f91

1006

STANFORD UNIVERSITY STANFORD, CALIFORNIA 94305-2140

DONALD E. KNUTH
Fletcher Jones Professor
Department of Computer Science
Telephone [415] 723-4367

January 18, 1989

Prof. Louis W. Shapiro
Department of Mathematics
Howard University
Washington, DC 20059

1006

Dear Prof. Shapiro,

Thanks for sending the reprint of your paper with Donaghey that surveys the Motzkin numbers [*JCT* **A23** (1977), 291–301].

I happened to notice that the sequence γ_n you mention on page 293 is precisely half of the sequence that Euler called a "memorable failure of induction" in 1765 (see the answer to exercise 7.56 in *Concrete Mathematics*, and reference 91 in the bibliography): The numbers are $F_{n-1}(F_{n-1}+1)/2$ for $n = 0, 1, 2, 3, \ldots, 8$, but then the pattern stops!

I found this by noticing that $(1-x)^2 - 4x^2 = (1-3x)(1+x)$, hence the generating function in your equation (7) can be divided by $1+x$ and you still get essentially a polynomial multiple of the generating function for the β's. Indeed,

$$\gamma_n = \tfrac{1}{2}(3\beta_n - \beta_{n+1}).$$

From this, or from your equation for M_n at the bottom of page 293,

$$m_n = \tfrac{1}{2}(3\beta_n + 2\beta_{n+1} - \beta_{n+2}).$$

Thus Euler's numbers β_n give a nice "basis" for both the Motzkin numbers and their γ relations. Since the generating function for $\langle\beta_n\rangle$ is

$$\begin{aligned}\frac{1}{\sqrt{(1-3x)(1+x)}} &= \frac{\sqrt{3}}{2}\,\frac{1}{\sqrt{1-3x}}\,\frac{1}{\sqrt{1-(1-3x)/4}}\\ &= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\infty} \binom{k-1/2}{n}\frac{\sqrt{3}}{2}\left(-\frac{1}{4}\right)^k (-3)^n\\ &= \sum_{n=0}^{\infty} x^n \frac{3^{n+1/2}}{2} \sum_{k=0}^{\infty} \binom{n-k-1/2}{n}\left(-\frac{1}{4}\right)^k\\ &= \sum_{n=0}^{\infty} x^n \frac{3^{n+1/2}}{\sqrt{4\pi}} \sum_{k=0}^{\infty} \frac{k!}{(2k)!\,n^{\underline{k+1/2}}},\end{aligned}$$

we have the asymptotic formula

$$\beta_n = 3^n\sqrt{\frac{3}{4\pi n}}\left(1 + \frac{5}{8n} + O(n^{-2})\right).$$

(See the answers to exercises 9.44 and 9.60.) This gives asymptotics for m_n and γ_n.

Cordially,

Donald E. Knuth
Professor

DEK/pw

cc: R. K. Guy, N. J. A. Sloane, R. P. Stanley, H. Wilf