Some S-fractions related to the expansions of sin(ax)/cos(bx) and cos(ax)/cos(bx)

Peter Bala, May 11 2017

1. An S-fraction (also known as a fraction of Stieltjes-type) is a continued fraction of the form

$$S(x) = \cfrac{1}{1 - \cfrac{d_1 x}{1 - \cfrac{d_2 x}{1 - \cfrac{d_3 x}{1 - \cdots}}}}. \tag{1}$$

We say that it corresponds to the formal power series

$$f(x) = 1 + c_1 x + c_2 x^2 + \cdots \tag{2}$$

if the expansion of its n^{th} approximant in ascending powers of x agrees with the power series (2) up to and including the term in $x^{n-1}, n = 1, 2, 3, \ldots$.

Recall the **even part** of a continued fraction is the continued fraction whose n-th approximant is the $2n$-th approximant of the given continued fraction. The even part of the generic S-fraction (1) is given by [2, Chapter 1, Section 4]

$$\cfrac{1}{1 - d_1 x - \cfrac{d_1 d_2 x^2}{1 - (d_2 + d_3)\,x - \cfrac{d_3 d_4 x^2}{1 - (d_4 + d_5)\,x - \cfrac{d_5 d_6 x^2}{1 - (d_6 + d_7)\,x - \cdots}}}}. \tag{3}$$

(3) is an example of a J-fraction (J stands for Jacobi). The general form of a J-fraction is

$$J(x) = \cfrac{a_0}{1 - b_1 x - \cfrac{a_1 x^2}{1 - b_2 x - \cfrac{a_2 x^2}{1 - b_3 x - \ddots}}}$$

We say that the this J-fraction is **asociated** with the formal power series (2), denoted by $J(x) = f(x)$, if the expansion of its n^{th} approximant in ascending powers of x agrees with the power series (2) up to and including the term in $x^{2n-1}, n = 1, 2, 3, \ldots$.

By making use of an equivalence transformation it is easy to see that the m^{th} binomial transform of the generic J-fraction $J(x)$ also has the form of a J-fraction:

$$\frac{1}{1-mx}J\left(\frac{x}{1-mx}\right) = \cfrac{a_0}{1-(b_1+m)\,x-\cfrac{a_1x^2}{1-(b_2+m)\,x-\cfrac{a_2x^2}{1-(b_3+m)\,x-\ddots}}}. \tag{4}$$

Now in general, a J-fraction will not be the even part of an S-fraction. The purpose of this note is to give some examples of J-fractions whose m^{th} binomial transform, for particular values of m, is equal to the even part of an S-fraction. In Section 2 we consider the J-fraction associated with the trigonometric function $\sin(ax)/a\cos(bx)$. The form of the J-fraction is due to Stieltjes. We show there are two values of m such that the m^{th} binomial transform of Stieltjes' J-fraction equals the even part of an S-fraction. There are similar results for the trigonometric function $\cos(ax)/\cos(bx)$, which we outline in Section 3.

2. Consider the exponential generating function $\sin(ax)/a\cos(bx)$, with complex parameters a and b. The Taylor expansion of the function about $x = 0$ begins

$$\begin{aligned}\frac{1}{a}\frac{\sin(ax)}{\cos(bx)} &= x-\left(a^2-3b^2\right)\frac{x^3}{3!}+\left(a^4-10a^2b^2+25b^4\right)\frac{x^5}{5!}\\ &\quad -\left(a^6-21a^4b^2+175a^2b^4-427b^6\right)\frac{x^7}{7!}+\cdots.\end{aligned} \tag{5}$$

The coefficients in the expansion are homogeneous polynomials in a and b. See A104033 for information about these polynomials. Let $A_{a,b}(x)$ denote the ordinary generating function for this sequence of polynomials (taken with an offset of 0):

$$\begin{aligned}A_{a,b}(x) &= 1-\left(a^2-3b^2\right)x+\left(a^4-10a^2b^2x+25b^4\right)x^2\\ &\quad -\left(a^6-21a^4b^2+175a^2b^4-427b^6\right)x^3+\cdots.\end{aligned}$$

The J-fraction associated with $A_{a,b}(x)$ is given by

$$A_{a,b}(x) = \cfrac{1}{1-b_1x-\cfrac{a_1x^2}{1-b_2x-\cfrac{a_2x^2}{1-b_3x-\ddots}}}, \tag{6}$$

where

$$\begin{aligned} a_n &= 4n^2b^2\left(4n^2b^2 - a^2\right), \\ b_n &= 4\left(2n^2 - 2n + 1\right)b^2 - a^2 - b^2. \end{aligned}$$

This is a particular case of a result of Stieltjes (see [1, equation 18, p. 386] with $c = 2b$).

Proposition 1. *Let $a, b \in \mathbb{C}$ and let $m = (b-a)^2$. Let $A_{a,b}(x)$ be the J-fraction given by (6). Then the m^{th} binomial transform of $A_{a,b}(x)$ is the even part of the S-fraction*

$$\cfrac{1}{1 - \cfrac{2b(2b-a)x}{1 - \cfrac{2b(2b+a)x}{1 - \cfrac{4b(4b-a)x}{1 - \cfrac{4b(4b+a)x}{1 - \cdots}}}}}. \tag{7}$$

Proof. By (3), the even part of (7) is the J-fraction

$$\cfrac{1}{1 - (4b^2 - 2ab)x - \cfrac{2b(2b-a)2b(2b+a)x^2}{1 - \left(2b(2b+a) + 4b(4b-a)\right)x - \cfrac{4b(4b-a)(4b+a)x^2}{1 - (4b(4b+a) + 6b(6b-a))x - \cdots}}}, \tag{8}$$

where the n^{th} numerator equals

$$2nb(2nb-a)2nb(2nb+a) \quad = \quad 4n^2b^2(4n^2b^2 - a^2)$$

and the n^{th} denominator is given by

$$2nb(2nb+a) + 2(n+1)b(2(n+1)b - a) \quad = \quad 4\left(2n^2 + 2n + 1\right)b^2 - 2ab.$$

On the other hand, by applying (4) to (6), we find the m^{th} binomial transform of $A_{a,b}(x)$ has the J-fraction representation

$$\frac{1}{1-mx}A_{a,b}\left(\frac{x}{1-mx}\right) = \cfrac{1}{1 - \left(4b^2 - 2ab\right)x - \cfrac{a_1x^2}{1 - \left(b_2 + m\right)x - \cfrac{a_2x^2}{1 - \left(b_3 + m\right)x - \ddots}}}, \tag{9}$$

where the n^{th} numerator is given by

$$a_n = 4n^2b^2(4n^2b^2 - a^2),$$

and the n^{th} denominator is given by

$$b_{n+1}+m=4\left(2n^2+2n+1\right)b^2-2ab.$$

Thus (9) equals the even part of (7) as claimed. □

The function $\sin(ax)/a\cos(bx)$ is unchanged on replacing a with $-a$. Hence the associated ordinary generating function $A_{a,b}(x)$ satisfies $A_{-a,b}(x)=A_{a,b}(x)$. As an immediate consequence we have the following companion result to Proposition 1.

Proposition 2. *Let $a,b\in\mathbb{C}$ and let $M=(b+a)^2$. Let $A_{a,b}(x)$ be the J-fraction given by (6). Then the M^{th} binomial transform of $A_{a,b}(x)$ is the even part of the S-fraction*

$$\cfrac{1}{1-\cfrac{2b(2b+a)x}{1-\cfrac{2b(2b-a)x}{1-\cfrac{4b(4b+a)x}{1-\cfrac{4b(4b-a)x}{1-\cdots}}}}}. \tag{10}$$

□

Corollary 1. *The following continued fraction identity holds:*

$$\cfrac{1}{1-\cfrac{2b(2b+a)x}{1-\cfrac{2b(2b-a)x}{1-\cfrac{4b(4b+a)x}{1-\cfrac{4b(4b-a)x}{1-\cdots}}}}}=\cfrac{1}{1-4abx-\cfrac{2b(2b-a)x}{1-\cfrac{2b(2b+a)x}{1-4abx-\cfrac{4b(4b-a)x}{1-\cfrac{4b(4b+a)x}{1-4abx-\cdots}}}}},$$

or equivalently, changing a to $-a$,

$$\cfrac{1}{1-\cfrac{2b(2b-a)x}{1-\cfrac{2b(2b+a)x}{1-\cfrac{4b(4b-a)x}{1-\cfrac{4b(4b+a)x}{1-\cdots}}}}}=\cfrac{1}{1+4abx-\cfrac{2b(2b+a)x}{1-\cfrac{2b(2b-a)x}{1+4abx-\cfrac{4b(4b+a)x}{1-\cfrac{4b(4b-a)x}{1+4abx-\cdots}}}}}.$$

Proof. Comparing Proposition 1 with Proposition 2, we see that the S-fraction (10) is the $(M-m)^{th}=(a+b)^2-(a-b)^2=(4ab)^{th}$ binomial transform of the S-fraction (7). Making use of (4) we arrive at the desired result. □

Example 1. A000182 is the sequence of tangent numbers [1, 2, 16, 272, 7936, ...]. The e.g.f. is $\tan(x) = \sin(x)/\cos(x)$. Applying Proposition 1 with $a = 1$ and $b = 1$ gives the (well-known) result that the o.g.f. for the tangent numbers corresponds to the S-fraction

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{6x}{1-\cfrac{12x}{1-\cfrac{20x}{1-\cdots}}}}},$$

where the (unsigned) partial numerators are given by $n(n+1), n = 1, 2, ...$

By Proposition 2, the 4^{th} binomial transform of the o.g.f. for the tangent numbers corresponds to the S-fraction

$$\cfrac{1}{1-\cfrac{6x}{1-\cfrac{2x}{1-\cfrac{20x}{1-\cfrac{12x}{1-\cdots}}}}}.$$

Corollary 1 gives the continued fraction identity

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{6x}{1-\cfrac{12x}{1-\cfrac{20x}{1-\cdots}}}}} = \cfrac{1}{1+4x-\cfrac{6x}{1-\cfrac{2x}{1+4x-\cfrac{20x}{1-\cfrac{12x}{1+4x-\cdots}}}}}.$$

Example 2. A002439 is the sequence of Glaisher's T-numbers [1, 23, 1681, 257543, ...]. The e.g.f. is $1/2 \times \sin(2x)/\cos(3x)$. By (6), the J-fraction associated with the generating function of this sequence begins

$$J(x) = \cfrac{1}{1-23x-\cfrac{(1\times 2)(24x)^2}{1-167x-\cfrac{(5\times 7)(24x)^2}{1-455x-\ddots}}}.$$

Applying Proposition 1 with $a = 2$ and $b = 3$ we find that the binomial transform of the ordinary generating function of A002439 corresponds to the elegant S-fraction

$$\cfrac{1}{1-\cfrac{1\times 24x}{1-\cfrac{2\times 24x}{1-\cfrac{5\times 24x}{1-\cfrac{7\times 24x}{1-\cdots}}}}},$$

where the multiplicands 1,2,5,7,... in the partial numerators are the generalized pentagonal numbers A001318. Thus the o.g.f. of Glaisher's T-numbers has the continued fraction representation

$$\cfrac{1}{1+x-\cfrac{1\times 24x}{1-\cfrac{2\times 24x}{1+x-\cfrac{5\times 24x}{1-\cfrac{7\times 24x}{1+x-\cdots}}}}}.$$

By Proposition 2, the 25^{th} binomial transform of the ordinary generating function of A002439 corresponds to the S-fraction

$$\cfrac{1}{1-\cfrac{2\times 24x}{1-\cfrac{1\times 24x}{1-\cfrac{7\times 24x}{1-\cfrac{5\times 24x}{1-\cdots}}}}}.$$

where now where the multiplicands 2,1,7,5,... in the partial numerators are obtained by swapping adjacent generalized pentagonal numbers. Thus we have an alternative representation for the o.g.f. of Glaisher's T-numbers as

$$\cfrac{1}{1+25x-\cfrac{2\times 24x}{1-\cfrac{1\times 24x}{1+25x-\cfrac{7\times 24x}{1-\cfrac{5\times 24x}{1+25x-\cdots}}}}}.$$

Corollary 1 in this case leads to the continued fraction identity

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{2x}{1-\cfrac{5x}{1-\cfrac{7x}{1-\cdots}}}}} = \cfrac{1}{1+x-\cfrac{2x}{1-\cfrac{x}{1+x-\cfrac{7x}{1-\cfrac{5x}{1+x-\cdots\,.}}}}} \tag{11}$$

Remark 1. It follows from a formula of Zagier for the terms of A079144 that the S-fraction on the left-hand side of (11) is a generating function for the number of labeled interval orders on n elements (the number of $(\mathbf{2}+\mathbf{2})$-free posets).

3. Consider now the exponential generating function $\cos(ax)/\cos(bx)$, with a and b constants. The Taylor expansion of the function about $x = 0$ begins

$$\begin{aligned}\frac{\cos(ax)}{\cos(bx)} &= 1-\left(b^2-a^2\right)\frac{x^2}{2!}+\left(5b^4-16b^2a^2+a^4\right)\frac{x^4}{4!} \\ &\quad -\left(61b^6-75b^4a^2+15b^2a^4-a^6\right)\frac{x^6}{6!}+\cdots\,. \end{aligned} \tag{12}$$

The coefficients in the expansion are homogeneous polynomials in a and b. See A086646 for information about these polynomials. Let $C_{a,b}(x)$ denote the ordinary generating function for this sequence of polynomials (taken with an offset of 0):

$$\begin{aligned} C_{a,b}(x) &= 1-\left(b^2-a^2\right)x+\left(5b^4-16b^2a^2+a^4\right)x^2 \\ &\quad -\left(61b^6-75b^4a^2+15b^2a^4-a^6\right)x^3+\cdots\,. \end{aligned}$$

The J-fraction associated to $C_{a,b}(x)$, which turns out to be an S-fraction, is essentially due to Stieltjes and can be found by applying Stieltjes' expansion theorem, [2, Chapter 11, Section 53] to an addition formula satisfied by $\cos(ax)/\cos(bx)$. The result is

$$C_{a,b}(x) = \cfrac{1}{1-\cfrac{a_1x}{1-\cfrac{a_2x}{1-\cfrac{a_3x}{1-\cdots\,,}}}} \tag{13}$$

where

$$\begin{aligned} a_{2n} &= (2nb)^2, \\ a_{2n+1} &= (2n+1)^2b^2-a^2. \end{aligned}$$

The same method used in Section 2 to prove Proposition 1 can be used to establish the following result.

Proposition 3. *Let $a, b \in \mathbb{C}$. Let $C_{a,b}(x)$ be given by (13).*

(i) *Define $m = (b-a)^2$. Then the m^{th} binomial transform of $C_{a,b}(x)$ equals the S-fraction*

$$\cfrac{1}{1-\cfrac{2b(b-a)x}{1-\cfrac{2b(b+a)x}{1-\cfrac{4b(3b-a)x}{1-\cfrac{4b(3b+a)x}{1-\cfrac{6b(5b-a)x}{1-\cfrac{6b(5b+a)x}{1-\cdots}}}}}}}. \tag{14}$$

(ii) *Define $M = (b+a)^2$. Then the M^{th} binomial transform of $C_{a,b}(x)$ equals the S-fraction*

$$\cfrac{1}{1-\cfrac{2b(b+a)x}{1-\cfrac{2b(b-a)x}{1-\cfrac{4b(3b+a)x}{1-\cfrac{4b(3b-a)x}{1-\cfrac{6b(5b+a)x}{1-\cfrac{6b(5b-a)x}{1-\cdots}}}}}}}. \tag{15}$$

□

Since (15) is the $(M-m)^{th} = (4ab)^{th}$ binomial transform of (14) then by (4) we have the following identity:

Corollary 2.

$$\cfrac{1}{1-\cfrac{2b(b+a)x}{1-\cfrac{2b(b-a)x}{1-\cfrac{4b(3b+a)x}{1-\cfrac{4b(3b-a)x}{1-\cdots}}}}} = \cfrac{1}{1-4abx-\cfrac{2b(b-a)x}{1-\cfrac{2b(b+a)x}{1-4abx-\cfrac{4b(3b-a)x}{1-\cfrac{4b(3b+a)x}{1-4abx-\cdots}}}}}.$$

□

Remark 2. The continued fraction identities in Corollaries 1 and 2 appear to be particular cases of a more general identity

$$\cfrac{1}{1-\cfrac{(\lambda_1+a)x}{1-\cfrac{(\lambda_1-a)x}{1-\cfrac{2(\lambda_2+a)x}{1-\cfrac{2(\lambda_2-a)x}{1-\cdots}}}}} = \cfrac{1}{1-2ax-\cfrac{(\lambda_1-a)x}{1-\cfrac{(\lambda_1+a)x}{1-2ax-\cfrac{2(\lambda_2-a)x}{1-\cfrac{2(\lambda_2+a)x}{1-2ax-\cdots}}}}}. \tag{16}$$

where $\{\lambda_n\}_{n\geq 1}$ is an arbitrary sequence. Indeed, the $2n$-th approximants of the left and right sides of (16) seem to be identically equal.

References

[1] T. J. Stieltjes, Sur quelques intégrales définies et leur développement en fraction continues, *Oeuvres*, vol. 2, 378-394.
- available online at https://archive.org/details/oeuvresthomasja02stierich

[2] H. S. Wall, *Analytic Theory of Continued Fractions.* Reprinted by AMS Chelsea Publishing, 2000 .