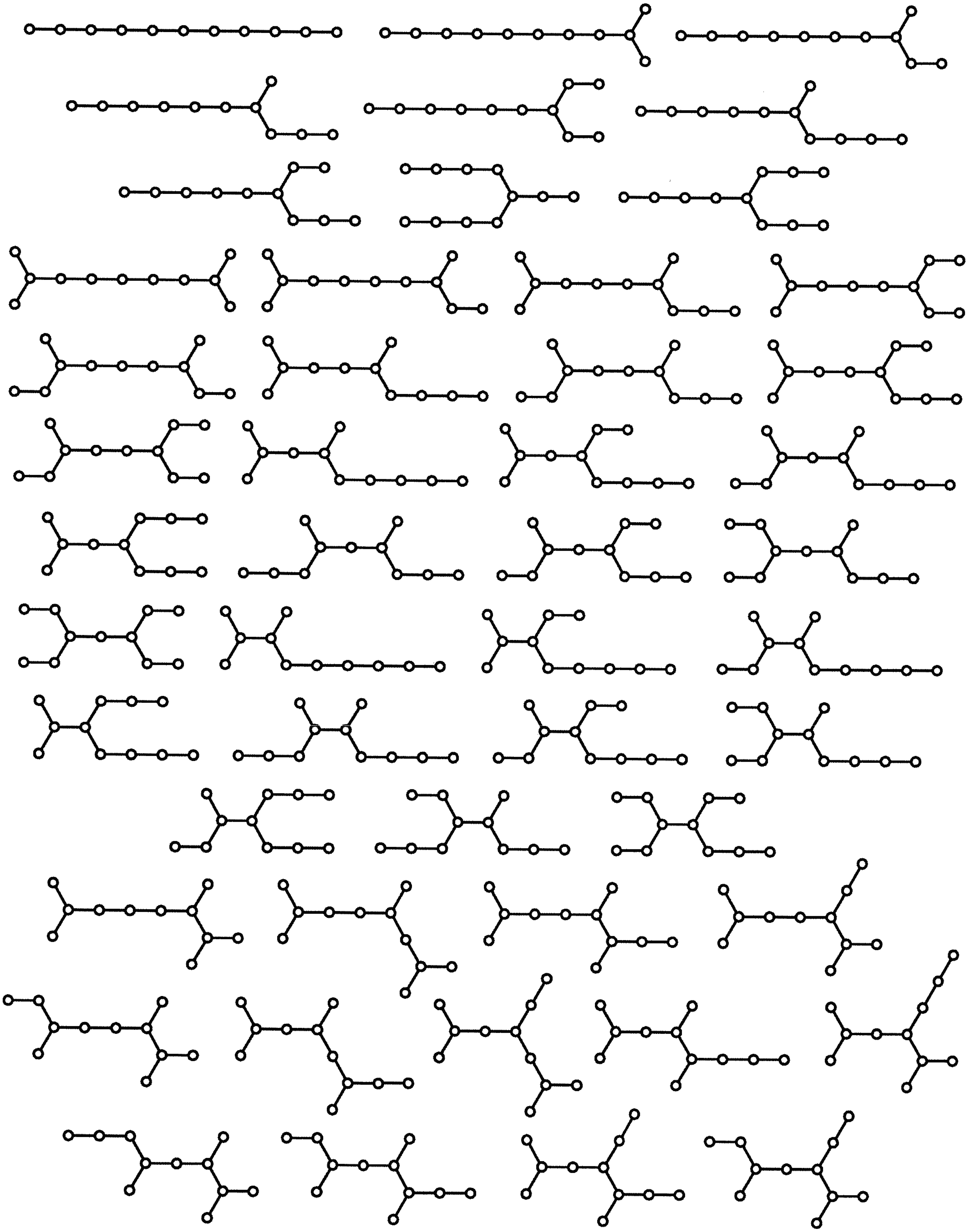
v = 10



v = 10



v = 11



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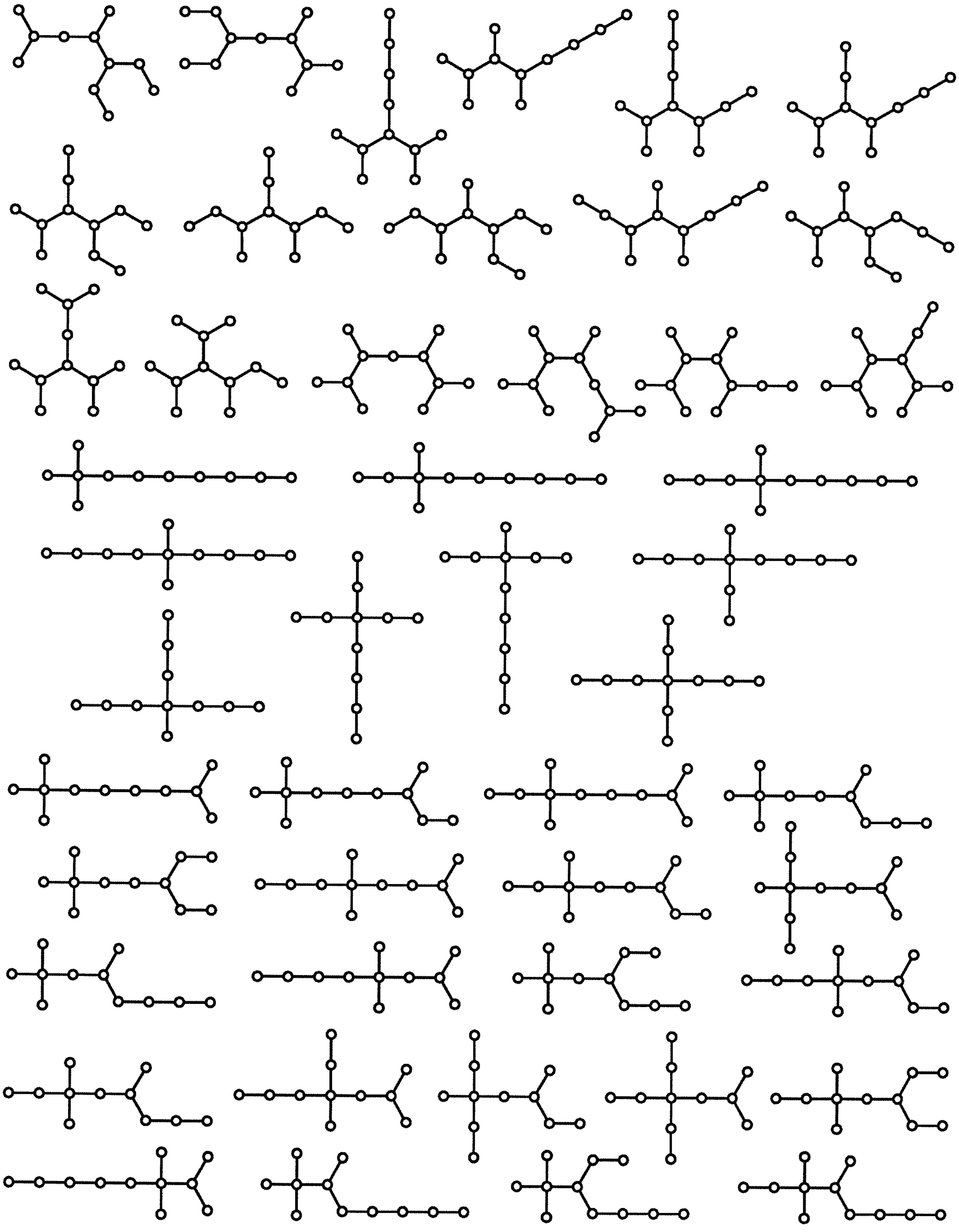
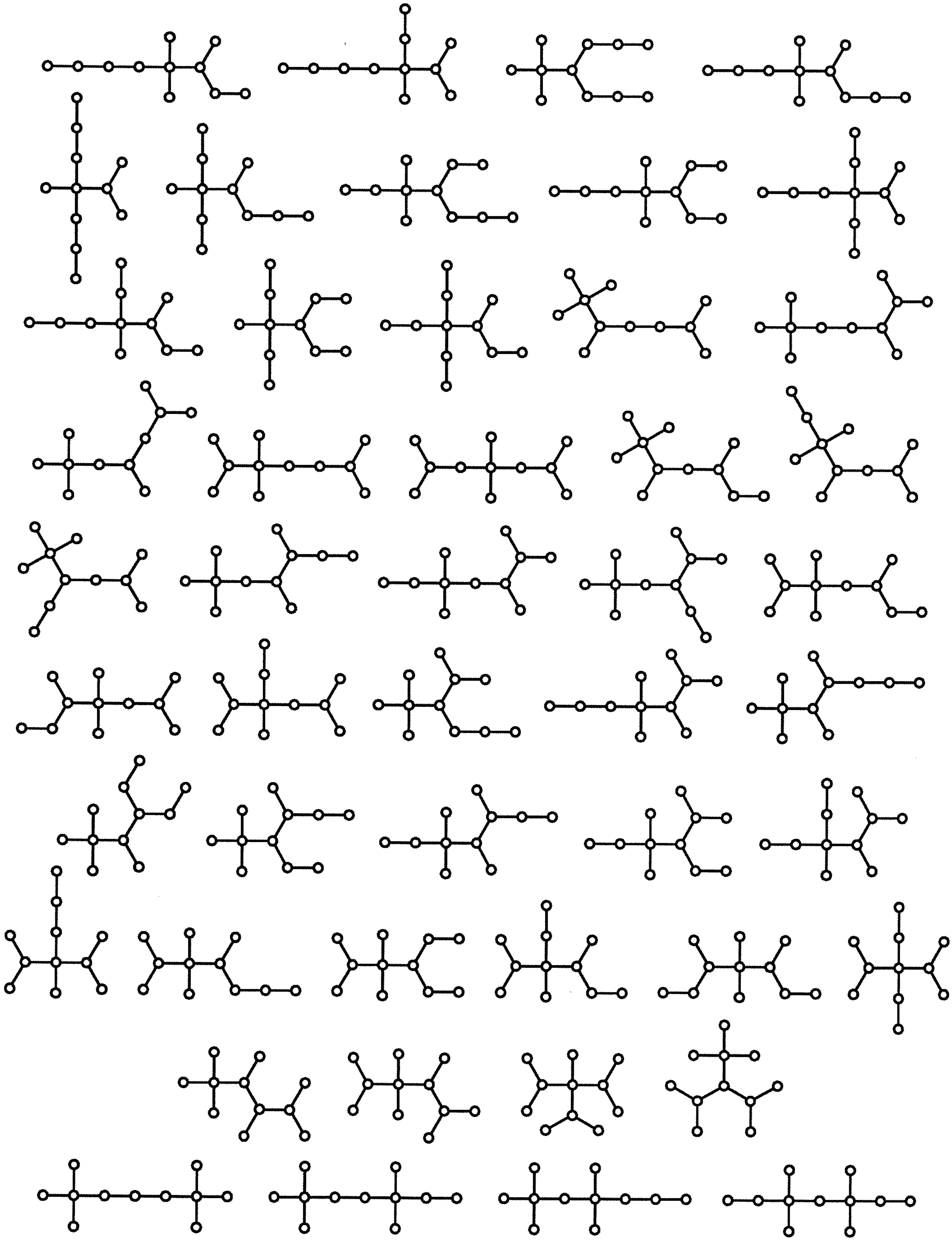
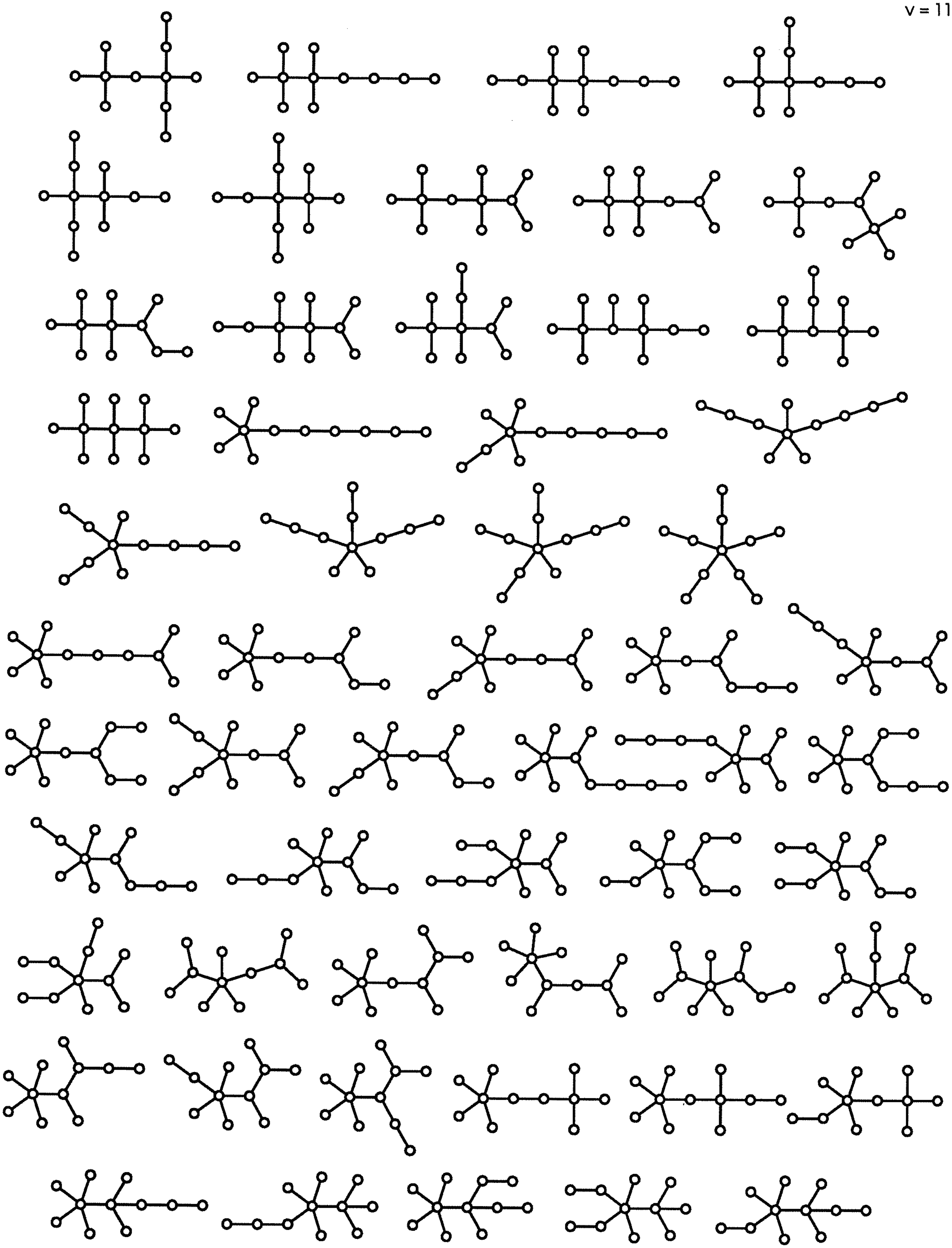


v = 11





v = 11