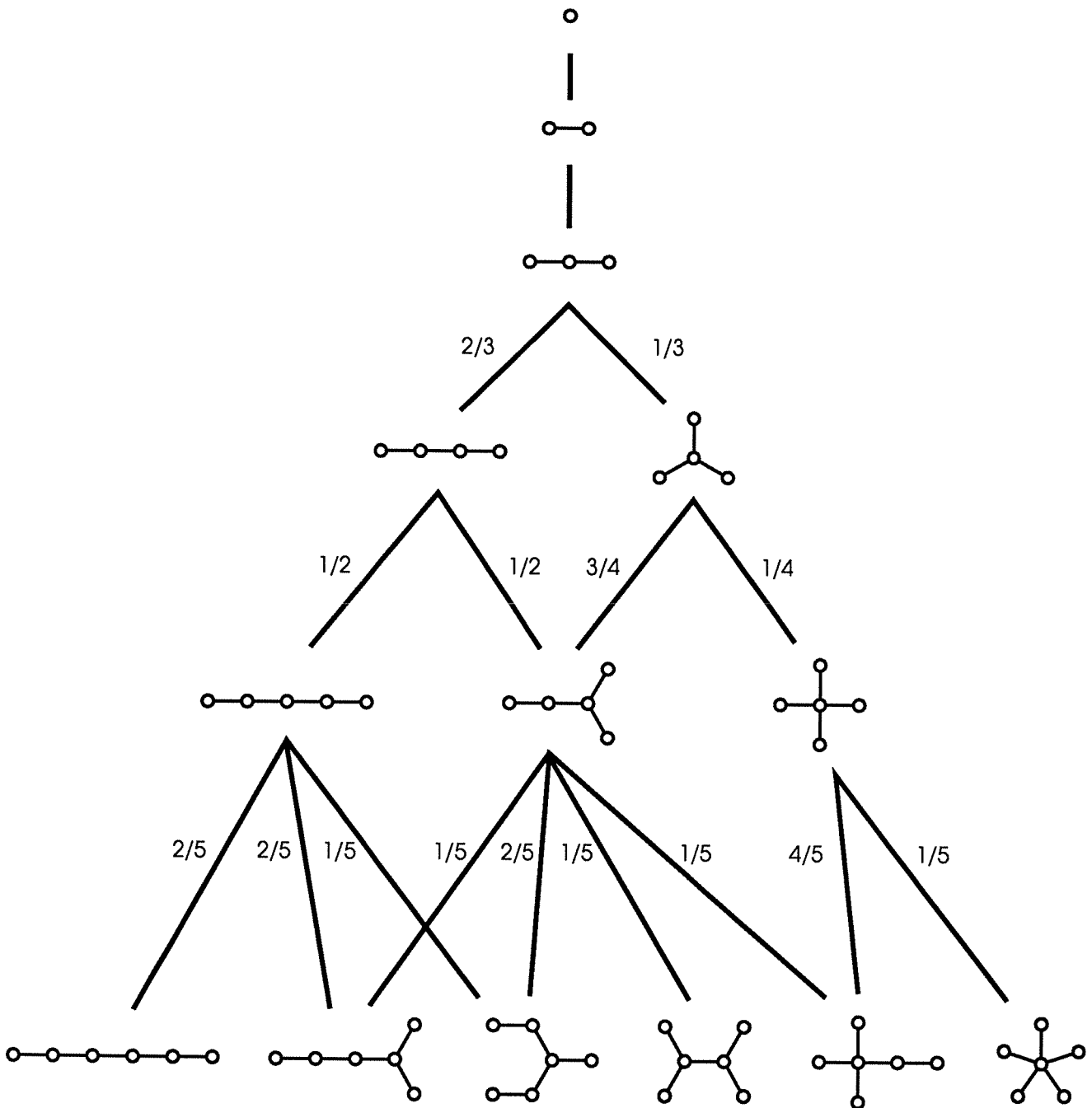


3

Probabilities of Random Trees ($v \leq 11$)



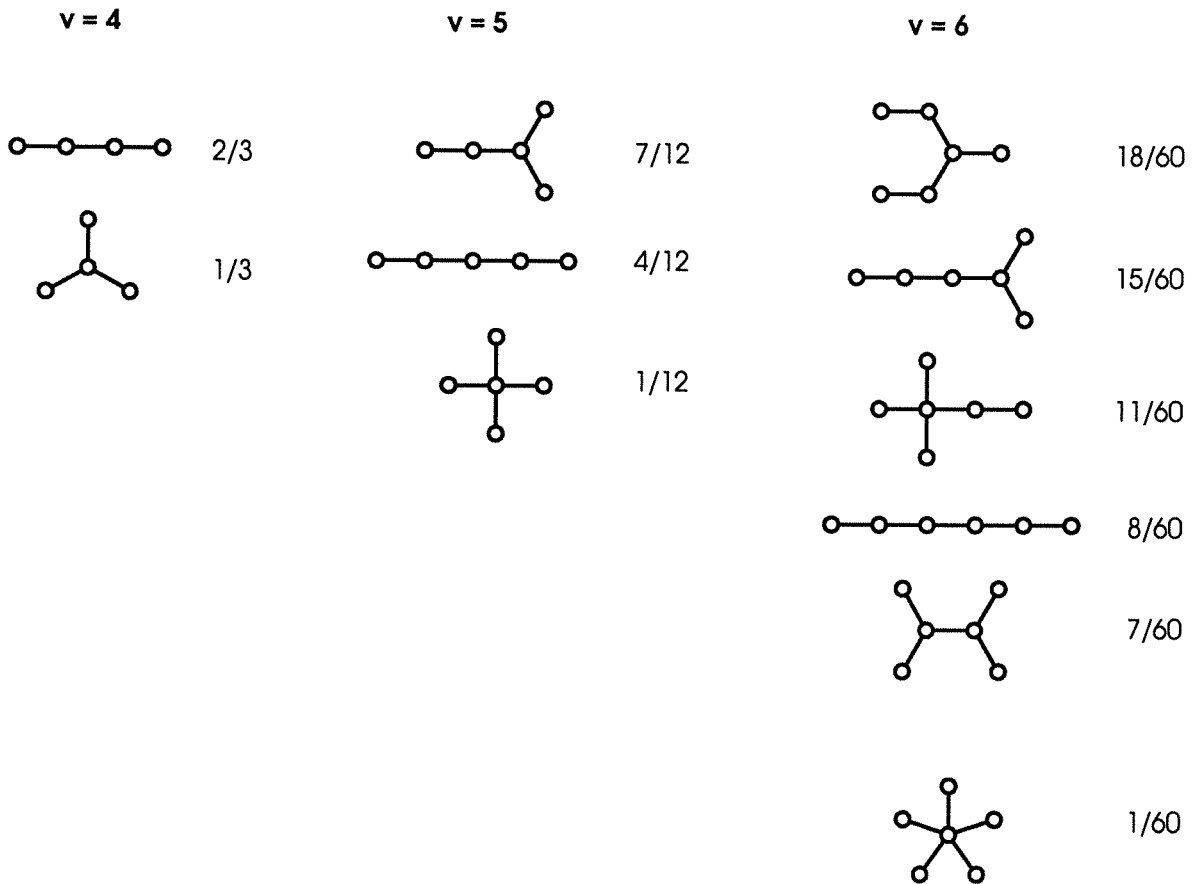
Begin with a single vertex (see chart at left). Add a leaf (a $\text{---}\bullet$) and the result is certainly a complete graph on two vertices. Add another leaf to either vertex and, certainly, we have a 3-path. But the next leaf could be attached to an end of the 3-path — in two ways — or to the middle in one way. If the leaf is added randomly, with all three vertices equally likely, then the probability of a 4-path is $2/3$, while the probability of a star is $1/3$.

The chart on the previous page shows the beginning of the possibilities for growing trees by random addition of leaves. It is a POSET (partially-ordered set) in which a heavy line connects two trees if one can be obtained from the other by adding a leaf. A fraction on a heavy line indicates the probability of making the lower tree from the upper tree (among all trees that the upper tree can produce).

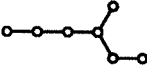
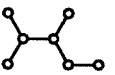
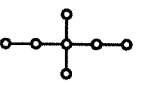
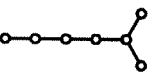
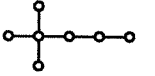
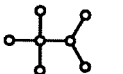
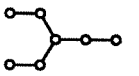

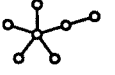
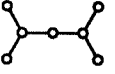

For each tree up to 10 vertices, the tables on the next five pages show the probability (among all trees of the same number of vertices) of generation by random leaf addition *from scratch*. These are obtained by multiplying partial probabilities on paths through the POSET; a tree may be producible by many paths, and the probabilities of all such paths are added.

Trees are tabulated in order of decreasing probability, from common to rare. Notice the rough correlation between decreasing probability and increasing symmetry, and the rough correlation between increasing probability and increasing variety of vertex degrees.

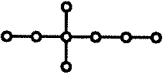
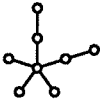
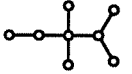
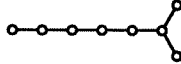
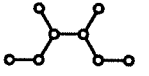
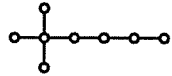
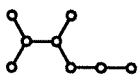
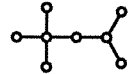
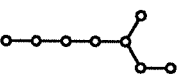

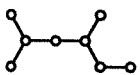
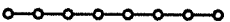
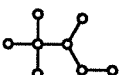
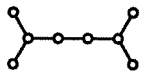
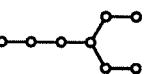
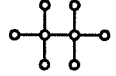
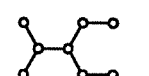
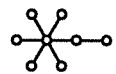
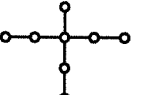

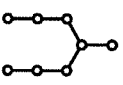
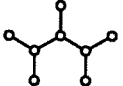
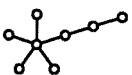
For $v = 10$ and 11 the denominator of probabilities is given once per page.



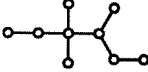
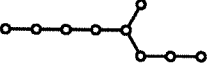

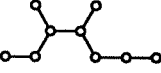
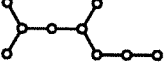
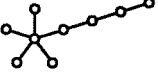
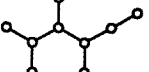
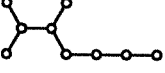
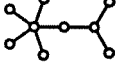
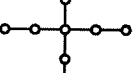
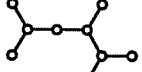
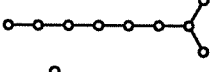
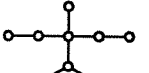
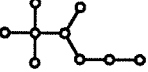
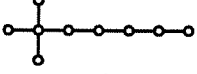
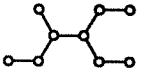
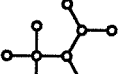
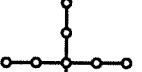
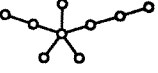
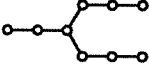
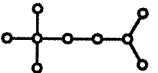
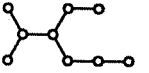
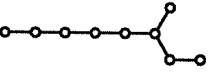
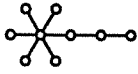
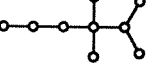
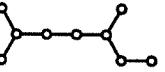
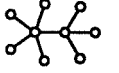
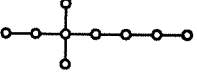
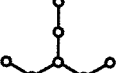
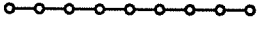
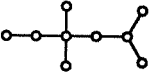
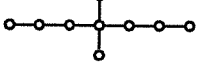
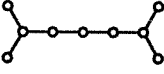
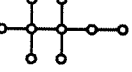
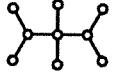


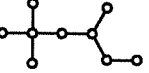

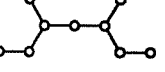
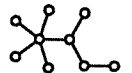

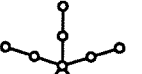

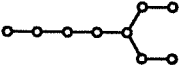
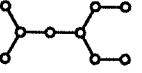
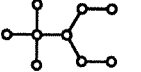
v = 7

	82/360
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	51/360
	31/360
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	16/360
	15/360
	1/360

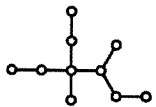
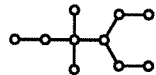
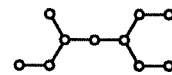
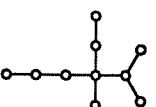
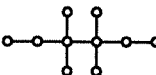
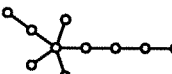
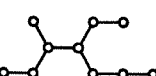
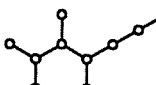
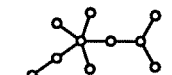
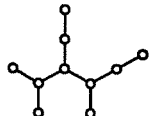
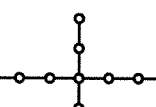
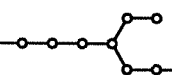
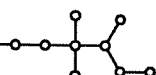
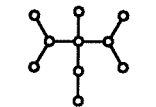
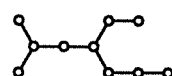
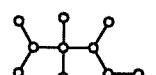
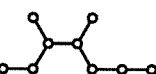
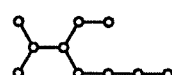
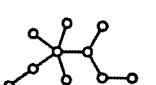
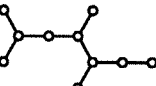
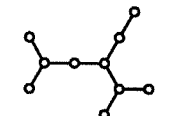
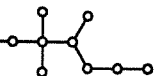
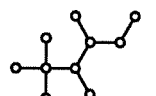
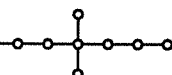
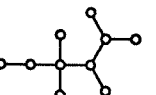
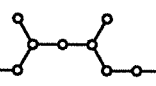
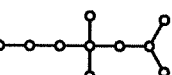
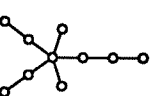
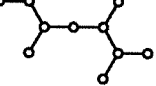
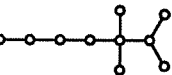
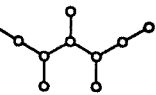
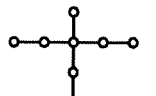
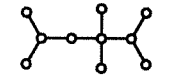
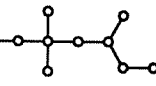
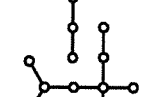
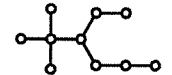

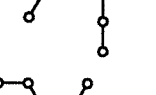
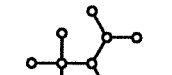
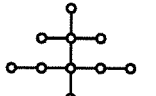
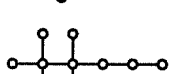
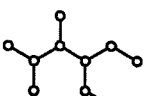
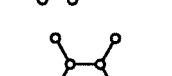
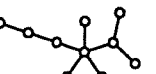
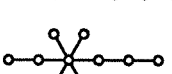
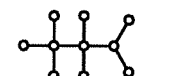
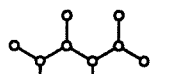
v = 8

	262/2520		67/2520
	256/2520		63/2520
	240/2520		57/2520
	192/2520		56/2520
	176/2520		41/2520
	173/2520		32/2520
	155/2520		31/2520
	136/2520		25/2520
	133/2520		22/2520
	120/2520		1/2520
	98/2520		
	94/2520		
	90/2520		

$v = 9$

	1719/20,160		436/20,160		177/20,160
	1236/20,160		432/20,160		147/20,160
	1221/20,160		431/20,160		146/20,160
	1020/20,160		427/20,160		127/20,160
	1005/20,160		404/20,160		120/20,160
	882/20,160		399/20,160		120/20,160
	756/20,160		370/20,160		119/20,160
	730/20,160		366/20,160		112/20,160
	710/20,160		363/20,160		91/20,160
	609/20,160		360/20,160		64/20,160
	603/20,160		360/20,160		63/20,160
	561/20,160		350/20,160		63/20,160
	554/20,160		342/20,160		56/20,160
	522/20,160		327/20,160		29/20,160
			321/20,160		1/20,160
			312/20,160		
			309/20,160		
			288/20,160		

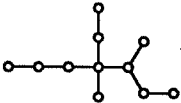
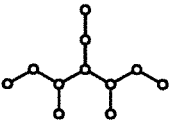
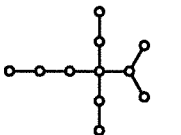
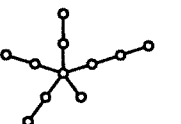
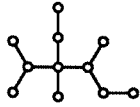
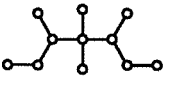
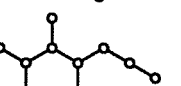
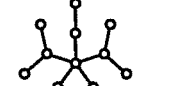
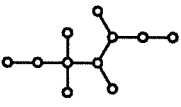
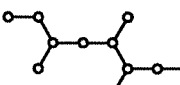
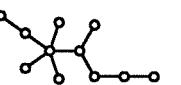
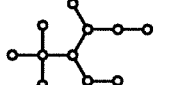

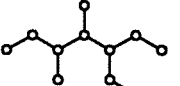
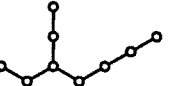
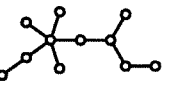
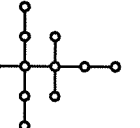
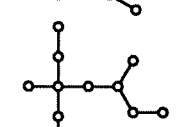
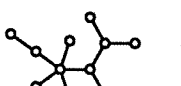
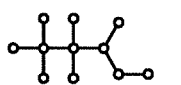
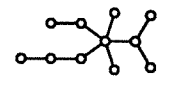
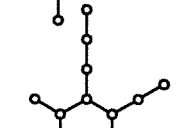
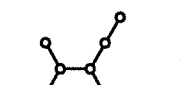
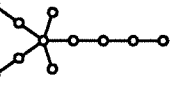
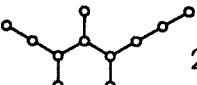
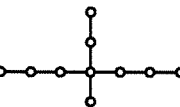
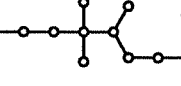
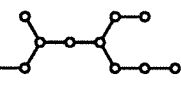
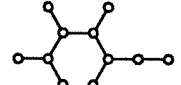
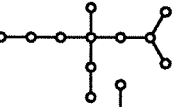
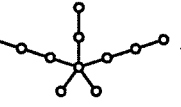
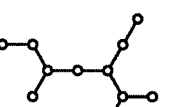
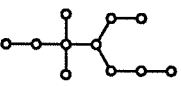
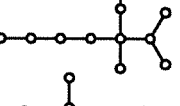
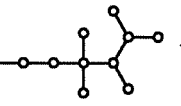
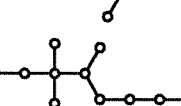
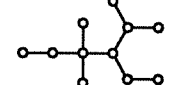
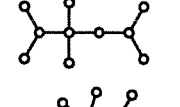
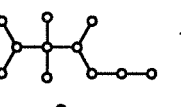
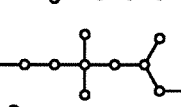
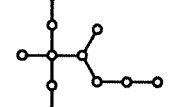
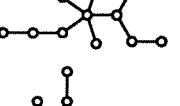
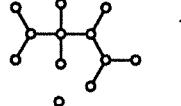
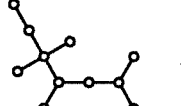
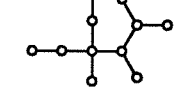
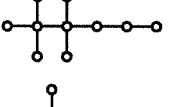
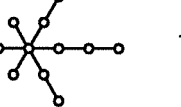
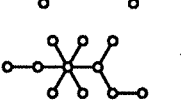
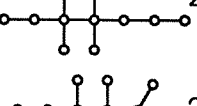
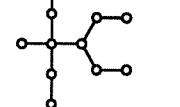
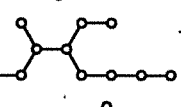
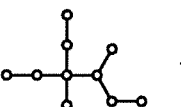
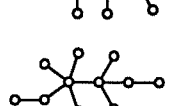
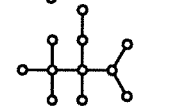
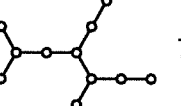
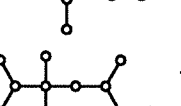

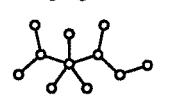
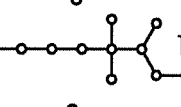
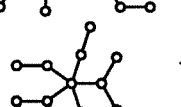
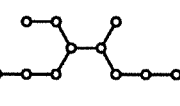
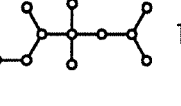
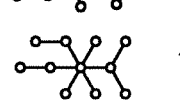
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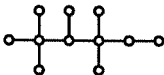
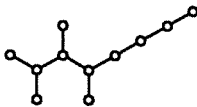
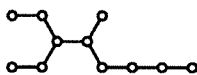
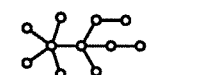
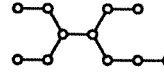
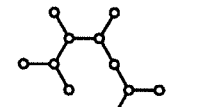
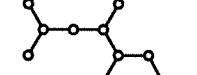
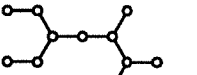
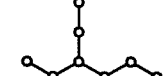
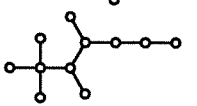
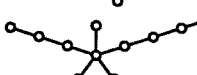

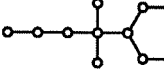
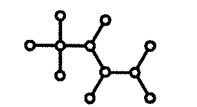
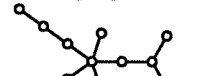
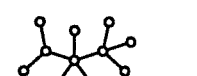
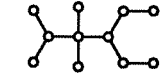
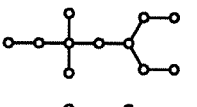
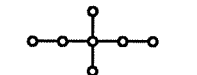
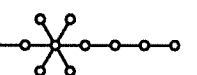
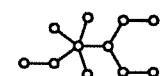
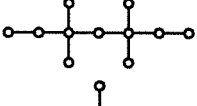
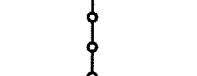
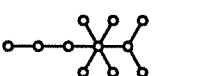
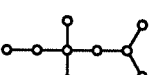
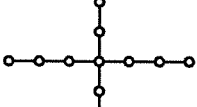
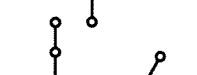
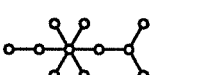
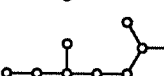
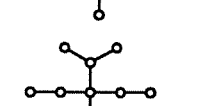

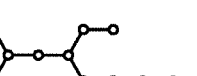
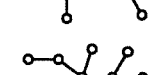
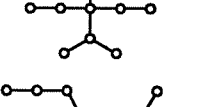
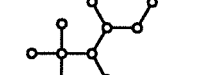
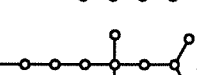
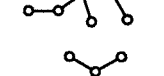
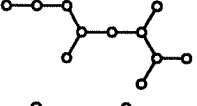
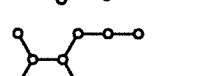
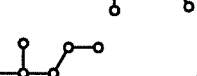
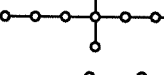
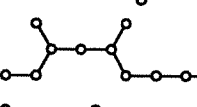

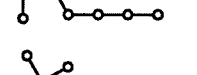
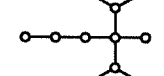
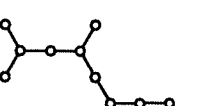
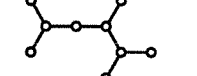
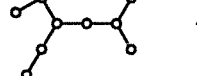

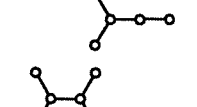
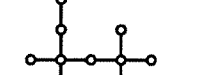
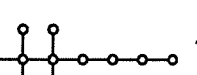
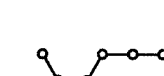
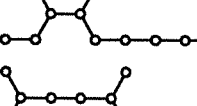
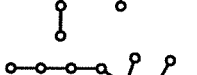
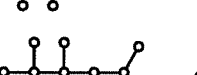
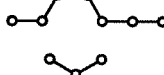
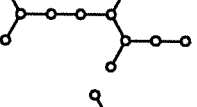
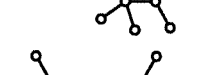
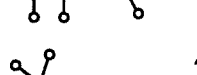
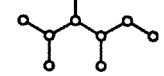
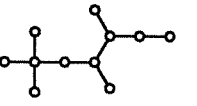
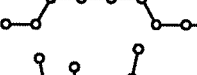
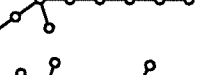
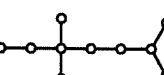
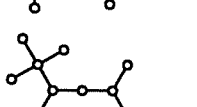
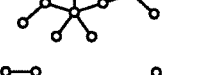
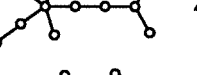
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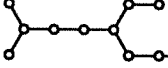
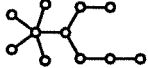
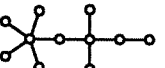
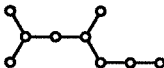
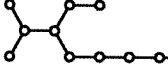
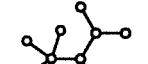
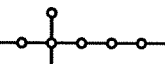
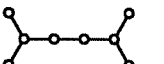
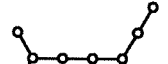
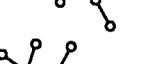
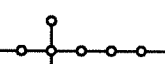
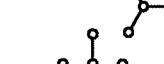

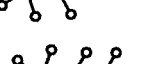
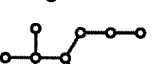
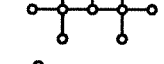
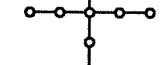
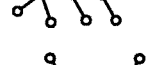
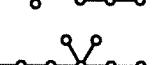
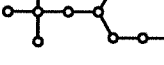

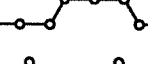
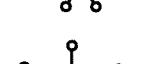

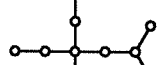
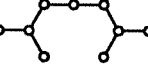

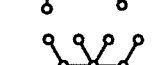
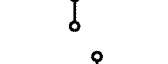
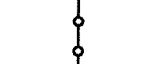
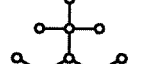
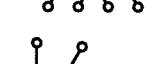
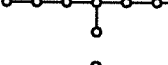
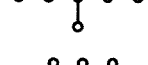
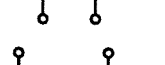
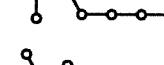
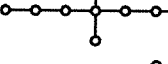
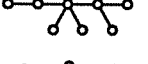
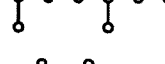
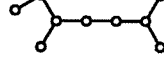
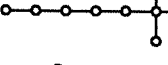
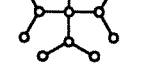


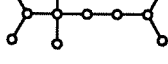
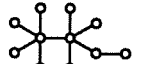
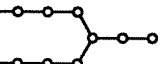
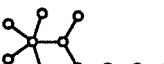
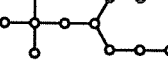
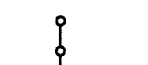
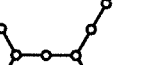
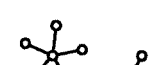
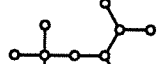

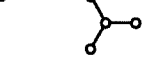
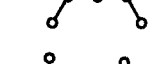

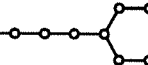
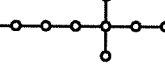
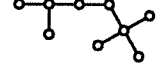
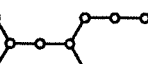
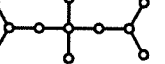
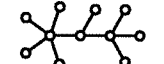

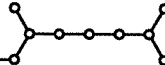
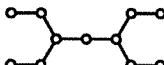
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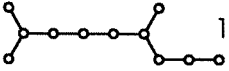


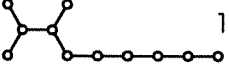
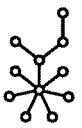
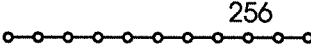
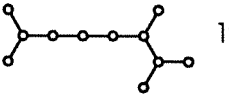
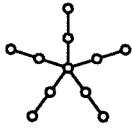
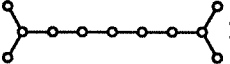
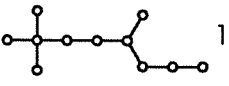

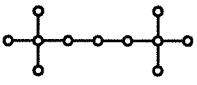
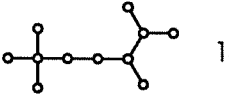
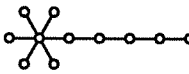
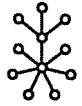
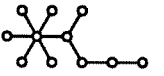
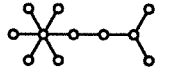
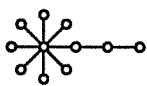

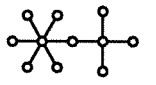
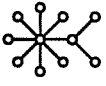
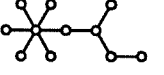
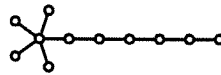

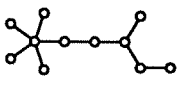
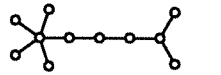

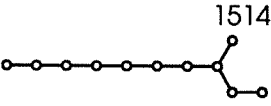
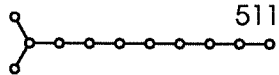
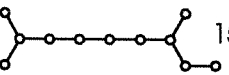
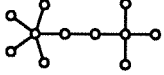
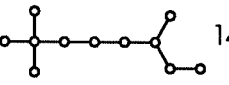
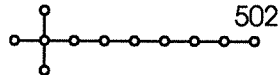
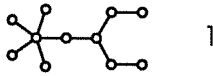
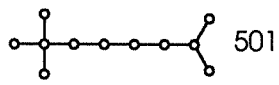
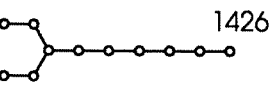
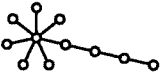
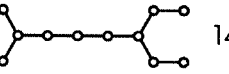

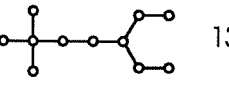

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 21 700	 16 044	 13 470	 11 820
 21 574	 15 985	 13 425	 11 807
 20 776	 15 577	 13 260	 11 793
	 15 204	 13 215	 11 609

	$\frac{11\ 516}{1\ 814\ 400}$		8139		6008		5352
	11 158		8115		5990		5341
	11 095		7934		5950		5174
	10 990		7899		5930		4947
	10 927		7812		5898		4918
	10 816		7728		5883		4913
	10 612		7630		5828		4585
	10 570		7525		5785		4515
	10 482		7252		5765		4450
	10 430		6448		5748		4435
	10 360		6430		5723		4380
	8510		6426		5714		4365
	8405		6411		5703		4356
	8340		6276		5693		4346
	8174		6268		5585		4256
	8150		6168		5362		4212
							4086

v = 11

	$\frac{4076}{1\ 814\ 400}$		3818		2836		2180
	4065		3797		2824		2175
	4055		3714		2818		2150
	4050		3693		2810		2136
	4035		3640		2786		2130
	4030		3612		2784		2127
	4020		3500		2775		2126
	4009		3472		2764		2121
	3999		3430		2477		2115
	3986		3368		2300		2100
	3965		3218		2285		2094
	3930		2910		2265		2076
	3909		2900		2250		2040
	3900		2875		2238		1974
	3849		2865		2188		1942
					2186		1935
					2181		1934

	1930		1180		258
	1927		890		256
	1923		762		255
	1894		610		246
	1887		526		246
	1702		525		178
	1695		516		129
	1556		514		46
	1520		513		$\frac{1}{1\ 814\ 400}$
	1514		511		
	1511		504		
	1484		502		
	1432		501		
	1426		400		
	1423		399		
	1396		336		