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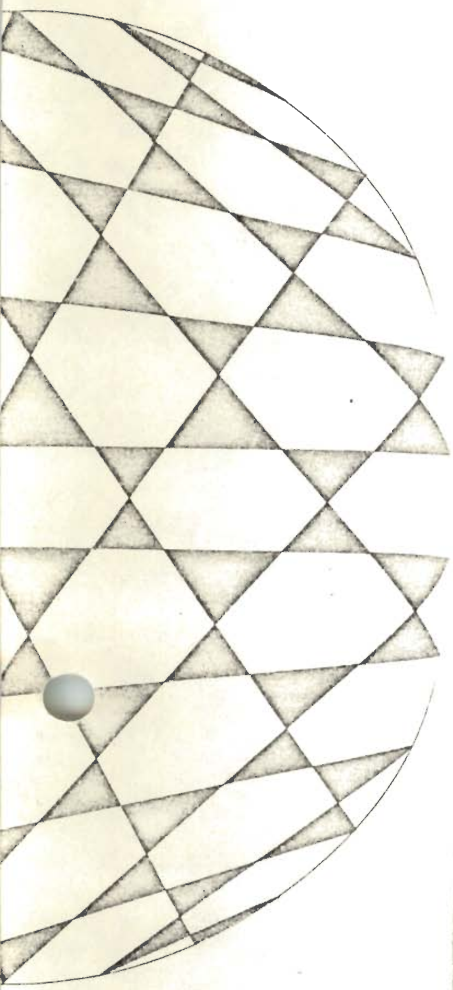

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ON PRIME NUMBERS AND PERFECT NUMBERS*

By JACQUES TOUCHARD

1. Introduction

IN AN important paper[3], Balth. van der Pol has shown that the function

$$\alpha(t) = 1 - 24 \sum_{1}^{\infty} \sigma_k e^{-kt}, \tag{1}$$

where σ_k is the sum of the divisors of k, including 1 and k (cf. Hardy and Wright[1]), satisfies the differential equation

$$2\frac{d^3\alpha}{dt^3} + 2\alpha\frac{d^2\alpha}{dt^2} - 3\left(\frac{d\alpha}{dt}\right)^2 = 0 \tag{2}$$

and, by substitution of (1) into (2) he shows that the numbers σ_k satisfy the recurrence relation

$$\frac{n^2(n-1)}{12}\sigma_n = \sum_{k=1}^{n-1}[5k(n-k) - n^2]\sigma_k\sigma_{n-k}, \qquad n > 1. \tag{3}$$

The first few instances of this, after simplification, are

$$\begin{aligned} \sigma_2 &= 3\sigma_1^2 \\ 3\sigma_3 &= 4\sigma_1\sigma_2 \\ 2\sigma_4 &= -\sigma_1\sigma_3 + 2\sigma_2^2 \\ 5\sigma_5 &= -6\sigma_1\sigma_4 + 6\sigma_2\sigma_3. \end{aligned}$$

Recently, van der Pol has found that relation (3) may also be written as

$$\frac{n^2(n-1)}{6}\sigma_n = \sum_{k=1}^{n-1}(3n^2 - 10k^2)\sigma_k\sigma_{n-k}. \tag{4}$$

This is the first instance of a family of modifications of (3) which may be obtained as follows: put

$$S_p \equiv S_p(n-1) = \sum_{k=1}^{n-1} k^p\sigma_k\sigma_{n-k}, \tag{5}$$

and change k to $n - k$, so that

* Translated from the French manuscript by John Riordan.

$$S_p = n^p S_0 - pn^{p-1} S_1 + \binom{p}{2} n^{p-2} S_2 - \cdots + (-1)^p S_p.$$

This entails a number of relations, the simplest of which are

$$2S_1 = nS_0$$
$$4S_3 = -n^3 S_0 + 6nS_2.$$

By means of these (3) may be given various forms, of which we notice the following

$$n^2(n-1)\sigma_n - 18n^2 S_0 + 60S_2 = 0 \qquad (6)$$
$$n^3(n-1)\sigma_n - 48nS_2 + 72S_3 = 0. \qquad (7)$$

The first of these is in fact eq. (4) in different notation.

Noting that for prime numbers, p, $\sigma_p = 1 + p$ and for perfect numbers $\sigma_n = 2n$, eqs. (3), (6), and (7) lead to results which are examined below.

2. Prime Numbers

Writing

$$u_n = \sigma_n - 1 - n, \qquad u_1 = -1 \qquad (8)$$

and inserting (8) into (3) gives

$$\frac{n^2(n-1)}{12} u_n + \frac{n(n^2-1)(n-4)}{4} =$$
$$\sum_{1}^{n-1} [5k(n-k) - n^2]\,[u_k u_{n-k} + 2(n-k+1)u_k], \qquad n > 1. \qquad (9)$$

Thus (9) is an expression, without any auxiliary arithmetic functions, which completely determines u_n and it is clear from (8) that u_n is zero for n prime, and a positive integer for n composite. The first few instances are

$$u_2 - 9 = 3u_1^2 + 12u_1$$
$$3u_3 - 12 = 4u_1u_2 + 12u_1 + 8u_2$$
$$4u_4 = -2u_1u_3 + 4u_2^2 - 8u_1 + 24u_2 - 4u_3.$$

Now consider (6) and suppose first that n is composite. Replacing σ_n by the smaller number $1 + n$, we have

$$n^2(n^2-1) - 18n^2 S_0 + 60S_2 < 0.$$

The biquadratic equation

$$z^4 - [18S_0(n-1) +$$

then has two positive roots, one s

Next take n prime. Then n i

smaller of the two positive roots.

To see this, take $z = \sqrt{n^2+1}$

to $2n^2 - 18S_0(n-1)$ and this is n

$$n^2 < 9.$$

for n a prime. This may be veri

$n > 3$

$$S_0(n-1) = 2\sigma_{n-1} +$$
$$> 2n + \sum_k^n$$
$$> 2n + ($$

since $\sigma_k \geq k + 1$, $\sigma_{n-1} > n$ for n

$$n^2 < 18n + \frac{3}{2}(n-3)$$

and (11) is proved generally.

Thus the result of the substituti

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necessary and sufficient conditio

may be given as:

$$\Delta(n-1) = [18S_0(n -$$

as well as

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must be squares.

3. Perfe

By definition a number n is perf

visors, n itself excluded, or, as no

$2n$. Write

$$v_n = \sigma_n - 2$$

The biquadratic equation

$$z^4 - [18S_0(n-1) + 1]z^2 + 60S_2(n-1) = 0 \qquad (10)$$

then has two positive roots, one smaller, one greater than n.

Next take n prime. Then n must satisfy (10) and is in fact the smaller of the two positive roots.

To see this, take $z = \sqrt{n^2+1}$ in (10); the left-hand side reduces to $2n^2 - 18S_0(n-1)$ and this is negative if

$$n^2 < 9S_0(n-1) \qquad (11)$$

for n a prime. This may be verified directly for $n = 2$ and 3 and for $n > 3$

$$S_0(n-1) = 2\sigma_{n-1} + \sum_{2}^{n-2} \sigma_k\sigma_{n-k}$$

$$> 2n + \sum_{k=2}^{n-2} (k+1)(n-k+1)$$

$$> 2n + (n-3)(n^2+9n+2)/6$$

since $\sigma_k \geq k+1$, $\sigma_{n-1} > n$ for n prime. Hence it is clear that

$$n^2 < 18n + \frac{3}{2}(n-3)(n^2+9n+2), \qquad n > 3$$

and (11) is proved generally.

Thus the result of the substitution $z = \sqrt{n^2+1}$ is negative; hence $\sqrt{n^2+1}$ lies between the two positive roots and n is the smaller. The necessary and sufficient conditions that n be prime in consequence may be given as:

$$\Delta(n-1) = [18S_0(n-1)+1]^2 - 240S_2(n-1)$$

as well as

$$\frac{1}{2}[18S_0(n-1) + 1 - \sqrt{\Delta(n-1)}]$$

must be squares.

3. Perfect Numbers

By definition a number n is perfect when it equals the sum of its divisors, n itself excluded, or, as noted in the introduction, when $\sigma_n = 2n$. Write

$$v_n = \sigma_n - 2n, \qquad v_1 = -1 \qquad (12)$$

so that v_n is zero for all perfect numbers, and otherwise positive or negative according as the number is abundant or deficient.

Using (12) in (3), it is found that

$$\frac{n^2(n-1)}{12} v_n + \frac{n(n-1)}{6} (n^2 - 4n - 4) = \sum_{1}^{n-1} [5k(n-k) - n^2]\,(v_k + 4k)v_{n-k}, \qquad n > 1. \quad (13)$$

It is known[1] that the even perfect numbers are all of the form

$$n = 2^a(2^{a+1} - 1),$$

where $2^{a+1} - 1$ is prime. No odd perfect numbers are known but it has not been proved that they do not exist. Certain properties follow from (13).

Suppose n an odd perfect number so that $v_n = 0$. Then from (13)

$$n(n-1)(n^2 - 4n - 4)/6 = \sum_{1}^{n-1} [5k(n-k) - n^2]\,(v_k + 4k)v_{n-k}. \quad (14)$$

Because of the symmetry of $5k(n-k)v_kv_{n-k}$ in k and $n - k$ and because n is odd, the right of (14) is even. For $n = 12m + 5$ or $12m + 11$, the left of (14) is a fraction of denominator 3, and for $n = 12m + 3$ or $12m + 7$, it is an odd integer. In any of these four cases (14) is impossible and we are left with the only possibilities $n = 12m + 1$ and $12m + 9$.

For the last it is easy to show that m must be a multiple of 3. For if $n = 12m + 9 = 3\,(4m + 3)$ and if 3 and $4m + 3$ are relatively prime

$$\sigma_{3(4m+3)} = \sigma_3\sigma_{4m+3} = 4\sigma_{4m+3}$$

and, since n is perfect

$$\sigma_{3(4m+3)} = 2.3(4m + 3),$$

so that

$$4\sigma_{4m+3} = 6(4m + 3)$$

which is impossible since the right-hand side is not divisible by 4.

By similar but longer arguments, it may be shown that perfect numbers of the form $12m + 9$, if they exist, must be of one of the forms $81(4m + 1)$, $9\cdot13(12m + 1)$ or $9\cdot13(12m + 5)$. Further results are possible, but seem not to be leading to a decisive result.

All this is summarized in the

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…$v_k + 4k)v_{n-k}, \qquad n > 1.$ (13)

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…$12m + 5)$. Further results are
…a decisive result.

THEOREM: *If odd perfect numbers exist, they are of the forms $12m + 1$ or $36m + 9$.*

4. Consequences of Eq. (7)

The results of the theorem are much more easily obtained from eq. (7). For n perfect ($\sigma_n = 2n$) this may be written as

$$n^4(n - 1)/12 = 2nS_2 - 3S_3$$

and the right-hand side is an integer.

Then if n is odd and divisible by 3, $n - 1$ must be divisible by 4, so that $n = 12m + 9$. On the other hand, for n odd and prime to 3, $n - 1$ must be divisible by 12, and $n = 12m + 1$.

Turn now to n prime ($\sigma_n = 1 + n$); (7) becomes

$$n^3(n^2 - 1)/24 = 2nS_2 - 3S_2.$$

If $n > 3$, $n^2 - 1$ must be divisible by 24; hence $n = 12m + 1$, $12m + 5$, $12m + 7$, or $12m + 11$. It is known that an infinity of primes of each of these forms exists, but of course we may not conclude by analogy that perfect numbers $12m + 1$ or $12m + 9$ also exist.

For numerical concreteness, I give below values of σ_n and $S_p(n)$, $p = 0\,(1)\,3$ for $n = 1\,(1)\,10$.

VALUES OF σ_n AND $S_p(n)$

n	σ_n	$S_0(n)$	$S_1(n)$	$S_2(n)$	$S_3(n)$
1	1	1	1	1	1
2	3	6	9	15	27
3	4	17	34	76	184
4	7	38	95	275	875
5	6	70	210	720	2700
6	12	116	406	1666	7546
7	8	185	740	3440	17600
8	15	258	1161	6129	35721
9	13	384	1920	11250	72750
10	18	490	2695	17545	126445
		385	441	477	499

Notation: $S_p(n) = \sum_{1}^{n} k^p \sigma_k \sigma_{n-k+1}$

REFERENCES

[1] Hardy and Wright, *An Introduction to the Theory of Numbers*, Oxford, 1938, p. 240.
[2] H. Hasse, *Vorlesungen uber Zahlentheorie*, Springer, Berlin, 1950, p. 32.
[3] Balth. van der Pol, "On a Non-linear Partial Differential Equation Satisfied by the Logarithm of the Jacobian Theta Functions, with Arithmetical Applications I, II," *Nederl. Akad. Wetensch. Proc.*, Ser. A, v. 54, 1951, p. 261–271, 272–284.

$$\sigma_n = \sum_{\substack{d \mid n \\ 1 \le d \le n}} d$$

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